A note on extensions of algebraic and formal groups, V

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Abstract. We give an explicit description of homomorphisms and extensions of group schemes and formal groups, related to deformations connecting the multiplicative group and the additive group.

Introduction

Throughout the article, p denotes a prime number. Let W (resp. $\widehat{W}$) denote the group scheme (resp. the formal group scheme) of Witt vectors over $\mathbb{Z}$, and let $\mathbb{G}_m$ (resp. $\widehat{\mathbb{G}}_m$) denote the multiplicative group scheme (resp. the multiplicative formal group scheme) over $\mathbb{Z}$. Let F be the Frobenius endomorphism of W or of $\widehat{W}$ (for the definition see 1.2).

In the previous paper [13], we gave an explicit description of the groups $\mathrm{Ext}_A(\mathcal{G}^{(\lambda)}, \mathbb{G}_{m,A})$ and $\mathrm{Ext}_A(\widehat{\mathcal{G}}^{(\lambda)}, \widehat{\mathbb{G}}_{m,A})$ when A is a $\mathbb{Z}_{(p)}$ -algebra. Here $\lambda \in A$ and $\mathcal{G}^{(\lambda)} = \mathrm{Spec} A[T, 1/(1 + \lambda T)]$ is a group scheme giving a deformation of $\mathbb{G}_a$ to $\mathbb{G}_m$ (for the definition, see 3.1). More precisely, we constructed isomorphisms

$$\begin{aligned} \mathrm{Ker}[F - [\lambda^{p-1}] : W(A) \rightarrow W(A)] &\simeq \mathrm{Hom}(\widehat{\mathcal{G}}^{(\lambda)}, \widehat{\mathbb{G}}_{m,A}), \\ \mathrm{Coker}[F - [\lambda^{p-1}] : W(A) \rightarrow W(A)] &\simeq H_0^2(\widehat{\mathcal{G}}^{(\lambda)}, \widehat{\mathbb{G}}_{m,A}), \end{aligned}$$

using deformations of the Artin-Hasse exponential series. If λ is nilpotent, we have also isomorphisms

$$\begin{aligned} \mathrm{Ker}[F - [\lambda^{p-1}] : \widehat{W}(A) \rightarrow \widehat{W}(A)] &\simeq \mathrm{Hom}(\mathcal{G}^{(\lambda)}, \mathbb{G}_{m,A}), \\ \mathrm{Coker}[F - [\lambda^{p-1}] : \widehat{W}(A) \rightarrow \widehat{W}(A)] &\simeq H_0^2(\mathcal{G}^{(\lambda)}, \mathbb{G}_{m,A}). \end{aligned}$$

Our aim of this article is to generalize the isomorphisms to that for $\widehat{\mathcal{G}}^{(\mu)}$ instead of $\widehat{\mathbb{G}}_{m,A}$. Precisely, our result is as follows.

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THEOREM. *Let A be a $\mathbb{Z}_{(p)}$ -algebra and $\lambda, \mu \in A$. Then there exist isomorphisms:*

$$\begin{aligned} \operatorname{Ker}[F^{(\mu)} - [\lambda^{p-1}] : W^{(\mu)}(A) \rightarrow W^{(\mu)}(A)] &\simeq \operatorname{Hom}(\widehat{\mathcal{G}}^{(\lambda)}, \widehat{\mathcal{G}}^{(\mu)}), \\ \operatorname{Coker}[F^{(\mu)} - [\lambda^{p-1}] : W^{(\mu)}(A) \rightarrow W^{(\mu)}(A)] &\simeq H_0^2(\widehat{\mathcal{G}}^{(\lambda)}, \widehat{\mathcal{G}}^{(\mu)}). \end{aligned}$$

Moreover, if λ is nilpotent, there exist isomorphism:

$$\begin{aligned} \operatorname{Ker}[F^{(\mu)} - [\lambda^{p-1}] : \widehat{W}^{(\mu)}(A) \rightarrow \widehat{W}^{(\mu)}(A)] &\simeq \operatorname{Hom}(\mathcal{G}^{(\lambda)}, \mathcal{G}^{(\mu)}), \\ \operatorname{Coker}[F^{(\mu)} - [\lambda^{p-1}] : \widehat{W}^{(\mu)}(A) \rightarrow \widehat{W}^{(\mu)}(A)] &\simeq H_0^2(\mathcal{G}^{(\lambda)}, \mathcal{G}^{(\mu)}). \end{aligned}$$

(Theorem 3.5.1 and Remark 3.12. For the definition of $W^{(\mu)}(A)$ and $F^{(\mu)}$, see the Section 1.)

It is crucial to define variants of Witt vectors and to construct deformations of the Artin-Hasse exponential series for an explicit description of the isomorphisms.

Now we explain the contents of the article.

The first two sections are devoted to prepare necessary tools to state the main theorem. In the Section 1, we define variants of Witt vectors $W^{(M)}(A)$ for a $\mathbb{Z}[M]$ -algebra A , paraphrasing the classical theory of Witt vectors. $W^{(M)}(A)$ is interpreted as the A -valued points of a group scheme $W^{(M)}$ over $\mathbb{Z}[M]$. At the end of the section, we construct an exact sequence of group schemes over $\mathbb{Z}[M]$

$$0 \rightarrow W^{(M)} \rightarrow \prod_{B/A} W_B \rightarrow W_A \rightarrow 0,$$

where $A = \mathbb{Z}[M]$ and $B = A[t]/(t^2 - Mt)$.

In the Section 2, we define variants of the Artin-Hasse exponential series

$$E_p^{(M)}(U, A; T) \in \mathbb{Z}_{(p)}[U_0, U_1, U_2, \dots, A, M][[T]],$$

modifying the power series

$$E_p(U, A; T) \in \mathbb{Z}_{(p)}[U_0, U_1, U_2, \dots, A][[T]]$$

presented in [13]. The definition of $E_p^{(M)}(U, A; T)$ is parallel to that of $W^{(M)}$ in a sense. It would be worthy of note that we have a formula

$$E_p^{(\Lambda^p)}(p[1]_A, A; T) = \frac{(1 + \Lambda T)^p - 1}{\Lambda^p}$$

(Example 3.9) because $\{(1 + \lambda T)^p - 1\}/\lambda^p$ defines the isogeny Ψ in the Kummer-Artin-Schreier theory:

$$0 \rightarrow \mathbb{Z}/p\mathbb{Z} \rightarrow \mathcal{G}^{(\lambda)} \xrightarrow{\Psi} \mathcal{G}^{(\lambda^p)} \rightarrow 0$$

(cf. [15]).

In the Section 3, we prove the main result, after reviewing the Hochschild cohomology in our case. The theorem can be reduced to the main result of [13] thanks to an exact sequence of formal groups

$$0 \rightarrow \widehat{\mathcal{G}}_A^{(M)} \rightarrow \left(\prod_{B/A} \widehat{\mathbb{G}}_{m,B} \right) \rightarrow \widehat{\mathbb{G}}_{m,A} \rightarrow 0,$$

where A is a $\mathbb{Z}[M]$ -algebra and $B = A[t]/(t^2 - Mt)$.

The last section is devoted to verify a compatibility of the main theorem with the Cartier theory on p -typical curves of formal groups. After a short review on the Dieudonné-Cartier algebra D_A and the Cartier module of p -typical curves, we prove that $W^{(M)}(A)$ is isomorphic to the Cartier module of the formal group $\widehat{\mathcal{G}}^{(M)}$, giving an explicit description of p -typical curves of $\widehat{\mathcal{G}}^{(M)}$ (Theorem 4.15.1 and Proposition 4.19). It would be remarkable that the D_A -module $W^{(M)}(A)$ has a free resolution, written in a simple form (Theorem 4.4). We conclude the article, interpreting the main result mentioned above in the framework of the Cartier theory (Theorem 4.24.1 and Theorem 4.40.1). We have to introduce a formal power series

$$G_p^{(A)}(\mathbf{Z}, \mathbf{U}; \mathbf{T}) \in \mathbb{Z}_{(p)}[A, U_0, U_1, U_2, \dots, Z_0, Z_1, Z_2, \dots][[T_0, T_1, T_2, \dots]]$$

to proceed with the argument for Theorem 4.40.1.

In this article we have established formalisms necessary to describe systematically the theory unifying the Kummer and Artin-Schreier-Witt theories, which permits us to explain a little more pleasantly the argument developed in [14]. Detailed accounts will appear elsewhere as a sequel to this work.

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Notation

Throughout the paper, p denotes a prime integer, $\mathbb{Z}_{(p)}$ the localization of $\mathbb{Z}$ at the prime ideal (p) , and A a $\mathbb{Z}_{(p)}$ -algebra. All rings are commutative with a unit element 1, unless otherwise stated.

- $\mathbb{G}_{a,A}$: the additive group scheme over A
 $\mathbb{G}_{m,A}$: the multiplicative group scheme over A
 $W_{n,A}$: the group scheme of Witt vectors of length n over A
 W_A : the group scheme of Witt vectors over A
 $\widehat{\mathbb{G}}_{a,A}$: the additive formal group scheme over A
 $\widehat{\mathbb{G}}_{m,A}$: the multiplicative formal group scheme over A
 $\widehat{W}_{n,A}$: the formal group scheme of Witt vectors of length n over A
 $\widehat{W}_A$: the formal group scheme of Witt vectors over A
 F : the Frobenius endomorphism of W_A
 V : the Verschiebung endomorphism of W_A
 $[a]$: the Teichmüller lifting $(a, 0, 0, \dots) \in W(A)$ of $a \in A$

$H_0^2(G, H)$ denotes the Hochschild cohomology group consisting of symmetric 2-cocycles of G with coefficients in H for group schemes or formal group schemes G and H .

For a commutative ring B , $B^\times$ denotes the multiplicative group $\mathbb{G}_m(B)$.

List of formal power series

- $E_p(T)$: the Artin-Hasse exponential series, recalled in 2.1
 $E_p(U, \Lambda; T)$: recalled in 2.2
 $E_p(\mathbf{U}, \Lambda; T)$: recalled in 2.2
 $E_p^{(M)}(\mathbf{U}, \Lambda; T)$: defined in 2.3
 $F_p(\mathbf{U}, \Lambda; X, Y)$: recalled in 2.5
 $F_p^{(M)}(\mathbf{U}, \Lambda; X, Y)$: defined in 2.7
 $E_p(\mathbf{U}; \mathbf{T})$: recalled in 4.14
 $E_p^{(M)}(\mathbf{U}; \mathbf{T})$: defined in 4.17
 $G_p^{(\Lambda)}(\mathbf{Z}, \mathbf{U}; \mathbf{T})$: defined in 4.29
 $E_p^{(\Lambda, M)}(\mathbf{Z}, \mathbf{U}, \mathbf{V}; \mathbf{T})$: defined in 4.35

Contents

1. Around Witt vectors
2. Around the Artin-Hasse exponential series
3. Statement and proof of the theorem
4. Around the Cartier theory

1. Around Witt vectors

We start with reviewing necessary facts on Witt vectors. For details, see [2, Chap. V] or [4, Chap. III].

1.1. For each $r \geq 0$, we denote by $\Phi_r(\mathbf{T}) = \Phi_r(T_0, T_1, \dots, T_r)$ the so-called Witt polynomial

$$\Phi_r(\mathbf{T}) = T_0^{p^r} + pT_1^{p^{r-1}} + \dots + p^r T_r$$

in $\mathbb{Z}[\mathbf{T}] = \mathbb{Z}[T_0, T_1, \dots]$. We define polynomials

$$S_r(\mathbf{X}, \mathbf{Y}) = S_r(X_0, \dots, X_r, Y_0, \dots, Y_r)$$

and

$$P_r(\mathbf{X}, \mathbf{Y}) = P_r(X_0, \dots, X_r, Y_0, \dots, Y_r)$$

in $\mathbb{Z}[\mathbf{X}, \mathbf{Y}] = \mathbb{Z}[X_0, X_1, \dots, Y_0, Y_1, \dots]$ inductively by

$$\Phi_r(S_0(\mathbf{X}, \mathbf{Y}), S_1(\mathbf{X}, \mathbf{Y}), \dots, S_r(\mathbf{X}, \mathbf{Y})) = \Phi_r(\mathbf{X}) + \Phi_r(\mathbf{Y})$$

and

$$\Phi_r(P_0(\mathbf{X}, \mathbf{Y}), P_1(\mathbf{X}, \mathbf{Y}), \dots, P_r(\mathbf{X}, \mathbf{Y})) = \Phi_r(\mathbf{X})\Phi_r(\mathbf{Y}).$$

Put

$$\begin{aligned} S(\mathbf{X}, \mathbf{Y}) &= (S_0(\mathbf{X}, \mathbf{Y}), S_1(\mathbf{X}, \mathbf{Y}), S_2(\mathbf{X}, \mathbf{Y}), \dots), \\ P(\mathbf{X}, \mathbf{Y}) &= (P_0(\mathbf{X}, \mathbf{Y}), P_1(\mathbf{X}, \mathbf{Y}), P_2(\mathbf{X}, \mathbf{Y}), \dots). \end{aligned}$$

Then as is well-known, the ring structure of the scheme of Witt vectors

$$W_{\mathbb{Z}} = \operatorname{Spec} \mathbb{Z}[T_0, T_1, T_2, \dots]$$

is given by the addition

$$\begin{aligned} \mathbf{T} &= (T_0, T_1, T_2, \dots) \mapsto S(\mathbf{T} \otimes 1, 1 \otimes \mathbf{T}) \\ &= (S_0(\mathbf{T} \otimes 1, 1 \otimes \mathbf{T}), S_1(\mathbf{T} \otimes 1, 1 \otimes \mathbf{T}), S_2(\mathbf{T} \otimes 1, 1 \otimes \mathbf{T}), \dots) \end{aligned}$$

and the multiplication

$$\begin{aligned} \mathbf{T} &= (T_0, T_1, T_2, \dots) \mapsto P(\mathbf{T} \otimes 1, 1 \otimes \mathbf{T}) \\ &= (P_0(\mathbf{T} \otimes 1, 1 \otimes \mathbf{T}), P_1(\mathbf{T} \otimes 1, 1 \otimes \mathbf{T}), P_2(\mathbf{T} \otimes 1, 1 \otimes \mathbf{T}), \dots). \end{aligned}$$

Define a morphism of schemes

$$\Phi : W = \operatorname{Spec} \mathbb{Z}[T_0, T_1, T_2, \dots] \rightarrow \mathbb{A}^{\mathbb{N}} = \operatorname{Spec} \mathbb{Z}[\Phi_0, \Phi_1, \Phi_2, \dots]$$

by

$$(\Phi_0, \Phi_1, \Phi_2, \dots) \mapsto (\Phi_0(\mathbf{T}), \Phi_1(\mathbf{T}), \Phi_2(\mathbf{T}), \dots).$$

Then Φ is a homomorphism of ring schemes. Let A be a ring. If p is invertible in A , $\Phi : W(A) \rightarrow A^{\mathbb{N}}$ is bijective. Hence, if the homomorphism by p is injective on A , $\Phi : W(A) \rightarrow A^{\mathbb{N}}$ is injective. For $\mathbf{a} \in W(A)$, $\Phi(\mathbf{a}) = (\Phi_0(\mathbf{a}), \Phi_1(\mathbf{a}), \Phi_2(\mathbf{a}), \dots)$ is called the phantom of $\mathbf{a}$.

We denote by $\widehat{W}_{\mathbb{Z}}$ the formal completion of $W_{\mathbb{Z}}$ along the zero section. $\widehat{W}_{\mathbb{Z}}$ is considered as a subfunctor of $W_{\mathbb{Z}}$. Indeed, if A is a ring,

$$\widehat{W}(A) = \left\{ (a_0, a_1, a_2, \dots) \in W_n(A); \begin{array}{l} a_i \text{ is nilpotent for all } i \text{ and} \\ a_i = 0 \text{ for all but a finite number of } i \end{array} \right\}.$$

1.2. The Verschiebung homomorphism $V : W_{\mathbb{Z}} \rightarrow W_{\mathbb{Z}}$ is defined by

$$T_0 \mapsto 0, T_1 \mapsto T_0, T_2 \mapsto T_1, \dots : \mathbb{Z}[T_0, T_1, T_2, \dots] \rightarrow \mathbb{Z}[T_0, T_1, T_2, \dots].$$

Define now polynomials

$$F_r(\mathbf{T}) = F_r(T_0, \dots, T_r, T_{r+1}) \in \mathbb{Q}[T_0, \dots, T_r, T_{r+1}]$$

inductively by

$$\Phi_r(F_0(\mathbf{T}), \dots, F_r(\mathbf{T})) = \Phi_{r+1}(T_0, \dots, T_r, T_{r+1})$$

for $r \geq 0$. Then

$$F_r(\mathbf{T}) = F_r(T_0, \dots, T_r, T_{r+1}) \in \mathbb{Z}[T_0, \dots, T_r, T_{r+1}]$$

for each $r \geq 0$. We denote by $F : W_{\mathbb{Z}} \rightarrow W_{\mathbb{Z}}$ the morphism defined by

$$T_0 \mapsto F_0(\mathbf{T}), T_1 \mapsto F_1(\mathbf{T}), T_2 \mapsto F_2(\mathbf{T}), \dots : \mathbb{Z}[T_0, T_1, T_2, \dots] \rightarrow \mathbb{Z}[T_0, T_1, T_2, \dots].$$

Then it is verified without difficulty that F is a homomorphism of ring schemes. It is obvious that $\widehat{W}_{\mathbb{Z}}$ is stable under F (cf. [1, Ch. 9.1.3] or [3, Ch. O.1.3]). It is readily seen that $F_r(\mathbf{T}) \equiv T_r^p \pmod{p}$ for each $r \geq 0$. Therefore, if A is an $\mathbb{F}_p$ -algebra, $F : W_A \rightarrow W_A$ is nothing but the usual Frobenius endomorphism.

Let A be a ring. Then we can verify without difficulty that:

- (1) $FV = p$;
- (2) $V(F(\mathbf{a})\mathbf{b}) = \mathbf{a}V(\mathbf{b})$ for $\mathbf{a}, \mathbf{b} \in W(A)$.

1.3. Let A be a ring and $a \in A$. We denote the Witt vector $(a, 0, 0, \dots)$ with components in A by $[a]$. $[a]$ is called the Teichmüller lifting of a . It is readily seen:

- (1) $[a][b] = [ab]$;
- (2) $F[a] = [a^p]$;
- (3) $(a_0, a_1, a_2, \dots) = \sum_{k=0}^{\infty} V^k[a_k]$.

1.4. Now we introduce variants of the Witt calculus. Define

$$\Phi_r^{(M)}(\mathbf{T}) = \Phi_r^{(M)}(T_0, \dots, T_r) \in \mathbb{Z}[M][T_0, \dots, T_r]$$

by

$$\begin{aligned} \Phi_r^{(M)}(\mathbf{T}) &= \frac{1}{M} \Phi_r(MT_0, \dots, MT_r) \\ &= M^{p^r-1} T_0^{p^r} + p M^{p^{r-1}-1} T_1^{p^{r-1}} + \dots + p^{r-1} M^{p-1} T_{r-1} + p^r T_r. \end{aligned}$$

Furthermore, we define

$$\begin{aligned} S_r^{(M)}(\mathbf{X}, \mathbf{Y}) &= S_r^{(M)}(X_0, \dots, X_r, Y_0, \dots, Y_r) \in \mathbb{Z}[M][X_0, \dots, X_r, Y_0, \dots, Y_r], \\ P_r^{(M)}(\mathbf{X}, \mathbf{Y}) &= P_r^{(M)}(X_0, \dots, X_r, Y_0, \dots, Y_r) \in \mathbb{Z}[M][X_0, \dots, X_r, Y_0, \dots, Y_r], \\ F_r^{(M)}(\mathbf{T}) &= F_r^{(M)}(T_0, \dots, T_r, T_{r+1}) \in \mathbb{Z}[M][T_0, \dots, T_r, T_{r+1}] \end{aligned}$$

by

$$\begin{aligned} S_r^{(M)}(X_0, \dots, X_r, Y_0, \dots, Y_r) &= \frac{1}{M} S_r(MX_0, \dots, MX_r, MY_0, \dots, MY_r), \\ P_r^{(M)}(X_0, \dots, X_r, Y_0, \dots, Y_r) &= \frac{1}{M} P_r(X_0, \dots, X_r, MY_0, \dots, MY_r), \\ F_r^{(M)}(\mathbf{T}) &= F_r^{(M)}(T_0, \dots, T_r, T_{r+1}) = \frac{1}{M} F_r(MT_0, \dots, MT_r, MT_{r+1}), \end{aligned}$$

respectively. For example, we have

$$\begin{aligned} S_0^{(M)}(\mathbf{X}, \mathbf{Y}) &= X_0 + Y_0, \\ S_1^{(M)}(\mathbf{X}, \mathbf{Y}) &= X_1 + Y_1 + M^{p-1} \frac{X_0^p + Y_0^p - (X_0 + Y_0)^p}{p}, \\ P_0^{(M)}(\mathbf{X}, \mathbf{Y}) &= X_0 Y_0, \\ P_1^{(M)}(\mathbf{X}, \mathbf{Y}) &= X_0^p Y_1 + M^{p-1} X_1 Y_0^p + p X_1 Y_1, \\ F_0^{(M)}(\mathbf{T}) &= M^{p-1} T_0^p + p T_1. \end{aligned}$$

We can verify easily that

- (1) $\Phi_r^{(M)}(S_0^{(M)}(\mathbf{X}, \mathbf{Y}), S_1^{(M)}(\mathbf{X}, \mathbf{Y}), \dots, S_r^{(M)}(\mathbf{X}, \mathbf{Y})) = \Phi_r^{(M)}(\mathbf{X}) + \Phi_r^{(M)}(\mathbf{Y});$
- (2) $\Phi_r^{(M)}(P_0^{(M)}(\mathbf{X}, \mathbf{Y}), P_1^{(M)}(\mathbf{X}, \mathbf{Y}), \dots, P_r^{(M)}(\mathbf{X}, \mathbf{Y})) = \Phi_r(\mathbf{X}) \Phi_r^{(M)}(\mathbf{Y});$
- (3) $\Phi_r^{(M)}(F_0^{(M)}(\mathbf{T}), F_1^{(M)}(\mathbf{T}), \dots, F_r^{(M)}(\mathbf{T})) = \Phi_{r+1}^{(M)}(\mathbf{T});$
- (4) $F_r^{(M)}(\mathbf{T}) \equiv M^{p-1} T_r^p \pmod{p}.$

Put

$$\begin{aligned} S^{(M)}(\mathbf{X}, \mathbf{Y}) &= (S_0^{(M)}(\mathbf{X}, \mathbf{Y}), S_1^{(M)}(\mathbf{X}, \mathbf{Y}), S_2^{(M)}(\mathbf{X}, \mathbf{Y}), \dots), \\ P^{(M)}(\mathbf{X}, \mathbf{Y}) &= (P_0^{(M)}(\mathbf{X}, \mathbf{Y}), P_1^{(M)}(\mathbf{X}, \mathbf{Y}), P_2^{(M)}(\mathbf{X}, \mathbf{Y}), \dots), \\ F^{(M)}(\mathbf{T}) &= (F_0^{(M)}(\mathbf{T}), F_1^{(M)}(\mathbf{T}), F_2^{(M)}(\mathbf{T}), \dots). \end{aligned}$$

Then it follows from (1) (2) (3) that

- (5) $S^{(M)}(S^{(M)}(\mathbf{X}, \mathbf{Y}), \mathbf{Z}) = S^{(M)}(\mathbf{X}, S^{(M)}(\mathbf{Y}, \mathbf{Z}));$
- (6) $P^{(M)}(P^{(M)}(\mathbf{X}, \mathbf{Y}), \mathbf{Z}) = P^{(M)}(\mathbf{X}, P^{(M)}(\mathbf{Y}, \mathbf{Z}));$
- (7) $F^{(M)}(S^{(M)}(\mathbf{X}, \mathbf{Y})) = S^{(M)}(F^{(M)}(\mathbf{X}), F^{(M)}(\mathbf{Y})).$

Furthermore, noting that $\Phi_r^{(M)}(\mathbf{T}) \equiv p^r T_r \pmod{M^{p-1}}$, we can conclude that

- (8) $S_r^{(M)}(\mathbf{X}, \mathbf{Y}) \equiv X_r + Y_r \pmod{M^{p-1}};$
- (9) $P_r^{(M)}(\mathbf{X}, \mathbf{Y}) \equiv \Phi_r(\mathbf{X})Y_r \pmod{M^{p-1}};$
- (10) $F_r^{(M)}(\mathbf{T}) \equiv pT_{r+1} \pmod{M^{p-1}}.$

1.5. We define a commutative group scheme $W^{(M)}$ over $\mathbb{Z}[M]$ by

$$W^{(M)} = \operatorname{Spec} \mathbb{Z}[M][T_0, T_1, T_2, \dots]$$

with the addition

$$\begin{aligned} \mathbf{T} &= (T_0, T_1, T_2, \dots) \\ &\mapsto (S_0^{(M)}(\mathbf{T} \otimes 1, 1 \otimes \mathbf{T}), S_1^{(M)}(\mathbf{T} \otimes 1, 1 \otimes \mathbf{T}), S_2^{(M)}(\mathbf{T} \otimes 1, 1 \otimes \mathbf{T}), \dots). \end{aligned}$$

The unit is given by

$$\mathbf{T} = (T_0, T_1, T_2, \dots) \mapsto \mathbf{0} = (0, 0, 0, \dots),$$

and, if $p > 2$, the inverse is given by

$$\mathbf{T} = (T_0, T_1, T_2, \dots) \mapsto -\mathbf{T} = (-T_0, -T_1, -T_2, \dots).$$

Furthermore, we define a morphism $\cdot : W_{\mathbb{Z}[M]} \times_{\mathbb{Z}[M]} W^{(M)} \rightarrow W^{(M)}$ by

$$\begin{aligned} \mathbf{T} &= (T_0, T_1, T_2, \dots) \\ &\mapsto (P_0^{(M)}(\mathbf{U} \otimes 1, 1 \otimes \mathbf{T}), P_1^{(M)}(\mathbf{U} \otimes 1, 1 \otimes \mathbf{T}), P_2^{(M)}(\mathbf{U} \otimes 1, 1 \otimes \mathbf{T}), \dots). \end{aligned}$$

Then $W^{(M)}$ is a $W_{\mathbb{Z}[M]}$ -module.

Define a morphism of $\mathbb{Z}[M]$ -schemes

$$\Phi^{(M)} : W^{(M)} = \operatorname{Spec} \mathbb{Z}[M][T_0, T_1, T_2, \dots] \rightarrow \mathbb{G}_{a, \mathbb{Z}[M]}^{\mathbb{N}} = \operatorname{Spec} \mathbb{Z}[M][\Phi_0, \Phi_1, \Phi_2, \dots]$$

by

$$(\Phi_0, \Phi_1, \Phi_2, \dots) \mapsto (\Phi_0^{(M)}(T), \Phi_1^{(M)}(T), \Phi_2^{(M)}(T), \dots).$$

We have a commutative diagram of $\mathbb{Z}[M]$ -schemes

$$\begin{array}{ccc} W_{\mathbb{Z}[M]} \times_{\mathbb{Z}[M]} W^{(M)} & \longrightarrow & W^{(M)} \\ \Phi \times \Phi^{(M)} \downarrow & & \downarrow \Phi^{(M)} \\ \mathbb{A}_{\mathbb{Z}[M]}^{\mathbb{N}} \times_{\mathbb{Z}[M]} \mathbb{G}_{a, \mathbb{Z}[M]}^{\mathbb{N}} & \longrightarrow & \mathbb{G}_{a, \mathbb{Z}[M]}^{\mathbb{N}}, \end{array}$$

where $\mathbb{A}^{\mathbb{N}} \times_{\mathbb{Z}} \mathbb{G}_a^{\mathbb{N}} \rightarrow \mathbb{G}_a^{\mathbb{N}}$ is the usual multiplication.

REMARK 1.6. Let A be a $\mathbb{Z}[M]$ -algebra. If p is invertible in A , $\Phi : W^{(M)}(A) \rightarrow A^{\mathbb{N}}$ is bijective; if p is a non-zero divisor of A , $\Phi : W^{(M)}(A) \rightarrow A^{\mathbb{N}}$ is injective.

REMARK 1.7. Let A be a $\mathbb{Z}[M]$ -algebra. Let $a, b \in W^{(M)}(A)$ and $c \in W(A)$. We will denote sometimes $a + b$, $c \cdot a$ by $a + {}^{(M)}b$, $c \cdot {}^{(M)}a$, respectively, to avoid confusion.

