The endomorphism rings of some abelian varieties II

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§ 0. Introduction

In our previous paper [3], we studied the endomorphism rings of abelian varieties and proved, in particular, that the Jacobian variety J of the "generic" hyperelliptic curve of arbitrary genus (in every characteristic) has the property $\widetilde{\text{End}} J \cong \mathbb{Z}$ (for an abelian variety A defined over a field k , $\widetilde{\text{End}} A$ denotes the ring of endomorphisms of A which are defined over the algebraic closure $\bar{k}$ of k).

In this paper, for an arbitrary natural number g , we construct Jacobian varieties J of dimension g defined over $\mathbb{Q}$ such that $\widetilde{\text{End}} J \cong \mathbb{Z}$.

Let us consider the curve defined over $\mathbb{Q}$:

$$C = \text{Proj } \mathbb{Q}[X, Y, Z]/(F(X, Y, Z)),$$

where $F(X, Y, Z) = Y^2 - X^{2g+1}Z + bXZ^{2g+1} + (pc/4)Z^{2g+2}$, $\deg X = \deg Z = 1$, $\deg Y = g+1$, and b, c, p are integers with the following properties:

- α) p is an odd prime number (> 1) such that some positive power of $(p-1)/2$ belongs to $g\mathbb{Z}$,
- β) $b \bmod p$ is a primitive root of $F_p = \mathbb{Z}/p\mathbb{Z}$,
- γ) $\text{G.C.M. } \{b, c\} = \text{G.C.M. } \{b, 2g+1\} = \text{G.C.M. } \{c, g\} = 1$,
- δ) $c \equiv -p \bmod 4$.

It is easy to show that C is a non-singular hyperelliptic curve of genus g with a $\mathbb{Q}$ -rational point $P = V_+(Y, Z)$. The Jacobian variety $J(C)$ defined over $\mathbb{Q}$ of C has the required property:

THEOREM 1. $\widetilde{\text{End}} J(C) \cong \mathbb{Z}$.

For example, $(b, c, p, g) = (-2, -1, 5, 2^n), (-2, 1, 7, 3^n)$ ($n \in \mathbb{N}$) enjoy the properties (α) , (β) , (γ) and (δ) . In general, it is proved by Dirichlet's theorem on prime numbers in arithmetic progressions that there exist triples (b, c, p) with the properties (α) , (β) , (γ) and (δ) for every g . Thus the required Jacobian varieties J are obtained for every g .

§ 1. Proof of Theorem 1

First, we need :

LEMMA 2. *$J(C)$ has everywhere semi-stable reduction over Z (for the definition, see [2, Exposé IX, Definition 3.4]).*

PROOF. By a change of coordinates $Y_1 = Y - Z^{g+1}/2$, we see that

$$F(X, Y, Z) = G(X, Y_1, Z) = Y_1^2 + Y_1 Z^{g+1} - X^{2g+1} Z + b X Z^{2g+1} + \frac{pc+1}{4} Z^{2g+2}$$

belongs to $Z[X, Y_1, Z]$ by the properties (α) and (δ) . It is obvious that F , and hence G , are irreducible polynomials of $\mathbf{Q}[X, Y_1, Z]$. Since G is primitive in $Z[X, Y_1, Z]$, G is a prime element of $Z[X, Y_1, Z]$. Thus $D = \text{Proj } Z[X, Y_1, Z]/(G(X, Y_1, Z))$ is irreducible and reduced, and $D_q \cong C$. In particular D is flat over Z . We claim that every geometric fiber of D over Z is irreducible and reduced, and has at most ordinary double points (i.e. nodes of multiplicity 2) as singularities, and hence D is a stable curve over Z , [1, Definition (1.1)]. First, it is easy to check that D is smooth over Z at every point of $B = V_+(\{2, Y_1, Z\})$, and that

$$D - B \cong \text{Spec } Z[1/2][U, V]/(V^2 - H(U)),$$

where U (or, V) corresponds to X/Z (or, $Y/Z^{g+1} + 1/2$, resp.), and $H(U) = V^2 - F(U, V, 1) \in Z[1/2][U]$. Let $q (> 2)$ be a prime number, and K an overfield of F_q which is algebraically closed. Let $\bar{H}(U) (\in F_q[U])$ be $H(U) \bmod q$, then $\bar{H}(U)$ has no triple factors in $K[U]$. Indeed, it follows immediately, from the properties (α) , (β) and (γ) , that $\bar{H}$, $d\bar{H}/dU$ and $d^2\bar{H}/dU^2$ have no common zeros. Hence $(D - B)_K \cong \text{Spec } K[U, V]/(V^2 - \bar{H}(U))$ is irreducible and reduced, and has at most ordinary double points as singularities (the irreducibility follows from $\deg \bar{H} \equiv 1 \pmod{2}$). Thus D is a stable curve over Z , and hence D_{R_s} is also a stable curve over R_s , where R_s denotes the strict henselization of $Z_{(s)}$ (s is an arbitrary prime number). By [1, Theorem (2.4)], $J(C)$ has semi-stable reduction over R_s . Hence $J(C)$ has everywhere semi-stable reduction over Z , by [2, Exposé IX, (3.4.0)]. q.e.d.

