On the equivalence of algebraic formulations of knot cobordism

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Let $C_m$ be the knot cobordism group of PL locally flat knotted $m$-spheres in $(m+2)$-spheres. It is known that $C_{2n} = \{0\}$ (see [2]). Several authors studied the algebraic structure of $C_{2n-1}$. Levine [2] proved that $C_{2n-1} (n \geq 3)$ is isomorphic to the “cobordism group” $G_{(-1)^n}$ of integral square matrices with property $(-1)^n$. Here an integral square matrix $A$ is said to have property $(-1)^n$ iff $A + (-1)^n A'$ is unimodular, where $A'$ is the transposed matrix of $A$. With this formulation, together with a result of Milnor [7], he studied the structure of $C_{2n-1}$ in detail [3]. The present author [4], [5] gave another description of $C_{2n-1}$: $C_{2n-1}$ is isomorphic to the group denoted by $P_{2n} (Z \rightarrow 1)$ which is a “Witt group” of $(-1)^n t$-hermitian forms that are defined over the Laurent polynomial ring $Z[t, t^{-1}]$ and become nonsingular over $Z$ after the substitution $t = 1$ (cf. (1)).

Since these groups, $G_{(-1)^n}$ and $P_{2n} (Z \rightarrow 1)$, are different formulations of the same $C_{2n-1}$, they should be mutually isomorphic. Moreover, by a geometric observation, it is known that the isomorphism $G_{(-1)^n} \rightarrow P_{2n} (Z \rightarrow 1)$ is given by the correspondence $A \rightarrow A + (-1)^n t A'$, see Remark after Proposition 6.2 of [5]. However, as we remarked there, we had then no direct algebraic proof of the fact that $G_{(-1)^n} \approx P_{2n} (Z \rightarrow 1)$ nor the fact that the isomorphism is given by $A \rightarrow A + (-1)^n t A'$.

The purpose of the present paper is to give algebraic proofs of these facts. We shall consider any principal ideal domain $R$ with involution rather than the ring of integers $Z$. Thus our proof will be slightly more general than it was originally required.

As it was shown by Levine [3], $G_{(-1)^n}$ is closely related to the classification of isometries of inner product spaces (cf. Milnor [7]). While our $P$-group is a variant of Wall’s $L$-group [8], which classifies “relatively nonsingular” hermitian forms. Thus the isomorphism of $G_{(-1)^n}$ and $P_{2n} (Z \rightarrow 1)$ shown in this paper may be of some algebraic interest.

1) In this paper, we shall use the notation $P_{2n}(\kappa \rightarrow \kappa')$ instead of $P_{2n}(I \rightarrow C \rightarrow \kappa \rightarrow \kappa' \rightarrow 1)$ which was used in [5].
REMARK. Cappell and Shaneson [1] gave another formulation of $C_{2n-1}$ in terms of their $\Gamma$-groups: $\tilde{G}_{2n-1}$ isomorphic to $\Gamma_{2n-1}(Z-1)$, where $\tilde{G}_{2n-1}$ is the kernel of the index (or the Arf-Kervaire) homomorphism $C_{2n-1} \to L_{2n}(1)$. An algebraic proof of the isomorphism $\tilde{G}_{(1)n} \cong \tilde{\Gamma}_{2n+2}(Z-1)$ remains to be found.

§ 1. Preliminary definitions and main results

Let $R$ be a principal ideal domain. Suppose we are given an involutory automorphism denoted by $\bar{-}$: $R \to R$. We are also given a unit $u \in R^\times$ such that $\bar{u} = u^{-1}$. We shall fix the triple $(R, \bar{-}, u)$ throughout.

A matrix $A$ whose entries are elements of $R$ is called an $R$-matrix.

DEFINITION 1.1. An $R$-matrix is $u$-admissible iff $A + uA'$ is nonsingular, i.e., $\det(A + uA') \in R^\times$. Here $A'$ denotes the conjugate transposed matrix of $A$.

DEFINITION 1.2. (Levine [2]). A $u$-admissible matrix $A$ is null-cobordant iff it is congruent$^2$ to a matrix of the form $(0 \ N_1 \ N_2)$, where $N_1$ and $N_2$ are square matrices of the same size.

Following Levine [2], we define the block sum $A_1 \oplus A_2$ of two matrices $A_1, A_2$ to be $A_1 \oplus 0$. We define the Levine group $G_u(R, \bar{-})$ to be the quotient of the Grothendieck group of all congruence classes of $u$-admissible matrices with addition $\oplus$, by the subgroup generated by all null-cobordant matrices. The original $G_{(-1)n}$ introduced by Levine [2] is nothing other than $G_{(-1)n}(Z, \text{id})$.

Next we shall give the definition of the group $P_u^R(Z-1)$. Let $A = R[t, t^{-1}]$, the Laurent polynomial ring of an indeterminate variable $t$. The involution $\bar{-}$: $R \to R$ can be extended to the involution $\bar{-}$: $A \to A$ defined by $\bar{a_m t^m} = \bar{a_m} t^{-m}$ ($a_m \in R$). Let $M$ be a free $A$-module of finite rank. (The structure of $A$-modules will be written as left $A$-modules.)

DEFINITION 1.3. A $ut$-hermitian form is a function $\lambda: M \times M \to A$ which is $A$-linear in the first variable, anti-$A$-linear in the second and satisfies the 'symmetry' relation: $\lambda(x, y) = ut\lambda(y, x)$, for any $x, y \in M$.

Let $\{e_1, \ldots, e_m\}$ be a basis of $M$. To a $ut$-hermitian form $\lambda: M \times M \to A$ corresponds a $A$-matrix $B = (b_{ij})$ whose $(i, j)$-entry $b_{ij} = \lambda(e_i, e_j)$. We shall say that $B$ represents a $ut$-hermitian form $\lambda$ (with respect to $\{e_1, \ldots, e_m\}$).

$^2$ Two matrices $A$ and $B$ are congruent iff there exists a non-singular matrix $P$ such that $B = PAP^\dagger$. 
DEFINITION 1.4. Suppose $B = (b_{ij})$ represents $\lambda : M \times M \to A$. $\lambda$ is said to be $R$-nonsingular if the image of the matrix $B$, $\varepsilon B = (\varepsilon(b_{ij}))$, is nonsingular, where $\varepsilon : A \to R$ is the augmentation: $\varepsilon(\sum a_m b^m) = \sum a_m$.

In other words, $\lambda : M \times M \to A$ is $R$-nonsingular iff the change of rings gives a nonsingular $u$-hermitian form $\varepsilon \circ \lambda : R \otimes R M \times R \otimes R M \to R$.

DEFINITION 1.5. A $u$-hermitian form $\lambda : M \times M \to A$ is of $u$-even type$^3$ if for any $x \in M$, there exists $\alpha \in A$ such that $\lambda(x, x) = \alpha + ut\beta$. Such $\alpha$ is determined up to the elements of the form $\beta - ut\bar{\beta}, \beta \in A$.

For convenience, we shall refer to an $R$-nonsingular $u$-hermitian form of $u$-even type simply as a $u$-form. Let $\lambda_1, \lambda_2$ be $u$-forms. The orthogonal sum of $\lambda_1$ and $\lambda_2$ is defined as usual and will be denoted $\lambda_1 \perp \lambda_2$. $\lambda_1 \perp \lambda_2$ is again a $u$-form.

DEFINITION 1.6. A $u$-form $\lambda : M \times M \to A$ is null-cobordant if there exists a $A$-submodule $H \subset M$ such that (i) $\lambda(H \times H) = \{0\}$, (ii) $H$ is mapped onto a free direct summand $H'$ of $R \otimes A M$ by the canonical onto homomorphism $\varepsilon \otimes 1 : M = A \otimes A M \to R \otimes A M$ and (iii) $\text{rank}_R H' = \frac{1}{2} \text{rank}_R R \otimes A M$.

REMARK. Following Cappell and Shaneson [1] we call such $H$ a pre-subkernel of $\lambda$. We do not assume that a pre-subkernel is always a direct summand of $M$.

A standard plane is the simplest example of a null-cobordant $u$-form, which is defined by

$$\sigma : A \{e, f\} \times A \{e, f\} \to A,$$

$$\sigma(e, e) = a(f, f) = 0,$$

$$\sigma(e, f) = 1, \quad a(f, e) = ut,$$

where $A \{e, f\}$ denotes a free $A$-module with a basis $\{e, f\}$. A $u$-form $\lambda : M \times M \to A$ is said to be stably null-cobordant if an orthogonal sum $\lambda \perp \sigma \perp \cdots \perp \sigma$, of $\lambda$ and a finite number of standard planes $\sigma, \cdots, \sigma$, is null-cobordant.