1.8. Let A be a $\mathbb{Z}[M]$ -algebra, and let μ denote the image of M in A . We denote sometimes $W^{(M)} \otimes_{\mathbb{Z}[M]} A$ by $W^{(\mu)}$. We define also

$$\begin{aligned} S_r^{(\mu)}(\mathbf{X}, \mathbf{Y}) &= S_r^{(\mu)}(X_0, \dots, X_r, Y_0, \dots, Y_r) \in A[X_0, \dots, X_r, Y_0, \dots, Y_r], \\ P_r^{(\mu)}(\mathbf{X}, \mathbf{Y}) &= P_r^{(\mu)}(X_0, \dots, X_r, Y_0, \dots, Y_r) \in A[X_0, \dots, X_r, Y_0, \dots, Y_r], \\ F_r^{(\mu)}(T) &= F_r^{(\mu)}(T_0, \dots, X_r, T_{r+1}) \in A[T_0, \dots, T_r, T_{r+1}] \end{aligned}$$

by substituting M by μ in $S_r^{(M)}(\mathbf{X}, \mathbf{Y})$, $P_r^{(M)}(\mathbf{X}, \mathbf{Y})$, $F_r^{(M)}(T)$, respectively.

EXAMPLE 1.9.1. It is clear that

$$S_r^{(1)}(\mathbf{X}, \mathbf{Y}) = S_r(\mathbf{X}, \mathbf{Y}), \quad P_r^{(1)}(\mathbf{X}, \mathbf{Y}) = P_r(\mathbf{X}, \mathbf{Y}), \quad F_r^{(1)}(T) = F_r(T),$$

and therefore $W_{\mathbb{Z}}^{(1)}$ is nothing but the scheme of Witt vectors $W_{\mathbb{Z}}$.

EXAMPLE 1.9.2. It follows from 1.4 (6) (7) (8) that

$$S_r^{(0)}(\mathbf{X}, \mathbf{Y}) = X_r + Y_r, \quad P_r^{(0)}(\mathbf{X}, \mathbf{Y}) = \Phi_r(\mathbf{X})Y_r, \quad F_r^{(0)}(T) = pT_{r+1}.$$

Hence the group scheme $W_{\mathbb{Z}}^{(0)}$ is isomorphic to the direct product $\mathbb{G}_{a, \mathbb{Z}}^{\mathbb{N}}$.

1.10. We define homomorphisms $V : W^{(M)} \rightarrow W^{(M)}$ and $F^{(M)} : W^{(M)} \rightarrow W^{(M)}$ by

$$(T_0, T_1, T_2, \dots) \mapsto (0, T_0, T_1, \dots) : \mathbb{Z}[M][T_0, T_1, T_2, \dots] \rightarrow \mathbb{Z}[M][T_0, T_1, T_2, \dots]$$

and

$$(T_0, T_1, T_2, \dots) \mapsto (F_0^{(M)}(\mathbf{T}), F_1^{(M)}(\mathbf{T}), F_2^{(M)}(\mathbf{T}), \dots) : \\ \mathbb{Z}[M][T_0, T_1, T_2, \dots] \rightarrow \mathbb{Z}[M][T_0, T_1, T_2, \dots],$$

respectively.

PROPOSITION 1.11. *Let A be a $\mathbb{Z}[M]$ -algebra, $\mathbf{a} \in W^{(M)}(A)$ and $\mathbf{c} \in W(A)$.*

Then:

- (1) $F^{(M)} \circ V = p$;
- (2) $F^{(M)}(\mathbf{c} \cdot \mathbf{a}) = F(\mathbf{c}) \cdot F^{(M)}(\mathbf{a})$;
- (3) $V(F(\mathbf{c}) \cdot \mathbf{a}) = \mathbf{c} \cdot V(\mathbf{a})$;
- (4) $V(\mathbf{c} \cdot F^{(M)}(\mathbf{a})) = V(\mathbf{c}) \cdot \mathbf{a}$.

PROOF. To verify the assertion, we may assume

$$A = \mathbb{Z}[M][U_0, U_1, U_2, \dots, T_0, T_1, T_2, \dots], \quad \mathbf{c} = \mathbf{U} = (U_0, U_1, U_2, \dots), \\ \mathbf{a} = \mathbf{T} = (T_0, T_1, T_2, \dots).$$

- (1) Note that, for $r \geq 0$,

$$\Phi_r^{(M)}(F^{(M)}(V(\mathbf{T}))) = \Phi_{r+1}^{(M)}(V(\mathbf{T})) = p\Phi_r^{(M)}(\mathbf{T}).$$

- (2) Note that, for $r \geq 0$,

$$\Phi_r^{(M)}(F^{(M)}(\mathbf{U} \cdot \mathbf{T})) = \Phi_{r+1}^{(M)}(\mathbf{U} \cdot \mathbf{T}) = \Phi_{r+1}(\mathbf{U})\Phi_{r+1}^{(M)}(\mathbf{T})$$

and

$$\Phi_r^{(M)}(F(\mathbf{U}) \cdot F^{(M)}(\mathbf{T})) = \Phi_r(F(\mathbf{U}))\Phi_r(F^{(M)}(\mathbf{T})) = \Phi_{r+1}(\mathbf{U})\Phi_{r+1}^{(M)}(\mathbf{T}).$$

- (3) Note that, if $r > 0$,

$$\Phi_r^{(M)}(V(F(\mathbf{U}) \cdot \mathbf{T})) = p\Phi_{r-1}^{(M)}(F(\mathbf{U}) \cdot \mathbf{T}) \\ = p\Phi_{r-1}(F(\mathbf{U}))\Phi_{r-1}^{(M)}(\mathbf{T}) = p\Phi_r(\mathbf{U})\Phi_{r-1}^{(M)}(\mathbf{T})$$

and

$$\Phi_r^{(M)}(\mathbf{U} \cdot V(\mathbf{T})) = \Phi_r(\mathbf{U})\Phi_r^{(M)}(V(\mathbf{T})) = \Phi_r(\mathbf{U})p\Phi_{r-1}^{(M)}(\mathbf{T}).$$

- (4) Note that, if $r > 0$,

$$\Phi_r(V(\mathbf{U} \cdot F^{(M)}(\mathbf{T}))) = p\Phi_{r-1}^{(M)}(\mathbf{U} \cdot F^{(M)}(\mathbf{T})) \\ = p\Phi_{r-1}(\mathbf{U})\Phi_{r-1}^{(M)}(F^{(M)}(\mathbf{T})) = p\Phi_{r-1}(\mathbf{U})\Phi_r^{(M)}(\mathbf{T})$$

and

$$\Phi_r^{(M)}(V(\mathbf{U}) \cdot \mathbf{T}) = \Phi_r(V(\mathbf{U}))\Phi_r^{(M)}(\mathbf{T}) = p\Phi_{r-1}(\mathbf{U})\Phi_r^{(M)}(\mathbf{T}).$$

1.12. Let A be a $\mathbb{Z}[M]$ -algebra. By 1.11 (3) $\text{Im } V^k = \text{Im}[V^k : W^{(M)}(A) \rightarrow W^{(M)}(A)]$ is a sub- $W(A)$ -module of $W^{(M)}(A)$. $W^{(M)}(A)$ has a structure of a topological group with the V -adic topology defined by the family of subgroups $\{\text{Im } V^k\}_{k>0}$. $W^{(M)}(A)$ is complete and separated for the V -adic topology.

1.13. Let A be a $\mathbb{Z}[M]$ -algebra and $a \in A$. We denote $(a, 0, 0, \dots) \in W^{(M)}(A)$ by $[a]$.

LEMMA 1.13.1. *Let A be $\mathbb{Z}[M]$ -algebra, $(a_0, a_1, a_2, \dots) \in W^{(M)}(A)$, $(c_0, c_1, c_2, \dots) \in W(A)$ and $a, c \in A$. Then:*

- (1) $[c] \cdot (a_0, a_1, a_2, \dots) = (ca_0, c^p a_1, c^{p^2} a_2, \dots)$;
- (2) $(c_0, c_1, c_2, \dots) \cdot [a] = (c_0 a, M^{p-1} c_1 a^p, M^{p^2-1} c_2 a^{p^2}, \dots)$;
- (3) $F^{(M)}([a]) = [M^{p-1} a^p]$.

PROOF. (1) We may assume that $A = \mathbb{Z}[M][U, T_0, T_1, T_2, \dots]$, $c = U$ and $(a_0, a_1, a_2, \dots) = (T_0, T_1, T_2, \dots)$. Now note that

$$\begin{aligned} \Phi_r^{(M)}([U] \cdot (T_0, T_1, T_2, \dots)) &= \Phi_r(U, 0, \dots, 0) \Phi_r^{(M)}(T_0, T_1, T_2, \dots, T_r) \\ &= U^{p^r} \{ M^{p^r-1} T_0^{p^r} + p M^{p^{r-1}-1} (T_1)^{p^{r-1}} + p^2 M^{p^{r-2}-1} (T_2)^{p^{r-2}} + \dots + p^r T_r \} \end{aligned}$$

and

$$\begin{aligned} \Phi_r^{(M)}(UT_0, U^p T_1, U^{p^2} T_2, \dots, U^{p^r} T_r) \\ = M^{p^r-1} (UT_0)^{p^r} + p M^{p^{r-1}-1} (U^p T_1)^{p^{r-1}} + p^2 M^{p^{r-2}-1} (U^{p^2} T_2)^{p^{r-2}} \\ + \dots + p^r (U^{p^r} T_r). \end{aligned}$$

Hence the result.

(2) We may assume that $A = \mathbb{Z}[M][U_0, U_1, U_2, \dots, T]$, $(c_0, c_1, c_2, \dots) = (U_0, U_1, U_2, \dots)$ and $a = T$. Now note that

$$\begin{aligned} \Phi_r^{(M)}((U_0, U_1, U_2, \dots) \cdot [T]) &= \Phi_r(U_0, U_1, U_2, \dots, U_r) \Phi_r^{(M)}(T, 0, \dots, 0) \\ &= (U_0^{p^r} + p U_1^{p^{r-1}} + p^2 U_2^{p^{r-2}} + \dots + p^r U_r) (M^{p^r-1} T^{p^r}) \end{aligned}$$

and

$$\begin{aligned} \Phi_r^{(M)}(U_0 T, M^{p-1} U_1 T^p, M^{p^2-1} U_2 T^{p^2}, \dots, M^{p^r-1} U_r T^{p^r}) \\ = M^{p^r-1} (U_0 T)^{p^r} + p M^{p^{r-1}-1} (M^{p-1} U_1 T^p)^{p^{r-1}} \\ + p^2 M^{p^{r-2}-1} (M^{p^2-1} U_2 T^{p^2})^{p^{r-2}} + \dots + p^r (M^{p^r-1} U_r T^{p^r}). \end{aligned}$$

Hence the result.

(3) Note that, for $r \geq 0$,

$$\begin{aligned}\Phi_r^{(M)}(F^{(M)}([T])) &= \Phi_{r+1}^{(M)}([T]) = M^{p^{r+1}-1}T^{p^{r+1}} \\ &= M^{p^r-1}(M^{p-1}T^p)^{p^r} = \Phi_r^{(M)}([M^{p-1}T^p]).\end{aligned}$$

REMARK 1.13.2. We will denote sometimes $[a] \in W^{(M)}(A)$ by $[a]_M$ to avoid confusion.

LEMMA 1.14. Let A be a $\mathbb{Z}[M]$ -algebra and $\mathbf{a} = (a_0, a_1, a_2, \dots) \in W^{(M)}(A)$. Then we have

$$\mathbf{a} = \sum_{k=0}^{\infty} V^k[a_k].$$

PROOF. We may assume that $A = \mathbb{Z}[M][T_0, T_1, T_2, \dots]$ and $(a_0, a_1, a_2, \dots) = (T_0, T_1, T_2, \dots)$. Then we can verify the equality, noting that

$$\begin{aligned}\Phi_r^{(M)}\left(\sum_{k=0}^{\infty} V^k[T_k]\right) &= \sum_{k=0}^r \Phi_r^{(M)}(V^k[T_k]) = \sum_{k=0}^r p^k \Phi_{r-k}^{(M)}([T_k]) \\ &= \Phi_r^{(M)}(T_0, T_1, \dots, T_r).\end{aligned}$$

1.15. We define a morphism $\alpha^{(M)} : W^{(M)} \rightarrow W_{\mathbb{Z}[M]}$ by

$$\begin{aligned}(T_0, T_1, T_2, \dots) &\mapsto (MT_0, MT_1, MT_2, \dots) : \\ \mathbb{Z}[M][T_0, T_1, T_2, \dots] &\rightarrow \mathbb{Z}[M][T_0, T_1, T_2, \dots].\end{aligned}$$

By 1.4 (1),

$$\begin{aligned}M\Phi_r^{(M)}(S_0^{(M)}(\mathbf{X}, \mathbf{Y}), S_1^{(M)}(\mathbf{X}, \mathbf{Y}), \dots, S_r^{(M)}(\mathbf{X}, \mathbf{Y})) \\ = \Phi_r(S_0(MX_0, MY_0), S_1(MX_0, MX_1, MY_0, MY_1), \dots, \\ S_r(MX_0, \dots, MX_r, MY_0, \dots, MY_r))\end{aligned}$$

for $r \geq 0$. Therefore $\alpha^{(M)} : W^{(M)} \rightarrow W_{\mathbb{Z}[M]}$ is a homomorphism of group schemes. Furthermore we have gotten a commutative diagram of group schemes over $\mathbb{Z}[M]$:

$$\begin{array}{ccc}W^{(M)} & \xrightarrow{\alpha^{(M)}} & W_{\mathbb{Z}[M]} \\ \Phi^{(M)} \downarrow & & \downarrow \Phi \\ \mathbb{G}_{a, \mathbb{Z}[M]}^{\mathbb{N}} & \xrightarrow{M} & \mathbb{G}_{a, \mathbb{Z}[M]}^{\mathbb{N}}.\end{array}$$

Here

$$M : \mathbb{G}_{a, \mathbb{Z}[M]}^{\mathbb{N}} = \operatorname{Spec} \mathbb{Z}[M][\Phi_0, \Phi_1, \Phi_2, \dots] \rightarrow \mathbb{G}_{a, \mathbb{Z}[M]}^{\mathbb{N}} = \operatorname{Spec} \mathbb{Z}[M][\Phi_0, \Phi_1, \Phi_2, \dots]$$

is defined by

$$(\Phi_0, \Phi_1, \Phi_2, \dots) \mapsto (M\Phi_0, M\Phi_1, M\Phi_2, \dots).$$

PROPOSITION 1.16. *Let A be a $\mathbb{Z}[M]$ -algebra and $\mathbf{a} \in W^{(M)}(A)$, $\mathbf{c} \in W(A)$. Then:*

- (1) $\alpha^{(M)}(\mathbf{c} \cdot \mathbf{a}) = \mathbf{c} \cdot \alpha^{(M)}(\mathbf{a})$;
- (2) $\alpha^{(M)}(F^{(M)}(\mathbf{a})) = F(\alpha^{(M)}(\mathbf{a}))$;
- (3) $\alpha^{(M)}(V(\mathbf{a})) = V(\alpha^{(M)}(\mathbf{a}))$.

PROOF. (3) is trivial. To prove (1) and (2), we may assume

$$A = \mathbb{Z}[M][U_0, U_1, U_2, \dots, T_0, T_1, T_2, \dots], \quad \mathbf{c} = \mathbf{U} = (U_0, U_1, U_2, \dots),$$

$$\mathbf{a} = \mathbf{T} = (T_0, T_1, T_2, \dots).$$

- (1) Note that, for $r \geq 0$,

$$\Phi_r(\alpha^{(M)}(\mathbf{U} \cdot \mathbf{T})) = M\Phi_r^{(M)}(\mathbf{U} \cdot \mathbf{T}) = M\Phi_r(\mathbf{U})\Phi_r^{(M)}(\mathbf{T})$$

and

$$\Phi_r(\mathbf{U} \cdot \alpha^{(M)}(\mathbf{T})) = \Phi_r(\mathbf{U})\Phi_r(\alpha^{(M)}(\mathbf{T})) = \Phi_r(\mathbf{U})M\Phi_r^{(M)}(\mathbf{T}).$$

- (2) Note that, for $r \geq 0$,

$$\begin{aligned} \Phi_r(\alpha^{(M)}(F^{(M)}(\mathbf{T}))) &= M\Phi_r^{(M)}(F^{(M)}(\mathbf{T})) = M\Phi_{r+1}^{(M)}(\mathbf{T}) \\ &= \Phi_{r+1}(\alpha^{(M)}(\mathbf{T})) = \Phi_r(F(\alpha^{(M)}(\mathbf{T}))). \end{aligned}$$

REMARK 1.17. As in the classical theory of Witt vectors, we can define $W_n^{(M)}$ by

$$W_n^{(M)} = \operatorname{Coker}[V^n : W^{(M)} \rightarrow W^{(M)}].$$

We have gotten systems of morphisms V and $F^{(M)}$ and several formulae. For example, we have commutative diagrams

$$\begin{array}{ccc} W_{n+2}^{(M)} & \xrightarrow{F^{(M)}} & W_{n+1}^{(M)} \\ R \downarrow & & \downarrow R \\ W_{n+1}^{(M)} & \xrightarrow{F^{(M)}} & W_n^{(M)} \end{array}$$

and

$$\begin{array}{ccc} W_{n+1}^{(M)} & \xrightarrow{V} & W_{n+2}^{(M)} \\ R \downarrow & & \downarrow R \\ W_n^{(M)} & \xrightarrow{V} & W_{n+1}^{(M)}, \end{array}$$

where R denotes the canonical surjection.

We conclude the section, giving another description of $W^{(M)}$.

1.18. Let A be a $\mathbb{Z}[M]$ -algebra and $B = A[t]/(t^2 - Mt)$. Let ε denote the image of t in B . Then we have $\varepsilon^2 = M\varepsilon$. Defining a ring homomorphism $B \rightarrow A$ by $\varepsilon \mapsto 0$, we have also a ring homomorphism $W(B) \rightarrow W(A)$ and

$$\text{Ker}[W(B) \rightarrow W(A)] = \{(\varepsilon a_0, \varepsilon a_1, \varepsilon a_2, \dots); a_0, a_1, a_2, \dots \in A\}.$$

LEMMA 1.19. Let $B = \mathbb{Z}[M, t]/(t^2 - Mt)$, and let ε denote the image of t in B . Then:

- (1) $\Phi_r(\varepsilon T_0, \varepsilon T_1, \dots, \varepsilon T_r) = \varepsilon \Phi_r^{(M)}(T_0, T_1, \dots, T_r)$;
- (2) $\varepsilon S_r^{(M)}(\mathbf{X}, \mathbf{Y}) = S_r(\varepsilon X_0, \varepsilon X_1, \dots, \varepsilon X_r, \varepsilon Y_0, \varepsilon Y_1, \dots, \varepsilon Y_r)$;
- (3) $\varepsilon P_r^{(M)}(\mathbf{X}, \mathbf{Y}) = P_r(X_0, X_1, \dots, X_r, \varepsilon Y_0, \varepsilon Y_1, \dots, \varepsilon Y_r)$;
- (4) $\varepsilon F_r^{(M)}(\mathbf{T}) = F_r(\varepsilon T_0, \dots, \varepsilon T_r, \varepsilon T_{r+1})$.

PROOF. The assertion is a direct consequence of the following lemma.

LEMMA 1.20. Let A be a ring and $B = A[M, t]/(t^2 - Mt)$, and let ε denote the image of t in B . Let $f(T_1, T_2, \dots, T_n) \in A[T_1, T_2, \dots, T_n]$ with $f(0, 0, \dots, 0) = 0$, and put

$$f^{(M)}(T_1, T_2, \dots, T_n) = \frac{1}{M} f(MT_1, MT_2, \dots, MT_n).$$

Then $f^{(M)}(T_1, T_2, \dots, T_n) \in A[M][T_1, T_2, \dots, T_n]$ and

$$\varepsilon f^{(M)}(T_1, T_2, \dots, T_n) = f(\varepsilon T_1, \varepsilon T_2, \dots, \varepsilon T_n).$$

PROOF. It is sufficient to verify the assertion in the case where

$$f(T_1, T_2, \dots, T_n) = T_1^{i_1} T_2^{i_2} \dots T_n^{i_n}$$

with $i_1 + i_2 + \dots + i_n > 0$. In this case, we have

$$f^{(M)}(T_1, T_2, \dots, T_n) = M^{i_1 + i_2 + \dots + i_n - 1} T_1^{i_1} T_2^{i_2} \dots T_n^{i_n},$$

and therefore,

$$\begin{aligned}\varepsilon f^{(M)}(T_1, T_2, \dots, T_n) &= \varepsilon M^{i_1+i_2+\dots+i_n-1} T_1^{i_1} T_2^{i_2} \dots T_n^{i_n} \\ &= \varepsilon^{i_1+i_2+\dots+i_n} T_1^{i_1} T_2^{i_2} \dots T_n^{i_n} = (\varepsilon T_1)^{i_1} (\varepsilon T_2)^{i_2} \dots (\varepsilon T_n)^{i_n}\end{aligned}$$

since $M^{k-1}\varepsilon = \varepsilon^k$ for $k \geq 1$.

Combining 1.18 and 1.19, we obtain the following:

THEOREM 1.21. *Let $A = \mathbb{Z}[M]$, $B = \mathbb{Z}[M, t]/(t^2 - Mt)$. Then $W^{(M)}$ is isomorphic to $\text{Ker} [\prod_{B/A} W_B \rightarrow W_A]$, where $\prod_{B/A}$ denotes the Weil restriction functor. More precisely,*

(1) $(a_0, a_1, a_2, \dots) \mapsto (\varepsilon a_0, \varepsilon a_1, \varepsilon a_2, \dots)$ gives rise to a $W(A)$ -isomorphism

$$W^{(M)}(A) \xrightarrow{\sim} \text{Ker}[W(B) \rightarrow W(A)];$$

(2) $F : W(B) \rightarrow W(B)$ induces $F^{(M)}$ on $W^{(M)}(A)$;

(3) $V : W(B) \rightarrow W(B)$ induces V on $W^{(M)}(A)$.

2. Around the Artin-Hasse exponential series

We first recall the necessary facts on the Artin-Hasse exponential series. For details, see [2, Ch. V] or [4].

2.1. The Artin-Hasse exponential series $E_p(U) \in \mathbb{Z}_{(p)}[[U]]$ is defined by

$$E_p(T) = \exp \left(\sum_{r \geq 0} \frac{T^{p^r}}{p^r} \right).$$

For $\mathbf{U} = (U_r)_{r \geq 0}$, we put

$$E_p(\mathbf{U}; T) = \prod_{r \geq 0} E_p(U_r T^{p^r}) = \exp \left(\sum_{r \geq 0} \frac{\Phi_r(\mathbf{U}) T^{p^r}}{p^r} \right).$$

It is readily seen that

$$E_p(S(\mathbf{U}, \mathbf{V}); T) = E_p(\mathbf{U}; T) E_p(\mathbf{V}; T).$$

2.2. In [13, Sect. 2] we defined a formal power series $E_p(U, A; T) \in \mathbb{Z}_{(p)}[[U, A]][[T]]$ by

$$E_p(U, A; T) = (1 + AT)^{\frac{U}{A}} \prod_{r=1}^{\infty} (1 + A^{p^r} T^{p^r})^{\frac{1}{p^r}} \left\{ \left(\frac{U}{A} \right)^{p^r} - \left(\frac{U}{A} \right)^{p^{r-1}} \right\}.$$

Then we have the equalities:

$$\log E_p(U, \Lambda; T) = \sum_{r=0}^{\infty} \frac{1}{p^r} \left(\frac{U}{\Lambda} \right)^{p^r} \left\{ \log(1 + \Lambda^{p^r} T^{p^r}) - \frac{1}{p} \log(1 + \Lambda^{p^{r+1}} T^{p^{r+1}}) \right\}$$

and

$$E_p(U, \Lambda; T) = \begin{cases} \prod_{(k,p)=1} E_p(U \Lambda^{k-1} T^k)^{(-1)^{k-1}/k} & \text{if } p > 2 \\ \prod_{(k,2)=1} E_p(U \Lambda^{k-1} T^k)^{1/k} \left[\prod_{(k,2)=1} E_p(U \Lambda^{2k-1} T^{2k})^{1/k} \right]^{-1} & \text{if } p = 2 \end{cases}$$

([13, Th. 2.4]).

For $U = (U_r)_{r \geq 0}$, we put

$$\begin{aligned} E_p(U, \Lambda; T) &= \prod_{r \geq 0} E_p(U_r, \Lambda^{p^r}; T^{p^r}) \\ &= (1 + \Lambda T)^{\frac{\Phi_0(U)}{\Lambda}} \prod_{k=1}^{\infty} (1 + \Lambda^{p^k} T^{p^k})^{\frac{1}{p^k \Lambda^{p^k}} \{ \Phi_k(U) - \Lambda^{p^{k-1}(p-1)} \Phi_{k-1}(U) \}}. \end{aligned}$$

Then we obtain

$$\log E_p(U, \Lambda; T) = \sum_{r=0}^{\infty} \frac{\Phi_r(U)}{p^r \Lambda^{p^r}} \left\{ \log(1 + \Lambda^{p^r} T^{p^r}) - \frac{1}{p} \log(1 + \Lambda^{p^{r+1}} T^{p^{r+1}}) \right\}$$

and

$$\begin{aligned} E_p(U, \Lambda; T) &= \begin{cases} \prod_{(k,p)=1} E_p(U; \Lambda^{k-1} T^k)^{(-1)^{k-1}/k} & \text{if } p > 2 \\ \prod_{(k,2)=1} E_p(U; \Lambda^{k-1} T^k)^{1/k} \left[\prod_{(k,2)=1} E_p(U; \Lambda^{2k-1} T^{2k})^{1/k} \right]^{-1} & \text{if } p = 2. \end{cases} \end{aligned}$$

It is verified without difficulty that

$$E_p(S(U, V), \Lambda; T) = E_p(U, \Lambda; T) E_p(V, \Lambda; T)$$

([13, Cor. 2.9]).

EXAMPLE 2.2.1. $E_p(U, 0; T) = E_p(U; T)$.

EXAMPLE 2.2.2. $E_p([A], \Lambda; T) = 1 + \Lambda T$.

2.3. For $\mathbf{U} = (U_0, U_1, U_2, \dots)$, we define $E_p^{(M)}(\mathbf{U}, \Lambda; T) \in \mathbb{Z}_{(p)}[\Lambda, M, U_0, U_1, U_2, \dots][[T]]$ by

$$E_p^{(M)}(\mathbf{U}, \Lambda; T) = \frac{1}{M} [E_p(\alpha^{(M)}\mathbf{U}, \Lambda; T) - 1].$$

EXAMPLE 2.3.1. $E_p^{(1)}(\mathbf{U}, \Lambda; T) = E_p(\mathbf{U}, \Lambda; T) - 1.$

EXAMPLE 2.3.2.

$$E_p^{(0)}(\mathbf{U}, \Lambda; T) = \sum_{r=0}^{\infty} \frac{U_r}{\Lambda^{p^r}} \left\{ \log(1 + \Lambda^{p^r} T^{p^r}) - \frac{1}{p} \log(1 + \Lambda^{p^{r+1}} T^{p^{r+1}}) \right\}.$$

Indeed, by the definition we have

$$1 + ME_p^{(M)}(\mathbf{U}, \Lambda; T) = E_p(\alpha^{(M)}\mathbf{U}, \Lambda; T)$$

and

$$\begin{aligned} & \log E_p(\alpha^{(M)}\mathbf{U}, \Lambda; T) \\ &= \sum_{r=0}^{\infty} \frac{\Phi_r(MU_0, MU_1, \dots, MU_r)}{p^r \Lambda^{p^r}} \left\{ \log(1 + \Lambda^{p^r} T^{p^r}) - \frac{1}{p} \log(1 + \Lambda^{p^{r+1}} T^{p^{r+1}}) \right\}. \end{aligned}$$

Now note that, for $r \geq 0$,

$$\Phi_r(MU_0, MU_1, \dots, MU_r) \equiv p^r MU_r \pmod{M^p}.$$

Hence we have

$$\log E_p(\alpha^{(M)}\mathbf{U}, \Lambda; T) \equiv \sum_{r=0}^{\infty} \frac{MU_r}{\Lambda^{p^r}} \left\{ \log(1 + \Lambda^{p^r} T^{p^r}) - \frac{1}{p} \log(1 + \Lambda^{p^{r+1}} T^{p^{r+1}}) \right\} \pmod{M^p},$$

and therefore

$$E_p(\alpha^{(M)}\mathbf{U}, \Lambda; T) \equiv 1 + \sum_{r=0}^{\infty} \frac{MU_r}{\Lambda^{p^r}} \left\{ \log(1 + \Lambda^{p^r} T^{p^r}) - \frac{1}{p} \log(1 + \Lambda^{p^{r+1}} T^{p^{r+1}}) \right\} \pmod{M^2}.$$

Thus we obtain

$$E_p^{(M)}(\mathbf{U}, \Lambda; T) \equiv \sum_{r=0}^{\infty} \frac{U_r}{\Lambda^{p^r}} \left\{ \log(1 + \Lambda^{p^r} T^{p^r}) - \frac{1}{p} \log(1 + \Lambda^{p^{r+1}} T^{p^{r+1}}) \right\} \pmod{M}.$$

EXAMPLE 2.3.3. $E_p^{(M)}(U, 0; T) = \frac{1}{M} [\prod_{r=0}^{\infty} E_p(MU_r; T^{p^r}) - 1]$.