It is proved by K. A. Ribet [6, Corollary 1.4] that every endomorphism of an abelian variety A over $\mathbf{Q}$ is defined over $\mathbf{Q}$ if A has everywhere semi-stable reduction over Z . Hence $\widetilde{\text{End}} J(C) = \text{End}_{\mathbf{Q}} J(C)$ by Lemma 2, where $\text{End}_{\mathbf{Q}} J(C)$ denotes the ring of endomorphisms defined over $\mathbf{Q}$ of $J(C)$. Since $\widetilde{\text{End}} J(C)$ is a Z -algebra which is free of finite rank as a Z -module [4, § 19, Corollary 1], Theorem 1 is reduced to the assertion :

$$\mathrm{rk} \mathrm{End}_{\mathbf{Q}} J(C) = 1,$$

where $\mathrm{rk} \mathrm{End}_{\mathbf{Q}} J(C)$ denotes the rank of $\mathrm{End}_{\mathbf{Q}} J(C)$. Hence, by Proposition 3 below, it suffices to show: (1) $H(U)$ (given in the proof of Lemma 2) is irreducible in $\mathbf{Q}[U]$, and (2) the Galois group G of the minimal splitting field of $H(U)$ over $\mathbf{Q}$ acts doubly transitively on the set $\{\alpha \in \bar{\mathbf{Q}} \mid H(\alpha) = 0\}$. (1) follows immediately from the consideration of the Newton polygon of

$$2^{2g+1}H(U/2) = U^{2g+1} - 2^{2g}bU - 2^{2g-1}pc$$

with respect to the 2-adic valuation of $\mathbf{Q}$. As for (2), it suffices to show that G (considered, canonically, as a subgroup of the symmetric group $\mathfrak{S}_{2g+1}$) contains a cycle of length $2g$. Since $H(U) \equiv U(U^{2g} - b) \pmod{p}$, the above assertion is reduced to the irreducibility of $U^{2g} - b$ in $F_p[U]$ (cf. [7, § 61]). On the other hand, by [5, Theorem 428],¹⁾ $U^{2g} - b$ is irreducible in $F_p[U]$ if (i) $X^d \equiv b \pmod{p}$ has no solution X in F_p for every divisor d (> 1) of $2g$, and (ii) $-4X^4 \equiv b \pmod{p}$ has no solution X in F_p if g is even. Now, (i) follows from the property (β) , and (ii) follows from the properties (α) and (β) .

q.e.d.

PROPOSITION 3. *Let k be a field of characteristic $p \neq 2$, and let $f(X)$ be an irreducible polynomial of degree $2g+1$ in $k[X]$ which is squarefree in $\bar{k}[X]$ ($\bar{k}$ is the algebraic closure of k). The Jacobian variety $J(V)$ defined over k of the non-singular curve*

$$V = \mathrm{Proj} \, k[X, Y, Z] / (Y^2 - f(X/Z)Z^{2g+2})$$

($\deg X = \deg Z = 1$ and $\deg Y = g+1$) satisfies the condition:

$$\mathrm{rk} \mathrm{End}_k J(V) \leq \#\{\text{prime divisors of } f(X)/(X-\theta) \text{ in } k(\theta)[X]\},$$

where $\theta (\in \bar{k})$ is a zero of $f(X)$.

PROOF. The non-singularity of V is easily checked, and we omit it. Let $\bar{V}$ be $V_{\bar{k}}$, G the Galois group of $\bar{k}$ over k , and $\theta = \theta_1, \theta_2, \dots, \theta_{2g+1}$ the zeros ($\in \bar{k}$) of $f(X)$. Let $P_0, P_1, \dots, P_{2g+1}$ be the $\bar{k}$ -rational points of $\bar{V}$ defined by $Y=Z=0$, $Y=X-\theta_1Z=0$, $\dots$, $Y=X-\theta_{2g+1}Z=0$, respectively. Since the group $J(\bar{V})(\bar{k})$ of $\bar{k}$ -rational points of $J(\bar{V})$ is isomorphic to the group of divisor classes of degree 0 of $\bar{V}$, the group $J(\bar{V})_2$ of 2-section points of $J(\bar{V})(\bar{k})$ is expressed as:

$$(3.1) \quad \varphi: \bigoplus_{i=1}^{2g+1} (Z/2Z)(P_i - P_0) / (Z/2Z) \sum_{i=1}^{2g+1} (P_i - P_0) \xrightarrow{\sim} J(\bar{V})_2.$$

1) This reference was pointed out by Professor T. Shioda.