The following lemma was proved in [6].

LEMMA 1.7 ([6], p. 586). Let $\lambda_i : M_i \times M_i \to A$ ($i = 1, 2$) be $u$-forms. Suppose $\lambda_2$ and the orthogonal sum $\lambda_1 \perp \lambda_2$ are stably null-cobordant, then $\lambda_i$ is also stably null-cobordant.

We shall say that $u$-forms $\lambda_i, \lambda_2$ are stably cobordant if the orthogonal sum $\lambda_1 \perp (-\lambda_2)$ is stably null-cobordant. From Lemma 1.7 and the fact that $\lambda \perp (-\lambda)$ is null-cobordant, one can prove

3) As a matter of fact, any $u$-hermitian form in our situation is proved to be of $u$-even type.
LEMMA 1.8. The stable cobordism relation is an equivalence relation.

Now we define the group $P_u(Z \rightarrow 1)$: Let $P_u(Z \rightarrow 1)$ be the set of all stable cobordism classes of $ut$-forms. This set is in fact an abelian group with respect to the orthogonal sum $\perp$.

REMARK. A more general formulation of "Witt groups" of the above type was given in [6]. There we started with an onto homomorphism $A \rightarrow B$ of (not necessarily commutative) rings with involution and a central unit $\alpha \in A^\times$ such that $\overline{\alpha} = \alpha^{-1}$. An analogous construction gave an abelian group denoted by $\mathcal{L}_u(A \rightarrow B)$. $P$-groups of [5] and $\mathcal{L}$-groups of [1] are respectively written as follows: $P_n(\pi \rightarrow \pi') = \mathcal{L}(\pi \rightarrow \pi')$ and $\mathcal{L}_n(\pi \rightarrow \pi') = \mathcal{L}(\pi \rightarrow \pi')$. (When we speak of $P_n(\pi \rightarrow \pi')$, we are assuming that $\pi \rightarrow \pi'$ is an onto homomorphism of groups whose kernel is generated by a central element $t$.) $P_u(Z \rightarrow 1)$ in this paper can be written as $P_u(Z \rightarrow 1) = \mathcal{L}_u(\pi : A \rightarrow R)$.

Let $A$ be an admissible $R$-matrix. To the matrix $A$ corresponds a $ut$-form which is represented by a $A$-matrix $A + utA'$. This correspondence induces a homomorphism of abelian groups

$$\rho: G_u(R, -) \rightarrow P_u(Z \rightarrow 1).$$

The main purpose of this paper is to prove algebraically the following

**THEOREM A.** $\rho$ is an isomorphism.

Theorem A immediately follows from Theorems B, C below. Two $ut$-forms $\lambda$ and $\lambda_1$ are said to be stably isomorphic if there exist integers $r$, $s \geq 0$ such that $\lambda_1 \perp \sigma_1 \perp \cdots \perp \sigma_r \cong \lambda \perp \sigma_1 \perp \cdots \perp \sigma_s$, where $\sigma_i$ are standard planes and "$\cong$" means "is isomorphic to".

**THEOREM B.** Let $\lambda: M \times M \rightarrow A$ be any $ut$-form. Then there exists an admissible $R$-matrix $A$ such that $\lambda$ is stably isomorphic to the $ut$-form which is represented by $A + utA'$.

**THEOREM C.** An admissible $R$-matrix $A$ is null-cobordant iff the $ut$-form represented by $A + utA'$ is null-cobordant.

From Theorems B, C we can also deduce another consequence on the structure of null-cobordant $ut$-forms:

**THEOREM D.** Suppose a $ut$-form $\lambda$ is null-cobordant. Then $\lambda$ is stably isomorphic to a null-cobordant $ut$-form $\lambda_1: M_1 \times M_1 \rightarrow A$ of which a pre-subkernel is a $A$-free direct summand of $M_1$. 

PROOF. By Theorem B, $\lambda$ is stably isomorphic to $A + utA'$: $M_1 \times M_1 \rightarrow A$ with some admissible $R$-matrix $A$. By the hypothesis we may suppose that $A + utA'$ is null-cobordant. Then by Theorem C, the matrix $A$ is null-cobordant, and by Definition 1.2, one may suppose that $A$ is of the form
\[
\begin{pmatrix}
0 & N_1 \\
N_2 & N_3
\end{pmatrix}.
\]
Suppose that $A + utA'$ represents a $ut$-form $\lambda$, with respect to a basis $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ of $M_1$. Then the submodule $A\{e_1, \ldots, e_n\}$ is a pre-subkernel of $\lambda$, which is clearly a $A$-free direct summand. Q.E.D.

REMARK. The assumption that $R$ be a principal ideal domain will not be used until § 4. One can prove Theorem B without this assumption.

§ 2. Elementary enlargement

In this section we shall provide technical lemmas (Lemma 2.3 and Corollary 2.4) which will be used in the proof of Theorem B.

If $B=(b_{pq})$ represents a $ut$-form $\lambda: M \times M \rightarrow A$ with respect to a basis $\{e_1, \ldots, e_n\}$, we shall use the following notation to emphasize the basis $\{e_1, \ldots, e_n\}$:

\[
\begin{pmatrix}
\vdots \\
e_1 & \cdots & e_q & \cdots & e_n \\
\vdots
\end{pmatrix}
\]

\[
\lambda = e_p \begin{pmatrix}
\vdots \\
\vdots \\
\vdots
\end{pmatrix}
\]

\[
\begin{pmatrix}
\vdots \\
\vdots \\
\vdots
\end{pmatrix}
\]

\[
\begin{pmatrix}
\vdots \\
\vdots \\
\vdots
\end{pmatrix}
\]

\[
\begin{pmatrix}
\vdots \\
\vdots \\
\vdots
\end{pmatrix}
\]

\[
\begin{pmatrix}
\vdots \\
\vdots \\
\vdots
\end{pmatrix}
\]

We shall also use the notation $A\{e_1, \ldots, e_n\}$ to denote a free $A$-module with a distinguished basis $\{e_1, \ldots, e_n\}$.

The key concept in this section is that of elementary enlargement of a given $ut$-form $\lambda$, which we shall now introduce. Let $\lambda: M \times M \rightarrow A$ be a $ut$-form with $M=A\{e_1, \ldots, e_n\}$, and suppose that $\lambda$ is represented by $B=(b_{pq})$ with respect to $\{e_1, \ldots, e_n\}$. Let $i, j$ be two fixed integers such that $1 \leq i \leq j \leq n$. Let $r \in \mathbb{Z}$ be an integer, $c$ an element of $R$.

DEFINITION 2.1. The elementary enlargement of $\lambda$ of type $E(e_i, e_j, r; c)$ means a $ut$-form $\hat{\lambda}: N \times N \rightarrow A$ which is defined as follows:

(i) If $r=0$ or 1, then $N=M$ and $\hat{\lambda}=\lambda$.

(ii) If $r \geq 2$, then $N=M \oplus A\{h_1, \ldots, h_{r-1}, g_1, \ldots, g_{r-1}\}$, the direct sum of $M$ and a free $A$-module of rank $2(r-1)$ with the basis $\{h_1, \ldots, h_{r-1}, g_1, \ldots, g_{r-1}\}$, and $\hat{\lambda}$ is represented by the following matrix.
(iii) If $r < 0$, then $N = M \oplus A[h_1, \ldots, h_{|r|}, g_1, \ldots, g_{|r|}]$, the direct sum of $M$ and a free $A$-module of rank $2|r|$ with the basis \{h_1, \ldots, h_{|r|}, g_1, \ldots, g_{|r|}\}, and $\lambda$ is represented by the following matrix

\[
\begin{pmatrix}
    e_1 & \cdots & e_j & \cdots & e_n & h_1 & \cdots & h_{|r|} & g_1 & \cdots & g_{|r|} \\
    (b_{pq}) & 0 & uc(1-t), \cdots, uc(1-t) & 0 & \theta_i & \theta_j \\
    h_1 & 0 & 0 & 1 & (1-t), \cdots, (1-t) & 0 \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
    h_{|r|} & 0 & 0 & 0 & \theta_i & \theta_j \\
    g_1 & 0 & -u(1-t) & ut, -u(1-t), \cdots, -u(1-t) & 0 \\
    \vdots & \vdots & \vdots & \vdots & \vdots \\
    g_{|r|} & 0 & 0 & 0 & ut \\
\end{pmatrix}
\]