PROPOSITION 2.4. Let $U = (U_0, U_1, U_2, \dots)$, $V = (V_0, V_1, V_2, \dots)$. Then we have

$$\begin{aligned} E_p^{(M)}(U + {}^{(M)}V, \Lambda; T) \\ = E_p^{(M)}(U, \Lambda; T) + E_p^{(M)}(V, \Lambda; T) + ME_p^{(M)}(U, \Lambda; T)E_p^{(M)}(V, \Lambda; T) \end{aligned}$$

PROOF. It is sufficient to prove that

$$[1 + ME_p^{(M)}(U, \Lambda; T)][1 + ME_p^{(M)}(V, \Lambda; T)] = 1 + ME_p^{(M)}(U + {}^{(M)}V, \Lambda; T),$$

that is to say,

$$E_p(\alpha^{(M)}U, \Lambda; T)E_p(\alpha^{(M)}V, \Lambda; T) = E_p(\alpha^{(M)}(U + {}^{(M)}V), \Lambda; T).$$

This is a consequence of the functional equation for $E_p(U, \Lambda; T)$ since $\alpha^{(M)}(U) + \alpha^{(M)}(V) = \alpha^{(M)}(U + {}^{(M)}V)$.

We conclude the section by introducing formal power series to describe 2-cocycles.

2.5. In [13, 2.15] we defined a formal power series

$$F_p(U, \Lambda; X, Y) \in \mathbb{Z}_{(p)}[U_0, U_1, U_2, \dots, \Lambda][[X, Y]]$$

by

$$F_p(U, \Lambda; X, Y) = \prod_{r=0}^{\infty} \left[\frac{(1 + \Lambda^{p^{r+1}} X^{p^{r+1}})(1 + \Lambda^{p^{r+1}} Y^{p^{r+1}})}{1 + \Lambda^{p^{r+1}}(X + Y + \Lambda XY)^{p^{r+1}}} \right]^{\Phi_r(U)/p^{r+1}\Lambda^{p^{r+1}}}.$$

It is readily seen that

- (1) $F_p(U, \Lambda; X, Y)F_p(U, \Lambda; X + Y + \Lambda XY, Z)$
 $= F_p(U, \Lambda; X, Y + Z + \Lambda YZ)F_p(U, \Lambda; Y, Z),$
- (2) $F_p(U, \Lambda; X, Y) = F_p(U, \Lambda; Y, X).$

Moreover, we have

- (3) $F_p(S(T, U), \Lambda; X, Y) = F_p(T, \Lambda; X, Y)F_p(U, \Lambda; X, Y).$

REMARK 2.6. In [12, Sect. 2] we defined a formal power series

$$F_p(U; X, Y) = F_{p,1}(U; X, Y) \in \mathbb{Z}_{(p)}[U_0, U_1, U_2, \dots][[X, Y]]$$

by

$$F_p(\mathbf{U}; X, Y) = \exp \left(\sum_{r=0}^{\infty} \Phi_r(\mathbf{U}) \frac{X^{p^{r+1}} + Y^{p^{r+1}} - (X + Y)^{p^{r+1}}}{p^{r+1}} \right).$$

We can verify without difficulty that

$$F_p(\mathbf{U}, 0; X, Y) = F_{p,1}(\mathbf{U}; X, Y).$$

2.7. Now define a formal power series

$$F_p^{(M)}(\mathbf{U}, \Lambda; X, Y) \in \mathbb{Z}_{(p)}[\Lambda, M, U_0, U_1, U_2, \dots][[X, Y]]$$

by

$$F_p^{(M)}(\mathbf{U}, \Lambda; X, Y) = \frac{1}{M} [F_p(\alpha^{(M)} \mathbf{U}, \Lambda; X, Y) - 1].$$

EXAMPLE 2.7.1. $F_p^{(1)}(\mathbf{U}, \Lambda; X, Y) = F_p(\mathbf{U}, \Lambda; X, Y) - 1$. In particular, $F_p^{(1)}(\mathbf{U}, 0; X, Y) = F_p(\mathbf{U}; X, Y) - 1$.

EXAMPLE 2.7.2.

$$F_p^{(0)}(\mathbf{U}, \Lambda; X, Y) = \sum_{r=0}^{\infty} \frac{U_r}{p\Lambda^{p^{r+1}}} \log \frac{(1 + \Lambda^{p^{r+1}} X^{p^{r+1}})(1 + \Lambda^{p^{r+1}} Y^{p^{r+1}})}{1 + \Lambda^{p^{r+1}}(X + Y + \Lambda XY)^{p^{r+1}}}.$$

Indeed, by the definition we have

$$1 + MF_p^{(M)}(\mathbf{U}, \Lambda; X, Y) = F_p(\alpha^{(M)} \mathbf{U}, \Lambda; X, Y)$$

and

$$\begin{aligned} & \log F_p(\alpha^{(M)} \mathbf{U}, \Lambda; X, Y) \\ &= \sum_{r=0}^{\infty} \frac{\Phi_r(MU_0, MU_1, \dots, MU_r)}{p^{r+1}\Lambda^{p^{r+1}}} \log \left[\frac{(1 + \Lambda^{p^{r+1}} X^{p^{r+1}})(1 + \Lambda^{p^{r+1}} Y^{p^{r+1}})}{1 + \Lambda^{p^{r+1}}(X + Y + \Lambda XY)^{p^{r+1}}} \right]. \end{aligned}$$

Hence, as in Example 2.3.2, we obtain

$$\begin{aligned} & \log F_p(\alpha^{(M)} \mathbf{U}, \Lambda; X, Y) \\ & \equiv \sum_{r=0}^{\infty} \frac{MU_r}{p\Lambda^{p^{r+1}}} \log \left[\frac{(1 + \Lambda^{p^{r+1}} X^{p^{r+1}})(1 + \Lambda^{p^{r+1}} Y^{p^{r+1}})}{1 + \Lambda^{p^{r+1}}(X + Y + \Lambda XY)^{p^{r+1}}} \right] \pmod{M^p}, \end{aligned}$$

which implies the assertion.

In particular, putting $\mathbf{U} = [1]_0$ and $\Lambda = 0$, we obtain

$$F_p^{(0)}([1]_0, 0; X, Y) = \frac{X^p + Y^p - (X + Y)^p}{p},$$

which is the 2-cocycle of $Z^2(\widehat{\mathbb{G}}_a, \widehat{\mathbb{G}}_a)$ defining the extension $\widehat{W}_2$ or the 2-cocycle of $Z^2(\mathbb{G}_a, \mathbb{G}_a)$ defining the extension W_2 .

PROPOSITION 2.8. *Let $\mathbf{U} = (U_0, U_1, U_2, \dots)$. Then*

$$\begin{aligned} & F_p^{(M)}(\mathbf{U}, \Lambda; X, Y) + F_p^{(M)}(\mathbf{U}, \Lambda; X + Y + \Lambda XY, Z) \\ & + MF_p^{(M)}(\mathbf{U}, \Lambda; X, Y)F_p^{(M)}(\mathbf{U}, \Lambda; X + Y + \Lambda XY, Z) \\ & = F_p^{(M)}(\mathbf{U}, \Lambda; X, Y + Z + \Lambda YZ) + F_p^{(M)}(\mathbf{U}, \Lambda; Y, Z) \\ & + MF_p^{(M)}(\mathbf{U}, \Lambda; X, Y + Z + \Lambda YZ)F_p^{(M)}(\mathbf{U}, \Lambda; Y, Z) \end{aligned}$$

and

$$F_p^{(M)}(\mathbf{U}, \Lambda; X, Y) = F_p^{(M)}(\mathbf{U}, \Lambda; Y, X).$$

PROOF. To verify the first assertion, it is sufficient to prove that

$$\begin{aligned} & [1 + MF_p^{(M)}(\mathbf{U}, \Lambda; X, Y)][1 + MF_p^{(M)}(\mathbf{U}, \Lambda; X + Y + \Lambda XY, Z)] \\ & = [1 + MF_p^{(M)}(\mathbf{U}, \Lambda; X, Y + Z + \Lambda YZ)][1 + MF_p^{(M)}(\mathbf{U}, \Lambda; Y, Z)], \end{aligned}$$

that is to say,

$$\begin{aligned} & F_p(\alpha^{(M)}\mathbf{U}, \Lambda; X, Y)F_p(\alpha^{(M)}\mathbf{U}, \Lambda; X + Y + \Lambda XY, Z) \\ & = F_p(\alpha^{(M)}\mathbf{U}, \Lambda; X, Y + Z + \Lambda YZ)F_p(\alpha^{(M)}\mathbf{U}, \Lambda; Y, Z). \end{aligned}$$

This is a consequence of 2.5(1). The second assertion follows immediately from 2.5(2).

PROPOSITION 2.9. *Let $\mathbf{U} = (U_0, U_1, U_2, \dots)$ and $\mathbf{V} = (V_0, V_1, V_2, \dots)$. Then*

$$\begin{aligned} F_p^{(M)}(\mathbf{U} + {}^{(M)}\mathbf{V}, \Lambda; X, Y) & = F_p^{(M)}(\mathbf{U}, \Lambda; X, Y) + F_p^{(M)}(\mathbf{V}, \Lambda; X, Y) \\ & + MF_p^{(M)}(\mathbf{U}, \Lambda; X, Y)F_p^{(M)}(\mathbf{V}, \Lambda; X, Y). \end{aligned}$$

PROOF. It is sufficient to prove that

$$\begin{aligned} & [1 + MF_p^{(M)}(\mathbf{U}, \Lambda; X, Y)][1 + MF_p^{(M)}(\mathbf{V}, \Lambda; X, Y)] \\ & = 1 + MF_p^{(M)}(\mathbf{U} + {}^{(M)}\mathbf{V}, \Lambda; X, Y), \end{aligned}$$

that is to say,

$$F_p(\alpha^{(M)}\mathbf{U}, \Lambda; X, Y)F_p(\alpha^{(M)}\mathbf{V}, \Lambda; X, Y) = F_p(\alpha^{(M)}(\mathbf{U} + {}^{(M)}\mathbf{V}), \Lambda; X, Y).$$

This is a consequence of 2.5(3).

3. Statement and proof of the theorem

3.1. Let A be a $\mathbb{Z}[A]$ -algebra. We define a group scheme $\mathcal{G}_A^{(\Lambda)}$ over A by

$$\mathcal{G}_A^{(\Lambda)} = \operatorname{Spec} A \left[T, \frac{1}{1 + \Lambda T} \right]$$

with

- (1) the multiplication: $T \mapsto T \otimes 1 + 1 \otimes T + \Lambda T \otimes T$;
- (2) the unit: $T \mapsto 0$;
- (3) the inverse: $T \mapsto -T/(1 + \Lambda T)$.

Moreover, we define an A -homomorphism $\alpha_A^{(\Lambda)} : \mathcal{G}_A^{(\Lambda)} \rightarrow \mathbb{G}_{m,A}$ by

$$U \mapsto 1 + \Lambda T : A[U, U^{-1}] \rightarrow A \left[T, \frac{1}{1 + \Lambda T} \right].$$

If Λ is invertible in A , $\alpha_A^{(\Lambda)}$ is an A -isomorphism. On the other hand, if $\Lambda = 0$ in A , $\mathcal{G}_A^{(\Lambda)}$ is nothing but the additive group $\mathbb{G}_{a,A}$.

REMARK 3.2. Let $B = A[t]/(t^2 - \Lambda t)$, and let ε denote the image of t in B . Then we have $\varepsilon^2 = \Lambda \varepsilon$. Defining a ring homomorphism $B \rightarrow A$ by $\varepsilon \mapsto 0$, we have

$$\operatorname{Ker}[B^\times \rightarrow A^\times] = \{1 + \varepsilon a ; a \in A, 1 + \Lambda a \text{ is invertible in } A\}.$$

Hence $\mathcal{G}_A^{(\Lambda)}$ is isomorphic to $\operatorname{Ker} [\prod_{B/A} \mathbb{G}_{m,B} \rightarrow \mathbb{G}_{m,A}]$, where $\prod_{B/A}$ denotes the Weil restriction functor. Furthermore, the inclusion $A \rightarrow B$ defines a section of $\prod_{B/A} \mathbb{G}_{m,B} \rightarrow \mathbb{G}_{m,A}$, and therefore, the exact sequence

$$0 \rightarrow \mathcal{G}_A^{(\Lambda)} \rightarrow \prod_{B/A} \mathbb{G}_{m,B} \rightarrow \mathbb{G}_{m,A} \rightarrow 0$$

splits.

Now we recall some results of [13]. For generalities of Hochschild cohomology, see [2, Ch. II.3 and Ch. III.6].

3.3. Let A be a $\mathbb{Z}_{(p)}[A]$ -algebra. We define a complex

$$C^1(\widehat{\mathcal{G}}_A^{(\Lambda)}, \widehat{\mathbb{G}}_{m,A}) \xrightarrow{\partial} C^2(\widehat{\mathcal{G}}_A^{(\Lambda)}, \widehat{\mathbb{G}}_{m,A})$$

by

$$C^1(\widehat{\mathcal{G}}_A^{(\Lambda)}, \widehat{\mathbb{G}}_{m,A}) = \{F(T) \in A[[T]] ; F(T) \equiv 1 \pmod{\deg 1}\} \subset A[[T]]^\times$$

and

$$C^2(\widehat{\mathcal{G}}_A^{(\Lambda)}, \widehat{\mathbb{G}}_{m,A}) = \{G(X, Y) \in A[[X, Y]]; G(X, Y) \equiv 1 \pmod{\deg 1}\} \subset A[[X, Y]]^\times.$$

The boundary map $\partial : C^1(\widehat{\mathcal{G}}_A^{(\Lambda)}, \widehat{\mathbb{G}}_{m,A}) \rightarrow C^2(\widehat{\mathcal{G}}_A^{(\Lambda)}, \widehat{\mathbb{G}}_{m,A})$ is given by

$$\partial : F(T) \mapsto \frac{F(X)F(Y)}{F(X+Y+\Lambda XY)}.$$

A formal power series $G(X, Y) \in C^2(\widehat{\mathcal{G}}_A^{(\Lambda)}, \widehat{\mathbb{G}}_{m,A})$ is called a symmetric 2-cocycle if $G(X, Y)$ satisfies the following functional equations:

- (1) $G(X, Y)G(X+Y+\Lambda XY, Z) = G(X, Y+Z+\Lambda YZ)G(Y, Z),$
- (2) $G(X, Y) = G(Y, X).$

We denote by $Z^2(\widehat{\mathcal{G}}_A^{(\Lambda)}, \widehat{\mathbb{G}}_{m,A})$ the subgroup of $C^2(\widehat{\mathcal{G}}_A^{(\Lambda)}, \widehat{\mathbb{G}}_{m,A})$, formed by the symmetric 2-cocycles. $\text{Im } \partial$ is denoted by $B^2(\widehat{\mathcal{G}}_A^{(\Lambda)}, \widehat{\mathbb{G}}_{m,A})$ and we put

$$H_0^2(\widehat{\mathcal{G}}_A^{(\Lambda)}, \widehat{\mathbb{G}}_{m,A}) = Z^2(\widehat{\mathcal{G}}_A^{(\Lambda)}, \widehat{\mathbb{G}}_{m,A}) / B^2(\widehat{\mathcal{G}}_A^{(\Lambda)}, \widehat{\mathbb{G}}_{m,A}).$$

By abbreviation we denote by $\tilde{C}^*(\widehat{\mathcal{G}}_A^{(\Lambda)}, \widehat{\mathbb{G}}_{m,A})$ the cochain complex

$$C^1(\widehat{\mathcal{G}}_A^{(\Lambda)}, \widehat{\mathbb{G}}_{m,A}) \xrightarrow{\partial} Z^2(\widehat{\mathcal{G}}_A^{(\Lambda)}, \widehat{\mathbb{G}}_{m,A}),$$

concentrated in the degrees 1 and 2. We define also a cochain complex

$$W(A) \xrightarrow{F-[A^{p-1}]} W(A),$$

denoted by $\tilde{D}^*(\widehat{\mathcal{G}}_A^{(\Lambda)}, \widehat{\mathbb{G}}_{m,A})$. We defined a morphism of complexes

$$\xi : \tilde{D}^*(\widehat{\mathcal{G}}_A^{(\Lambda)}, \widehat{\mathbb{G}}_{m,A}) \rightarrow \tilde{C}^*(\widehat{\mathcal{G}}_A^{(\Lambda)}, \widehat{\mathbb{G}}_{m,A})$$

by

$$\begin{array}{ccc} W(A) & \xrightarrow{\xi^0} & C^1(\widehat{\mathcal{G}}_A^{(\Lambda)}, \widehat{\mathbb{G}}_{m,A}) \\ F-[A^{p-1}] \downarrow & & \downarrow \partial \\ W(A) & \xrightarrow[\xi^1]{} & Z^2(\widehat{\mathcal{G}}_A^{(\Lambda)}, \widehat{\mathbb{G}}_{m,A}), \end{array}$$

where ξ^0 is given by

$$a \mapsto E_p(a, \Lambda; T) : W(A) \rightarrow C^1(\widehat{\mathcal{G}}_A^{(\Lambda)}, \widehat{\mathbb{G}}_{m,A}) \subset A[[T]]^\times$$

and ξ^1 by

$$\mathbf{a} \mapsto F_p(\mathbf{a}, \Lambda; X, Y) : W(A) \rightarrow Z^2(\widehat{\mathcal{G}}_A^{(\Lambda)}, \widehat{\mathbb{G}}_{m,A}) \subset A[[X, Y]]^\times$$

([13, 2.19]).

We proved that $\xi : \tilde{D}^*(\widehat{\mathcal{G}}_A^{(\Lambda)}, \widehat{\mathbb{G}}_{m,A}) \rightarrow \tilde{C}^*(\widehat{\mathcal{G}}_A^{(\Lambda)}, \widehat{\mathbb{G}}_{m,A})$ induces isomorphism

$$\xi^0 : \text{Ker}[F - [\Lambda^{p-1}] : W(A) \rightarrow W(A)] \xrightarrow{\sim} \text{Hom}_{A\text{-gr}}(\widehat{\mathcal{G}}_A^{(\Lambda)}, \widehat{\mathbb{G}}_{m,A})$$

and

$$\xi^1 : \text{Coker}[F - [\Lambda^{p-1}] : W(A) \rightarrow W(A)] \xrightarrow{\sim} H_0^2(\widehat{\mathcal{G}}_A^{(\Lambda)}, \widehat{\mathbb{G}}_{m,A})$$

([13, Th. 2.19.1]).

REMARK 3.3.1. In [13, 2.2], we defined a complex

$$C^1(\widehat{\mathcal{G}}_A^{(\Lambda)}, \widehat{\mathbb{G}}_{m,A}) \xrightarrow{\partial} C^2(\widehat{\mathcal{G}}_A^{(\Lambda)}, \widehat{\mathbb{G}}_{m,A})$$

by

$$C^1(\widehat{\mathcal{G}}_A^{(\Lambda)}, \widehat{\mathbb{G}}_{m,A}) = A[[T]]^\times$$

and

$$C^2(\widehat{\mathcal{G}}_A^{(\Lambda)}, \widehat{\mathbb{G}}_{m,A}) = A[[X, Y]]^\times$$

with the boundary map

$$\partial : F(T) \mapsto \frac{F(X)F(Y)}{F(X+Y+\Lambda XY)}.$$

It is readily seen that the inclusions

$$\{F(T) \in A[[T]] ; F(T) \equiv 1 \bmod \deg 1\} \subset A[[T]]^\times$$

and

$$\{G(X, Y) \in A[[X, Y]] ; G(X, Y) \equiv 1 \bmod \deg 1\} \subset A[[X, Y]]^\times$$

induce a quasi-isomorphism of the complex defined in 3.3 to the complex defined in [13, 2.2].

3.4. Let A be a $\mathbb{Z}_{(p)}[\Lambda, M]$ -algebra. We shall define a complex

$$C^1(\widehat{\mathcal{G}}_A^{(\Lambda)}, \widehat{\mathcal{G}}_A^{(M)}) \xrightarrow{\partial} C^2(\widehat{\mathcal{G}}_A^{(\Lambda)}, \widehat{\mathcal{G}}_A^{(M)})$$

by

$$C^1(\widehat{\mathcal{G}}_A^{(\Lambda)}, \widehat{\mathcal{G}}_A^{(M)}) = \{F(T) \in A[[T]] ; F(T) \equiv 0 \bmod \deg 1\}$$

and

$$C^2(\widehat{\mathcal{G}}_A^{(\Lambda)}, \widehat{\mathcal{G}}_A^{(M)}) = \{G(X, Y) \in A[[X, Y]] ; G(X, Y) \equiv 0 \bmod \deg 1\}.$$

The group laws of $C^1(\widehat{\mathcal{G}}_A^{(\Lambda)}, \widehat{\mathcal{G}}_A^{(M)})$ and $C^2(\widehat{\mathcal{G}}_A^{(\Lambda)}, \widehat{\mathcal{G}}_A^{(M)})$ are given by

$$(F_1(T), F_2(T)) \mapsto F_1(T) + F_2(T) + MF_1(T)F_2(T)$$

and

$$(G_1(X, Y), G_2(X, Y)) \mapsto G_1(X, Y) + G_2(X, Y) + MG_1(X, Y)G_2(X, Y),$$

respectively. Moreover, the boundary map $\partial : C^1(\widehat{\mathcal{G}}_A^{(\Lambda)}, \widehat{\mathcal{G}}_A^{(M)}) \rightarrow C^2(\widehat{\mathcal{G}}_A^{(\Lambda)}, \widehat{\mathcal{G}}_A^{(M)})$ is defined by

$$\partial : F(T) \mapsto \frac{F(X) + F(Y) + MF(X)F(Y) - F(X + Y + \Lambda XY)}{1 + MF(X + Y + \Lambda XY)}.$$

A formal power series $G(X, Y) \in C^2(\widehat{\mathcal{G}}_A^{(\Lambda)}, \widehat{\mathcal{G}}_A^{(M)})$ is called a symmetric 2-cocycle if $G(X, Y)$ satisfies the following functional equations:

- (1) $G(X, Y) + G(X + Y + \Lambda XY, Z) + MG(X, Y)G(X + Y + \Lambda XY, Z)$
 $= G(X, Y + Z + \Lambda YZ) + G(Y, Z) + MG(X, Y + Z + \Lambda YZ)G(Y, Z),$
- (2) $G(X, Y) = G(Y, X).$

For example, $F_p^{(M)}(a, A; X, Y)$ is a symmetric 2-cocycle, as is shown in Proposition 2.8. We denote by $Z^2(\widehat{\mathcal{G}}_A^{(\Lambda)}, \widehat{\mathcal{G}}_A^{(M)})$ the subgroup of $C^2(\widehat{\mathcal{G}}_A^{(\Lambda)}, \widehat{\mathcal{G}}_A^{(M)})$, formed by the symmetric 2-cocycles. $\text{Im } \partial$ is denoted by $B^2(\widehat{\mathcal{G}}_A^{(\Lambda)}, \widehat{\mathcal{G}}_A^{(M)})$ and we put

$$H_0^2(\widehat{\mathcal{G}}_A^{(\Lambda)}, \widehat{\mathcal{G}}_A^{(M)}) = Z^2(\widehat{\mathcal{G}}_A^{(\Lambda)}, \widehat{\mathcal{G}}_A^{(M)}) / B^2(\widehat{\mathcal{G}}_A^{(\Lambda)}, \widehat{\mathcal{G}}_A^{(M)}).$$

Again by abbreviation we denote by $\tilde{C}^*(\widehat{\mathcal{G}}_A^{(\Lambda)}, \widehat{\mathcal{G}}_A^{(M)})$ the cochain complex

$$0 \longrightarrow C^1(\widehat{\mathcal{G}}_A^{(\Lambda)}, \widehat{\mathcal{G}}_A^{(M)}) \xrightarrow{\partial} Z^2(\widehat{\mathcal{G}}_A^{(\Lambda)}, \widehat{\mathcal{G}}_A^{(M)}) \longrightarrow 0,$$

concentrated in the degrees 1 and 2.

3.5. Let A be a $\mathbb{Z}[\Lambda, M]$ -algebra and $B = A[t]/(t^2 - Mt)$. Let ε denote the image of t in B . Then we have $\varepsilon^2 = M\varepsilon$. The splitting exact sequence of formal groups

$$0 \rightarrow \widehat{\mathcal{G}}_A^{(M)} \rightarrow \left(\prod_{B/A} \widehat{\mathbb{G}}_{m,B} \right) \rightarrow \widehat{\mathbb{G}}_{m,A} \rightarrow 0$$

induces a splitting exact sequence of complexes

$$0 \rightarrow \tilde{C}^*(\hat{\mathcal{G}}_A^{(A)}, \hat{\mathcal{G}}_A^{(M)}) \rightarrow \tilde{C}^*(\hat{\mathcal{G}}_B^{(A)}, \hat{\mathbb{G}}_{m,B}) \rightarrow \tilde{C}^*(\hat{\mathcal{G}}_A^{(A)}, \hat{\mathbb{G}}_{m,A}) \rightarrow 0.$$

More precisely,

$$C^1(\hat{\mathcal{G}}_B^{(A)}, \hat{\mathbb{G}}_{m,B}) \rightarrow C^1(\hat{\mathcal{G}}_A^{(A)}, \hat{\mathbb{G}}_{m,A})$$

and

$$Z^2(\hat{\mathcal{G}}_B^{(A)}, \hat{\mathbb{G}}_{m,B}) \rightarrow Z^2(\hat{\mathcal{G}}_A^{(A)}, \hat{\mathbb{G}}_{m,A})$$

are induced from the ring homomorphism $B \rightarrow A$. Moreover

$$C^1(\hat{\mathcal{G}}_A^{(A)}, \hat{\mathcal{G}}_A^{(M)}) \rightarrow C^1(\hat{\mathcal{G}}_B^{(A)}, \hat{\mathbb{G}}_{m,B})$$

and

$$Z^2(\hat{\mathcal{G}}_A^{(A)}, \hat{\mathcal{G}}_A^{(M)}) \rightarrow Z^2(\hat{\mathcal{G}}_B^{(A)}, \hat{\mathbb{G}}_{m,B})$$

are defined by

$$F(T) \mapsto 1 + \varepsilon F(T) : A[[T]] \rightarrow B[[T]]^\times$$

and

$$G(X, Y) \mapsto 1 + \varepsilon G(X, Y) : A[[X, Y]] \rightarrow B[[X, Y]]^\times,$$

respectively.

On the other hand, we have a commutative diagram with splitting exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & W^{(M)}(A) & \longrightarrow & W(B) & \longrightarrow & W(A) \longrightarrow 0 \\ & & \downarrow F^{(M)} - [A^{p-1}] & & \downarrow F - [A^{p-1}] & & \downarrow F - [A^{p-1}] \\ 0 & \longrightarrow & W^{(M)}(A) & \longrightarrow & W(B) & \longrightarrow & W(A) \longrightarrow 0, \end{array}$$

by Theorem 1.21. Obviously the diagram of complexes

$$\begin{array}{ccc} \tilde{D}^*(\hat{\mathcal{G}}_B^{(A)}, \hat{\mathbb{G}}_{m,B}) & \longrightarrow & \tilde{D}^*(\hat{\mathcal{G}}_A^{(A)}, \hat{\mathbb{G}}_{m,A}) \\ \xi \downarrow & & \downarrow \xi \\ \tilde{C}^*(\hat{\mathcal{G}}_B^{(A)}, \hat{\mathbb{G}}_{m,B}) & \longrightarrow & \tilde{C}^*(\hat{\mathcal{G}}_A^{(A)}, \hat{\mathbb{G}}_{m,A}) \end{array}$$

is commutative. Hence we obtain a morphism of complexes

$$\xi : \tilde{D}^*(\hat{\mathcal{G}}_A^{(A)}, \hat{\mathcal{G}}_A^{(M)}) \rightarrow \tilde{C}^*(\hat{\mathcal{G}}_A^{(A)}, \hat{\mathcal{G}}_A^{(M)}),$$

where $\tilde{D}^*(\hat{\mathcal{G}}_A^{(A)}, \hat{\mathcal{G}}_A^{(M)})$ denote the cochain complex

$$0 \longrightarrow W^{(M)}(A) \xrightarrow{F^{(M)} - [A^{p-1}]} W^{(M)}(A) \rightarrow 0.$$

To sum up, we obtain a commutative diagram of cochain complexes with splitting exact rows

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \tilde{D}^*(\widehat{\mathcal{G}}_A^{(A)}, \widehat{\mathcal{G}}_A^{(M)}) & \longrightarrow & \tilde{D}^*(\widehat{\mathcal{G}}_B^{(A)}, \widehat{\mathbb{G}}_{m,B}) & \longrightarrow & \tilde{D}^*(\widehat{\mathcal{G}}_A^{(A)}, \widehat{\mathbb{G}}_{m,A}) \longrightarrow 0 \\
 & & \downarrow \xi & & \downarrow \xi & & \downarrow \xi \\
 0 & \longrightarrow & \tilde{C}^*(\widehat{\mathcal{G}}_A^{(A)}, \widehat{\mathcal{G}}_A^{(M)}) & \longrightarrow & \tilde{C}^*(\widehat{\mathcal{G}}_B^{(A)}, \widehat{\mathbb{G}}_{m,B}) & \longrightarrow & \tilde{C}^*(\widehat{\mathcal{G}}_A^{(A)}, \widehat{\mathbb{G}}_{m,A}) \longrightarrow 0.
 \end{array}$$

Hence we have the following theorem, noting that

$\xi : \tilde{D}^*(\widehat{\mathcal{G}}_B^{(A)}, \widehat{\mathbb{G}}_{m,B}) \rightarrow \tilde{C}^*(\widehat{\mathcal{G}}_B^{(A)}, \widehat{\mathbb{G}}_{m,B})$ and $\xi : \tilde{D}^*(\widehat{\mathcal{G}}_A^{(A)}, \widehat{\mathbb{G}}_{m,A}) \rightarrow \tilde{C}^*(\widehat{\mathcal{G}}_A^{(A)}, \widehat{\mathbb{G}}_{m,A})$ are quasi-isomomorphisms ([13, Th. 2.19.1]).