First the homomorphism φ can be defined because $\text{div}(Y/Z^{g+1}) = \sum_{i=1}^{2g+1} (P_i - P_0)$ and $\text{div}(X/Z - \theta_i) = 2(P_i - P_0)$ ($i=1, \dots, 2g+1$).²⁾ If $\text{Ker } \varphi \neq 0$, then, after renumbering θ_i 's, we may assume that $\text{Ker } \varphi$ contains $\sum_{i=1}^j (P_i - P_0) \bmod 2$ for some j such that $1 \leq j \leq 2g$. Thus $\sum_{i=1}^j P_i + (2g-2-j)P_0$ is a member of $|\mathcal{O}((2g-2)P_0)|$. Since $|\mathcal{O}(2P_0)|$ is a pencil of degree 2 on the hyperelliptic curve $\bar{V}$, $\mathcal{O}((2g-2)P_0)$ is the canonical bundle $K_{\bar{V}}$ of $\bar{V}$ and $|K_{\bar{V}}|$ is composite with $|\mathcal{O}(2P_0)|$. This is a contradiction because $P_1 + P_n \notin |\mathcal{O}(2P_0)|$ if $n \neq 1$. Hence, it follows that φ is an injection. Since $J(\bar{V})_2$ is of order 2^{2g} , φ is necessarily an isomorphism. Next, let

$$\psi: (Z/2Z) \otimes \text{End}_k J(V) \rightarrow J(\bar{V})_2$$

be defined by $\psi(f \bmod 2) = f(P_1 - P_0)$ ($f \in \text{End}_k J(V)$). ψ is an injection because $\{(P_1 - P_0)^g | g \in G\}$ generates $J(\bar{V})_2$. Since ψ is compatible with the action of $H = \{g \in G | \theta^g = \theta\}$, $\text{Im } \psi$ is contained in the set $J(\bar{V})_2^H$ of H -invariant elements of $J(\bar{V})_2$. Hence Lemma 3 is reduced to the assertion:

$$\dim_{Z/2Z} J(\bar{V})_2^H = \text{the number of } H\text{-orbits of } \{P_2, \dots, P_{2g+1}\},$$

which is obvious by (3.1).

q.e.d.

Before closing this paper, we give an example in characteristic $p > 0$. Let $F_p(t)$ be a purely transcendental extension of F_p , and let J be the Jacobian variety over $F_p(t)$ constructed as in Lemma 3 from

$$f(X) = X^{2g+1} + X + t \in F_p[X].$$

Then:

THEOREM 4. $\widetilde{\text{End}} J \cong Z$ if $p \nmid 2g(2g+1)$.

PROOF. By a direct calculation, it is proved that

(4.1) J_k has everywhere semi-stable reduction over $P_{\bar{F}_p}$ if $p \nmid 2g(2g+1)$,

where $P_{\bar{F}_p}$ has the function field $k = \bar{F}_p(t^{1/2(2g+1)})$,

(4.2) J has semistable reduction over $\bar{F}_p[t]_{(t)}$ if $p \nmid 2g$.

Let k' be the field of definition for an endomorphism $\varphi \in \text{End } J_{\bar{F}_p(t)}$. Since $k(k')$ is a finite unramified extension field of k by (4.1) [6, Theorem 1.3], it follows that $\bar{F}_p(t) \subseteq k' \subseteq k$. Similarly, by (4.2), k' is unramified over $\bar{F}_p(t)$ at

2) For a rational function f of V , $\text{div}(f)$ denotes the divisor of f .

$t=0$ [6, Theorem 1.1]. Hence $k'=\bar{F}_p(t)$ because every intermediate field ($\neq \bar{F}_p(t)$) of $k/\bar{F}_p(t)$ is ramified at $t=0$. Thus $\widetilde{\text{End}} J_{\bar{F}_p(t)} = \text{End}_{\bar{F}_p(t)} J_{\bar{F}_p(t)}$. Hence, in view of Proposition 3, Theorem 4 is reduced to (i) $f(X)$ is irreducible in $\bar{F}_p(t)[X]$, and (ii) $g(X)=f(X)/(X-\theta)$ is irreducible in $\bar{F}_p(t, \theta)[X]$, where $\theta (\in \bar{k})$ is a zero of $f(X)$. (i) follows from:

$$\bar{F}_p[X] \xrightarrow{\sim} \bar{F}_p[t, X]/(f(X)),$$

and (ii) follows from:

$$\bar{F}_p[\theta, X]/(X^{2g} + X^{2g-1}\theta + \dots + \theta^{2g} + 1) \xrightarrow{\sim} \bar{F}_p[t, \theta, X]/(g(X)),$$

because $X^{2g} + X^{2g-1}\theta + \dots + \theta^{2g} + 1$ is an irreducible polynomial in θ and X if $p \nmid 2g$. q.e.d.

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