The elementary enlargement of $\lambda$ of type $E(e_i, e_j, r; c)$ will be denoted by $E(e_i, e_j, r; c)(\lambda)$. 
DEFINITION 2.2. Let \( \lambda : M \times M \to \Lambda \) be as before, where \( M = \Lambda \{e_1, \ldots, e_n\} \). We introduce the following notation for convenience: Let \( i, j \) be fixed integers such that \( 1 \leq i \leq j \leq n \), \( \alpha \) an element of \( \text{Ker}(\varepsilon : \Lambda \to R) \). Then the notation \( \lambda + [\alpha]_{ij} \) denotes the ut-form \( M \times M \to \Lambda \) defined as follows:

(i) if \( i \neq j \),

\[
(\lambda + [\alpha]_{ij})(e_p, e_q) = \begin{cases} 
\lambda(e_p, e_q), & (p, q) \neq (i, j), (j, i), \\
\lambda(e_i, e_j) + \alpha, & (p, q) = (i, j), \\
\lambda(e_j, e_i) + ut\alpha, & (p, q) = (j, i), 
\end{cases}
\]

and

(ii) if \( i = j \),

\[
(\lambda + [\alpha]_{ii})(e_p, e_q) = \begin{cases} 
\lambda(e_p, e_q), & (p, q) \neq (i, i), \\
\lambda(e_i, e_i) + \alpha + ut\alpha, & (p, q) = (i, i). 
\end{cases}
\]

Obviously, \( (\lambda + [\alpha]_{ij}) + [\beta]_{ij} = \lambda + [\alpha + \beta]_{ij} \).

LEMMA 2.3. (i) If \( r \geq 2 \), \( E(e_i, e_j, r; c)(\lambda) \) is stably isomorphic to \( E(e_i, e_j, r-1; ct^{-1})(\lambda + \{-cu(1-t^{-1})(1-t\})_{ij}) \).

(ii) If \( r < 0 \), \( E(e_i, e_j, r; c)(\lambda) \) is stably isomorphic to \( E(e_i, e_j, r+1; ct) \cdot (\lambda + \{-cu(1-t)(1-t\})_{ij}) \).

COROLLARY 2.4. (i) If \( r \geq 2 \), \( E(e_i, e_j, r; c)(\lambda) \) is stably isomorphic to \( \lambda + \{-cu(1-t^{-1})(1-t\})_{ij} \).

(ii) If \( r < 0 \), \( E(e_i, e_j, r; c)(\lambda) \) is stably isomorphic to \( \lambda + \{-cu(1-t^{-1})(1-t\})_{ij} \).

PROOF. By “\( \simeq \)" we mean “is stably isomorphic to". Suppose \( r \geq 2 \), then by applying Lemma 2.3 (i) inductively, we have

\[
E(e_i, e_j, r; c)(\lambda) \simeq E(e_i, e_j, r-1; ct^{-1})(\lambda + \{-cu(1-t^{-1})(1-t\})_{ij}) 
\]

\[
\simeq E(e_i, e_j, r-2; ct^{-2})(\lambda + \{-cu(1-t^{-1})(1-t\})_{ij}) 
\]

\[
\simeq \ldots
\]

\[
\simeq E(e_i, e_j, 1; ct^{-r-1})(\lambda + \{-cu(1-t^{-1}) + \cdots + t^{-r-2})(1-t\})_{ij}) 
\]

\[
= \lambda + \{-cu(1-t^{-1})(1-t\})_{ij}.
\]

One can compute similarly in the case \( r < 0 \).

Q.E.D.

Proof of Lemma 2.3. The proof is based on elementary calculations.

Case (i): \( r \geq 2, i \neq j \). Make the elementary basis transformation:
\[ h'_1 = h_1, \quad g'_1 = g_1, \]
\[ h'_2 = h_k - (1-t)h_i \quad (k \geq 2), \quad g'_k = g_k \quad (k \geq 2), \]
\[ e'_p = e_p \quad (p \neq i, j), \]
\[ e'_i = e_i + c(1-t^{-1})g_i, \]
\[ e'_j = e_j - (1-t)h_i. \]

With the new basis \( \{ e'_i, \ldots, e'_n, h'_2, \ldots, h'_{r-1}, g'_2, \ldots, g'_{r-1}, h'_1, g'_1 \} \), one can easily see the following (in what follows \( \lambda = \lambda(e_i, e_j, r; c(\lambda)) \)).

(1) \[ \lambda(h'_i, h'_j) = \lambda(g'_i, g'_j) = 0, \quad \lambda(h'_i, g'_j) = 1, \quad \lambda(g'_i, h'_j) = ut. \]

Therefore, \( \lambda[H, g'_1] \) is a standard plane. (We use the notation \( \lambda|H \) to denote the \( ut\)-form \( H \times H \rightarrow \Lambda \) obtained by restricting a \( ut\)-form \( \lambda : M \times M \rightarrow \Lambda \) to a \( \Lambda\)-submodule \( H \subseteq M \).)

(2) \[
\begin{align*}
\lambda(h'_i, e'_j) &= \lambda(h_i, e_i + c(1-t^{-1})g_i) \\
&= \lambda(h_i, e_i) + c(1-t)\lambda(h_i, g_i) \\
&= -c(1-t) + c(1-t) \\
&= 0,
\end{align*}
\]
\[
\begin{align*}
\lambda(g'_i, e'_j) &= \lambda(g_i, e_j - (1-t)h_i) \\
&= \lambda(g_i, e_j) - (1-t)\lambda(g_i, h_i) \\
&= -u(1-t) - (1-t^{-1})ut \\
&= 0.
\end{align*}
\]

Therefore, \( \Lambda[H, g'_1] \) is orthogonal to \( \Lambda[e'_i, \ldots, e'_n, h'_2, \ldots, h'_{r-1}, g'_2, \ldots, g'_{r-1}] \) with respect to \( \lambda \).

(3) \[
\begin{align*}
\lambda(h'_i, h'_j) &= \lambda(g'_i, g'_j) = 0 \quad (k, l \geq 2), \\
\lambda(h'_i, g'_j) &= \lambda(h_k, g_i) \quad (k, l \geq 2), \\
\lambda(e'_p, h'_j) &= 0 \quad (p \neq i), \\
\lambda(e'_i, h'_k) &= \lambda(e_i + c(1-t^{-1})g_i, h_k - (1-t)h_i) \\
&= \lambda(e_i, h_k) + c(1-t^{-1})\lambda(g_i, h_k) - (1-t^{-1})\lambda(e_i, h_k) \\
&- c(1-t^{-1})(1-t^{-1})\lambda(g_i, h_k) \\
&= uc(1-t) - c(1-t^{-1})u(1-t) - (1-t^{-1})uc(1-t) \\
&- c(1-t^{-1})(1-t^{-1})ut \\
&= ct^{-1}u(1-t) \quad (k \geq 2),
\end{align*}
\]
Therefore, $AJA{\{e_i, \cdots, e_n, h_2, \cdots, h_{r-1}, g_2, \cdots, g_{r-1}\}}$ is the elementary enlargement of $\lambda|A{\{e_i, \cdots, e_n\}}$ of type $E(e'_1, e'_j, r-1; ct^{-1})$.

(4)

Therefore, $\lambda|A{\{e_i, \cdots, e_n\}}$ is isomorphic to $\lambda + \{-cu(1-t^{-1})(1-t)\}$. From (1) to (4), $\lambda$ is isomorphic to the orthogonal sum of a standard plane $\lambda|A{\{h'_i, g'_i\}}$ and $E(e_i, e_j, r-1; ct^{-1})(\lambda + \{-cu(1-t^{-1})(1-t)\})$. This is what we want to prove.

Case (ii): $r \geq 2, i=j$. We make the following elementary transformation:

$h'_i = h_i$, \quad $g'_i = g_i$,
$h'_k = h_k - (1-t)h_1$ \quad ($k \geq 2$), \quad $g'_k = g_k$ \quad ($k \geq 2$),
$e'_p = e_p$ \quad ($p \neq i$),
$e'_i = e_i + c(1-t^{-1})g_1 - (1-t)h_1$.

With this transformation, we can calculate similarly.

Case (iii): $r < 0$. We make the following transformation:

$h'_i = h_i$, \quad $g'_i = g_i$,
$h'_k = h_k - (1-t^{-1})h_1$ \quad ($k \geq 2$), \quad $g'_k = g_k$ \quad ($k \geq 2$),
$e'_p = e_p$ \quad ($p \neq i, j$),
$e'_i = e_i - cu(1-t)g_j$ \quad \{ if $i \neq j$, \}
$e'_j = e_j - (1-t^{-1})h_1$ \quad \{ if $i = j$, \}
$e'_i = e_i - cu(1-t)g_1 - (1-t^{-1})h_1$.