THEOREM 3.5.1. *Let A be $\mathbb{Z}_{(p)}[A, M]$ -algebra. Then there exist isomorphisms*

$$\text{Ker}[F^{(M)} - [A^{p-1}] : W^{(M)}(A) \rightarrow W^{(M)}(A)] \xrightarrow{\sim} \text{Hom}_{A\text{-gr}}(\widehat{\mathcal{G}}_A^{(A)}, \widehat{\mathcal{G}}_A^{(M)})$$

and

$$\text{Coker}[F^{(M)} - [A^{p-1}] : W^{(M)}(A) \rightarrow W^{(M)}(A)] \xrightarrow{\sim} H_0^2(\widehat{\mathcal{G}}_A^{(A)}, \widehat{\mathcal{G}}_A^{(M)}).$$

REMARK 3.6. We can describe explicitly the isomorphisms

$$\xi^0 : \text{Ker}[F^{(M)} - [A^{p-1}] : W^{(M)}(A) \rightarrow W^{(M)}(A)] \xrightarrow{\sim} \text{Hom}_{A\text{-gr}}(\widehat{\mathcal{G}}_A^{(A)}, \widehat{\mathcal{G}}_A^{(M)})$$

and

$$\xi^1 : \text{Coker}[F^{(M)} - [\lambda^{p-1}] : W^{(M)}(A) \rightarrow W^{(M)}(A)] \xrightarrow{\sim} H_0^2(\widehat{\mathcal{G}}_A^{(A)}, \widehat{\mathcal{G}}_A^{(M)})$$

induced from

$$\xi : \tilde{D}^*(\widehat{\mathcal{G}}_A^{(A)}, \widehat{\mathcal{G}}_A^{(M)}) \rightarrow \tilde{C}^*(\widehat{\mathcal{G}}_A^{(A)}, \widehat{\mathcal{G}}_A^{(M)}).$$

Indeed,

(1) $\mathbf{a} \mapsto E_p^{(M)}(\mathbf{a}, A; T)$ gives rise to an isomorphism

$$\xi^0 : \text{Ker}[F^{(M)} - [A^{p-1}] : W^{(M)}(A) \rightarrow W^{(M)}(A)] \xrightarrow{\sim} \text{Hom}_{A\text{-gr}}(\widehat{\mathcal{G}}_A^{(A)}, \widehat{\mathcal{G}}_A^{(M)});$$

(2) $\mathbf{a} \mapsto F_p^{(M)}(\mathbf{a}, A; X, Y)$ gives rise to an isomorphism

$$\xi^1 : \text{Coker}[F^{(M)} - [A^{p-1}] : W^{(M)}(A) \rightarrow W^{(M)}(A)] \xrightarrow{\sim} H_0^2(\widehat{\mathcal{G}}_A^{(A)}, \widehat{\mathcal{G}}_A^{(M)}).$$

This is a direct consequence of the following lemma.

LEMMA 3.7. *Let A be a $\mathbb{Z}_{(p)}[A, M]$ -algebra and $B = A[t]/(t^2 - Mt)$, and let ε denote the image of t in B . Let $\mathbf{a} \in W^{(M)}(A)$, and put $\varepsilon \mathbf{a} = (\varepsilon a_0, \varepsilon a_1, \varepsilon a_2, \dots)$. Then:*

- (1) $E_p(\varepsilon \mathbf{a}, \Lambda; T) = 1 + \varepsilon E_p^{(M)}(\mathbf{a}, \Lambda; T);$
- (2) $F_p(\varepsilon \mathbf{a}, \Lambda; X, Y) = 1 + \varepsilon F_p^{(M)}(\mathbf{a}, \Lambda; X, Y).$

PROOF. We may assume that $A = \mathbb{Z}_{(p)}[\Lambda, M][U_0, U_1, U_2, \dots]$ and $\mathbf{a} = \mathbf{U} = (U_0, U_1, U_2, \dots)$. Put

$$E_p(\mathbf{U}, \Lambda; T) = 1 + \sum_{k=1}^{\infty} a_k(\mathbf{U}) T^k,$$

where $a_k(\mathbf{U}) \in \mathbb{Z}_{(p)}[\Lambda][U_0, U_1, U_2, \dots]$. Put

$$a_k^{(M)}(\mathbf{U}) = \frac{1}{M} a_k(MU_0, MU_1, MU_2, \dots).$$

Then $a_k^{(M)}(\mathbf{U}) \in \mathbb{Z}_{(p)}[\Lambda, M][U_0, U_1, U_2, \dots]$ since $a_k(\mathbf{U})$ has no constant term. Furthermore

$$E_p^{(M)}(\mathbf{U}, \Lambda; T) = \sum_{k=1}^{\infty} a_k^{(M)}(\mathbf{U}) T^k.$$

Now by Lemma 1.20 we have

$$\varepsilon a_k^{(M)}(\mathbf{U}) = a_k(\varepsilon U_0, \varepsilon U_1, \varepsilon U_2, \dots)$$

for $k > 0$. This implies that

$$E_p(\varepsilon \mathbf{a}, \Lambda; T) = 1 + \varepsilon E_p^{(M)}(\mathbf{a}, \Lambda; T).$$

We can prove (2) similarly.

EXAMPLE 3.8. Let $A = \mathbb{Z}[M, \Lambda/M]$. Then

$$\left[\frac{\Lambda}{M} \right]_M \in \text{Ker}[F^{(M)} - [\Lambda^{p-1}] : W^{(M)}(A) \rightarrow W^{(M)}(A)]$$

and

$$E_p^{(M)}\left(\left[\frac{\Lambda}{M} \right]_M, \Lambda; T\right) = \frac{\Lambda}{M} T.$$

(Cf. Example 2.2.1 and 2.2.2.)

EXAMPLE 3.9. Put

$$p^n[1] = (q_0, q_1, q_2, \dots) \in W(\mathbb{Z}).$$

Then, by Lemma 1.13.1,

$$p^n[1]_A = (q_0, q_1 A^{p-1}, q_2 A^{p^2-1}, \dots) \in W^{(\Lambda)}(\mathbb{Z}[A]).$$

Let

$$A = \mathbb{Z} \left[\Lambda, M, \frac{p^n \Lambda}{M}, \frac{p^{n-1} \Lambda^p}{M}, \dots, \frac{\Lambda^{p^n}}{M} \right],$$

and put

$$\mathbf{q} = p^n \left[\frac{\Lambda}{M} \right]_M = \left(\frac{q_0 \Lambda}{M}, \frac{q_1 \Lambda^p}{M}, \frac{q_2 \Lambda^{p^2}}{M}, \dots \right) \in W^{(M)}(A).$$

Then

$$\mathbf{q} \in \text{Ker} [F^{(M)} - [\Lambda^{p-1}] : W^{(M)}(A) \rightarrow W^{(M)}(A)]$$

and

$$E_p^{(M)}(\mathbf{q}, \Lambda; T) = \frac{(1 + \Lambda T)^{p^n} - 1}{M}.$$

EXAMPLE 3.10. Let A be a $\mathbb{Z}_{(p)}[\Lambda]$ -algebra. As is remarked in 1.9.2, the additive group $W^{(0)}(A)$ is isomorphic to the direct product $A^{\mathbb{N}}$ and $F^{(0)} - [\Lambda^{p-1}] : W^{(0)}(A) \rightarrow W^{(0)}(A)$ is given by

$$\Phi : (a_0, a_1, a_2, \dots) \mapsto (pa_1 - \Lambda^{p-1}a_0, pa_2 - \Lambda^{p(p-1)}a_1, pa_3 - \Lambda^{p^2(p-1)}a_2, \dots).$$

Then:

- (1) $\xi^0 : \mathbf{a} \mapsto E_p^{(0)}(\mathbf{a}, \Lambda; T)$ gives rise to an isomorphism

$$\text{Ker}[\Phi : A^{\mathbb{N}} \rightarrow A^{\mathbb{N}}] \xrightarrow{\sim} \text{Hom}_{A\text{-gr}}(\widehat{\mathcal{G}}_A^{(\Lambda)}, \widehat{\mathbb{G}}_{a,A});$$

- (2) $\xi : \mathbf{a} \mapsto F_p^{(0)}(\mathbf{a}, \Lambda; X, Y)$ gives rise to an isomorphism

$$\text{Coker}[\Phi : A^{\mathbb{N}} \rightarrow A^{\mathbb{N}}] \xrightarrow{\sim} H_0^2(\widehat{\mathcal{G}}_A^{(\Lambda)}, \widehat{\mathbb{G}}_{a,A}),$$

as was mentioned in Yato [16].

Assume now that A is a $\mathbb{Z}_{(p)}[\Lambda, \Lambda^{p-1}/p]$ -algebra. Let $a \in A$, and put

$$\mathbf{a} = \left(a, \frac{\Lambda^{p-1}}{p}a, \frac{\Lambda^{p^2-1}}{p^2}a, \frac{\Lambda^{p^3-1}}{p^3}a, \dots \right) \in A^{\mathbb{N}}.$$

Then $\mathbf{a} \in \text{Ker}[\Phi : A^{\mathbb{N}} \rightarrow A^{\mathbb{N}}]$, and we have

$$E_p^{(0)}(\mathbf{a}, \Lambda; T) = \frac{a}{\Lambda} \log(1 + \Lambda T).$$

EXAMPLE 3.11. Let A be a $\mathbb{Z}_{(p)}[\Lambda, M]$ -algebra. The homomorphism of formal groups $\alpha^{(M)} : \widehat{\mathcal{G}}_A^{(M)} \rightarrow \widehat{\mathbb{G}}_{m,A}$ induces a morphism of cochain complexes

$$\alpha^{(M)} : \tilde{C}^*(\widehat{\mathcal{G}}_A^{(\Lambda)}, \widehat{\mathcal{G}}_A^{(M)}) \rightarrow \tilde{C}^*(\widehat{\mathcal{G}}_A^{(\Lambda)}, \widehat{\mathbb{G}}_{m,A}).$$

We can easily verify the commutativity of the diagrams

$$\begin{array}{ccc} W^{(M)}(A) & \xrightarrow{\alpha^{(M)}} & W(A) \\ \xi^0 \downarrow & & \downarrow \xi^0 \\ C^1(\widehat{\mathcal{G}}_A^{(A)}, \widehat{\mathcal{G}}_A^{(M)}) & \xrightarrow{\alpha^{(M)}} & C^1(\widehat{\mathcal{G}}_A^{(A)}, \widehat{\mathbb{G}}_{m,A}) \end{array}$$

and

$$\begin{array}{ccc} W^{(M)}(A) & \xrightarrow{\alpha^{(M)}} & W(A) \\ \xi^1 \downarrow & & \downarrow \xi^1 \\ Z^2(\widehat{\mathcal{G}}_A^{(A)}, \widehat{\mathcal{G}}_A^{(M)}) & \xrightarrow{\alpha^{(M)}} & Z^2(\widehat{\mathcal{G}}_A^{(A)}, \widehat{\mathbb{G}}_{m,A}) . \end{array}$$

Moreover, we obtain a commutative diagram of cochain complexes

$$\begin{array}{ccc} \tilde{D}^*(\widehat{\mathcal{G}}_A^{(A)}, \widehat{\mathcal{G}}_A^{(M)}) & \xrightarrow{\alpha^{(M)}} & \tilde{D}^*(\widehat{\mathcal{G}}_A^{(A)}, \widehat{\mathbb{G}}_{m,A}) \\ \xi \downarrow & & \downarrow \xi \\ \tilde{C}^*(\widehat{\mathcal{G}}_A^{(A)}, \widehat{\mathcal{G}}_A^{(M)}) & \xrightarrow{\alpha^{(M)}} & \tilde{C}^*(\widehat{\mathcal{G}}_A^{(A)}, \widehat{\mathbb{G}}_{m,A}) , \end{array}$$

and therefore commutative diagrams

$$\begin{array}{ccc} \text{Ker}[F^{(M)} - [A^{p-1}] : W^{(M)}(A) \rightarrow W^{(M)}(A)] & \xrightarrow{\alpha^{(M)}} & \text{Ker}[F - [A^{p-1}] : W(A) \rightarrow W(A)] \\ \downarrow \wr \xi^0 & & \downarrow \wr \xi^0 \\ \text{Hom}_{A-\text{gr}}(\widehat{\mathcal{G}}_A^{(A)}, \widehat{\mathcal{G}}_A^{(M)}) & \xrightarrow{\alpha^{(M)}} & \text{Hom}_{A-\text{gr}}(\widehat{\mathcal{G}}_A^{(A)}, \widehat{\mathbb{G}}_{m,A}) \end{array}$$

and

$$\begin{array}{ccc} \text{Coker}[F^{(M)} - [A^{p-1}] : W^{(M)}(A) \rightarrow W^{(M)}(A)] & \xrightarrow{\alpha^{(M)}} & \text{Coker}[F - [A^{p-1}] : W(A) \rightarrow W(A)] \\ \downarrow \wr \xi^1 & & \downarrow \wr \xi^1 \\ H_0^2(\widehat{\mathcal{G}}_A^{(A)}, \widehat{\mathcal{G}}_A^{(M)}) & \xrightarrow{\alpha^{(M)}} & H_0^2(\widehat{\mathcal{G}}_A^{(A)}, \widehat{\mathbb{G}}_{m,A}) . \end{array}$$

Assume now the homothety by M is not bijective but injective on A , and put $A_0 = A/(M)$. Then we have a commutative diagram of cochain complexes with exact rows

$$\begin{array}{ccccccc} 0 \longrightarrow & \tilde{D}^*(\widehat{\mathcal{G}}_A^{(A)}, \widehat{\mathcal{G}}_A^{(M)}) & \xrightarrow{\alpha^{(M)}} & \tilde{D}^*(\widehat{\mathcal{G}}_A^{(A)}, \widehat{\mathbb{G}}_{m,A}) & \longrightarrow & \tilde{D}^*(\widehat{\mathcal{G}}_{A_0}^{(A)}, \widehat{\mathbb{G}}_{m,A_0}) & \longrightarrow 0 \\ & \downarrow \xi & & \downarrow \xi & & \downarrow \xi & \\ 0 \longrightarrow & \tilde{C}^*(\widehat{\mathcal{G}}_A^{(A)}, \widehat{\mathcal{G}}_A^{(M)}) & \xrightarrow{\alpha^{(M)}_*} & \tilde{C}^*(\widehat{\mathcal{G}}_A^{(A)}, \widehat{\mathbb{G}}_{m,A}) & \longrightarrow & \tilde{C}^*(\widehat{\mathcal{G}}_{A_0}^{(A)}, \widehat{\mathbb{G}}_{m,A_0}) . & \end{array}$$

Hence it is readily seen that the exact sequence

$$\begin{aligned} 0 \rightarrow \operatorname{Hom}_{A-\operatorname{gr}}(\widehat{\mathcal{G}}_A^{(\Lambda)}, \widehat{\mathcal{G}}_A^{(M)}) &\rightarrow \operatorname{Hom}_{A-\operatorname{gr}}(\widehat{\mathcal{G}}_A^{(\Lambda)}, \widehat{\mathbb{G}}_{m,A}) \rightarrow \operatorname{Hom}_{A_0-\operatorname{gr}}(\widehat{\mathcal{G}}_{A_0}^{(\Lambda)}, \widehat{\mathbb{G}}_{m,A_0}) \\ &\rightarrow H_0^2(\widehat{\mathcal{G}}_A^{(\Lambda)}, \widehat{\mathcal{G}}_A^{(M)}) \rightarrow H_0^2(\widehat{\mathcal{G}}_A^{(\Lambda)}, \widehat{\mathbb{G}}_{m,A}) \rightarrow H_0^2(\widehat{\mathcal{G}}_{A_0}^{(\Lambda)}, \widehat{\mathbb{G}}_{m,A_0}) \rightarrow 0 \end{aligned}$$

is deduced from the commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & W^{(M)}(A) & \xrightarrow{\alpha^{(M)}} & W(A) & \longrightarrow & W(A_0) \longrightarrow 0 \\ & & \downarrow F^{(M)} - [\Lambda^{p-1}] & & \downarrow F - [\Lambda^{p-1}] & & \downarrow F - [\Lambda^{p-1}] \\ 0 & \longrightarrow & W^{(M)}(A) & \xrightarrow{\alpha^{(M)}} & W(A) & \longrightarrow & W(A_0) \longrightarrow 0 \end{array}$$

by the snake lemma.

REMARK 3.12. Let A be a $\mathbb{Z}_{(p)}[\Lambda]$ -algebra. Assume that Λ is nilpotent in A . We proved that

(1) $\mathbf{a} \mapsto E_p(\mathbf{a}, \Lambda; T)$ gives rise to an isomorphism

$$\operatorname{Ker}[F - [\Lambda^{p-1}] : \widehat{W}(A) \rightarrow \widehat{W}(A)] \xrightarrow{\sim} \operatorname{Hom}_{A-\operatorname{gr}}(\mathcal{G}_A^{(\Lambda)}, \mathbb{G}_{m,A});$$

(2) $\mathbf{a} \mapsto F_p(\mathbf{a}, \Lambda; X, Y)$ gives rise to an isomorphism

$$\operatorname{Coker}[F - [\Lambda^{p-1}] : \widehat{W}(A) \rightarrow \widehat{W}(A)] \xrightarrow{\sim} H_0^2(\mathcal{G}_A^{(\Lambda)}, \mathbb{G}_{m,A})$$

([13, Th. 2.7.1]).

Now let A be a $\mathbb{Z}_{(p)}[\Lambda, M]$ -algebra. Assume that Λ is nilpotent in A . We can conclude similarly as in Theorem 3.5.1 that

(1) $\mathbf{a} \mapsto E_p^{(M)}(\mathbf{a}, \Lambda; T)$ gives rise to an isomorphism

$$\operatorname{Ker}[F^{(M)} - [\Lambda^{p-1}] : \widehat{W}^{(M)}(A) \rightarrow \widehat{W}^{(M)}(A)] \xrightarrow{\sim} \operatorname{Hom}_{A-\operatorname{gr}}(\mathcal{G}_A^{(\Lambda)}, \mathcal{G}_A^{(M)});$$

(2) $\mathbf{a} \mapsto F_p^{(M)}(\mathbf{a}, \Lambda; X, Y)$ gives rise to an isomorphism

$$\operatorname{Coker}[F^{(M)} - [\Lambda^{p-1}] : \widehat{W}^{(M)}(A) \rightarrow \widehat{W}^{(M)}(A)] \xrightarrow{\sim} H_0^2(\mathcal{G}_A^{(\Lambda)}, \mathcal{G}_A^{(M)}).$$

Here $\widehat{W}^{(M)}$ denote the functor defined by

$$\widehat{W}^{(M)}(A) = \left\{ (a_0, a_1, a_2, \dots) \in W^{(M)}(A); \begin{array}{l} Ma_i \text{ is nilpotent for all } i \text{ and} \\ a_i = 0 \text{ for all but a finite number of } i \end{array} \right\}.$$

Then we have a splitting exact sequence

$$0 \rightarrow \widehat{W}^{(M)}(A) \rightarrow \widehat{W}(B) \rightarrow \widehat{W}(A) \rightarrow 0,$$

where $B = A[t]/(t^2 - Mt)$.

We conclude the section by giving some remarks on a case of extensions over a discrete valuation ring [6, 7, 8].

REMARK 3.13. Let A be a discrete valuation ring and $\mathfrak{m}$ (resp. k , K) the maximal ideal (resp. the residue field, the field of fractions) of A . Take a uniformizing parameter π of A . We denote by v the valuation of A normalized by $v(\pi) = 1$.

Let $\lambda, \mu \in \mathfrak{m} - \{0\}$. Put $n = v(\mu)$ and $A_0 = A/\mathfrak{m}^n$. Let $F(T) \in A[T]$, satisfying

$$1) \quad F(0) \equiv 1 \pmod{\mu}; \quad 2) \quad F(X)F(Y) \equiv F(X + Y + \lambda XY) \pmod{\mu}.$$

We defined a smooth affine group scheme $\mathcal{E}^{(\lambda, \mu; F)}$ over A by

$$\mathcal{E}^{(\lambda, \mu; F)} = \text{Spec } A \left[T_0, T_1, \frac{1}{1 + \lambda T_0}, \frac{1}{F(T_0) + \mu T_1} \right]$$

with the multiplication:

$$\begin{aligned} T_0 &\mapsto \lambda T_0 \otimes T_0 + T_0 \otimes 1 + 1 \otimes T_0, \\ T_1 &\mapsto \mu T_1 \otimes T_1 + T_1 \otimes F(T_0) + F(T_0) \otimes T_1 \\ &\quad + \frac{1}{\mu} [F(T_0) \otimes F(T_0) - F(\lambda T_0 \otimes T_0 + T_0 \otimes 1 + 1 \otimes T_0)]. \end{aligned}$$

Furthermore we proved that $F \mapsto [\mathcal{E}^{(\lambda, \mu; F)}]$ gives rise to a surjective homomorphism

$$\partial : \text{Hom}_{A_0\text{-gr}}(\mathcal{G}^{(\lambda)}, \mathbb{G}_{m, A_0}) \rightarrow \text{Ext}_A^1(\mathcal{G}^{(\lambda)}, \mathcal{G}^{(\mu)}),$$

and $\text{Ker } \partial$ is generated by the class of $1 + \lambda T$ (cf. [6, Sect. 3] or [8, II.1.2]).

Now note that $F(T)$ is invertible in $A[[T]]$. Then

$$(T_0, T_1) \mapsto \left(T_0, \frac{T_1}{F(T_0)} \right)$$

defines an isomorphism of formal groups

$$\widehat{\mathcal{E}}^{(\lambda, \mu; F)} = \text{Spf } A[[T_0, T_1]] \xrightarrow{\sim} \widehat{\mathcal{E}} = \text{Spf } A[[T_0, T_1]],$$

where $\widehat{\mathcal{E}}$ is the extension of $\widehat{\mathcal{G}}^{(\lambda)}$ by $\widehat{\mathcal{G}}^{(\mu)}$ defined by the 2-cocycle

$$(\partial F)(X, Y) = \frac{1}{\mu} \left[\frac{F(X)F(Y)}{F(\lambda XY + X + Y)} - 1 \right] \in Z^2(\widehat{\mathcal{G}}^{(\lambda)}, \widehat{\mathcal{G}}^{(\mu)}).$$

Furthermore, defining a homomorphism

$$\mathrm{Ext}_A^1(\mathcal{G}^{(\lambda)}, \mathcal{G}^{(\mu)}) \rightarrow H_0^2(\widehat{\mathcal{G}}^{(\lambda)}, \widehat{\mathcal{G}}^{(\mu)})$$

by

$$[\mathcal{E}^{(\lambda, \mu; F)}] \mapsto [(\partial F)(X, Y)],$$

we obtain a commutative diagram with exact rows

$$\begin{array}{ccccccc} \mathrm{Hom}_{A-\mathrm{gr}}(\mathcal{G}^{(\lambda)}, \mathbb{G}_{m,A}) & \longrightarrow & \mathrm{Hom}_{A_0-\mathrm{gr}}(\mathcal{G}_{A_0}^{(\lambda)}, \mathbb{G}_{m,A_0}) & \xrightarrow{\partial} & \mathrm{Ext}_A^1(\mathcal{G}^{(\lambda)}, \mathcal{G}^{(\mu)}) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \mathrm{Hom}_{A-\mathrm{gr}}(\widehat{\mathcal{G}}^{(\lambda)}, \widehat{\mathbb{G}}_{m,A}) & \longrightarrow & \mathrm{Hom}_{A_0-\mathrm{gr}}(\widehat{\mathcal{G}}_{A_0}^{(\lambda)}, \widehat{\mathbb{G}}_{m,A_0}) & \xrightarrow{\partial} & H_0^2(\widehat{\mathcal{G}}^{(\lambda)}, \widehat{\mathcal{G}}^{(\mu)}) & & \end{array}$$

(cf. [6] or [7]).

Assume now that A is of mixed characteristics 0 and p . Let $\mathbf{a} = (a_0, a_1, a_2, \dots) \in \mathrm{Ker}[F - [\lambda p^{-1}] : W(K) \rightarrow W(K)]$. Then a_r ($r > 0$) is determined inductively by

$$\Phi_r(a_0, \dots, a_{r-1}, a_r) = \lambda^{p^{r-1}(p-1)} \Phi_{r-1}(a_0, \dots, a_{r-1}),$$

and we obtain

$$\Phi_r(\mathbf{a}) = \lambda^{p^r-1} a_0 \quad (r > 0)$$

and

$$E_p(\mathbf{a}, \lambda; T) = (1 + \lambda T)^{a_0/\lambda}.$$

Furthermore, $\mathbf{a} \in W(A)$ if and only if $(1 + \lambda T)^{a_0/\lambda} \in A[[T]]$.

PROPOSITION 3.13.1. *If $v(\lambda) \geq \frac{v(p)}{p-1} \geq v(\mu)$, then the reduction map*

$$\mathrm{Hom}_{A-\mathrm{gr}}(\widehat{\mathcal{G}}^{(\lambda)}, \widehat{\mathbb{G}}_{m,A}) \rightarrow \mathrm{Hom}_{A_0-\mathrm{gr}}(\widehat{\mathcal{G}}_{A_0}^{(\lambda)}, \widehat{\mathbb{G}}_{m,A_0})$$

is zero, and therefore, the canonical map

$$\mathrm{Ext}_A^1(\mathcal{G}^{(\lambda)}, \mathcal{G}^{(\mu)}) \rightarrow \mathrm{Ext}_A^1(\widehat{\mathcal{G}}^{(\lambda)}, \widehat{\mathcal{G}}^{(\mu)})$$

is injective.

PROOF. Let $a \in A$, and put $(1 + \lambda T)^{a/\lambda} = \sum_{j=0}^{\infty} c_j T^j$. It is verified without difficulty that, if $v(a) \geq \frac{v(p)}{p-1}$, then $v(c_k) \geq \frac{v(p)}{p-1}$ for $k > 0$, which implies that $(1 + \lambda T)^{a/\lambda} \in A[[T]]$. Furthermore, $(1 + \lambda T)^{a/\lambda} \equiv 1 \pmod{\mu}$ since $\frac{v(p)}{p-1} \geq v(\mu)$. On the other hand, it is verified that, if $v(a) < \frac{v(p)}{p-1}$, then $\lim_{r \rightarrow \infty} v(c_{p^r}) = -\infty$, which implies that $(1 + \lambda T)^{a/\lambda} \notin A[[T]]$. Hence the result.

EXAMPLE 3.13.2. If $v(\lambda) \geq v(\mu) > \frac{v(p)}{p-1} + 1$, then the canonical map

$$\mathrm{Ext}_A^1(\mathcal{G}^{(\lambda)}, \mathcal{G}^{(\mu)}) \rightarrow \mathrm{Ext}_A^1(\widehat{\mathcal{G}}^{(\lambda)}, \widehat{\mathcal{G}}^{(\mu)})$$

is not injective.

Indeed, take $a \in A$ with $v(\mu) > v(a) > \frac{v(p)}{p-1}$ and put $(1 + \lambda T)^{a/\lambda} = \sum_{j=0}^{\infty} c_j T^j$. Then $(1 + \lambda T)^{a/\lambda} \in A[[T]]$ and $\lim_{j \rightarrow \infty} v(c_j) = \infty$ since $v(\lambda) > v(a) > \frac{v(p)}{p-1}$. Therefore, $(1 + \lambda T)^{a/\lambda} \bmod \mu \in A_0[T]$. On the other hand, $(1 + \lambda T)^{a/\lambda} \not\equiv 1 \bmod \mu$ since $v(a) < v(\mu)$.

These imply that

- (1) the class $[\mathcal{E}^{(\lambda, \mu; F)}]$ is not trivial in $\mathrm{Ext}_A^1(\mathcal{G}^{(\lambda)}, \mathcal{G}^{(\mu)})$;
- (2) the image of $[\mathcal{E}^{(\lambda, \mu; F)}]$ is trivial in $\mathrm{Ext}_A^1(\widehat{\mathcal{G}}^{(\lambda)}, \widehat{\mathcal{G}}^{(\mu)})$.