The rest of the proof is similar to the previous cases. Q.E.D.
§ 3. Proof of Theorem B

Let $M = A[e_1, \ldots, e_n]$. Let $\lambda: M \times M \to A$ be any $ut$-form. In order to prove Theorem B, we have to find an admissible $R$-matrix $A$ such that the $ut$-form $\lambda$ is stably isomorphic to a $ut$-form which is represented by the $A$-matrix $A + utA'$.

First we give a lemma. Let $A$ be an admissible $n \times n$ $R$-matrix and suppose that a $ut$-form $\lambda_0: M \times M \to A$ is represented by the $A$-matrix $A + utA'$. Let $\lambda_i = E(e_i, e_j, r; c)(\lambda_0)$ be an elementary enlargement of $\lambda_0$.

**Lemma 3.1.** $\lambda_i$ is also represented by an $A$-matrix of the form $A_i + utA'_i$ where $A_i$ is an admissible $R$-matrix.

**Proof.** First suppose $r \geq 2$. Define the $(n+2(r-1)) \times (n+2(r-1))$ $R$-matrix $A_i$ as follows.

\[
A_i = \begin{pmatrix}
A & uc, & \cdots, & uc & 0 & 0 \\
-\bar{c} & 0 & \cdots & 0 & 1 & \cdots \cdots & 1 \\
0 & \cdots & 0 & 1 & & \cdots & \cdots & 1 \\
-\bar{c} & 0 & \cdots & 0 & 1 & \cdots \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \cdots & 1 \\
0 & \cdots & 0 & 1 & \cdots \cdots & 1 & 1 \\
\bar{i} & \bar{j} & \bar{u} & & & & & \\
\end{pmatrix}
\]

Then $A_i + utA'_i$ clearly represents $\lambda_i = E(e_i, e_j, r; c)(\lambda_0)$. It is easy to verify that $A_i + utA'_i$ is nonsingular, so that $A_i$ is admissible. One can similarly deal with the remaining case $r < 0$. Q.E.D.

Here we introduce the following notation: Let $\alpha = \sum a_mt^m$ be an element of $A$. For any integer $r$, we define

$$\left[\alpha\right]_r := \begin{cases} 
\sum_{m \geq r} a_m, & r > 0 \\
-\sum_{m < r} a_m, & r < 0 \\
0, & r = 0.
\end{cases}$$
We also define
\[ \frac{d}{dt} \alpha := \sum ma_m t^{m-1}. \]

Note that \([\alpha], \in R \) and \( \frac{d}{dt} \alpha \in A \). Let \( \varepsilon : A \to R \) be the augmentation. Clearly,
\[ \varepsilon \left( \frac{d}{dt} \alpha \right) = \sum ma_m. \]

**Lemma 3.2.** (i) \( \frac{d}{dt} (\alpha \beta) = \left( \frac{d}{dt} \alpha \right) \beta + \alpha \left( \frac{d}{dt} \beta \right) \). (ii) \( \varepsilon \left( \frac{d}{dt} \alpha \right) = -\varepsilon \left( \frac{d}{dt} \alpha \right) \).

(iii) \( \varepsilon \left( \frac{d}{dt} \alpha \right) = \sum_r [\alpha]_r \).

(iv) \( \sum_{r>0} [\alpha]_r t^r + \sum_{r>0} [\alpha]_r t^{-r} = \frac{\varepsilon(\alpha) - \alpha}{1-t} \).

These formulae are easy to verify.

Now suppose that \( \lambda : M \times M \to A \) be any ut-form with \( M = \{e_1, \ldots, e_n\} \). Since \( \lambda \) is of ut-even type, there exist \( \mu_1, \ldots, \mu_n \in A \) such that \( \lambda(e_i, e_i) = \mu_i + ut\mu_i, i=1, \ldots, n \). We fix such \( \mu_1, \ldots, \mu_n \) throughout. Let \( \lambda_{ij} = \lambda(e_i, e_j) \in A \).

Since \( \lambda \) is ut-hermitian, \( \lambda_{ij} = ut \lambda_{ji} \).

Define an \( n \times n ) R \)-matrix \( A_0 \):
\[ (A_0)_{ij} = u\varepsilon \left( \frac{d}{dt} \lambda_{ij} \right), \]

where \((A_0)_{ij}\) is the \((i, j)\)-entry of \( A_0 \).

We have

\[ (A_0 + utA_0)_{ij} = \left[ u\varepsilon \left( \frac{d}{dt} \lambda_{ij} \right) + ut \left\{ u^{-1} \varepsilon \left( \frac{d}{dt} \lambda_{ij} \right) \right\} \right] \]

\[ = \left[ u\varepsilon \left( \frac{d}{dt} \lambda_{ij} \right) + ut \left( \frac{d}{dt} ut \lambda_{ij} \right) \right] \]

\[ = \left[ u\varepsilon \left( \frac{d}{dt} \lambda_{ij} \right) + ut \left( \frac{d}{dt} \lambda_{ij} \right) \right] \]

\[ = u(1-t)\varepsilon \left( \frac{d}{dt} \lambda_{ij} \right) + ut \lambda_{ij}. \]

From this formula (\( \ast \)), \( (A_0 + utA_0)_{ij} = (A_0 + utA_0)_{ij} \big|_{t=1} = u\varepsilon(\lambda_{ij}) \). Therefore, \( A_0 \) is admissible, for \( \lambda \) is \( R \)-nonsingular.

Let \( I = [\alpha, \beta, \ldots] \) be a finite set of indices. Successive applications of elementary enlargements starting with a ut-form \( \lambda' : M \times M \to A \) will be denoted by
In other words,
\[
\prod_{\ell \in I} E(e_{i\ell}, e_{j\ell}, r_{\ell}; c_{\ell})(\lambda')
= E(e_{ia}, e_{jb}, r_a; c_a)(E(e_{ia}, e_{jb}, r_b; c_b)(\cdots(\lambda') \cdots)).
\]

With this notation we define a ut-hermitian form \( \lambda_i \) as follows:
\[
\lambda_i = \prod_{(i, j, r), i < j} E(e_i, e_j, r; [\lambda_{j i} r])(\prod_{(i, r)} E(e_i, e_i, r; [\mu_{i r}]) (\lambda_i)),
\]
where \((i, j, r)\) (or \((i, r)\)) runs over all triples (or pairs) such that \([\lambda_{j i} r] \neq 0\) (or \([\mu_{i r}] \neq 0\)), and \( \lambda_0 : M \times M \to A \) is the ut-form represented by \( A_0 + ut A'_0 \).

Inductively applying Lemma 3.1, we know that \( \lambda_i \) is a ut-form that is represented by a \( A \)-matrix of the form \( A_1 + ut A'_1 \) with an admissible \( R \)-matrix \( A_1 \). Therefore, to complete the proof of Theorem B, we have only to show that the original ut-form \( \lambda : M \times M \to A \) is stably isomorphic to this \( \lambda_i \).

Applying Corollary 2.4 inductively, \( \lambda_i \) is stably isomorphic to the following form (which will be denoted by \( \lambda_0 \)):
\[
\lambda_0 + \sum_{r \geq 1, i < j \leq n} \{-[\lambda_{j i} r], u(1 - t^{-(r-1)})(1 - t)\}_{ij}
+ \sum_{r < 0, 1 \leq i < j \leq n} \{-[\lambda_{j i}], u(1 - t^r)(1 - t)\}_{ij}
+ \sum_{r \geq 1, 1 \leq i \leq n} \{-[\mu_{i i}], u(1 - t^{-(r-1)})(1 - t)\}_{ii}
+ \sum_{r < 0, 1 \leq i \leq n} \{-[\mu_{i i}], u(1 - t^r)(1 - t)\}_{ii}.
\]

This form \( \lambda_i \) is in fact equal to \( \lambda \):
Suppose \( i < j \), then
\[
\lambda_i(e_i, e_j) = (A_0 + ut A'_0)_{ij} - \sum_{r \geq 1} [\lambda_{j i}], u(1 - t^{-(r-1)})(1 - t)
- \sum_{r < 0} [\lambda_{j i}], u(1 - t^r)(1 - t)
= u(1 - t) \left( \frac{d}{dt} \lambda_{j i} \right) + ut \lambda_{j i},
\]
Formula (ii)
\[
- \left\{ \sum_{r \geq 1} [\lambda_{j i}], t^{(r-1)} + \sum_{r < 0} [\lambda_{j i}], t^r \right\} u(1 - t)
= u(1 - t) \left( \frac{d}{dt} \lambda_{j i} \right) + ut \lambda_{j i},
\]
Lemma 3.2 (iii), (iv)
\[
- \left\{ (e \left( \frac{d}{dt} \lambda_{j i} \right) - [\lambda_{j i}], - (\frac{e \lambda_{j i} - \lambda_{j i}}{1 - t} - [\lambda_{j i}],) \right\} u(1 - t)
\]


We can show similarly that $\lambda_2(e_i, e_i) = \mu_i + u\mu_i = \lambda_{ii}$. Therefore, $\lambda_2 = \lambda$. This completes the proof of Theorem B.

§ 4. Proof of Theorem C

First we fix the notation. Let $X = R \{d_1, \ldots, d_m\}$, a free $R$-module with a fixed basis $\{d_1, \ldots, d_m\}$. Let $A = (a_{ij})$ be an admissible $m \times m$ $R$-matrix. With $A$ is associated a sesquilinear form $\varphi: X \otimes X \rightarrow R$ as follows:

$$\varphi(\sum r_i d_i, \sum s_j d_j) = \sum r_i a_{ij} s_j \quad (r_i, s_j \in R).$$

Let $M = A \otimes_R X$, i.e., $M = A \{d_1, \ldots, d_m\}$. Let $\lambda: M \times M \rightarrow A$ be the ut-form associated with $A$: $\lambda(\sum a_{ij} d_i, \sum b_{ij} d_j) = \sum a_{ij}(a_{ij} + ut\beta_{ij})\beta_{ij}$, $(a_{ij}, \beta_{ij} \in A)$, i.e., the ut-form represented by $A + utA'$. Symbolically, $\lambda = \varphi + ut\varphi'$.

In this section, we shall show that if $\lambda$ is null-cobordant in the sense of Definition 1.6 and further if $R$ is a principal ideal domain then there exists an $R$-free direct summand $Y$ of $X$ such that

(i) $\varphi(Y \times Y) = 0$,
(ii) $\text{rank}_R Y = \frac{1}{2} \text{rank}_R X$.

This will suffice to prove Theorem C, because if we take a basis $\{e_1, \ldots, e_n\}$ of $Y$ and extend it to a basis of $X, \{e_1, \ldots, e_n, f_1, \ldots, f_n\}$, then with this basis $\varphi$ will be represented by the matrix of the form $\begin{pmatrix} 0 & N_1 \\ N_2 & N_3 \end{pmatrix}$. Therefore, $A$ is congruent to $\begin{pmatrix} 0 & N_1 \\ N_2 & N_3 \end{pmatrix}$.

In what follows, we need another form denoted by $\lambda: X \times X \rightarrow R$:

$$\lambda(\sum r_i d_i, \sum s_j d_j) = \sum r_i (a_{ij} + ut\beta_{ij})\beta_{ij}, \quad r_i, s_j \in R.$$  
Symbolically, $\lambda = \varepsilon \circ \lambda = \varphi + ut\varphi'$, where $\varepsilon: A \rightarrow R$ is the augmentation. Since $A$ is admissible, $\lambda$ is a non-singular $u$-hermitian form.

Lemma 4.1. Suppose $\lambda: M \times M \rightarrow A$ is null-cobordant. Then there exists a basis $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ of $X$ such that

$\lambda(e_i, e_j) = \lambda(f_i, f_j) = 0,$
$\lambda(e_i, f_j) = \delta_{ij}$ (Kronecker’s delta) and $\lambda(f_j, e_i) = u\delta_{ij},$
where $2n = m = \text{rank}_R X$.

**Proof.** The proof is essentially the same as the one in [8, p. 48]. Let $H \subset M$ be a pre-subkernel of $\lambda$. Let $H = \varepsilon \otimes 1(H)$. By (ii), (iii) of Definition 1.6, $H'$ is an $R$-free direct summand of $X$ with rank$_R H' = \frac{1}{2} \text{rank}_R X$. By (i) of Definition 1.6, $\lambda(H' \times H') = \varepsilon(\lambda(H \times H)) = \{0\}$. We have a splitting $X = H' \oplus H''$ into a direct sum of two free $R$-modules of the same rank. Let \{e$_1$, \ldots, e$_n$\} be an $R$-basis of $H'$. Since $\lambda$ is nonsingular, one can find a basis \{f'_1, \ldots, f'_n\} of $H''$ such that $\lambda(e_i, f'_j) = u\lambda(f'_j, e_i) = \delta_{ij}$. By the definition, $\lambda$ is of $u$-even type, so we have $\lambda(f'_i, f'_j) = m_i + u\overline{m}_i (i = 1, \ldots, n)$ with some $m_i \in R$. Make the elementary basis transformation: $f'_j = f'_j - (m_1 e_j + \sum_{i > j} \lambda_i \cdot (f'_j, f'_i) e_i)$. It is easy to check that \{e$_1$, \ldots, e$_n$, f$_1$, \ldots, f$_n$\} is the required basis. Q.E.D.

We shall fix \{e$_1$, \ldots, e$_n$, f$_1$, \ldots, f$_n$\} of Lemma 4.1 throughout. Let $e_1, \ldots, e_n$ be elements of $H$ such that $e \otimes (\varepsilon) = e_i$. Since $M = \Lambda \otimes_R X$, any element $y$ of $M$ is written as a "Laurent polynomial with coefficients in $X$":

$$y = \sum_k x_k t^k, \quad x_k \in X,$$

so that we can write

$$\varepsilon_i = \sum_k x_{ik} t^k, \quad x_{ik} \in X.$$

We introduce the notation:

$$a_{ik, jl} = \varphi(x_{ik}, x_{jl}) \in R,$$

$$b_{ik, jl} = u\varphi(x_{jl}, x_{ik}) \in R.$$

(Recall that $\varphi$ was given at the beginning of this section.) Clearly, $b_{ik, jl} = u\overline{a}_{jl, ik}$.

If we regard $X$ as a subset of $M$ consisting of "constant Laurent polynomials", we can consider $\lambda(x_{ik}, x_{jl})$. We have

$$\lambda(x_{ik}, x_{jl}) = \varphi(x_{ik}, x_{jl}) + u\varphi(x_{jl}, x_{ik}) = a_{ik, jl} + b_{ik, jl} t.$$

There is a large integer $L > 0$ such that $x_{ik} = 0$ if $|k| > L$. We fix such an $L$. Obviously, $a_{ik, jl} = 0$ and $b_{ik, jl} = 0$ if $|k| > L$ or $|l| > L$.

Let

$$A_{ik, jl} = a_{ik, jl} + b_{ik-1, jl} + a_{ik-1, jl-1} + b_{ik-2, jl-1} + \cdots,$$

$$B_{ik, jl} = b_{ik, jl} + a_{ik, jl-1} + b_{ik-1, jl-1} + a_{ik-1, jl-2} + \cdots.$$

These are in fact finite sums.
LEMMA 4.2. $A_{ik,jl} = 0, B_{ik,jl} = 0$ if $|k| > L$ or $|l| > L$.

PROOF. Obviously, $A_{ik,jl} = 0, B_{ik,jl} = 0$ if $k < -L$ or $l < -L$. Since $a_{ik,jl} = 0, b_{ik,jl} = 0$ if $k > L$ or $l > L$, the values of $A_{ik,jl}$ and $B_{ik,jl}$ are independent of $k$ and $l$ as long as $k > L$ or $l > L$. In other words, if $k > L$ or $l > L$ we have $A_{ik,jl} = A_{ik+1,jl+1}$ and $B_{ik,jl} = B_{ik+1,jl+1}$. We denote these “stable” values by $A_{ij}^m, B_{ij}^m$ respectively, i.e., $A_{ij}^m = \lim_{N \to \infty} A_{im+N,jN}, B_{ij}^m = \lim_{N \to \infty} B_{im+N,jN}$. Then,

$$\lambda \left( \sum_{|i|,|l| \leq L} x_{ik} t^k, \sum_{|i|,|l| \leq L} x_{jl} t^l \right)$$

$$= \sum_{k,l} t^{k-l} \lambda(x_{ik}, x_{jl})$$

$$= \sum_{k,l} t^{k-l}(a_{ik,jl} + b_{ik,jl} t)$$

$$= \sum_{m} t^{m} \left( \sum_{k-l=m} a_{ik,jl} + \sum_{k-l=m-1} b_{ik,jl} \right)$$

$$= \sum_{m} t^{m} A_{ij}^m.$$

Since $\lambda(\sum_{|i|,|l| \leq L} x_{ik} t^k, \sum_{|i|,|l| \leq L} x_{jl} t^l) = \lambda(\bar{e}_i, \bar{e}_j) = 0$, we have $A_{ij}^m = 0$ for any $m, i, j$.