Here $F(T) = (1 + \lambda T)^{a/\lambda} \bmod \mu \in A_0[T]$.

4. Around the Cartier theory

4.1. Let A be a ring. Let R denote the ring, generated by $\mathbb{V}$ over $W(A)$, with the relation:

- (1) $\mathbf{a}\mathbb{V} = \mathbb{V}F(\mathbf{a})$.

Any element of R is expressed uniquely in the form

$$\mathbf{a}_0 + \mathbb{V}\mathbf{a}_1 + \cdots + \mathbb{V}^n \mathbf{a}_n, \quad \mathbf{a}_j \in W(A).$$

By the relation (1), $\mathbb{V}^n R$ is a two-sided ideal of R . Put $\widehat{R} = \varprojlim_n R/\mathbb{V}^n R$.

Now we denote by D_A the ring, generated by $\mathbb{F}$ over $\widehat{R}$, with the relations:

- (2) $\mathbb{F}\mathbf{a} = F(\mathbf{a})\mathbb{F}$,
- (3) $\mathbb{F}\mathbb{V} = p$,
- (4) $\mathbb{V}\mathbf{a}\mathbb{F} = V(\mathbf{a})$.

Any element of D_A is expressed uniquely in the form

$$\sum_{j=1}^m \mathbf{a}_{-j} \mathbb{F}^j + \mathbf{a}_0 + \sum_{j=1}^{\infty} \mathbb{V}^j \mathbf{a}_j, \quad \mathbf{a}_j \in W(A).$$

D_A is called the Cartier-Dieudonné algebra or the Dieudonné algebra over A . (Cf. [4, 28.1].)

EXAMPLE 4.2.1. Let A be a ring. Let $\mathbf{a} \in W(A)$. Then

$$\mathbf{T} = (T_0, T_1, T_2, \dots) \mapsto \mathbf{a}\mathbf{T} = (P_0(\mathbf{a}, \mathbf{T}), P_1(\mathbf{a}, \mathbf{T}), P_2(\mathbf{a}, \mathbf{T}), \dots)$$

defines an endomorphism of the formal group $\widehat{W}_A$. Moreover,

$$T \mapsto \left(\sum_{j=1}^m V^j a_{-j} + a_0 + \sum_{j=1}^{\infty} a_j F^j \right) T, \quad a_j \in W(A)$$

defines an endomorphism of $\widehat{W}_A$. $V \mapsto \mathbb{F}$ and $F \mapsto \mathbb{V}$ give rise to an anti-isomorphism of $\text{End}_{A\text{-gr}} \widehat{W}_A$ to D_A (cf. [4, 27.2.12]).

EXAMPLE 4.2.2. Let A be a ring. $W(A)$ is a left D_A -module, equipped with $\mathbb{F} = F$ and $\mathbb{V} = V$.

EXAMPLE 4.2.3. Let A be a $\mathbb{Z}[M]$ -algebra. $W^{(M)}(A)$ is a left D_A -module, equipped with $\mathbb{F} = F^{(M)}$ and $\mathbb{V} = V$. Furthermore, $\alpha^{(M)} : W^{(M)}(A) \rightarrow W(A)$ is a D_A -homomorphism.

REMARK 4.2.4. Let A be a ring and $\mathbf{P} = \sum_{j=0}^{\infty} \mathbb{V}^j a_j \in D_A$. If a_0 is invertible in $W(A)$, then $\mathbf{P}$ is invertible in D_A .

Indeed, define $\mathbf{b}_k \in W(A)$ inductively by

$$\mathbf{b}_0 = a_0^{-1}$$

and

$$\mathbf{b}_k = -(F^k a_0)^{-1} \left(\sum_{i=1}^k (F^{k-i} a_i) \mathbf{b}_{k-i} \right).$$

Then $\mathbf{Q} = \sum_{j=0}^{\infty} \mathbb{V}^j \mathbf{b}_j$ is the inverse of $\mathbf{P} = \sum_{j=0}^{\infty} \mathbb{V}^j a_j$.

4.3. Let A be a $\mathbb{Z}_{(p)}[\Lambda]$ -algebra. Define a D_A -homomorphism $\varphi_\Lambda : D_A \rightarrow W^{(\Lambda)}(A)$ by

$$\begin{aligned} \varphi : \sum_{j=1}^m a_{-j} \mathbb{F}^j + a_0 + \sum_{j=1}^{\infty} \mathbb{V}^j a_j &\mapsto \left(\sum_{j=1}^m a_{-j} \mathbb{F}^j + a_0 + \sum_{j=1}^{\infty} \mathbb{V}^j a_j \right) [1]_\Lambda \\ &= \sum_{j=1}^m a_{-j} \cdot F^{(\Lambda)^j}([1]_\Lambda) + \sum_{j=0}^{\infty} V^j(a_j \cdot [1]_\Lambda) \\ &= \sum_{j=1}^m a_{-j} \cdot [\Lambda^{p^j-1}]_\Lambda + \sum_{j=0}^{\infty} V^j(a_j \cdot [1]_\Lambda). \end{aligned}$$

Since

$$\varphi_\Lambda(\mathbb{F} - [\Lambda^{p-1}]) = [\Lambda^{p-1}]_\Lambda - [\Lambda^{p-1}] \cdot [1]_\Lambda = 0,$$

φ_Λ induces a D_A -homomorphism

$$\tilde{\varphi}_\Lambda : D_A/D_A(\mathbb{F} - [\Lambda^{p-1}]) \rightarrow W^{(\Lambda)}(A).$$

THEOREM 4.4. Under the notation above, $\tilde{\varphi}_A$ is bijective.

PROOF. It is obvious that $\tilde{\varphi}$ is surjective since

$$\varphi\left(\sum_{j=0}^{\infty} \mathbb{V}^j[a_j]\right) = \left(\sum_{j=0}^{\infty} \mathbb{V}^j[a_j]\right)[1]_A = \sum_{j=0}^{\infty} V([a_j]_A).$$

Assume that

$$\varphi\left(\sum_{j=1}^m \mathbf{a}_{-j} \mathbb{F}^j + \mathbf{a}_0 + \sum_{j=1}^{\infty} \mathbb{V}^j \mathbf{a}_j\right) = 0.$$

Define $\mathbf{b}_{-m+1}, \dots, \mathbf{b}_0, \mathbf{r} \in W(A)$ inductively by

$$\begin{aligned} \mathbf{b}_{-m+1} &= \mathbf{a}_{-m}, \quad \mathbf{b}_{-m+i} = \mathbf{a}_{-m+i-1} + [\Lambda^{p^{m-i+1}(p-1)}] \mathbf{b}_{-m+i-1} \quad (1 < i \leq m), \\ \mathbf{r} &= \mathbf{a}_0 + [\Lambda^{p-1}] \mathbf{b}_0. \end{aligned}$$

Then we have

$$\sum_{j=0}^m \mathbf{a}_{-j} \mathbb{F}^j = \left(\sum_{j=0}^{m-1} \mathbf{b}_{-j} \mathbb{F}^j\right)(\mathbb{F} - [\Lambda^{p-1}]) + \mathbf{r}.$$

Therefore we may assume that

$$\varphi\left(\sum_{j=0}^{\infty} \mathbb{V}^j \mathbf{a}_j\right) = 0.$$

Now we verify that there exist $\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2, \dots \in W(A)$ such that $V\mathbf{b}_0 = \mathbf{a}_0$ and $V\mathbf{b}_j = \mathbf{a}_j + [\Lambda^{p-1}]\mathbf{b}_{j-1}$ for each $j > 0$.

In fact, since

$$\left(\sum_{j=0}^{\infty} \mathbb{V}^j \mathbf{a}_j\right)[1]_A = 0,$$

$\mathbf{a}_0 \cdot [1]_A \in \text{Im}[V : W^{(A)}(A) \rightarrow W^{(A)}(A)]$, and therefore, $\mathbf{a}_0 \in \text{Im}[V : W(A) \rightarrow W(A)]$. Now assume that there exist $\mathbf{b}_0, \mathbf{b}_1, \dots, \mathbf{b}_n \in W(A)$ such that $V\mathbf{b}_j = \mathbf{a}_j + [\Lambda^{p-1}]\mathbf{b}_{j-1}$ for $0 \leq j \leq n$. Here we set $\mathbf{b}_{-1} = \mathbf{0}$. Then we have

$$\sum_{j=0}^n \mathbb{V}^j \mathbf{a}_j = \left(\sum_{j=0}^n \mathbb{V}^{j+1} \mathbf{b}_j\right)(\mathbb{F} - [\Lambda^{p-1}]) + \mathbb{V}^{n+1}([\Lambda^{p-1}]\mathbf{b}_n),$$

and therefore,

$$V^{n+1}([\Lambda^{p-1}]\mathbf{b}_n \cdot [1]_A) + \sum_{j=n+1}^{\infty} V^j(\mathbf{a}_j \cdot [1]_A) = 0.$$

It follows that $V^{n+1}([A^{p-1}]b_n + a_{n+1}) \cdot [1]_A \in \text{Im}[V^{n+2} : W^{(A)}(A) \rightarrow W^{(A)}(A)]$ and that $[A^{p-1}]b_n + a_{n+1} \in \text{Im}[V : W(A) \rightarrow W(A)]$.

Hence we have obtained

$$\sum_{j=0}^{\infty} V^j a_j = \left(\sum_{j=0}^{\infty} V^{j+1} b_j \right) (\mathbb{F} - [A^{p-1}]),$$

and we can conclude that $\text{Ker } \varphi = D_A(\mathbb{F} - [A^{p-1}])$ and that $\tilde{\varphi}_A$ is injective.

COROLLARY 4.5. *The sequence $0 \rightarrow D_A \xrightarrow{\cdot(\mathbb{F} - [A^{p-1}])} D_A \xrightarrow{\varphi_A} W^{(A)}(A) \rightarrow 0$ is a free resolution of $W^{(A)}(A)$.*

COROLLARY 4.6. *Let E be a left D_A -module. Then:*

- (1) $\text{Hom}_{D_A}(W^{(A)}(A), E)$ is isomorphic to $\text{Ker}[\mathbb{F} - [A^{p-1}] : E \rightarrow E]$;
- (2) $\text{Ext}_{D_A}^1(W^{(A)}(A), E)$ is isomorphic to $\text{Coker}[\mathbb{F} - [A^{p-1}] : E \rightarrow E]$;
- (3) $\text{Ext}_{D_A}^j(W^{(A)}(A), E) = 0$ for $j > 1$.

EXAMPLE 4.7. Let A be a $\mathbb{Z}[A, M]$ -algebra. Then $\varphi \mapsto \varphi([1]_A)$ gives rise to an isomorphism

$$\text{Hom}_{D_A}(W^{(A)}(A), W^{(M)}(A)) \xrightarrow{\sim} \text{Ker}[F^{(M)} - [A^{p-1}] : W^{(M)}(A) \rightarrow W^{(M)}(A)].$$

Putting $a = \varphi([1])$, we obtain

$$\varphi(u) = \varphi\left(\sum_{k=0}^{\infty} V^k [u_k]\right) = \sum_{k=0}^{\infty} V^k \varphi([u_k]) = \sum_{k=0}^{\infty} V^k ([u_k] \cdot a)$$

for $u = (u_0, u_1, u_2, \dots) \in W^{(A)}(A)$.

REMARK 4.8. Let A be a $\mathbb{Z}[A, M]$ -algebra, and let $a = (a_0, a_1, a_2, \dots) \in W^{(A)}(A)$, $b = (b_0, b_1, b_2, \dots) \in W^{(M)}(A)$. We define $\langle a, b \rangle_{A, M} \in W^{(M)}(A)$ by

$$\langle a, b \rangle_{A, M} = \sum_{k=0}^{\infty} V^k ([a_k]b) = \sum_{k=0}^{\infty} V^k \left(\sum_{i=0}^k [a_i^{p^{k-i}} b_{k-i}] \right),$$

adopting the formula at the end of 4.7. Putting

$$A = \sum_{k=0}^{\infty} V^k [a_k], \quad B = \sum_{k=0}^{\infty} V^k [b_k],$$

we have

$$A[1]_A = a, \quad B[1]_M = b$$

and

$$(AB)[1]_M = \langle a, b \rangle_{A,M}.$$

It is readily seen that:

- (1) $\langle a, b + b' \rangle_{A,M} = \langle a, b \rangle_{A,M} + \langle a, b' \rangle_{A,M}$;
- (2) $\langle a + a', b \rangle_{A,M} = \langle a, b \rangle_{A,M} + \langle a', b \rangle_{A,M}$ if $F^{(M)}b = [A^{p-1}]b$.

EXAMPLE 4.9. Let A be a $\mathbb{Z}[A, M]$ -algebra. Then the isomorphism

$$\text{Ext}_{D_A}^1(W^{(A)}(A), W^{(M)}(A)) \xrightarrow{\sim} \text{Coker}[F^{(M)} - [A^{p-1}] : W^{(M)}(A) \rightarrow W^{(M)}(A)]$$

is described explicitly as follows.

Let $C \in D_A$, and define a left D_A -endomorphism $U_{A,M,C}$ of D_A^2 by

$$U_{A,M,C} : (P, Q) \mapsto (P, Q) \begin{pmatrix} \mathbb{F} - [A^{p-1}] & -C \\ 0 & \mathbb{F} - [M^{p-1}] \end{pmatrix}.$$

Then we have gotten a commutative diagram of left D_A -modules with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & D_A & \xrightarrow{i} & D_A \oplus D_A & \xrightarrow{j} & D_A \longrightarrow 0 \\ & & \downarrow \cdot(\mathbb{F} - [M^{p-1}]) & & \downarrow U_{A,M,C} & & \downarrow \cdot(\mathbb{F} - [A^{p-1}]) \\ 0 & \longrightarrow & D_A & \xrightarrow{i} & D_A \oplus D_A & \xrightarrow{j} & D_A \longrightarrow 0. \end{array}$$

Here i and j are defined by $i : Q \mapsto (0, Q)$ and $j : (P, Q) \mapsto P$. Taking the cokernels of the vertical arrows, we obtain an extension of D_A -modules

$$0 \rightarrow W^{(M)}(A) \rightarrow E \rightarrow W^{(A)}(A) \rightarrow 0.$$

PROPOSITION 4.9.1. Put $c = C[1]_M \in W^{(M)}(A)$. The class of E in $\text{Ext}_{D_A}^1(W^{(A)}(A), W^{(M)}(A))$ coincides with the class of c in $\text{Coker}[F^{(M)} - [A^{p-1}] : W^{(M)}(A) \rightarrow W^{(M)}(A)]$.

PROOF. Let $\varphi_{(M,c)} : D_A \rightarrow W^{(M)}(A)$ be a D_A -homomorphism defined by $P \mapsto Pc$. Then an extension representing

$$[c] \in \text{Coker}[F^{(M)} - [A^{p-1}] : W^{(M)}(A) \rightarrow W^{(M)}(A)] = H_0^2(\widehat{\mathcal{G}}_A^{(A)}, \widehat{\mathcal{G}}_A^{(M)})$$

is given by the push-down

$$\begin{array}{ccccccc} 0 & \longrightarrow & D_A & \xrightarrow{\cdot(\mathbb{F} - [A^{p-1}])} & D_A & \longrightarrow & W^{(A)}(A) \longrightarrow 0 \\ & & \downarrow \varphi_{(M,c)} & & \downarrow & & \parallel \\ 0 & \longrightarrow & W^{(M)}(A) & \longrightarrow & \tilde{E} & \longrightarrow & W^{(A)}(A) \longrightarrow 0. \end{array}$$

Define now $i' : D_A \rightarrow D_A \oplus D_A$ by

$$Q \mapsto (0, Q(\mathbb{F} - [M^{p-1}]))$$

and $j' : D_A \oplus D_A \rightarrow D_A \oplus W^{(M)}(A)$ by

$$(P, Q) \mapsto (P, \varphi_M(Q)) = (P, Q[1]_M).$$

Then we have a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & D_A & \xrightarrow{i} & D_A \oplus D_A & \xrightarrow{j} & D_A \longrightarrow 0 \\ & & \parallel & & \downarrow U_{\Lambda, M, c} & & \downarrow (\mathbb{F} - [\Lambda^{p-1}], -\varphi_{(M, c)}) \\ 0 & \longrightarrow & D_A & \xrightarrow{i'} & D_A \oplus D_A & \xrightarrow{j'} & D_A \oplus W^{(M)}(A) \longrightarrow 0. \end{array}$$

Taking the cokernels of the vertical arrows, we obtain the desired isomorphism $E \xrightarrow{\sim} \tilde{E}$.

REMARK 4.10. Let $\tilde{E}$ be the extension of $W^{(\Lambda)}(A)$ by $W^{(M)}(A)$ defined above. We can describe $\tilde{E}$ in terms of 2-cocycles as follows.

At first consider the extension of D_A -module

$$0 \rightarrow D_A \xrightarrow{(\mathbb{F} - [\Lambda^{p-1}])} D_A \xrightarrow{\varphi_\Lambda} W^{(\Lambda)}(A) \rightarrow 0.$$

Define a section $s : W^{(\Lambda)}(A) \rightarrow D_A$ of φ_Λ by

$$s(\mathbf{a}) = \sum_{k=0}^{\infty} \mathbb{V}^k[a_k],$$

where $\mathbf{a} = (a_0, a_1, a_2, \dots)$. Then the 2-cocycle $Q \in Z^2(W^{(\Lambda)}(A), D_A)$ associated to the section s is given by

$$Q(\mathbf{a}, \mathbf{a}')(\mathbb{F} - [\Lambda^{p-1}]) = s(\mathbf{a}) + s(\mathbf{a}') - s(\mathbf{a} + \mathbf{a}').$$

Note now that $\tilde{c} = \varphi_{(M, c)} \circ Q \in Z^2(W^{(\Lambda)}(A), W^{(M)}(A))$ is defined by

$$\tilde{c}(\mathbf{a}, \mathbf{a}') = Q(\mathbf{a}, \mathbf{a}')c.$$

Hence the extension $\tilde{E}$ of $W^{(\Lambda)}(A)$ by $W^{(M)}(A)$ is written as $\tilde{E} = W^{(\Lambda)}(A) \times W^{(M)}(A)$ with the addition

$$(\mathbf{a}, \mathbf{b}) + (\mathbf{a}', \mathbf{b}') = (\mathbf{a} + {}^{(\Lambda)}\mathbf{a}', \mathbf{b} + {}^{(M)}\mathbf{b}' + {}^{(M)}\tilde{c}(\mathbf{a}, \mathbf{a}')).$$

It is easily verified that, if $\mathbf{c} = (F^{(M)} - [\Lambda^{p-1}])\mathbf{b}$,

$$\tilde{\mathbf{c}}(\mathbf{a}, \mathbf{a}') = \langle \mathbf{a}, \mathbf{b} \rangle_{\Lambda, M} + \langle \mathbf{a}', \mathbf{b} \rangle_{\Lambda, M} - \langle \mathbf{a} + \mathbf{a}', \mathbf{b} \rangle_{\Lambda, M}.$$

EXAMPLE 4.11. Let

$$A = \mathbb{Z}[\Lambda, M, U_0, U_1, U_2, \dots, U'_0, U'_1, U'_2, \dots, V_0, V_1, V_2, \dots]$$

and

$$\mathbf{U} = (U_0, U_1, U_2, \dots), \mathbf{U}' = (U'_0, U'_1, U'_2, \dots) \in W^{(\Lambda)}, \mathbf{V} = (V_0, V_1, V_2, \dots) \in W^{(M)}.$$

We can describe explicitly $\tilde{\mathbf{V}}(\mathbf{U}, \mathbf{U}') \in W^{(M)}(A)$ as follows.

Put

$$\mathbf{U}'' = (U''_0, U''_1, U''_2, \dots) = \mathbf{U} + {}^{(\Lambda)}\mathbf{U}' \in W^{(\Lambda)}(A)$$

and

$$\mathbf{Q} = \mathbf{Q}(\mathbf{U}, \mathbf{U}') = \sum_{k=1}^{\infty} \mathbb{V}^k \mathbf{q}_k, \quad \mathbf{q}_k \in W(A).$$

Then we have

$$V\mathbf{q}_1 = [U_0] + [U'_0] - [U''_0]$$

and

$$V\mathbf{q}_k = [U_{k-1}] + [U'_{k-1}] - [U''_{k-1}] + [\Lambda^{p-1}]\mathbf{q}_{k-1}$$

for $k \geq 2$ since

$$\left(\sum_{k=1}^{\infty} \mathbb{V}^k \mathbf{q}_k \right) (\mathbb{F} - [\Lambda^{p-1}]) = \sum_{k=0}^{\infty} \mathbb{V}^k ([U_k] + [U'_k] - [U''_k])$$

and

$$\left(\sum_{k=1}^{\infty} \mathbb{V}^k \mathbf{q}_k \right) (\mathbb{F} - [\Lambda^{p-1}]) = V\mathbf{q}_1 + \sum_{k=1}^{\infty} \mathbb{V}^k (V\mathbf{q}_{k+1} - [\Lambda^{p-1}]\mathbf{q}_k)$$

(cf. the proof of Theorem 4.4). This implies that

$$\Phi_r(\mathbf{q}_1) = \frac{1}{p} \{ \Phi_{r+1}([U_0]) + \Phi_{r+1}([U'_0]) - \Phi_{r+1}([U''_0]) \}$$

and

$$\Phi_r(\mathbf{q}_k) = \frac{1}{p} \{ \Phi_{r+1}([U_{k-1}]) + \Phi_{r+1}([U'_{k-1}]) - \Phi_{r+1}([U''_{k-1}]) + \Phi_{r+1}([\Lambda^{p-1}]\mathbf{q}_{k-1}) \}.$$

Noting that

$$\begin{aligned} \Phi_{r+k}^{(\Lambda)}(U_0, \dots, U_{k-2}, 0, \dots, 0) + p^{k-1} \Phi_{r+1}^{(\Lambda)}([U_{k-1}]) \\ = \Phi_{r+k}^{(\Lambda)}(U_0, \dots, U_{k-2}, U_{k-1}, 0, \dots, 0), \end{aligned}$$

we obtain

$$\begin{aligned} \Phi_r(q_k) &= \frac{1}{p^k \Lambda^{p^{r+1}-1}} \{ \Phi_{r+k}^{(\Lambda)}(U_0, U_1, \dots, U_{k-1}, 0, \dots, 0) \\ &\quad + \Phi_{r+k}^{(\Lambda)}(U'_0, U'_1, \dots, U'_{k-1}, 0, \dots, 0) - \Phi_{r+k}^{(\Lambda)}(U''_0, U''_1, \dots, U''_{k-1}, 0, \dots, 0) \} \\ &= \frac{1}{p^k \Lambda^{p^{r+1}}} \{ \Phi_{r+k}(\Lambda U_0, \Lambda U_1, \dots, \Lambda U_{k-1}, 0, \dots, 0) \\ &\quad + \Phi_{r+k}(\Lambda U'_0, \Lambda U'_1, \dots, \Lambda U'_{k-1}, 0, \dots, 0) - \Phi_{r+k}(\Lambda U''_0, \Lambda U''_1, \dots, \Lambda U''_{k-1}, 0, \dots, 0) \} \end{aligned}$$

for $k \geq 1$ and $r \geq 0$. Put now

$$Z = \tilde{V}(U, U') = \sum_{k=1}^{\infty} V^k(q_k \cdot V) \in W^{(M)}(A).$$

Then we have

$$\begin{aligned} \Phi_r^{(M)}(Z) &= \sum_{k=1}^r p^k \Phi_{r-k}(q_k) \Phi_{r-k}^{(M)}(V) \\ &= \sum_{k=1}^r \frac{1}{\Lambda^{p^{r-k+1}}} \{ \Phi_r(\Lambda U_0, \Lambda U_1, \dots, \Lambda U_{k-1}, 0, \dots, 0) \\ &\quad + \Phi_r(\Lambda U'_0, \Lambda U'_1, \dots, \Lambda U'_{k-1}, 0, \dots, 0) \\ &\quad - \Phi_r(\Lambda U''_0, \Lambda U''_1, \dots, \Lambda U''_{k-1}, 0, \dots, 0) \} \Phi_{r-k}^{(M)}(V). \end{aligned}$$

REMARK 4.12. Let A be a $\mathbb{Z}[M]$ -algebra. Then the D_A -homomorphism $\alpha^{(M)} : W^{(M)}(A) \rightarrow W(A)$ is represented by a morphism of complexes

$$\begin{array}{ccccccc} 0 & \longrightarrow & D_A & \xrightarrow{\cdot(\mathbb{F}-[M^{p-1}])} & D_A & \longrightarrow & W^{(M)}(A) \longrightarrow 0 \\ & & \downarrow \cdot[M^p] & & \downarrow \cdot[M] & & \downarrow \alpha^{(M)} \\ 0 & \longrightarrow & D_A & \xrightarrow{\cdot(\mathbb{F}-1)} & D_A & \longrightarrow & W(A) \longrightarrow 0. \end{array}$$

From now, we give an interpretation of $W^{(M)}(A)$ as the Cartier module of the formal group $\hat{\mathcal{G}}_A^{(M)}$.

4.13. Let A be a $\mathbb{Z}_{(p)}$ -algebra, and let G be a commutative formal group over A . Then it is well known that $\text{Hom}_{A\text{-gr}}(\widehat{W}_A, G)$ is isomorphic to the Cartier module $C(G)$ of p -typical curves of G (the first theorem of the Cartier theory on p -typical curves, cf. [4, Th. 27.7.5]).

$C(G) = \text{Hom}_{A\text{-gr}}(\widehat{W}_A, G)$ has a structure of right $\text{End}_{A\text{-gr}} \widehat{W}_A$ -module by the composition of morphisms. Therefore $C(G)$ has a structure of left D_A -module since $\text{End}_{A\text{-gr}} \widehat{W}_A$ is anti-isomorphic to D_A , as is remarked in 4.2.1.

EXAMPLE 4.14. Let $\mathbf{U} = (U_0, U_1, U_2, \dots)$ and $\mathbf{T} = (T_0, T_1, T_2, \dots)$. Define a formal power series $E_p(\mathbf{U}; \mathbf{T}) \in \mathbb{Z}_{(p)}[U_0, U_1, U_2, \dots][[T_0, T_1, T_2, \dots]]$ by

$$E_p(\mathbf{U}; \mathbf{T}) = \exp \left(\sum_{r=0}^{\infty} \frac{\Phi_r(\mathbf{U}) \Phi_r(\mathbf{T})}{p^r} \right) = \exp \left(\sum_{r=0}^{\infty} \frac{\Phi_r(P(\mathbf{U}, \mathbf{T}))}{p^r} \right).$$

It is verified without difficulty that

- (1) $E_p(S(\mathbf{U}, \mathbf{V}); \mathbf{T}) = E_p(\mathbf{U}; \mathbf{T}) E_p(\mathbf{V}; \mathbf{T});$
- (2) $E_p(\mathbf{U}; S(\mathbf{V}, \mathbf{T})) = E_p(\mathbf{U}; \mathbf{V}) E_p(\mathbf{U}; \mathbf{T});$
- (3) $E_p(P(\mathbf{U}, \mathbf{V}); \mathbf{T}) = E_p(\mathbf{V}; P(\mathbf{U}, \mathbf{T}));$
- (4) $E_p(F(\mathbf{U}); \mathbf{T}) = E_p(\mathbf{U}; V(\mathbf{T}));$
- (5) $E_p(V(\mathbf{U}); \mathbf{T}) = E_p(\mathbf{U}; F(\mathbf{T})).$

Let A be a $\mathbb{Z}_{(p)}$ -algebra. Then it is known that $\mathbf{a} \mapsto E_p(\mathbf{a}; \mathbf{T})$ gives rise to a D_A -isomorphism $\eta: W(A) \xrightarrow{\sim} C(\widehat{\mathbb{G}}_{m,A}) = \text{Hom}_{A\text{-gr}}(\widehat{W}_A, \widehat{\mathbb{G}}_{m,A})$ (cf. [4, 17.2.12]).

4.15. Let A be a $\mathbb{Z}[A, M]$ -algebra and $B = A[t]/(t^2 - Mt)$. Let ε denote the image of t in B . Then we have $\varepsilon^2 = M\varepsilon$. The exact sequence of formal groups

$$0 \rightarrow \widehat{\mathcal{G}}_A^{(M)} \rightarrow \left(\prod_{B/A} \widehat{\mathbb{G}}_{m,B} \right) \rightarrow \widehat{\mathbb{G}}_{m,A} \rightarrow 0$$

induces an exact sequence of Cartier modules

$$0 \rightarrow C(\widehat{\mathcal{G}}_A^{(M)}) \rightarrow C(\widehat{\mathbb{G}}_{m,B}) \rightarrow C(\widehat{\mathbb{G}}_{m,A}) \rightarrow 0.$$

More precisely,

$$C(\widehat{\mathbb{G}}_{m,B}) = \text{Hom}_{A\text{-gr}}(\widehat{W}_B, \widehat{\mathbb{G}}_{m,B}) \rightarrow C(\widehat{\mathbb{G}}_{m,A}) = \text{Hom}_{A\text{-gr}}(\widehat{W}_A, \widehat{\mathbb{G}}_{m,A})$$

is induced from the ring homomorphism $B \rightarrow A$, and

$$C(\widehat{\mathcal{G}}^{(M)}) = \text{Hom}_{A\text{-gr}}(\widehat{W}_A, \widehat{\mathcal{G}}^{(M)}) \rightarrow C(\widehat{\mathbb{G}}_{m,B}) = \text{Hom}_{B\text{-gr}}(\widehat{W}_B, \widehat{\mathbb{G}}_{m,B})$$

is defined by

$$F(\mathbf{T}) \mapsto 1 + \varepsilon F(\mathbf{T}).$$

On the other hand, we have an exact sequence of left D_A -modules

$$0 \rightarrow W^{(M)}(A) \rightarrow W(B) \rightarrow W(A) \rightarrow 0$$

by Theorem 1.21. Since the diagram

$$\begin{array}{ccc} W(B) & \longrightarrow & W(A) \\ \eta \downarrow & & \downarrow \eta \\ C(\widehat{\mathbb{G}}_{m,B}) & \longrightarrow & C(\widehat{\mathbb{G}}_{m,A}) \end{array}$$

is commutative, we obtain a commutative diagram of left D_A -modules with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & W^{(M)}(A) & \longrightarrow & W(B) & \longrightarrow & W(A) \longrightarrow 0 \\ & & \downarrow \eta & & \downarrow \eta & & \downarrow \eta \\ 0 & \longrightarrow & C(\widehat{\mathcal{G}}_A^{(M)}) & \longrightarrow & C(\widehat{\mathbb{G}}_{m,B}) & \longrightarrow & C(\widehat{\mathbb{G}}_{m,A}) \longrightarrow 0. \end{array}$$

We have gotten the following theorem, noting $\eta : W(B) \rightarrow C(\widehat{\mathbb{G}}_{m,B})$ and $\eta : W(A) \rightarrow C(\widehat{\mathbb{G}}_{m,A})$ are bijective.

THEOREM 4.15.1. *Let A be a $\mathbb{Z}_{(p)}[M]$ -algebra. Then there exists a D_A -isomorphism*

$$\eta : W^{(M)}(A) \xrightarrow{\sim} C(\widehat{\mathcal{G}}^{(M)}) = \mathrm{Hom}_{A-\mathrm{gr}}(\widehat{W}_A, \widehat{\mathcal{G}}^{(M)}).$$

4.16. Now we introduce formal power series to describe explicitly the isomorphism $\eta : W^{(M)}(A) \xrightarrow{\sim} \mathrm{Hom}_{A-\mathrm{gr}}(\widehat{W}_A, \widehat{\mathcal{G}}^{(M)})$.

Let $\mathbf{U} = (U_0, U_1, U_2, \dots)$ and $\mathbf{T} = (T_0, T_1, T_2, \dots)$. We define a formal power series

$$E_p^{(M)}(\mathbf{U}; \mathbf{T}) \in \mathbb{Z}_{(p)}[M][U_0, U_1, U_2, \dots][[T_0, T_1, T_2, \dots]]$$

by

$$E_p^{(M)}(\mathbf{U}; \mathbf{T}) = \frac{1}{M} [E_p(\alpha^{(M)} \mathbf{U}; \mathbf{T}) - 1].$$

EXAMPLE 4.17.1. $E_p^{(1)}(\mathbf{U}; \mathbf{T}) = E_p(\mathbf{U}; \mathbf{T}) - 1$.

EXAMPLE 4.17.2. $E_p^{(0)}(\mathbf{U}; \mathbf{T}) = \sum_{r=0}^{\infty} U_r \Phi_r(\mathbf{T})$.

Indeed, by the definition, we have

$$\Phi_r(MU_0, \dots, MU_r) \equiv p^r MU_r \pmod{M^p}.$$

It follows that

$$\begin{aligned} \log E_p(\alpha^{(M)}U : T) &= \sum_{r=0}^{\infty} \frac{\Phi_r(MU_0, \dots, MU_r)\Phi_r(T)}{p^r} \\ &\equiv 1 + M \left(\sum_{r=0}^{\infty} U_r \Phi_r(T) \right) \pmod{M^p}. \end{aligned}$$

Hence we obtain

$$E_p(\alpha^{(M)}U : T) \equiv 1 + M \left(\sum_{r=0}^{\infty} U_r \Phi_r(T) \right) \pmod{M^2},$$

and

$$E_p^{(M)}(U; T) = \frac{1}{M} [E_p(\alpha^{(M)}U; T) - 1] \equiv \sum_{r=0}^{\infty} U_r \Phi_r(T) \pmod{M}.$$

PROPOSITION 4.18. *Let $U = (U_0, U_1, U_2, \dots)$, $V = (V_0, V_1, V_2, \dots)$ and $T = (T_0, T_1, T_2, \dots)$. Then:*

- (1) $E_p^{(M)}(U + {}^{(M)}V; T) = E_p^{(M)}(U; T) + E_p^{(M)}(V; T) + ME_p^{(M)}(U; T)E_p^{(M)}(V; T);$
- (2) $E_p^{(M)}(U; V + T) = E_p^{(M)}(U; V) + E_p^{(M)}(U; T) + ME_p^{(M)}(U; V)E_p^{(M)}(U; T);$
- (3) $E_p^{(M)}(V \cdot {}^{(M)}U; T) = E_p^{(M)}(U; VT);$
- (4) $E_p^{(M)}(F^{(M)}(U); T) = E_p^{(M)}(U; V(T));$
- (5) $E_p^{(M)}(V(U); T) = E_p^{(M)}(U; F(T)).$

PROOF. The assertion (2) follows immediately from 4.14(2). We can deduce (1), (3), (4) and (5) from 4.14(1), 4.14(3), 4.14(4) and 4.14(5), combining the formulas

$$\begin{aligned} \alpha^{(M)}(U + {}^{(M)}V) &= \alpha^{(M)}(U) + \alpha^{(M)}(V), \\ \alpha^{(M)}(V \cdot {}^{(M)}U) &= V \cdot \alpha^{(M)}(U), \\ \alpha^{(M)}(F^{(M)}(U)) &= F(\alpha^{(M)}(U)), \\ \alpha^{(M)}(V(U)) &= V(\alpha^{(M)}(U)), \end{aligned}$$

respectively.

PROPOSITION 4.19. *Let A be a $\mathbb{Z}_{(p)}[M]$ -algebra. Then the isomorphism $\eta : W^{(M)}(A) \xrightarrow{\sim} C(\widehat{\mathcal{G}}^{(M)}) = \text{Hom}_{A\text{-gr}}(\widehat{W}_A, \widehat{\mathcal{G}}^{(M)})$ is given by $a \mapsto E_p^{(M)}(a; T)$.*

PROOF. This is a direct consequence of the following lemma.

LEMMA 4.20. Let A be a $\mathbb{Z}_{(p)}[M]$ -algebra and $B = A[t]/(t^2 - Mt)$, and let ε denote the image of t in B . Let $\mathbf{a} \in W^{(M)}$, and put $\varepsilon\mathbf{a} = (\varepsilon a_0, \varepsilon a_1, \varepsilon a_2, \dots)$. Then:

$$1 + \varepsilon E_p^{(M)}(\mathbf{a}; \mathbf{T}) = E_p(\varepsilon\mathbf{a}; \mathbf{T}).$$

PROOF. We can prove the assertion similarly as in the proof of Lemma 3.7, and thus omit the details.

EXAMPLE 4.21. Let A be a $\mathbb{Z}_{(p)}[M]$ -algebra. Then the additive group $C(\widehat{\mathbb{G}}_{a,A}) = W^{(0)}(A)$ is isomorphic to the direct product $A^{\mathbb{N}}$, and $\mathbb{F}$ and $\mathbb{V}$ on $C(\widehat{\mathbb{G}}_{a,A})$ are given by

$$F^{(0)} : (a_0, a_1, a_2, \dots) \mapsto (pa_1, pa_2, pa_3, \dots)$$

and

$$V : (a_0, a_1, a_2, \dots) \mapsto (0, a_0, a_1, a_2, \dots),$$

respectively, as is remarked in 1.9.2. Moreover, we have

$$E_p^{(0)}(\mathbf{a}; \mathbf{T}) = \sum_{r=0}^{\infty} a_r \Phi_r(\mathbf{T})$$

for $\mathbf{a} \in W^{(0)}(A) = C(\widehat{\mathbb{G}}_{a,A})$ (Example 4.17.2).

In particular, $[1]_0 \in W^{(0)}(A)$ corresponds to the restriction map $R : \widehat{W}_A \rightarrow \widehat{W}_{1,A} = \widehat{\mathbb{G}}_{a,A}$ since $E_p^{(0)}([1]; \mathbf{T}) = T_0$.

Hereafter we investigate the isomorphisms presented in Theorem 3.5.1 via the Cartier theory.

LEMMA 4.22. Let A be a $\mathbb{Z}[A]$ -algebra, $\mathbf{a} = (a_0, a_1, a_2, \dots) \in W(A)$ and $\mathbf{u} = (u_0, u_1, u_2, \dots) \in W^{(\Lambda)}(A)$. If $F(\mathbf{a}) = [\Lambda^{p-1}]\mathbf{a}$,

$$E_p(\mathbf{a}, \Lambda; E_p^{(\Lambda)}(\mathbf{u}; \mathbf{T})) = E_p(\langle \mathbf{u}, \mathbf{a} \rangle_{\Lambda,1}; \mathbf{T}).$$

Here

$$\langle \mathbf{u}, \mathbf{a} \rangle_{\Lambda,1} = \sum_{k=0}^{\infty} V^k([u_k]\mathbf{a}) \in W(A).$$

PROOF. We may assume that A is flat over $\mathbb{Z}[A]$ since $\text{Ker}[F - [\Lambda^{p-1}] : W_{\mathbb{Z}[A]} \rightarrow W_{\mathbb{Z}[A]}]$ is flat over $\mathbb{Z}[A]$ ([13, Cor. 1.8]). Put now $a = \Phi_0(\mathbf{a})$. Then we have $\Phi_r(\mathbf{a}) = \Lambda^{p^r-1}a$ for each $r > 0$ and

$$E_p(\mathbf{a}, \Lambda; T) = (1 + \Lambda T)^{a/\Lambda}.$$

Noting

$$E_p^{(\Lambda)}(\mathbf{u}; \mathbf{T}) = \frac{1}{\Lambda} \left[\exp \left(\sum_{r=0}^{\infty} \frac{\Phi_r(\Lambda u_0, \Lambda u_1, \dots, \Lambda u_r) \Phi_r(\mathbf{T})}{p^r} \right) - 1 \right],$$

we see that

$$\begin{aligned} E_p(\mathbf{a}, \Lambda; E_p^{(\Lambda)}(\mathbf{u}; \mathbf{T})) &= \left[\exp \left(\sum_{r=0}^{\infty} \frac{\Phi_r(\Lambda u_0, \Lambda u_1, \dots, \Lambda u_r) \Phi_r(\mathbf{T})}{p^r} \right) \right]^{a/\Lambda} \\ &= \exp \left(\sum_{r=0}^{\infty} \frac{a}{\Lambda} \frac{\Phi_r(\Lambda u_0, \Lambda u_1, \dots, \Lambda u_r) \Phi_r(\mathbf{T})}{p^r} \right) \\ &= \exp \left(\sum_{r=0}^{\infty} \frac{(\Lambda^{p^r-1} u_0^{p^r} + p \Lambda^{p^{r-1}-1} u_1^{p^{r-1}} + \dots + p^r u_r) a \Phi_r(\mathbf{T})}{p^r} \right) \\ &= \exp \left(\sum_{r=0}^{\infty} \frac{\{\Phi_r([u_0]) \Phi_r(\mathbf{a}) + p \Phi_{r-1}([u_1]) \Phi_{r-1}(\mathbf{a}) + \dots + p^r \Phi_0([u_r]) \Phi_0(\mathbf{a})\} \Phi_r(\mathbf{T})}{p^r} \right) \\ &= \exp \left(\sum_{r=0}^{\infty} \frac{\Phi_r([u_0] \cdot \mathbf{a} + V([u_1] \cdot \mathbf{a}) + \dots + V^r([u_r] \cdot \mathbf{a})) \Phi_r(\mathbf{T})}{p^r} \right) \\ &= E_p \left(\sum_{k=0}^{\infty} V^k([u_k] \cdot \mathbf{a}); \mathbf{T} \right). \end{aligned}$$

COROLLARY 4.23. Let A be a $\mathbb{Z}_{(p)}[\Lambda, M]$ -algebra, $\mathbf{a} \in W^{(M)}(A)$ and $\mathbf{u} \in W^{(\Lambda)}(A)$. If $F^{(M)}(\mathbf{a}) = [\Lambda^{p-1}] \mathbf{a}$,

$$E_p^{(M)}(\mathbf{a}, \Lambda; E_p^{(\Lambda)}(\mathbf{u}; \mathbf{T})) = E_p^{(M)}(\langle \mathbf{u}, \mathbf{a} \rangle_{\Lambda, M}; \mathbf{T}).$$

Here

$$\langle \mathbf{u}, \mathbf{a} \rangle_{\Lambda, M} = \sum_{k=0}^{\infty} V^k([u_k] \mathbf{a}) \in W^{(M)}(A).$$

PROOF. Let $B = A[t]/(t^2 - Mt)$, and let ε denote the image of t in B . We have $F(\varepsilon \mathbf{a}) = [\Lambda^{p-1}](\varepsilon \mathbf{a})$ since $F^{(M)}(\mathbf{a}) = [\Lambda^{p-1}] \mathbf{a}$ (cf. 1.19, (4)). Therefore, we obtain

$$E_p(\varepsilon \mathbf{a}, \Lambda; E_p^{(\Lambda)}(\mathbf{u}; \mathbf{T})) = E_p \left(\varepsilon \left(\sum_{k=0}^{\infty} V^k([u_k] \cdot \mathbf{a}) \right); \mathbf{T} \right),$$

combining 4.22 with 1.21. Now note that

$$E_p(\varepsilon \mathbf{a}, \Lambda; E_p^{(\Lambda)}(\mathbf{u}; \mathbf{T})) = 1 + \varepsilon E_p^{(M)}(\mathbf{a}, \Lambda; E_p^{(\Lambda)}(\mathbf{u}; \mathbf{T}))$$

and

$$E_p\left(\varepsilon\left(\sum_{k=0}^{\infty} V^k([u_k] \cdot \mathbf{a})\right); \mathbf{T}\right) = 1 + \varepsilon E_p^{(M)}\left(\sum_{k=0}^{\infty} V^k([u_k] \cdot \mathbf{a}); \mathbf{T}\right)$$

(cf. 3.7).

Now we can state the first main result on the compatibility of the isomorphism ξ with the Cartier theory.

4.24. Let A be a $\mathbb{Z}_{(p)}[A, M]$ -algebra. By Theorem 3.5.1 we have an isomorphism

$$\xi : \text{Ker}[F^{(M)} - [A^{p-1}] : W^{(M)}(A) \rightarrow W^{(M)}(A)] \xrightarrow{\sim} \text{Hom}_{A\text{-gr}}(\widehat{\mathcal{G}}_A^{(\Lambda)}, \widehat{\mathcal{G}}_A^{(M)}).$$

On the other hand, the Cartier theory asserts that $\gamma \mapsto C(\gamma)$ gives rise to an isomorphism

$$\text{Hom}_{A\text{-gr}}(\widehat{\mathcal{G}}_A^{(\Lambda)}, \widehat{\mathcal{G}}_A^{(M)}) \xrightarrow{\sim} \text{Hom}_{D_A}(C(\widehat{\mathcal{G}}_A^{(\Lambda)}), C(\widehat{\mathcal{G}}_A^{(M)}))$$

(the second theorem of the Cartier theory on p -typical curves, cf. [4, Th. 27.7.10]). Now we have isomorphisms

$$\eta : W^{(\Lambda)}(A) \xrightarrow{\sim} C(\widehat{\mathcal{G}}_A^{(\Lambda)})$$

and

$$\eta : W^{(M)}(A) \xrightarrow{\sim} C(\widehat{\mathcal{G}}_A^{(M)})$$

by Theorem 4.15.1. Furthermore we have an isomorphism

$$\text{Hom}_{D_A}(W^{(\Lambda)}(A), W^{(M)}(A)) \xrightarrow{\sim} \text{Ker}[F^{(M)} - [A^{p-1}] : W^{(M)}(A) \rightarrow W^{(M)}(A)],$$

defined by $\varphi \mapsto \varphi([1]_A)$, as is remarked in 4.7.

THEOREM 4.24.1. Let $\mathbf{a} \in \text{Ker}[F^{(M)} - [A^{p-1}] : W^{(M)}(A) \rightarrow W^{(M)}(A)]$. Then we have

$$C(\xi(\mathbf{a})) = \mathbf{a}$$

under the identification

$$\text{Hom}_{D_A}(C(\widehat{\mathcal{G}}_A^{(\Lambda)}), C(\widehat{\mathcal{G}}_A^{(M)})) = \text{Ker}[F^{(M)} - [A^{p-1}] : W^{(M)}(A) \rightarrow W^{(M)}(A)].$$

PROOF. Let $\psi \in C(\widehat{\mathcal{G}}_A^{(\Lambda)}) = \text{Hom}_{A\text{-gr}}(\widehat{W}_A, \widehat{\mathcal{G}}_A^{(\Lambda)})$ and $\gamma \in \text{Hom}_{A\text{-gr}}(\widehat{\mathcal{G}}_A^{(\Lambda)}, \widehat{\mathcal{G}}_A^{(M)})$. Then $C(\gamma)(\psi) = \gamma \circ \psi$. Now recall that:

(1) $\eta(\mathbf{u}) \in C(\widehat{\mathcal{G}}_A^{(\Lambda)}) = \text{Hom}_{A\text{-gr}}(\widehat{W}_A, \widehat{\mathcal{G}}_A^{(\Lambda)})$ is defined by $T \mapsto E_p^{(\Lambda)}(\mathbf{u}; \mathbf{T})$, where $\mathbf{u} \in W^{(\Lambda)}(A)$ (Proposition 4.19);

(2) $\xi(\mathbf{a}) \in \text{Hom}_{A\text{-gr}}(\widehat{\mathcal{G}}_A^{(\Lambda)}, \widehat{\mathcal{G}}_A^{(M)})$ is defined by $T \mapsto E_p^{(M)}(\mathbf{a}, \Lambda; T)$ (Remark 3.6).

Hence $C(\xi(\mathbf{a}))(\eta(\mathbf{u}))$ is defined by $T \mapsto E_p^{(M)}(\mathbf{a}, \Lambda; E_p^{(\Lambda)}(\mathbf{u}; T))$. By Corollary 4.23, we have

$$E_p^{(M)}(\mathbf{a}, \Lambda; E_p^{(\Lambda)}(\mathbf{u}; T)) = E_p^{(M)}(\langle \mathbf{u}, \mathbf{a} \rangle_{\Lambda, M}; T).$$

Then we get the conclusion, noting that $\langle [1]_{\Lambda}, \mathbf{a} \rangle_{\Lambda, M} = \mathbf{a}$.

REMARK 4.25. Let A be a $\mathbb{Z}_{(p)}[A, M]$ -algebra and $\mathbf{a} \in \text{Ker}[F^{(M)} - [\Lambda^{p-1}] : W^{(M)}(A) \rightarrow W^{(M)}(A)]$. Then the assertion of 4.23 is equivalent to the commutativity of the diagram

$$\begin{array}{ccc} \text{Hom}_{A\text{-gr}}(\widehat{W}_A, \widehat{\mathcal{G}}_A^{(\Lambda)}) & \xrightarrow{\sim} & W^{(\Lambda)}(A) \\ C(\xi(\mathbf{a})) \downarrow & & \downarrow \langle \cdot, \mathbf{a} \rangle_{\Lambda, M} \\ \text{Hom}_{A\text{-gr}}(\widehat{W}_A, \widehat{\mathcal{G}}_A^{(M)}) & \xrightarrow{\sim} & W^{(M)}(A) \end{array}$$

EXAMPLE 4.26. Let A be a $\mathbb{Z}_{(p)}[A]$ -algebra. Then the diagram

$$\begin{array}{ccc} \text{Hom}_{A\text{-gr}}(\widehat{W}_A, \widehat{\mathcal{G}}_A^{(\Lambda)}) & \xrightarrow{\sim} & W^{(\Lambda)}(A) \\ C(\alpha^{(\Lambda)}) \downarrow & & \downarrow \alpha^{(\Lambda)} \\ \text{Hom}_{A\text{-gr}}(\widehat{W}_A, \widehat{\mathbb{G}}_{m, A}) & \xrightarrow{\sim} & W(A) \end{array}$$

is commutative. Note that

$$\alpha^{(\Lambda)} = \xi([A]) : \widehat{\mathcal{G}}_A^{(\Lambda)} \rightarrow \widehat{\mathbb{G}}_{m, A}$$

(cf. Example 3.8) and that

$$\alpha^{(\Lambda)} = \langle \cdot, [A] \rangle_{\Lambda, 1} : W^{(\Lambda)}(A) \rightarrow W(A).$$

COROLLARY 4.27. Let A be a $\mathbb{Z}_{(p)}[A, M, N]$ -algebra and

$$\mathbf{a} \in \text{Ker}[F^{(M)} - [\Lambda^{p-1}] : W^{(M)}(A) \rightarrow W^{(M)}(A)],$$

$$\mathbf{b} \in \text{Ker}[F^{(N)} - [M^{p-1}] : W^{(N)}(A) \rightarrow W^{(N)}(A)].$$

Then the element of $\text{Ker}[F^{(N)} - [\Lambda^{p-1}] : W^{(N)}(A) \rightarrow W^{(N)}(A)]$ corresponding to $\xi(\mathbf{b}) \circ \xi(\mathbf{a})$ is given by $\langle \mathbf{a}, \mathbf{b} \rangle_{M, N} \in W^{(N)}(A)$.

PROOF. There exists uniquely $\mathbf{c} \in \text{Ker}[F^{(N)} - [\Lambda^{p-1}] : W^{(N)}(A) \rightarrow W^{(N)}(A)]$ such that $\xi(\mathbf{c}) = \xi(\mathbf{b}) \circ \xi(\mathbf{a})$. Then we have

$$C(\xi(\mathbf{c})) = C(\xi(\mathbf{b})) \circ C(\xi(\mathbf{a}))$$

and, by the theorem,

$$C(\xi(\mathbf{a})) = \langle \cdot, \mathbf{a} \rangle_{A,M}, \quad C(\xi(\mathbf{b})) = \langle \cdot, \mathbf{b} \rangle_{M,N}, \quad C(\xi(\mathbf{c})) = \langle \cdot, \mathbf{c} \rangle_{A,N}.$$

Hence we obtain

$$\langle \cdot, \mathbf{c} \rangle_{A,N} = \langle \langle \cdot, \mathbf{a} \rangle_{A,M}, \mathbf{b} \rangle_{M,N}$$

and therefore

$$\mathbf{c} = \langle [1]_A, \mathbf{c} \rangle_{A,N} = \langle \langle [1]_A, \mathbf{a} \rangle_{A,M}, \mathbf{b} \rangle_{M,N} = \langle \mathbf{a}, \mathbf{b} \rangle_{M,N}.$$

We conclude the article by giving a description of the Cartier module of an extension of $\widehat{\mathcal{G}}_A^{(\Lambda)}$ by $\widehat{\mathcal{G}}_A^{(M)}$.

4.28. Let A be a $\mathbb{Z}_{(p)}[A, M]$ -algebra and $\mathbf{c} \in W^{(M)}(A)$. Let $\mathcal{E}$ denote the extension of $\widehat{\mathcal{G}}_A^{(\Lambda)}$ by $\widehat{\mathcal{G}}_A^{(M)}$ by the 2-cocycle

$$C(X, Y) = F_p^{(M)}(\mathbf{c}, A; X, Y) \in Z^2(\widehat{\mathcal{G}}_A^{(\Lambda)}, \widehat{\mathcal{G}}_A^{(M)}).$$

Then the group law of $\mathcal{E} = \mathrm{Spf} A[[T_0, T_1]]$ is given by

$$\begin{aligned} T_0 &\mapsto T_0 \otimes 1 + 1 \otimes T_0 + \Lambda T_0 \otimes T_0, \\ T_1 &\mapsto T_1 \otimes 1 + 1 \otimes T_1 + C(T_0 \otimes 1, 1 \otimes T_0) \\ &\quad + M\{T_1 \otimes T_1 + (T_1 \otimes 1 + 1 \otimes T_1)C(T_0 \otimes 1, 1 \otimes T_0)\} \\ &\quad + M^2(T_1 \otimes T_1)C(T_0 \otimes 1, 1 \otimes T_0). \end{aligned}$$

Hence a homomorphism of formal groups

$$\widehat{W}_A = \mathrm{Spf} A[[T_0, T_1, T_2, \dots]] \longrightarrow \mathcal{E} = \mathrm{Spf} A[[T_0, T_1]]$$

is given by

$$T_0 \mapsto F_0(\mathbf{T}), \quad T_1 \mapsto F_1(\mathbf{T}) : A[[T_0, T_1]] \rightarrow A[[T_0, T_1, T_2, \dots]],$$

where $F_0(\mathbf{T}) = F_0(T_0, T_1, T_2, \dots)$ and $F_1(\mathbf{T}) = F_1(T_0, T_1, T_2, \dots) \in A[[T_0, T_1, T_2, \dots]]$ satisfy the functional equations

$$\begin{aligned} F_0(\mathbf{X} + \mathbf{Y}) &= F_0(\mathbf{X}) + F_0(\mathbf{Y}) + \Lambda F_0(\mathbf{X})F_0(\mathbf{Y}), \\ F_1(\mathbf{X} + \mathbf{Y}) &= F_1(\mathbf{X}) + F_1(\mathbf{Y}) + C(F_0(\mathbf{X}), F_0(\mathbf{Y})) \\ &\quad + M\{F_1(\mathbf{X})F_1(\mathbf{Y}) + (F_1(\mathbf{X}) + F_1(\mathbf{Y}))C(F_0(\mathbf{X}), F_0(\mathbf{Y}))\} \\ &\quad + M^2 F_1(\mathbf{X})F_1(\mathbf{Y})C(F_0(\mathbf{X}), F_0(\mathbf{Y})). \end{aligned}$$

REMARK 4.28.1. Let A be a $\mathbb{Z}_{(p)}[A]$ -algebra and $c \in W(A)$. Let $\mathcal{E}$ denote the extension of $\widehat{\mathcal{G}}_A^{(A)}$ by $\widehat{\mathbb{G}}_{m,A}$ by the 2-cocycle

$$C(X, Y) = F_p(c, A; X, Y) \in Z^2(\widehat{\mathcal{G}}_A^{(A)}, \widehat{\mathbb{G}}_{m,A}).$$

Then a homomorphism of formal groups $\widehat{W}_A \rightarrow \mathcal{E}$ is given by $T_0 \mapsto F_0(\mathbf{T})$, $T_1 \mapsto F_1(\mathbf{T})$, where $F_0(\mathbf{T}) = F_0(T_0, T_1, T_2, \dots)$ and $F_1(\mathbf{T}) = F_1(T_0, T_1, T_2, \dots) \in A[[T_0, T_1, T_2, \dots]]$ satisfy the functional equations

$$\begin{aligned} F_0(\mathbf{X} + \mathbf{Y}) &= F_0(\mathbf{X}) + F_0(\mathbf{Y}) + \Lambda F_0(\mathbf{X})F_0(\mathbf{Y}), \\ F_1(\mathbf{X} + \mathbf{Y}) &= F_1(\mathbf{X})F_1(\mathbf{Y})C(F_0(\mathbf{X}), F_0(\mathbf{Y})). \end{aligned}$$

We now define formal power series $\tilde{G}_p^{(A)}(\mathbf{Z}, \mathbf{U}; \mathbf{T})$ and $G_p^{(A)}(\mathbf{Z}, \mathbf{U}; \mathbf{T})$ to describe explicitly homomorphisms of $\widehat{W}_A$ to $\mathcal{E}$. First we establish a formalism for the extensions of $\widehat{\mathcal{G}}_A^{(A)}$ by $\widehat{\mathbb{G}}_{m,A}$.