From this, $B_{ij}^m = 0$ for any $m, i, j$, (because $A_{ik,jl} = a_{ik,jl} + b_{ik,jl} t + \cdots = u b_{jl,ik} + u a_{jl,ik} + \cdots$).

This completes the proof.

Let $\{x_{jl}\}_{1 \leq j \leq n, ||l|| \leq L}$ be a set of indeterminates. Denote by $R(\alpha_{jl})$ a free $R$-module with the basis $\{x_{jl}\}_{1 \leq j \leq n, ||l|| \leq L}$. Take a direct sum $X \oplus R(\alpha_{jl})$ with $X$ defined earlier. Let $I$ be an $R$-submodule of $X \oplus R(\alpha_{jl})$ which is generated by the finite subset $\{x_{ik} + \sum_{j,l} A_{ik,jl} \alpha_{jl}\}_{1 \leq j \leq n, ||l|| \leq L}$. Let $X_1$ be the quotient $R$-module $X \oplus R(\alpha_{jl})/I$, and let $\nu : X \to X_1$ be the inclusion $X \subseteq X \oplus R(\alpha_{jl})$ followed by the quotient map $X \oplus R(\alpha_{jl}) \to X_1$. Let $Y' = \ker (\nu : X \to X_1)$.

LEMMA 4.3. For any elements $x, y \in Y'$, we have $\varphi(x, y) = 0$.

PROOF. For any $x \in Y'$, there are $\{m_{ik}\}_{1 \leq i \leq n, ||l|| \leq L}$ of $R$ such that

$$x = \sum_{ik} m_{ik} \left( x_{ik} + \sum_{j,l} A_{ik,jl} \alpha_{jl} \right).$$

Since $x$ belongs to $X \oplus \{0\} \subseteq X \oplus R(\alpha_{jl})$, we have

$$\sum_{ik} m_{ik} \left( \sum_{j,l} A_{ik,jl} \alpha_{jl} \right) = 0,$$

so that

$$\sum_{ik} m_{ik} A_{ik,jl} = 0 \quad \text{for } \nu_j, l.$$

Therefore, $x$ ($\in X$) belongs to $Y'$ iff $x = \sum_{ik} m_{ik} x_{ik}$ with coefficients $\{m_{ik}\}$.
satisfying \((**\)).

Let \(x = \sum_{i_k} m_{i_k} x_{i_k}, y = \sum_{j_l} n_{j_l} x_{j_l}\) be elements of \(Y'\). Then, regarding \(x, y\) as elements of \(M\), we have

\[
\lambda(x, y) = \sum_{i_k, j_l} m_{i_k} n_{j_l} \lambda(x_{i_k}, x_{j_l})
\]

\[
= \sum_{i_k, j_l} m_{i_k} n_{j_l} (a_{i_k, j_l} + b_{i_k, j_l} t)
\]

\[
= \sum_{i_k, j_l} m_{i_k} n_{j_l} \left\{ (A_{i_k, j_l} - B_{i_k, j_l}) + (B_{i_k, j_l} - A_{i_k, j_l-1})t \right\}
\]

\[
= \sum_{i_k, j_l} m_{i_k} n_{j_l} \left\{ (A_{i_k, j_l} - uA_{j_l, i_k-1}) + (uA_{j_l, i_k} - A_{i_k, j_l-1})t \right\}
\]

\[= 0. \quad \text{(Cf. \((**\))} \]

However, for \(x, y \in X(\subset M)\), \(\lambda(x, y) = \varphi(x, y) + u\overline{\varphi(y, x)}\) with \(\varphi(x, y), \overline{\varphi(y, x)} \in R\). Thus \(\varphi(x, y) = 0\) for \(\forall x, y \in Y'\). \(\text{Q.E.D.}\)

Let \(\{\beta_{i_k}\}_{1 \leq i \leq n}, \{\gamma_{j_l}\}_{1 \leq j \leq n}\) be sets of indeterminates, \(R[\beta_{i_k}], R[\gamma_{j_l}]\) free \(R\)-modules with respective bases. Define \(R\)-homomorphisms

\[
\partial_A: R[\beta_{i_k}] \oplus R[\gamma_{j_l}] \rightarrow X \oplus R[\alpha_{j_l}],
\]

\[
\partial_B: R[\beta_{i_k}] \oplus R[\gamma_{j_l}] \rightarrow R[\alpha_{j_l}]
\]

as follows:

\[
\partial_A(\beta_{i_k}) = x_{i_k} + \sum_{j_l} A_{i_k, j_l} \alpha_{j_l},
\]

\[
\partial_A(\gamma_{j_l}) = 0,
\]

\[
\partial_B(\beta_{i_k}) = \sum_{j_l} A_{i_k, j_l} \alpha_{j_l},
\]

\[
\partial_B(\gamma_{j_l}) = 0.
\]

We have defined \(X_1 = \text{Coker}(\partial_A)\). Define \(R\)-modules \(X_2, X_{-1}, X_{-2}\):

\[
X_2 = \text{Coker}(\partial_B),
\]

\[
X_{-1} = \text{Ker}(\partial_B),
\]

\[
X_{-2} = \text{Ker}(\partial_A).
\]

**Lemma 4.4.** There is an exact sequence of \(R\)-modules:

\[
0 \rightarrow X_{-2} \rightarrow X_{-1} \rightarrow X \rightarrow X_1 \rightarrow X_2 \rightarrow 0.
\]

**Proof.** Consider the commutative diagram:
where the rows are considered as two term chain complexes (referred to as $C_A, C_B$) and the columns are exact sequences. The required exact sequence is the homology long exact sequence associated with the above diagram.

Q.E.D.

**Lemma 4.5 (Duality Lemma).** There are isomorphisms:

$$X_{-2} \cong \text{Hom}_R (X_2, R)$$

and

$$X_1 \cong \text{Hom}_R (X_{-1}, R) \oplus \text{Ext}_R (X_1, R).$$

**Remark.** Let $P$ be an $R$-module. The (left) $R$-module structure on $\text{Hom}_R (P, R)$ is given by

$$(af)(x) = f(x)a$$

for $f \in \text{Hom}_R (P, R)$, $a \in R$, $x \in P$.

The $R$-module structure on $\text{Ext}_R (P, R)$ is induced from this. The isomorphisms of Lemma 4.5 are with respect to these $R$-module structures.

Lemma 4.5 will be proved in the next section. We shall prove Theorem C taking Lemma 4.5 for granted. Suppose that $R$ is a principal ideal domain. Let $K$ be the quotient field. Since $K$ is $R$-flat, we have the exact sequence

$$0 \longrightarrow K \otimes_R X_{-1} \longrightarrow K \otimes_R X_{-1} \longrightarrow K \otimes_R X_1 \longrightarrow K \otimes_R X_0 \longrightarrow 0.$$

By Lemma 4.5, $\dim_K K \otimes_R X_{-1} = \dim_K K \otimes_R X_1$ and $\dim_K K \otimes_R X_{-1} = \dim_K K \otimes_R X_0$. From this,

$$\dim_K (\text{Ker} 1 \otimes \nu) = \frac{1}{2} \dim_K K \otimes_R X = n.$$ 

Since $\text{Ker} (1 \otimes \nu) = K \otimes_R Y'$, we have $\dim_K K \otimes_R Y' = n$. Let $Y = \{x \in X; \ldots \}$.
There exists \( a \in R - \{0\} \) such that \( ax \in Y \). Then \( Y \) is a free direct summand of \( X \) of rank \( n \), because \( R \) is a principal ideal domain. Moreover, by Lemma 4.3, \( \varphi(Y \times Y) = \{0\} \). As we remarked at the beginning of this section, this completes the proof of Theorem C (taking Lemma 4.5 for granted).