4.29. Let

$$\mathbf{U} = (U_0, U_1, U_2, \dots), \quad \mathbf{Z} = (Z_0, Z_1, Z_2, \dots)$$

and

$$\mathbf{T} = (T_0, T_1, T_2, \dots).$$

We define formal power series

$$\tilde{G}_p^{(A)}(\mathbf{Z}, \mathbf{U}; \mathbf{T}), G_p^{(A)}(\mathbf{Z}, \mathbf{U}; \mathbf{T}) \in \mathbb{Q}[\Lambda, U_0, U_1, U_2, \dots, Z_0, Z_1, Z_2, \dots][[T_0, T_1, T_2, \dots]]$$

by

$$\tilde{G}_p^{(A)}(\mathbf{Z}, \mathbf{U}; \mathbf{T}) = \exp \left[\sum_{r=1}^{\infty} \left(\sum_{k=0}^{r-1} \frac{\Phi_r(\Lambda U_0, \dots, \Lambda U_k, 0, \dots, 0) \Phi_{r-k-1}(\mathbf{Z})}{\Lambda^{p^{r-k}}} \right) \frac{\Phi_r(\mathbf{T})}{p^r} \right]$$

and

$$G_p^{(A)}(\mathbf{Z}, \mathbf{U}; \mathbf{T}) = \frac{\tilde{G}_p^{(A)}(\mathbf{Z}, \mathbf{U}; \mathbf{T})}{\prod_{r=0}^{\infty} [1 + \Lambda^{p^{r+1}} E_p^{(A)}(\mathbf{U}; \mathbf{T})^{p^{r+1}}] \frac{\Phi_r(\mathbf{Z})}{p^{r+1} \Lambda^{p^{r+1}}}}.$$

It is readily seen that

$$\tilde{G}_p^{(A)}(\mathbf{Z}, \mathbf{U}; S(\mathbf{X}, \mathbf{Y})) = \tilde{G}_p^{(A)}(\mathbf{Z}, \mathbf{U}; \mathbf{X}) \tilde{G}_p^{(A)}(\mathbf{Z}, \mathbf{U}; \mathbf{Y})$$

for $\mathbf{X} = (X_0, X_1, X_2, \dots)$, $\mathbf{Y} = (Y_0, Y_1, Y_2, \dots)$. Here

$$S(\mathbf{X}, \mathbf{Y}) = (S_0(\mathbf{X}, \mathbf{Y}), S_1(\mathbf{X}, \mathbf{Y}), S_2(\mathbf{X}, \mathbf{Y}), \dots).$$

Moreover, we have

$$\begin{aligned} & \tilde{G}_p^{(\Lambda)}(\mathbf{Z}, \mathbf{U}; \mathbf{T}) \\ &= \exp \left[\sum_{r=0}^{\infty} \frac{\Phi_r(\mathbf{Z})}{p^{r+1} \Lambda^{p^{r+1}}} \left(\sum_{k=0}^{\infty} \frac{\Phi_{r+k+1}(\Lambda U_0, \dots, \Lambda U_k, 0, \dots, 0) \Phi_{r+k+1}(\mathbf{T})}{p^k} \right) \right]. \end{aligned}$$

It follows that

$$\tilde{G}_p^{(\Lambda)}(S(\mathbf{X}, \mathbf{Y}), \mathbf{U}; \mathbf{T}) = \tilde{G}_p^{(\Lambda)}(\mathbf{X}, \mathbf{U}; \mathbf{T}) \tilde{G}_p^{(\Lambda)}(\mathbf{Y}, \mathbf{U}; \mathbf{T})$$

and

$$G_p^{(\Lambda)}(S(\mathbf{X}, \mathbf{Y}), \mathbf{U}; \mathbf{T}) = G_p^{(\Lambda)}(\mathbf{X}, \mathbf{U}; \mathbf{T}) G_p^{(\Lambda)}(\mathbf{Y}, \mathbf{U}; \mathbf{T})$$

for $\mathbf{X} = (X_0, X_1, X_2, \dots)$, $\mathbf{Y} = (Y_0, Y_1, Y_2, \dots)$.

PROPOSITION 4.30. *Under the notation above, we have*

$$G_p^{(\Lambda)}(\mathbf{Z}, \mathbf{U}; \mathbf{T}) \in \mathbb{Z}_{(p)}[\Lambda, U_0, U_1, U_2, \dots, Z_0, Z_1, Z_2, \dots][[T_0, T_1, T_2, \dots]].$$

PROOF. Put

$$(\tilde{F}_0(\mathbf{W}), \tilde{F}_1(\mathbf{W}), \tilde{F}_2(\mathbf{W}), \dots) = (F - [\Lambda^{p-1}])\mathbf{W} \in W(\mathbb{Z}[\Lambda, W_0, W_1, W_2, \dots])$$

and

$$A = \mathbb{Z}_{(p)}[\Lambda, U_0, U_1, U_2, \dots, Z_0, Z_1, Z_2, \dots],$$

$$B = A[W_0, W_1, W_2, \dots] / (\tilde{F}_0(\mathbf{W}) - Z_0, \tilde{F}_1(\mathbf{W}) - Z_1, \tilde{F}_2(\mathbf{W}) - Z_2, \dots).$$

Then B is flat over A since $F - [\Lambda^{p-1}] : W_{\mathbb{Z}[\Lambda]} \rightarrow W_{\mathbb{Z}[\Lambda]}$ is faithfully flat ([13, Cor. 1.8]). Moreover,

$$B \cap \mathbb{Q}[\Lambda, U_0, U_1, U_2, \dots, Z_0, Z_1, Z_2, \dots] = A.$$

Then it is sufficient to prove that $G_p^{(\Lambda)}(\mathbf{Z}, \mathbf{U}; \mathbf{T}) \in B[[T_0, T_1, T_2, \dots]]$. We can verify this assertion thanks to the following two equalities:

$$\begin{aligned} & E_p(\mathbf{W}, \Lambda; E_p^{(\Lambda)}(\mathbf{U}, \mathbf{T})) \\ &= [1 + \Lambda E_p^{(\Lambda)}(\mathbf{U}; \mathbf{T})]^{\frac{\Phi_0(\mathbf{W})}{\Lambda}} \prod_{r=0}^{\infty} [1 + \Lambda^{p^{r+1}} E_p^{(\Lambda)}(\mathbf{U}; \mathbf{T})^{p^{r+1}}]^{\frac{\Phi_r(\mathbf{Z})}{p^{r+1} \Lambda^{p^{r+1}}}} \end{aligned}$$

and

$$E_p\left(\sum_{k=0}^{\infty} V^k([U_k]\mathbf{W}); \mathbf{T}\right) = [1 + \Lambda E_p^{(\Lambda)}(\mathbf{U}; \mathbf{T})]^{\frac{\Phi_0(\mathbf{W})}{\Lambda}} \tilde{G}_p^{(\Lambda)}(\mathbf{Z}, \mathbf{U}; \mathbf{T}).$$

The first equality is a direct consequence of the definition:

$$E_p(\mathbf{W}, \Lambda; \mathbf{T}) = (1 + \Lambda \mathbf{T})^{\frac{\Phi_0(\mathbf{W})}{\Lambda}} \prod_{r=0}^{\infty} (1 + \Lambda^{p^{r+1}} \mathbf{T}^{p^{r+1}})^{\frac{\Phi_{r+1}(\mathbf{W}) - \Lambda^{p^r(p-1)} \Phi_r(\mathbf{W})}{p^{r+1} \Lambda^{p^{r+1}}}},$$

since we have $\mathbf{Z} = (F - [\Lambda^{p-1}])\mathbf{W}$ and therefore $\Phi_r(\mathbf{Z}) = \Phi_{r+1}(\mathbf{W}) - \Lambda^{p^r(p-1)}\Phi_r(\mathbf{W})$ for each $r \geq 0$.

Now we prove the second equality. By the definition, we have

$$[1 + \Lambda E_p^{(\Lambda)}(\mathbf{U}; \mathbf{T})]^{\frac{\Phi_0(\mathbf{W})}{\Lambda}} = \exp \left[\frac{\Phi_0(\mathbf{W})}{\Lambda} \left(\sum_{r=0}^{\infty} \frac{\Phi_r(\Lambda U_0, \dots, \Lambda U_r) \Phi_r(\mathbf{T})}{p^r} \right) \right]$$

and

$$\begin{aligned} \log \tilde{G}_p^{(\Lambda)}(\mathbf{Z}, \mathbf{U}; \mathbf{T}) &= \sum_{r=1}^{\infty} \left(\sum_{k=0}^{r-1} \frac{\Phi_r(\Lambda U_0, \dots, \Lambda U_k, 0, \dots, 0) \Phi_{r-k-1}(\mathbf{Z})}{\Lambda^{p^{r-k}}} \right) \frac{\Phi_r(\mathbf{T})}{p^r} \\ &= \sum_{r=1}^{\infty} \left\{ \sum_{k=0}^{r-1} \frac{\Phi_r(\Lambda U_0, \dots, \Lambda U_k, 0, \dots, 0) (\Phi_{r-k}(\mathbf{W}) - \Lambda^{p^{r-k-1}(p-1)} \Phi_{r-k-1}(\mathbf{W}))}{\Lambda^{p^{r-k}}} \right\} \frac{\Phi_r(\mathbf{T})}{p^r}. \end{aligned}$$

Hence the coefficient of $\frac{\Phi_r(\mathbf{T})}{p^r}$ in

$$\log \tilde{G}_p^{(\Lambda)}(\mathbf{Z}, \mathbf{U}; \mathbf{T}) + \frac{\Phi_0(\mathbf{W})}{\Lambda} \left(\sum_{r=0}^{\infty} \frac{\Phi_r(\Lambda U_0, \dots, \Lambda U_r) \Phi_r(\mathbf{T})}{p^r} \right)$$

is given by

$$\begin{aligned} &\frac{\Phi_r(\Lambda U_0, 0, \dots, 0)}{\Lambda^{p^r}} \Phi_r(\mathbf{W}) \\ &+ \sum_{k=1}^r \frac{(\Phi_r(\Lambda U_0, \dots, \Lambda U_k, 0, \dots, 0) - \Phi_r(\Lambda U_0, \dots, \Lambda U_{k-1}, 0, \dots, 0))}{\Lambda^{p^{r-k}}} \Phi_{r-k}(\mathbf{W}). \end{aligned}$$

Noting that

$$\begin{aligned} &\Phi_r(\Lambda U_0, \dots, \Lambda U_{k-1}, \Lambda U_k, 0, \dots, 0) - \Phi_r(\Lambda U_0, \dots, \Lambda U_{k-1}, 0, 0, \dots, 0) \\ &= p^k (\Lambda U_k)^{p^{r-k}}, \end{aligned}$$

we obtain

$$\begin{aligned} & \log[1 + \Lambda E_p^{(\Lambda)}(U; T)]^{\frac{\Phi_0(W)}{\Lambda}} + \log \tilde{G}_p^{(\Lambda)}(Z, U; T) \\ &= \sum_{r=0}^{\infty} \left(\sum_{k=0}^r p^k U_k^{p^{r-k}} \Phi_{r-k}(W) \right) \frac{\Phi_r(T)}{p^r}. \end{aligned}$$

This implies the result since we have

$$\Phi_r \left(\sum_{k=0}^{\infty} V^k([U_k]W) \right) = \sum_{k=0}^r p^k U_k^{p^{r-k}} \Phi_{r-k}(W)$$

for each $r \geq 0$.

EXAMPLE 4.31. Specializing Λ to 0, we obtain

$$\begin{aligned} & G^{(0)}(Z, U; T) \\ &= \exp \left[\sum_{r=1}^{\infty} \left(\sum_{k=0}^{r-1} \frac{U_k^{p^{r-k}} \Phi_{r-k-1}(Z)}{p^{r-k}} \right) \Phi_r(T) - \sum_{k=0}^{\infty} \frac{\Phi_k(Z)}{p^{k+1}} \left(\sum_{r=0}^{\infty} U_r \Phi_r(T) \right)^{p^{k+1}} \right]. \end{aligned}$$

Indeed, we have

$$\begin{aligned} \log \tilde{G}^{(\Lambda)}(Z, U; T) &= \sum_{r=1}^{\infty} \left(\sum_{k=0}^{r-1} \frac{\Phi_r(\Lambda U_0, \dots, \Lambda U_k, 0, \dots, 0) \Phi_{r-k-1}(Z)}{\Lambda p^{r-k}} \right) \frac{\Phi_r(T)}{p^r} \\ &\equiv \sum_{r=1}^{\infty} \left(\sum_{k=0}^{r-1} \frac{U_k^{p^{r-k}} \Phi_{r-k-1}(Z)}{p^{r-k}} \right) \Phi_r(T) \pmod{\Lambda^p} \end{aligned}$$

since

$$\Phi_r(\Lambda U_0, \dots, \Lambda U_k, 0, \dots, 0) = (\Lambda U_0)^{p^r} + \dots + p^{k-1} (\Lambda U_{k-1})^{p^{r-k+1}} + p^k (\Lambda U_k)^{p^{r-k}}.$$

On the other hand, we have

$$\begin{aligned} & \log \left[\prod_{k=0}^{\infty} \left\{ 1 + \Lambda^{p^{k+1}} E_p^{(\Lambda)}(U; T)^{p^{k+1}} \right\}^{\frac{\Phi_k(Z)}{p^{k+1} \Lambda^{p^{k+1}}}} \right] \\ &= \sum_{k=0}^{\infty} \frac{\Phi_k(Z)}{p^{k+1} \Lambda^{p^{k+1}}} \log[1 + \Lambda^{p^{k+1}} E_p^{(\Lambda)}(U; T)^{p^{k+1}}] \\ &\equiv \sum_{k=0}^{\infty} \frac{\Phi_k(V)}{p^{k+1}} E_p^{(\Lambda)}(U; T)^{p^{k+1}} \equiv \sum_{k=0}^{\infty} \frac{\Phi_r(V)}{p^{r+1}} \left(\sum_{r=0}^{\infty} U_r \Phi_r(T) \right)^{p^{k+1}} \pmod{\Lambda} \end{aligned}$$

since

$$E_p^{(\Lambda)}(U; T) \equiv \sum_{r=0}^{\infty} U_r \Phi_r(T) \pmod{\Lambda}$$

as is remarked in Example 4.17.2. Combining the two congruence relations, we obtain

$$G^{(\Lambda)}(Z, U; T) \equiv \exp \left[\sum_{r=1}^{\infty} \left(\sum_{k=1}^r \frac{U_{r-k}^{p^k} \Phi_{k-1}(Z)}{p^k} \right) \Phi_r(T) - \sum_{k=0}^{\infty} \frac{\Phi_k(Z)}{p^{k+1}} \left(\sum_{r=0}^{\infty} U_r \Phi_r(T) \right)^{p^{k+1}} \right] \pmod{\Lambda}.$$

DEFINITION 4.32. Let A be a $\mathbb{Z}_{(p)}[A]$ -algebra, and let $\mathbf{a} \in W^{(\Lambda)}(A)$ and $\mathbf{c} \in W(A)$. We define a formal power series $G_p^{(\Lambda)}(\mathbf{c}, \mathbf{a}; T) \in A[[T_0, T_1, T_2, \dots]]$, substituting $Z = \mathbf{c}$ and $U = \mathbf{a}$ in $G_p^{(\Lambda)}(Z, U; T)$.

By the preceding argument on $G_p^{(\Lambda)}(Z, U; T)$, we have

- (1) $G_p^{(\Lambda)}(\mathbf{c} + \mathbf{c}', \mathbf{a}; T) = G_p^{(\Lambda)}(\mathbf{c}, \mathbf{a}; T) G_p^{(\Lambda)}(\mathbf{c}', \mathbf{a}; T);$
- (2) $G_p^{(\Lambda)}((F - [A^{p-1}])\mathbf{c}, \mathbf{a}; T) = \frac{E_p(\sum_{k=0}^{\infty} V^k([a_k]\mathbf{c}); T)}{E_p(\mathbf{c}, \Lambda; E_p^{(\Lambda)}(\mathbf{a}; T))}.$

LEMMA 4.33. Let A be a $\mathbb{Z}_{(p)}[A]$ -algebra and $\mathbf{a} \in W^{(\Lambda)}(A)$, $\mathbf{b}, \mathbf{c} \in W(A)$. Let $\mathcal{E}$ denote the extension of $\widehat{\mathcal{G}}_A^{(\Lambda)}$ by $\widehat{\mathbb{G}}_{m,A}$ defined by the 2-cocycle $F_p(\mathbf{c}, \Lambda; X, Y) \in Z^2(\widehat{\mathcal{G}}_A^{(\Lambda)}, \widehat{\mathbb{G}}_{m,A})$. Then

$$T_0 \mapsto E_p^{(\Lambda)}(\mathbf{a}; T), \quad T_1 \mapsto E_p(\mathbf{b}; T) G_p^{(\Lambda)}(\mathbf{c}, \mathbf{a}; T) : \\ \widehat{W}_A = \text{Spf } A[[T_0, T_1, T_2, \dots]] \longrightarrow \mathcal{E} = \text{Spf } A[[T_0, T_1 - 1]]$$

defines a homomorphism of formal groups $\widehat{W}_A \rightarrow \mathcal{E}$.

PROOF. By Remark 4.28.1, the assertion is equivalent to the equalities:

$$E_p^{(\Lambda)}(\mathbf{a}; S(\mathbf{X}, \mathbf{Y})) = E_p^{(\Lambda)}(\mathbf{a}; \mathbf{X}) + E_p^{(\Lambda)}(\mathbf{a}; \mathbf{Y}) + \Lambda E_p^{(\Lambda)}(\mathbf{a}; \mathbf{X}) E_p^{(\Lambda)}(\mathbf{a}; \mathbf{Y})$$

and

$$E_p(\mathbf{b}; S(\mathbf{X}, \mathbf{Y})) G_p^{(\Lambda)}(\mathbf{c}, \mathbf{a}; S(\mathbf{X}, \mathbf{Y})) = \{E_p(\mathbf{b}; \mathbf{X}) G_p^{(\Lambda)}(\mathbf{c}, \mathbf{a}; \mathbf{X})\} \\ \{E_p(\mathbf{b}; \mathbf{Y}) G_p^{(\Lambda)}(\mathbf{c}, \mathbf{a}; \mathbf{Y})\} F_p(\mathbf{c}, \Lambda; E_p^{(\Lambda)}(\mathbf{a}; \mathbf{X}), E_p^{(\Lambda)}(\mathbf{a}; \mathbf{Y})).$$

Here

$$S(\mathbf{X}, \mathbf{Y}) = (S_0(\mathbf{X}, \mathbf{Y}), S_1(\mathbf{X}, \mathbf{Y}), S_2(\mathbf{X}, \mathbf{Y}), \dots).$$

Then it is sufficient to verify the equality

$$\frac{G_p^{(\Lambda)}(\mathbf{c}, \mathbf{a}; S(\mathbf{X}, \mathbf{Y}))}{G_p^{(\Lambda)}(\mathbf{c}, \mathbf{a}; \mathbf{X})G_p^{(\Lambda)}(\mathbf{c}, \mathbf{a}; \mathbf{Y})} = F_p(\mathbf{c}, \Lambda; E_p^{(\Lambda)}(\mathbf{a}; \mathbf{X}), E_p^{(\Lambda)}(\mathbf{a}; \mathbf{Y}))$$

by 4.18, (2) and 4.14, (2).

We may assume that

$$A = \mathbb{Z}_{(p)}[\Lambda, U_0, U_1, U_2, \dots, Z_0, Z_1, Z_2, \dots]$$

and

$$\mathbf{a} = \mathbf{U} = (U_0, U_1, U_2, \dots), \quad \mathbf{c} = \mathbf{Z} = (Z_0, Z_1, Z_2, \dots).$$

First we have

$$\begin{aligned} \prod_{r=0}^{\infty} \left[\frac{(1 + \Lambda^{p^{r+1}} E_p^{(\Lambda)}(\mathbf{U}; \mathbf{X})^{p^{r+1}})(1 + \Lambda^{p^{r+1}} E_p^{(\Lambda)}(\mathbf{U}; \mathbf{Y})^{p^{r+1}})}{1 + \Lambda^{p^{r+1}} E_p^{(\Lambda)}(\mathbf{U}; S(\mathbf{X}, \mathbf{Y}))^{p^{r+1}}} \right]^{\frac{\Phi_r(\mathbf{Z})}{p^{r+1} \Lambda^{p^{r+1}}}} \\ = F_p(\mathbf{Z}, \Lambda; E_p^{(\Lambda)}(\mathbf{U}; \mathbf{X}), E_p^{(\Lambda)}(\mathbf{U}; \mathbf{Y})), \end{aligned}$$

combining the definition

$$F_p(\mathbf{Z}, \Lambda; X, Y) = \prod_{r=0}^{\infty} \left[\frac{(1 + \Lambda^{p^{r+1}} X^{p^{r+1}})(1 + \Lambda^{p^{r+1}} Y^{p^{r+1}})}{1 + \Lambda^{p^{r+1}} (X + Y + \Lambda XY)^{p^{r+1}}} \right]^{\frac{\Phi_r(\mathbf{Z})}{p^{r+1} \Lambda^{p^{r+1}}}}$$

and the equality

$$E_p^{(\Lambda)}(\mathbf{U}; S(\mathbf{X}, \mathbf{Y})) = E_p^{(\Lambda)}(\mathbf{U}; \mathbf{X}) + E_p^{(\Lambda)}(\mathbf{U}; \mathbf{Y}) + \Lambda E_p^{(\Lambda)}(\mathbf{U}; \mathbf{X}) E_p^{(\Lambda)}(\mathbf{U}; \mathbf{Y}).$$

On the other hand, we have

$$\tilde{G}^{(\Lambda)}(\mathbf{Z}, \mathbf{U}; \mathbf{X}) \tilde{G}^{(\Lambda)}(\mathbf{Z}, \mathbf{U}; \mathbf{Y}) = \tilde{G}^{(\Lambda)}(\mathbf{Z}, \mathbf{U}; S(\mathbf{X}, \mathbf{Y})).$$

Hence we obtain the required equality.

PROPOSITION 4.34. *Let A be a $\mathbb{Z}_{(p)}[\Lambda]$ -algebra and $\mathbf{c} \in W(A)$. Let $\mathcal{E}$ denote the extension of $\hat{\mathcal{G}}_A^{(\Lambda)}$ by $\hat{\mathbb{G}}_{m,A}$ defined by the 2-cocycle $F_p(\mathbf{c}, \Lambda; X, Y) \in Z^2(\hat{\mathcal{G}}_A^{(\Lambda)}, \hat{\mathbb{G}}_{m,A})$, and let E denote the extension of $W^{(\Lambda)}(A)$ by $W(A)$ defined by the class $\mathbf{c}$ in*

$$\text{Coker}[F - [\Lambda^{p-1}] : W(A) \rightarrow W(A)] = \text{Ext}_{D_A}^1(W^{(\Lambda)}(A), W(A)).$$

Then

$$(\mathbf{a}, \mathbf{b}) \mapsto ((T_0, T_1) \mapsto (E_p^{(\Lambda)}(\mathbf{a}; \mathbf{T}), E_p(\mathbf{b}; \mathbf{T}) G_p^{(\Lambda)}(\mathbf{c}, \mathbf{a}; \mathbf{T})))$$

gives rise to an isomorphism $\eta : E \rightarrow C(\mathcal{E}) = \text{Hom}_{A\text{-gr}}(\widehat{W}_A, \mathcal{E})$.

PROOF. Let $\tilde{c}$ be the 2-cocycle of $Z^2(W^{(\Lambda)}(A), W(A))$ defined by c as in Remark 4.10. We begin with verifying that η is a homomorphism. By Remark 4.10, this is equivalent to the equalities:

$$E_p^{(\Lambda)}(a + {}^{(\Lambda)}a'; T) = E_p^{(\Lambda)}(a; T) + E_p^{(\Lambda)}(a'; T) + \Lambda E_p^{(\Lambda)}(a; T)E_p^{(\Lambda)}(a'; T)$$

and

$$E_p(b + b' + \tilde{c}(a, a'); T)G_p^{(\Lambda)}(c, a + {}^{(\Lambda)}a'; T) = \{E_p(b; T)G_p^{(\Lambda)}(c, a; T)\} \\ \{E_p(b'; T)G_p^{(\Lambda)}(c, a'; T)\}F_p(c, \Lambda; E_p^{(\Lambda)}(a; T), E_p^{(\Lambda)}(a'; T)).$$

Then it is sufficient to verify the equality

$$\frac{G_p^{(\Lambda)}(c, a; T)G_p^{(\Lambda)}(c, a'; T)}{G_p^{(\Lambda)}(c, a + {}^{(\Lambda)}a'; T)} = \frac{E_p(\tilde{c}(a, a'); T)}{F_p(c, \Lambda; E_p^{(\Lambda)}(a; T), E_p^{(\Lambda)}(a'; T))},$$

by 4.18, (1) and 4.14, (1).

We may assume that

$$A = \mathbb{Z}_{(p)}[\Lambda, U_0, U_1, U_2, \dots, U'_0, U'_1, U'_2, \dots, Z_0, Z_1, Z_2, \dots]$$

and

$$a = U = (U_0, U_1, U_2, \dots), \quad a' = U' = (U'_0, U'_1, U'_2, \dots), \quad c = Z = (Z_0, Z_1, Z_2, \dots).$$

First we have

$$\prod_{k=0}^{\infty} \left[\frac{(1 + \Lambda^{p^{k+1}} E_p^{(\Lambda)}(U; T)^{p^{k+1}})(1 + \Lambda^{p^{k+1}} E_p^{(\Lambda)}(U; T)^{p^{k+1}})}{1 + \Lambda^{p^{k+1}} E_p^{(\Lambda)}(U + {}^{(\Lambda)}U'; T)^{p^{k+1}}} \right]^{\frac{\Phi_k(Z)}{p^{k+1} \Lambda^{p^{k+1}}}} \\ = F_p(Z, \Lambda; E_p^{(\Lambda)}(U; T), E_p^{(\Lambda)}(U'; T))$$

combining the definition

$$F_p(Z, \Lambda; X, Y) = \prod_{r=0}^{\infty} \left[\frac{(1 + \Lambda^{p^{r+1}} X^{p^{r+1}})(1 + \Lambda^{p^{r+1}} Y^{p^{r+1}})}{1 + \Lambda^{p^{r+1}} (X + Y + \Lambda XY)^{p^{r+1}}} \right]^{\frac{\Phi_r(Z)}{p^{r+1} \Lambda^{p^{r+1}}}}$$

and the equalities

$$E_p^{(\Lambda)}(U + {}^{(\Lambda)}U'; T) = E_p^{(\Lambda)}(U; T) + E_p^{(\Lambda)}(U'; T) + \Lambda E_p^{(\Lambda)}(U; T)E_p^{(\Lambda)}(U'; T)$$

(Proposition 4.18, (1)). On the other hand, by the definition,

$$\tilde{G}_p^{(\Lambda)}(\mathbf{Z}, \mathbf{U}; \mathbf{T}) = \exp \left[\sum_{r=1}^{\infty} \left(\sum_{k=0}^{r-1} \frac{\Phi_r(\Lambda U_0, \dots, \Lambda U_k, 0, \dots, 0) \Phi_{r-k-1}(\mathbf{Z})}{\Lambda^{p^{r-k}}} \right) \frac{\Phi_r(\mathbf{T})}{p^r} \right].$$

Moreover, as is shown in 4.11, we have

$$\begin{aligned} \Phi_r(\tilde{c}(\mathbf{U}, \mathbf{U}')) = & \left\{ \sum_{k=0}^{r-1} \Phi_r(\Lambda U_0, \dots, \Lambda U_k, 0, \dots, 0) + \Phi_r(\Lambda U'_0, \dots, \Lambda U'_k, 0, \dots, 0) \right. \\ & \left. - \Phi_r(\Lambda U''_0, \dots, \Lambda U''_k, 0, \dots, 0) \right\} \frac{\Phi_{r-k-1}(\mathbf{Z})}{\Lambda^{p^{r-k}}} \end{aligned}$$

if we put $\mathbf{U} + {}^{(\Lambda)}\mathbf{U}' = (U''_0, U''_1, U''_2, \dots)$. Hence we obtain

$$\frac{\tilde{G}_p^{(\Lambda)}(\mathbf{Z}, \mathbf{U}; \mathbf{T}) \tilde{G}_p^{(\Lambda)}(\mathbf{Z}, \mathbf{U}'; \mathbf{T})}{\tilde{G}_p^{(\Lambda)}(\mathbf{Z}, \mathbf{U}''; \mathbf{T})} = E_p(\tilde{c}(\mathbf{U}, \mathbf{U}'); \mathbf{T})$$

and the required equality.

Now we verify the bijectivity of $\eta : E \rightarrow \text{Hom}_{A-\text{gr}}(\widehat{W}_A, \mathcal{E})$. Consider the commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & W(A) & \longrightarrow & E & \longrightarrow & W^{(\Lambda)}(A) \longrightarrow 0 \\ & & \downarrow \eta & & \downarrow \eta & & \downarrow \eta \\ 0 & \longrightarrow & \text{Hom}_{A-\text{gr}}(\widehat{W}_A, \widehat{\mathbb{G}}_{m,A}) & \longrightarrow & \text{Hom}_{A-\text{gr}}(\widehat{W}_A, \mathcal{E}) & \longrightarrow & \text{Hom}_{A-\text{gr}}(\widehat{W}_A, \widehat{\mathcal{G}}_A^{(\Lambda)}) \longrightarrow 0, \end{array}$$

where $\eta : W(A) \rightarrow \text{Hom}_{A-\text{gr}}(\widehat{W}_A, \widehat{\mathbb{G}}_{m,A})$ and $\eta : W^{(\Lambda)}(A) \rightarrow \text{Hom}_{A-\text{gr}}(\widehat{W}_A, \widehat{\mathcal{G}}_A^{(\Lambda)})$ are defined by $\mathbf{b} \mapsto E_p(\mathbf{b}; \mathbf{T})$ and $\mathbf{a} \mapsto E_p^{(\Lambda)}(\mathbf{a}; \mathbf{T})$ respectively, and the row below is induced from the exact sequence of the formal groups

$$0 \rightarrow \widehat{\mathbb{G}}_{m,A} \rightarrow \mathcal{E} \rightarrow \widehat{\mathcal{G}}_A^{(\Lambda)} \rightarrow 0.$$

We can conclude that the middle vertical arrow is bijective since the two sided arrows are bijective (Example 4.14 and Proposition 4.19).