§ 5. Proof of Duality Lemma

In this section we shall prove Lemma 4.5. We continue to use the same notation as in § 4. Recall from Lemma 4.1 that \( X \) has a basis \( \{e_1, \ldots, e_n, f_1, \ldots, f_n\} \) with which \( \lambda: X \times X \to R \) is written as an orthogonal sum of "hyperbolic planes". Let \( \{\rho_{ij}\}_{1 \leq i \leq n, 1 \leq j \leq L} \) be a set of indeterminates. Let \( V \) be the direct sum

\[ X \oplus R[\alpha_{ij}] \oplus R[\rho_{ij}] \]

We define a \( u \)-hermitian form \( \chi': V \times V \to R \) as the orthogonal sum \( \lambda \perp \lambda'_0 \), where \( \lambda: X \times X \to R \) was given in § 4 and \( \lambda'_0: (R[\alpha_{ij}] \oplus R[\rho_{ij}]) \times (R[\alpha_{ij}] \oplus R[\rho_{ij}]) \to R \) is the orthogonal sum of hyperbolic planes: \( \lambda'_0(\alpha_{ik}, \alpha_{jl}) = \lambda'_0(\rho_{ik}, \rho_{jl}) = 0 \), \( \lambda'_0(\alpha_{ik}, \rho_{jl}) = \delta_{ik} \delta_{kl} \), \( \lambda'_0(\rho_{ij}, \alpha_{kl}) = u \delta_{ij} \delta_{kl} \), \( \delta_{ik} \delta_{kl} \) being the product of Kronecker's deltas.

Let \( \{r_{ik}\}_{1 \leq i \leq n, 1 \leq k \leq L+1} \) be elements of \( V \) defined by

\[
\begin{align*}
\begin{cases}
r_{i,-L} = x_{i,-L} + \sum_{pq} A_{i,-L,pq} x_{pq} - \rho_{i,-L}, \\
r_{ik} = x_{ik} + \sum_{pq} A_{ik,pq} x_{pq} - \rho_{ik} + \rho_{i,k-1}, \\
r_{i,L+1} = \rho_{ik}.
\end{cases}
\end{align*}
\]

Also let \( \{s_{jl}\}_{1 \leq j \leq n, -L \leq l \leq L+1} \) be defined by

\[
\begin{align*}
\begin{cases}
s_{jl} = f_j^* = u(\alpha_{jl} + \alpha_{jl+1} + \cdots + \alpha_{jL}) & (|l| \leq L), \\
s_{jL+1} = f_j^*,
\end{cases}
\end{align*}
\]

where

\[
f_j^* = f_j + \sum_{pq} \lambda(f_j, x_{pq} + x_{pq-1} + \cdots + x_{p-L}) x_{pq}.
\]

**Lemma 5.1.** The set \( \{r_{ik}\}_{1 \leq i \leq n, -L \leq k \leq L+1} \cup \{s_{jl}\}_{1 \leq j \leq n, -L \leq l \leq L+1} \) forms a basis of \( V \) and we have

\[
\begin{align*}
\begin{cases}
\chi'(r_{ik}, r_{jl}) = 0, \\
\chi'(s_{ik}, s_{jl}) = 0, \\
\chi'(r_{ik}, s_{jl}) = u \chi'(s_{jl}, r_{ik}) = \delta_{ij} \delta_{kl}, & \text{for } \forall i, j.
\end{cases}
\end{align*}
\]

**Proof.** Let \( V' \) be the submodule of \( V \) generated by \( \{r_{ik}\} \cup \{s_{jl}\} \). We
shall prove that $V' = V$. Since $s_{jl} - s_{jl+1} = -uq_{jl} (|l| \leq L)$, $V'$ contains $R\{q_{jl}\}$, so by the formula for $s_{jl}$ it also contains $\{f_{jl}\}_{1 \leq l \leq n}$. Moreover, by the formula for $r_{lk}$, $V'$ contains $\{x_{ik} - \rho_{ik} + \rho_{ik-1}\}_{1 \leq ik \leq n}$, $\{x_{ik} - \rho_{ik} - \rho_{ik-1}\}_{1 \leq ik \leq n}$, $\{x_{ik} - \rho_{ip} + \rho_{ip-1}\}_{1 \leq ik \leq n}$, $\{x_{ik} - \rho_{ip} - \rho_{ip-1}\}_{1 \leq ik \leq n}$ and $\{\rho_{ip}\}_{1 \leq ik \leq n}$. Note that

$$
(x_{ik} - \rho_{ik} + \rho_{ik-1}) + \sum_{1 \leq ik \leq n} (x_{ik} - \rho_{ik} + \rho_{ik-1}) + \rho_{ik} = x_{ik} = e \otimes 1 (\sum_{1 \leq ik \leq n} x_{ik} t^k) = e \otimes 1 (e_j) = e_j.
$$

Thus $V'$ contains $\{e_j\}_{1 \leq j \leq n}$. Since $V' \supseteq \{e_1, \ldots, e_n, f_1, \ldots, f_n\} \cup \{q_{ik}\}$, it contains $\{x_{ik}\}$ (because $\{x_{ik}\} \subset X = R\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$). Therefore, again by the formula for $r_{lk}$, $V'$ contains $\{q_{lk}\}$. Consequently $V'$ coincides with $V$, and $\{r_{lk}\} \cup \{s_{jl}\}$ generates $V$.

Next we shall verify (*)&. First suppose $-L < k \leq L$ and $-L < l \leq L$. Then we have

$$
\lambda'(r_{lk}, s_{jl}) = \lambda'(x_{ik} + \sum_{pq} A_{ik,pq} x_{pq} - \rho_{ik} + \rho_{ik-1}, x_{jl} + \sum_{pq} A_{jl,pq} x_{pq} - \rho_{jl} + \rho_{jl-1})
$$

$$
= \lambda(x_{ik}, x_{jl}) - A_{ik,jl} + A_{ik,jl-1} - A_{jl,ik} u + A_{jl,ik-1} u
$$

$$
= (a_{ik,jl} + b_{ik,jl}) - (A_{ik,jl} - B_{ik,jl}) = (B_{ik,jl} - A_{ik,jl-1}) = 0.
$$

Next since $s_{jl} = f_j + (R$-linear combination of $\alpha_{pq}$), we have $\lambda'(s_{ik}, s_{jl}) = 0$. Finally,

$$
\lambda'(r_{lk}, s_{jl})
$$

$$
= \lambda'(x_{ik} + \sum_{pq} A_{ik,pq} x_{pq} - \rho_{ik} + \rho_{ik-1}, f_j + \sum_{pq} \lambda(f_j, x_{pq} + \cdots + x_{p-L}) \alpha_{pq}
$$

$$
- u(\alpha_{jl} + \cdots + \alpha_{jL}))
$$

$$
= \lambda(x_{ik}, f_j) - \lambda(f_j, x_{ik} + \cdots + x_{i-L}) u + \lambda(f_j, x_{i-L} + \cdots + x_{i-L}) u
$$

$$
+ \lambda(\rho_{ik} - \rho_{ik-1}, u(\alpha_{jl} + \cdots + \alpha_{jL}))
$$

$$
= \lambda(\rho_{ik} - \rho_{ik+1}, u(\alpha_{jl} + \cdots + \alpha_{jL})).
$$

It is easy to observe that $\lambda(\rho_{ik} - \rho_{ik+1}, u(\alpha_{jl} + \cdots + \alpha_{jL})) = \delta_{ik} \delta_{jl}$. Thus we have $\lambda'(r_{lk}, s_{jl}) = \delta_{ik} \delta_{jl}$. The verification for the remaining cases $k, l = -L$ or $L + 1$ will be left to the reader. (The proof of $\lambda'(r_{lk}, r_{jL+1}) = 0$ will
need $A_{ik, L} = 0$ which follows from $A_{ik, L} = A_{ik, L+1} = 0$. The proof of

\[ \lambda'(r_i L+1, s_j) = \delta_{i, j} \delta_{L+1, 1} \]

needs $\sum_{|i| \leq L} x_{ik} = e_i$ which was verified in the present proof.) This completes the proof of Lemma 5.1.

Let $\{x_{ik}\}_{1 \leq |k| \leq L}$ be a set of indeterminates, and construct a two term chain complex $C_D$:

\[ C_D : 0 \rightarrow R[\beta_{ik}] \oplus R[\gamma_i] \oplus R[\pi_{ik}] \xrightarrow{\partial_D} X \oplus R[\alpha_{ij}] \oplus R[\rho_{ij}] \rightarrow 0 \]

which is defined by

\[
\begin{align*}
\partial_D(\beta_{ik}) &= r_{ik} & 1 \leq i \leq n, |k| \leq L, \\
\partial_D(\gamma_i) &= r_{i, L+1} & 1 \leq i \leq n, \\
\partial_D(\pi_{ik}) &= \rho_{ik} & 1 \leq i \leq n, |k| \leq L.
\end{align*}
\]

**Lemma 5.2.** The chain complex $C_D$ is chain homotopy equivalent to the chain complex $C_A : 0 \rightarrow R[\beta_{ik}] \oplus R[\gamma_i] \oplus R[\pi_{ik}] \rightarrow R[\alpha_{ij}] \oplus R[\rho_{ij}] \rightarrow 0$ given in § 4.

**Proof.** Define elements $\beta'_ik, \gamma'_i$ of $R[\beta_{ik}] \oplus R[\gamma_i] \oplus R[\pi_{ik}]$ as follows:

\[
\beta'_ik = \begin{cases} 
\beta_{ik} + \pi_{ik} - \pi_{ik-1} & -L < k \leq L, \\
\beta_{i, -L} + \pi_{i, -L} & k = -L.
\end{cases}
\]

\[
\gamma'_i = \gamma_i - \pi_{ik}.
\]

We have

\[
(\ast\ast) \quad \begin{cases}
\partial_D(\beta'_ik) = \left\{ 
\begin{align*}
& r_{ik} + \rho_{ik} - \rho_{ik-1} = x_{ik} + \sum_{pq} A_{ik, pq} \alpha_{pq} , & -L < k \leq L, \\
& r_{i, -L} + \rho_{i, -L} = x_{i, -L} + \sum_{pq} A_{i, -L, pq} \alpha_{pq} , & k = -L,
\end{align*}
\right.
\end{cases}
\]

Define isomorphisms $\psi_i : R[\beta_{ik}] \oplus R[\gamma_i] \oplus R[\pi_{ik}] \rightarrow X \oplus R[\alpha_{ij}] \oplus R[\rho_{ij}]$ by

\[
\psi_i(\beta_{ik}) = \beta'_ik, \quad \psi_i(\gamma_i) = \gamma'_i, \quad \psi_i(\pi_{ik}) = \pi_{ik},
\]

\[
\psi_0 : X \oplus R[\alpha_{ij}] \rightarrow \text{id. and } \psi_0(\pi_{ik}) = \rho_{ik}.
\]

Then by (\ast\ast), the following diagram commutes:
However, the upper chain complex (denoted by $C'_D$) is clearly chain homotopy equivalent to $C_A$. Thus so is $C'_D$ also. Q.E.D.

Let $C^*$ be the chain complex defined by

$$C^* : 0 \rightarrow R[\beta_{ik}] \oplus R[\gamma_i] \oplus R[\pi_{ik}] \rightarrow 0$$

in which $\partial^*(\pi_{ik}) = \sum_{i \leq j < n, -L \leq t < L+1} A_{jk,i} \pi_{jt}, 1 \leq i \leq n, |k| \leq L$. (Recall $A_{jL+1,ik} = 0$.)

**Lemma 5.3.** The chain complex $C_D$ is chain homotopy equivalent to $C^*$.

**Proof.** By Lemma 5.1, $V = R[\beta_{ik}] \oplus R[\gamma_i]$, so one can write the complex $C_D$ as follows:

$$0 \rightarrow R[\beta_{ik}] \oplus R[\gamma_i] \oplus R[\pi_{ik}] \rightarrow R[\gamma_i] \oplus R[\pi_{ik}] \rightarrow 0$$

Restricting to the submodule $R[\beta_{ik}] \oplus R[\gamma_i]$, $\partial_D$ gives rise to an isomorphism $R[\beta_{ik}] \oplus R[\gamma_i] \cong R[\gamma_i]$ (recall the definition of $\partial_D$).

We have the commutative diagram in which rows are chain complexes and columns are exact ($C_E$ denotes the quotient chain complex):

From the diagram it follows that $C_D$ is chain homotopy equivalent to $C_E$. We shall show $C_E \cong C^*$ by computing $\partial_E$ using the $\lambda'$-intersection on $V$: 

$$C^* : 0 \rightarrow R[\pi_{ik}] \rightarrow 0$$

$$C_E : 0 \rightarrow R[\pi_{ik}] \rightarrow 0$$
First, by the definition of $\partial$, $\partial_p(\pi_{ik}) = \rho_{ik}$. Suppose $\rho_{ik} = \sum_{jl} m_{ik,jl} r_{jl} + \sum_{jl} n_{ik,jl} s_{jl}$ with certain $m_{ik,jl}, n_{ik,jl} \in R$. By Lemma 5.1, $\chi'(\rho_{ik}, r_{jl}) = n_{ik,jl} u$. On the other hand, by the definition of $r_{jl}$,

$$\chi'(\rho_{ik}, r_{jl}) = \left\{ \begin{array}{lcl} \chi'(\rho_{ik}, x_{jl} + \sum_{pq} A_{jl,pq} \alpha_{pq} - \rho_{jl} - \rho_{jl-1}) = \overline{A}_{jl,ik} u, & l = L, \\
\chi'(\rho_{ik}, x_{jl} + \sum_{pq} A_{jl,pq} \alpha_{pq} - \rho_{jl} + \rho_{jl-1}) = \overline{A}_{jl,ik} u, & -L < l \leq L, \\
\chi'(\rho_{ik}, \rho_{jl}) = 0 = \overline{A}_{jl+1,ik} u, & l = L + 1. \end{array} \right.$$ 

Therefore, $\partial_p(\pi_{ik}) = \sum_{jl} n_{ik,jl} s_{jl} = \sum_{jl} \overline{A}_{jl,ik} s_{jl}$. Thus $C_\pi \cong C^*$. This completes the proof of Lemma 5.3.

**Lemma 5.4.** The chain complex $C^*$ is chain homotopy equivalent to the dual complex $C^*_B$ of $C_B$:

$$C_B^*: 0 \longrightarrow \text{Hom}_R (R[\alpha_{jl}], R) \overset{\partial_B^*}{\longrightarrow} \text{Hom}_R (R[\beta_{ik}] \oplus R[\gamma_{jl}], R) \longrightarrow 0.$$ 

**Proof.** Let $\alpha_{jl}^*, \beta_{ik}^*, \gamma_{jl}^*$ be the respective dual bases, $\langle , , \rangle$ the Kronecker-pairing. We have

$$\langle \partial_B^* (\alpha_{jl}^*), \beta_{ik}^* \rangle = \langle \alpha_{jl}^*, \partial_p (\beta_{ik}) \rangle = \langle \alpha_{jl}^*, \sum_{pq} A_{jl,pq} \alpha_{pq} \rangle = A_{jl,ik},$$

$$\langle \partial_B^* (\alpha_{jl}^*), \gamma_{jl}^* \rangle = \langle \alpha_{jl}^*, \partial_p (\gamma_{jl}) \rangle = 0.$$ 

Considering the remark after Lemma 4.5, we have

$$\partial_B^* (\alpha_{jl}^*) = \sum A_{jl,ik} \beta_{ik}^*.$$ 

Then the commutative diagram

$$C^*: 0 \longrightarrow R[\pi_{ik}] \overset{\partial^*}{\longrightarrow} R[\gamma_{jl}] \overset{\partial^*}{\longrightarrow} 0$$

$$C_B^*: 0 \longrightarrow \text{Hom}_R (R[\alpha_{jl}], R) \overset{\partial_B^*}{\longrightarrow} \text{Hom}_R (R[\beta_{ik}] \oplus R[\gamma_{jl}], R) \longrightarrow 0$$

in which

$$\psi_1(\pi_{ik}) = \alpha_{ik}^*,$$
$$\psi_0(\gamma_{jl}) = \beta_{jl}^*, \quad (|l| \leq L),$$
$$\psi_0(\gamma_{jl+1}) = \gamma_{jl}^*.$$
gives the required conclusion.

Combining Lemmas 5.2, 5.3 and 5.4, we have that $C_A$ is chain homotopy equivalent to $C^*_B$. Therefore,

$$X_2 = \ker (\partial_A) \cong \ker (\partial^*_B)$$

and

$$X_1 = \text{coker} (\partial_A) \cong \text{coker} (\partial^*_B).$$

Applying the universal coefficient theorem to $C^*_B$, we obtain

$$\ker (\partial^*_B) = \text{Hom}_R (\text{coker} \partial_B, R) = \text{Hom}_R (X_2, R)$$

and

$$\text{coker} (\partial^*_B) = \text{Hom}_R (\ker \partial_B, R) \oplus \text{Ext}_R (\text{coker} \partial_B, R)$$

$$= \text{Hom}_R (X_1, R) \oplus \text{Ext}_R (X_2, R).$$

Therefore, $X_2 \cong \text{Hom}_R (X_2, R)$ and $X_1 \cong \text{Hom}_R (X_1, R) \oplus \text{Ext}_R (X_2, R)$. This completes the proof of Lemma 4.5.

References