Now we establish a formalism for the extensions of $\widehat{\mathcal{G}}_A^{(\Lambda)}$ by $\widehat{\mathcal{G}}_A^{(M)}$, paraphrasing the preceding argument.

4.35. Let

$$\mathbf{U} = (U_0, U_1, U_2, \dots), \quad \mathbf{V} = (V_0, V_1, V_2, \dots), \quad \mathbf{Z} = (U_0, U_1, U_2, \dots)$$

and

$$\mathbf{T} = (T_0, T_1, T_2, \dots).$$

We define a formal power series

$$E_p^{(\Lambda, M)}(\mathbf{Z}, \mathbf{U}, \mathbf{V}; \mathbf{T}) \in \mathbb{Z}_{(p)}[\Lambda, M, U_0, U_1, U_2, \dots, V_0, V_1, V_2, \dots, Z_0, Z_1, Z_2, \dots][[T_0, T_1, T_2, \dots]]$$

by

$$E_p^{(\Lambda, M)}(\mathbf{Z}, \mathbf{U}, \mathbf{V}; \mathbf{T}) = \frac{1}{M} [E_p(\alpha^{(M)} \mathbf{V}; \mathbf{T}) G_p^{(\Lambda)}(\alpha^{(M)} \mathbf{Z}, \mathbf{U}; \mathbf{T}) - 1].$$

EXAMPLE 4.36. Specializing Λ and M to 0, we obtain

$$E^{(0,0)}(\mathbf{Z}, \mathbf{U}, \mathbf{V}; \mathbf{T}) = \sum_{r=0}^{\infty} V_r \Phi_r(\mathbf{Z}) + \frac{1}{p} \left[\sum_{r=1}^{\infty} \left(\sum_{k=0}^{r-1} U_k^{p^{r-k}} Z_{r-k-1} \right) \Phi_r(\mathbf{T}) - \sum_{k=0}^{\infty} Z_k \left(\sum_{r=0}^{\infty} U_r \Phi_r(\mathbf{T}) \right)^{p^{k+1}} \right].$$

Indeed, we have

$$E_p(\alpha^{(M)} \mathbf{V}; \mathbf{T}) = \exp \left[\sum_{r=0}^{\infty} \frac{\Phi_r(\alpha^{(M)} \mathbf{V}) \Phi_r(\mathbf{T})}{p^r} \right] \equiv 1 + M \left[\sum_{r=0}^{\infty} V_r \Phi_r(\mathbf{T}) \right] \pmod{M^2}$$

since

$$\Phi_r(\alpha^{(M)} \mathbf{V}) = \Phi_r(MV_0, \dots, MV_r) \equiv p^r MV_r \pmod{M^p}$$

for each $r \geq 0$. On the other hand, we have

$$\begin{aligned} G^{(\Lambda)}(\alpha^{(M)} \mathbf{Z}, \mathbf{U}; \mathbf{T}) &= 1 + \frac{M}{p} \left[\sum_{r=1}^{\infty} \left(\sum_{k=0}^{r-1} U_k^{p^{r-k}} Z_{r-k-1} \right) \Phi_r(\mathbf{T}) - \sum_{k=0}^{\infty} Z_k \left(\sum_{r=0}^{\infty} U_r \Phi_r(\mathbf{T}) \right)^{p^{k+1}} \right] \\ &\pmod{(\Lambda, M^2)} \end{aligned}$$

since

$$\begin{aligned} G^{(\Lambda)}(\alpha^{(M)} \mathbf{Z}, \mathbf{U}; \mathbf{T}) &\equiv \exp \left[\sum_{r=1}^{\infty} \left(\sum_{k=0}^{r-1} \frac{U_k^{p^{r-k}} \Phi_{r-k-1}(\alpha^{(M)} \mathbf{Z})}{p^{r-k}} \right) \Phi_r(\mathbf{T}) \right. \\ &\quad \left. - \sum_{k=0}^{\infty} \frac{\Phi_k(\alpha^{(M)} \mathbf{Z})}{p^{k+1}} \left(\sum_{r=0}^{\infty} U_r \Phi_r(\mathbf{T}) \right)^{p^{k+1}} \right] \pmod{\Lambda} \end{aligned}$$

and

$$\Phi_k(\alpha^{(M)} \mathbf{Z}) = \Phi_k(MZ_0, \dots, MZ_k) \equiv p^k MZ_k \pmod{M^p}$$

for each $k \geq 0$. Combining the two congruence relations, we obtain

$$\begin{aligned} & E^{(\Lambda, M)}(\mathbf{Z}, \mathbf{U}, \mathbf{V}; \mathbf{T}) \\ &= \frac{1}{M} [E_p(\alpha^{(M)} \mathbf{V}, \mathbf{T}) G_p^{(\Lambda)}(\alpha^{(M)} \mathbf{Z}, \mathbf{U}; \mathbf{T}) - 1] \equiv \sum_{r=0}^{\infty} V_r \Phi_r(\mathbf{Z}) \\ &+ \frac{1}{p} \left[\sum_{r=1}^{\infty} \left(\sum_{k=0}^{r-1} U_k^{p^{r-k}} Z_{r-k-1} \right) \Phi_r(\mathbf{T}) - \sum_{k=0}^{\infty} Z_k \left(\sum_{r=0}^{\infty} U_r \Phi_r(\mathbf{T}) \right)^{p^{k+1}} \right] \bmod (\Lambda, M). \end{aligned}$$

DEFINITION 4.37. Let A be a $\mathbb{Z}_{(p)}[\Lambda, M]$ -algebra, and let $\mathbf{a} \in W^{(\Lambda)}(A)$ and $\mathbf{b}, \mathbf{c} \in W^{(M)}(A)$. We define a formal power series $E_p^{(\Lambda, M)}(\mathbf{c}, \mathbf{a}, \mathbf{b}; \mathbf{T}) \in A[[T_0, T_1, T_2, \dots]]$, substituting $\mathbf{U} = \mathbf{a}$, $\mathbf{V} = \mathbf{b}$ and $\mathbf{Z} = \mathbf{c}$ in $E_p^{(\Lambda, M)}(\mathbf{Z}, \mathbf{U}, \mathbf{V}; \mathbf{T})$.

As in 4.32, we see

$$\begin{aligned} (1) \quad & E_p^{(\Lambda, M)}(\mathbf{c} + {}^{(M)}\mathbf{c}', \mathbf{a}, \mathbf{b} + {}^{(M)}\mathbf{b}'; \mathbf{T}) = E_p^{(\Lambda)}(\mathbf{c}, \mathbf{a}, \mathbf{b}; \mathbf{T}) \\ & + E_p^{(\Lambda, M)}(\mathbf{c}', \mathbf{a}, \mathbf{b}'; \mathbf{T}) + M E_p^{(\Lambda, M)}(\mathbf{c}, \mathbf{a}, \mathbf{b}; \mathbf{T}) E_p^{(\Lambda, M)}(\mathbf{c}', \mathbf{a}, \mathbf{b}'; \mathbf{T}); \\ (2) \quad & E_p^{(\Lambda, M)}((F^{(M)} - [\Lambda^{p-1}])\mathbf{c}, \mathbf{a}, \mathbf{b}; \mathbf{T}) \\ &= \frac{E_p^{(M)}(\mathbf{b} + {}^{(M)}\sum_{k=0}^{\infty} V^k([a_k]\mathbf{c}); \mathbf{T}) - E_p^{(M)}(\mathbf{c}, \Lambda; E_p^{(\Lambda)}(\mathbf{a}; \mathbf{T}))}{1 + M E_p^{(M)}(\mathbf{c}, \Lambda; E_p^{(\Lambda)}(\mathbf{a}; \mathbf{T}))}. \end{aligned}$$

LEMMA 4.38. Let A be a $\mathbb{Z}_{(p)}[\Lambda, M]$ -algebra and $\mathbf{a} \in W^{(\Lambda)}(A)$, $\mathbf{b}, \mathbf{c} \in W^{(M)}(A)$. Let $\mathcal{E}$ denote the extension of $\widehat{\mathcal{G}}_A^{(\Lambda)}$ by $\widehat{\mathcal{G}}_A^{(M)}$ defined by the 2-cocycle $F_p^{(M)}(\mathbf{c}, \Lambda; X, Y) \in Z^2(\widehat{\mathcal{G}}_A^{(\Lambda)}, \widehat{\mathcal{G}}_A^{(M)})$. Then

$$\begin{aligned} T_0 &\mapsto E_p^{(\Lambda)}(\mathbf{a}; \mathbf{T}), \quad T_1 \mapsto E_p^{(\Lambda, M)}(\mathbf{c}, \mathbf{a}, \mathbf{b}; \mathbf{T}) : \\ &\widehat{W}_A = \operatorname{Spf} A[[T_0, T_1, T_2, \dots]] \longrightarrow \mathcal{E} = \operatorname{Spf} A[[T_0, T_1]] \end{aligned}$$

defines a homomorphism of formal groups $\widehat{W}_A \rightarrow \mathcal{E}$.

PROOF. We may assume that

$$A = \mathbb{Z}_{(p)}[\Lambda, M, U_0, U_1, U_2, \dots, V_0, V_1, V_2, \dots, Z_0, Z_1, Z_2, \dots]$$

and

$$\mathbf{a} = \mathbf{U} = (U_0, U_1, U_2, \dots), \quad \mathbf{b} = \mathbf{V} = (V_0, V_1, V_2, \dots), \quad \mathbf{c} = \mathbf{Z} = (Z_0, Z_1, Z_2, \dots).$$

Put

$$F_0(\mathbf{T}) = E_p^{(\Lambda)}(\mathbf{U}; \mathbf{T}), \quad F_1(\mathbf{T}) = E_p^{(\Lambda, M)}(\mathbf{Z}, \mathbf{U}, \mathbf{V}; \mathbf{T})$$

and

$$C(X, Y) = F_p^{(M)}(\mathbf{c}, A; X, Y).$$

Then by Remark 4.28, it is sufficient to verify that

$$\begin{aligned} F_1(\mathbf{X} + \mathbf{Y}) &= F_1(\mathbf{X}) + F_1(\mathbf{Y}) + C(F_0(\mathbf{X}), F_0(\mathbf{Y})) \\ &\quad + M\{F_1(\mathbf{X})F_1(\mathbf{Y}) + (F_1(\mathbf{X}) + F_1(\mathbf{Y}))C(F_0(\mathbf{X}), F_0(\mathbf{Y}))\} \\ &\quad + M^2 F_1(\mathbf{X})F_1(\mathbf{Y})C(F_0(\mathbf{X}), F_0(\mathbf{Y})), \end{aligned}$$

which is a direct consequence of the equality proved in 4.33:

$$1 + MF_1(\mathbf{X} + \mathbf{Y}) = \{1 + MF_1(\mathbf{X})\}\{1 + MF_1(\mathbf{Y})\}\{1 + MC(F_0(\mathbf{X}), F_0(\mathbf{Y}))\}.$$

PROPOSITION 4.39. *Let A be a $\mathbb{Z}_{(p)}[A, M]$ -algebra and $\mathbf{c} \in W^{(M)}$. Let $\mathcal{E}$ denote the extension of $\widehat{\mathcal{G}}_A^{(\Lambda)}$ by $\widehat{\mathcal{G}}_A^{(M)}$ defined by the 2-cocycle $F_p^{(M)}(\mathbf{c}, A; X, Y) \in Z^2(\widehat{\mathcal{G}}_A^{(\Lambda)}, \widehat{\mathcal{G}}_A^{(M)})$, and let E denote the extension of $W^{(\Lambda)}(A)$ by $W^{(M)}(A)$ defined by the class $\mathbf{c}$ in*

$$\text{Coker}[F^{(M)} - [A^{p-1}] : W^{(M)}(A) \rightarrow W^{(M)}(A)] = \text{Ext}_{D_A}^1(W^{(\Lambda)}(A), W^{(M)}(A)).$$

Then

$$(\mathbf{a}, \mathbf{b}) \mapsto [(T_0, T_1) \mapsto (E_p^{(\Lambda)}(\mathbf{a}; \mathbf{T}), E_p^{(\Lambda, M)}(\mathbf{c}, \mathbf{a}, \mathbf{b}; \mathbf{T}))]$$

gives rise to an isomorphism $\eta : E \rightarrow C(\mathcal{E}) = \text{Hom}_{A\text{-gr}}(\widehat{W}_A, \mathcal{E})$.

PROOF. We can deduce the assertion from Proposition 4.34 similarly as Lemma 4.38 is deduced from Lemma 4.33, and thus omit the details.

At last we attain the second main result on the compatibility of the isomorphism ξ with the Cartier theory.

4.40. Let A be a $\mathbb{Z}_{(p)}[A, M]$ -algebra. By Theorem 3.5.1 we have an isomorphism

$$\xi : \text{Coker}[F^{(M)} - [A^{p-1}] : W^{(M)}(A) \rightarrow W^{(M)}(A)] \xrightarrow{\sim} H_0^2(\widehat{\mathcal{G}}_A^{(\Lambda)}, \widehat{\mathcal{G}}_A^{(M)}).$$

On the other hand, the functor $C : G \mapsto C(G) = \text{Hom}_{A\text{-gr}}(\widehat{W}_A, G)$ is exact since C is fully faithful (the second theorem of the Cartier theory, cf. [4, Th. 27.7.10]) and $C(\widehat{W}_A)$ is isomorphic to D_A . Hence from a commutative extension of formal groups

$$0 \rightarrow \widehat{\mathcal{G}}_A^{(M)} \rightarrow \mathcal{E} \rightarrow \widehat{\mathcal{G}}_A^{(\Lambda)} \rightarrow 0$$

we obtain an extension of D_A -modules

$$0 \rightarrow C(\widehat{\mathcal{G}}_A^{(M)}) \rightarrow C(\mathcal{E}) \rightarrow C(\widehat{\mathcal{G}}_A^{(\Lambda)}) \rightarrow 0.$$

The correspondence $\mathcal{E} \mapsto C(\mathcal{E})$ gives rise to an injective homomorphism

$$\mathrm{Ext}_A^1(\widehat{\mathcal{G}}_A^{(A)}, \widehat{\mathcal{G}}_A^{(M)}) \longrightarrow \mathrm{Ext}_{D_A}^1(C(\widehat{\mathcal{G}}_A^{(A)}), C(\widehat{\mathcal{G}}_A^{(M)}))$$

since $G \mapsto C(G)$ is fully faithful. Note that $H_0^2(\widehat{\mathcal{G}}_A^{(A)}, \widehat{\mathcal{G}}_A^{(M)})$ is the subgroup of $\mathrm{Ext}_A^1(\widehat{\mathcal{G}}_A^{(A)}, \widehat{\mathcal{G}}_A^{(M)})$ formed by the class of extensions with a section.

Now we have isomorphisms

$$\eta : W^{(A)}(A) \xrightarrow{\sim} C(\widehat{\mathcal{G}}_A^{(A)})$$

and

$$\eta : W^{(M)}(A) \xrightarrow{\sim} C(\widehat{\mathcal{G}}_A^{(M)})$$

by Theorem 4.15.1. Furthermore, as is remarked in 4.9, we have an isomorphism

$$\mathrm{Ext}_{D_A}^1(W^{(A)}(A), W^{(M)}(A)) \xrightarrow{\sim} \mathrm{Coker}[F^{(M)} - [A^{p-1}] : W^{(M)}(A) \rightarrow W^{(M)}(A)],$$

defined by $\partial \mathbf{a} \mapsto [\mathbf{a}]$. Here $\partial : W^{(M)}(A) = \mathrm{Hom}_{D_A}(D_A, W^{(M)}(A)) \rightarrow \mathrm{Ext}_{D_A}^1(W^{(A)}(A), W^{(M)}(A))$ is the boundary map induced by the exact sequence

$$0 \rightarrow D_A \xrightarrow{\mathbb{F} - [A^{p-1}]} D_A \rightarrow W^{(A)}(A) \rightarrow 0.$$

THEOREM 4.40.1. *Let $\mathbf{a} \in \mathrm{Coker}[F^{(M)} - [A^{p-1}] : W^{(M)}(A) \rightarrow W^{(M)}(A)]$. Then we have*

$$C(\xi(\mathbf{a})) = \mathbf{a}$$

under the identification

$$\mathrm{Ext}_{D_A}^1(C(\widehat{\mathcal{G}}_A^{(A)}), C(\widehat{\mathcal{G}}_A^{(M)})) = \mathrm{Coker}[F^{(M)} - [A^{p-1}] : W^{(M)}(A) \rightarrow W^{(M)}(A)].$$

PROOF. Take $\mathbf{c} \in W^{(M)}(A)$ representing $\mathbf{a}$. We define homomorphisms of formal groups

$$\tilde{\varphi}_A : \widehat{W}_A = \mathrm{Spf} A[[T_0, T_1, T_2, \dots]] \rightarrow \widehat{\mathcal{G}}_A^{(A)} = \mathrm{Spf} A[[T]]$$

and

$$\tilde{\varphi}_{(M, \mathbf{c})} : \widehat{W}_A = \mathrm{Spf} A[[T_0, T_1, T_2, \dots]] \rightarrow \widehat{\mathcal{G}}_A^{(M)} = \mathrm{Spf} A[[T]]$$

by

$$T \mapsto E_p^{(A)}([1]_A; \mathbf{T}) : A[[T]] \rightarrow A[[T_0, T_1, T_2, \dots]].$$

and

$$T \mapsto E_p^{(M)}(\mathbf{c}; \mathbf{T}) : A[[T]] \rightarrow A[[T_0, T_1, T_2, \dots]],$$

respectively. It is readily seen that $C(\tilde{\varphi}_A) : D_A = C(\widehat{W}_A) \rightarrow W^{(\Lambda)}(A) = C(\widehat{\mathcal{G}}_A^{(\Lambda)})$ is given by $\varphi_A : 1 \mapsto [1]_A$, and $C(\tilde{\varphi}_{(M,c)}) : D_A = C(\widehat{W}_A) \rightarrow W^{(M)}(A) = C(\widehat{\mathcal{G}}_A^{(M)})$ is given by $\varphi_{(M,c)} : 1 \mapsto c$.

Let $\mathcal{E}$ denote the extension of $\widehat{\mathcal{G}}_A^{(\Lambda)}$ by $\widehat{\mathcal{G}}_A^{(M)}$ defined by the 2-cocycle $F_p^{(M)}(c, A; X, Y) \in Z^2(\widehat{\mathcal{G}}_A^{(\Lambda)}, \widehat{\mathcal{G}}_A^{(M)})$. We define a homomorphism of formal groups

$$\tilde{\varphi} : \widehat{W}_A = \operatorname{Spf} A[[T_0, T_1, T_2, \dots]] \rightarrow \mathcal{E} = \operatorname{Spf} A[[T_0, T_1]]$$

by

$$(T_0, T_1) \mapsto (E_p^{(\Lambda)}([1]_A; \mathbf{T}), E_p^{(\Lambda, M)}(c, [1]_A, [0]_M; \mathbf{T}))$$

(cf. Lemma 4.38). Note that $\varphi = C(\tilde{\varphi}) : D_A = C(\widehat{W}_A) \rightarrow E = C(\mathcal{E})$ is given by $1 \mapsto ([1]_A, [0]_M)$ when E is identified with the set $W^{(\Lambda)}(A) \times W^{(M)}(A)$ as in 4.10 (cf. Proposition 4.39).

Consider now a diagram of formal groups with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \widehat{W}_A & \xrightarrow{V - [\Lambda^{p-1}]} & \widehat{W}_A & \xrightarrow{\tilde{\varphi}_A} & \widehat{\mathcal{G}}_A^{(\Lambda)} \longrightarrow 0 \\ & & \tilde{\varphi}_{(M,c)} \downarrow & & \tilde{\varphi} \downarrow & & \parallel \\ 0 & \longrightarrow & \widehat{\mathcal{G}}_A^{(M)} & \longrightarrow & \mathcal{E} & \longrightarrow & \widehat{\mathcal{G}}_A^{(\Lambda)} \longrightarrow 0. \end{array}$$

Then the corresponding diagram of the Cartier modules are isomorphic to the commutative diagram of left D_A -modules with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & D_A & \xrightarrow{F - [\Lambda^{p-1}]} & D_A & \xrightarrow{\varphi_A} & W^{(\Lambda)}(A) \longrightarrow 0 \\ & & \varphi_{(M,c)} \downarrow & & \varphi \downarrow & & \parallel \\ 0 & \longrightarrow & W^{(M)}(A) & \longrightarrow & E & \longrightarrow & W^{(\Lambda)}(A) \longrightarrow 0, \end{array}$$

which appears in the proof of Proposition 4.9.1. More precisely, the left square is a push-down, and therefore, E is the extension of $W^{(\Lambda)}(A)$ by $W^{(M)}(A)$ defined by the class c in

$$\operatorname{Coker}[F^{(M)} - [\Lambda^{p-1}] : W^{(M)}(A) \rightarrow W^{(M)}(A)] = \operatorname{Ext}_{D_A}^1(W^{(\Lambda)}(A), W^{(M)}(A)).$$

Thus we obtain the assertion of the theorem.

REMARK 4.41. We can deduce the commutativity of the first diagram above from the faithfulness of the Cartier functor and the commutativity of the second diagram above, and we obtain the following formula:

(1) Let A be a $\mathbb{Z}_{(p)}[A, M]$ -algebra and $c \in W^{(M)}(A)$.

$$E_p^{(\Lambda, M)}(c, [1]_A, [0]_M; (V - [\Lambda^{p-1}])\mathbf{T}) = E_p^{(M)}(c; \mathbf{T}).$$

In particular, we have:

(2) Let A be a $\mathbb{Z}_{(p)}[A]$ -algebra and $\mathbf{c} \in W(A)$. Then we have

$$G_p^{(\Lambda)}(\mathbf{c}, [1]_A; (V - [\Lambda^{p-1}])\mathbf{T}) = E_p(\mathbf{c}; \mathbf{T}).$$

We can verify the formula directly as follows.

To prove (2), we may assume that $A = \mathbb{Z}_{(p)}[Z_0, Z_1, Z_2, \dots]$ and $\mathbf{c} = \mathbf{Z} = (Z_0, Z_1, Z_2, \dots)$. By the definition we have

$$G_p^{(\Lambda)}(\mathbf{Z}, [1]_A; \mathbf{T}) = \frac{\tilde{G}_p^{(\Lambda)}(\mathbf{Z}, [1]_A; \mathbf{T})}{\prod_{r=0}^{\infty} [1 + \Lambda^{p^{r+1}} E_p^{(\Lambda)}([1]_A; \mathbf{T})^{p^{r+1}}]^{\frac{\Phi_r(\mathbf{Z})}{p^{r+1}\Lambda^{p^{r+1}}}}}.$$

First we obtain

$$E_p^{(\Lambda)}([1]_A; (V - [\Lambda^{p-1}])\mathbf{T}) = 0$$

since we have

$$E_p^{(\Lambda)}([1]_A; (V - [\Lambda^{p-1}])\mathbf{T}) = E_p^{(\Lambda)}((F^{(\Lambda)} - [\Lambda^{p-1}])[1]_A; \mathbf{T})$$

by Proposition 4.18, (5) and

$$(F^{(\Lambda)} - [\Lambda^{p-1}])[1]_A = [\Lambda^{p-1}] - [\Lambda^{p-1}] = 0$$

by Lemma 1.13.1. On the other hand, we have

$$\log \tilde{G}_p^{(\Lambda)}(\mathbf{Z}, [1]_A; \mathbf{T}) = \sum_{r=1}^{\infty} \left(\sum_{k=0}^{r-1} \frac{\Phi_r([A])\Phi_{r-k-1}(\mathbf{Z})}{\Lambda^{p^{r-k}}} \right) \frac{\Phi_r(\mathbf{T})}{p^r}$$

by the definition. Now we have

$$\Phi_r((V - [\Lambda^{p-1}])\mathbf{T}) = p\Phi_{r-1}(\mathbf{T}) - \Lambda^{p^r(p-1)}\Phi_r(\mathbf{T})$$

for each $r > 0$. It follows that

$$\begin{aligned} & \log \tilde{G}_p^{(\Lambda)}(\mathbf{W}, [1]_A; (V - [\Lambda^{p-1}])\mathbf{T}) \\ &= \sum_{r=1}^{\infty} \left(\sum_{k=0}^{r-1} \frac{\Phi_r([A])\Phi_{r-k-1}(\mathbf{Z})}{\Lambda^{p^{r-k}}} \right) \frac{\Phi_{r-1}(\mathbf{T})}{p^{r-1}} \\ & \quad - \sum_{r=1}^{\infty} \left(\sum_{k=0}^{r-1} \frac{\Phi_r([A])\Phi_{r-k-1}(\mathbf{Z})}{\Lambda^{p^{r-k}}} \right) \frac{\Lambda^{p^r(p-1)}\Phi_r(\mathbf{T})}{p^r} \end{aligned}$$

$$\begin{aligned}
 &= \sum_{r=1}^{\infty} \left(\sum_{k=0}^{r-1} \frac{\Lambda^{p^r} \Phi_{r-k-1}(\mathbf{Z})}{\Lambda^{p^{r-k}}} \right) \frac{\Phi_{r-1}(\mathbf{T})}{p^{r-1}} \\
 &\quad - \sum_{r=1}^{\infty} \left(\sum_{k=0}^{r-1} \frac{\Lambda^{p^r} \Phi_{r-k-1}(\mathbf{Z})}{\Lambda^{p^{r-k}}} \right) \frac{\Lambda^{p^r(p-1)} \Phi_r(\mathbf{T})}{p^r} \\
 &= \Phi_0(\mathbf{Z}) \Phi_0(\mathbf{T}) + \sum_{r=1}^{\infty} \left(\sum_{k=0}^r \frac{\Lambda^{p^{r+1}} \Phi_{r-k}(\mathbf{Z})}{\Lambda^{p^{r-k+1}}} - \sum_{k=1}^r \frac{\Lambda^{p^{r+1}} \Phi_{r-k}(\mathbf{Z})}{\Lambda^{p^{r-k+1}}} \right) \frac{\Phi_r(\mathbf{T})}{p^r} \\
 &= \sum_{r=0}^{\infty} \frac{\Phi_r(\mathbf{Z}) \Phi_r(\mathbf{T})}{p^r} = \log E_p(\mathbf{Z}; \mathbf{T}).
 \end{aligned}$$

Hence we obtain the required equality.

We can deduce (1) from (2) by the definition of $E_p^{(\Lambda, M)}(\mathbf{Z}, \mathbf{U}, \mathbf{V}; \mathbf{T})$ since we have

$$\alpha^{(M)}((F^{(M)} - [\Lambda^{p-1}])\mathbf{c}) = (F - [\Lambda^{p-1}])\alpha^{(M)}(\mathbf{c})$$

by Proposition 1.16.

REMARK 4.42. From the theorem, we can conclude the following assertion:
Let A be a $\mathbb{Z}_{(p)}[\Lambda, M]$ -algebra. Then we have

$$H_0^2(\widehat{\mathcal{G}}_A^{(\Lambda)}, \widehat{\mathcal{G}}_A^{(M)}) = \text{Ext}_A^1(\widehat{\mathcal{G}}_A^{(\Lambda)}, \widehat{\mathcal{G}}_A^{(M)}).$$

EXAMPLE 4.43. The class of $\widehat{W}_2$ in $\text{Ext}_A^1(\widehat{\mathbb{G}}_{a,A}, \widehat{\mathbb{G}}_{a,A})$ is given by the class of $[1]_0$ under the identification

$$H_0^2(\widehat{\mathbb{G}}_{a,A}, \widehat{\mathbb{G}}_{a,A}) = \text{Coker}[F^{(0)} : W^{(0)}(A) \rightarrow W^{(0)}(A)],$$

which follows from Remark 2.7.2. Moreover, we have

$$E^{(0,0)}([1], \mathbf{U}, \mathbf{V}; \mathbf{T}) = \sum_{r=0}^{\infty} V_r \Phi_r(\mathbf{T}) + \frac{1}{p} \left[\sum_{r=0}^{\infty} U_r^p \Phi_{r+1}(\mathbf{T}) - \left(\sum_{r=0}^{\infty} U_r \Phi_r(\mathbf{T}) \right)^p \right]$$

as is remarked in Example 4.36. In particular, we obtain $E^{(0,0)}([1], [1], [0]; \mathbf{T}) = T_1$. Hence $([1], [0]) \in W^{(0)}(A) \times W^{(0)}(A)$ corresponds to the restriction map $R : \widehat{W}_A \rightarrow \widehat{W}_{2,A}$ since we have $E^{(0)}([1]; \mathbf{T}) = T_0$.

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