Finite groups with a standard subgroup isomorphic to $Sp(4, 2^n)$

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Introduction

A subgroup K of a finite group G is said to be *tightly embedded* if K has even order and $K \cap K^g$ has odd order for $g \in G - N_G(K)$. A quasisimple subgroup L of G is said to be a *standard* subgroup of G if $K = C_G(L)$ is tightly embedded, $N_G(K) = N_G(L)$, and $[L, L^g] \neq 1$ for $g \in G$. In this paper we prove the following theorem:

THEOREM. *Let G be a finite group and suppose L is a standard subgroup of G with $L/Z(L) \cong Sp(4, q)$, where $q = 2^n > 2$. Furthermore assume that $C_G(L)$ has a cyclic Sylow 2-subgroup and let X denote the normal closure of L in G . Then either $LO(G) \triangleleft G$ or $X/Z(X)$ is isomorphic to one of the following groups:*

$$\begin{aligned} &PSU(4, q), PSU(5, q), PSL(4, q), PSL(5, q) \\ &Sp(4, q^2), Sp(4, q) \times Sp(4, q). \end{aligned}$$

Since a fundamental result of M. Aschbacher appeared [1], much work has been done toward the classification of groups G with a standard subgroup L of known type. In particular, Aschbacher and Seitz [3] have treated the case where L is an arbitrary known quasisimple group and $C_G(L)$ has 2-rank at least 2. Also, Aschbacher [2] has classified groups with a tightly embedded subgroup having a generalized quaternion Sylow 2-subgroup. Thus it remains to investigate the case where $C_G(L)$ has a cyclic Sylow 2-subgroup. To complete this major undertaking we must in particular consider the case where $\bar{L} = L/Z(L)$ is a Chevalley group over a field F_q of characteristic 2 with $|Z(L)|$ odd. In this case our objective is to show that if $O(G) = 1$ and $L \not\triangleleft G$, then either $\langle L^g \rangle \cong L \times L$ or $\langle L^g \rangle$ is isomorphic to a Chevalley group of even characteristic (or possibly to some exceptional groups when both the BN-pair rank of $\bar{L}$ and q are small). Let k denote the BN-pair rank of $\bar{L}$. Griess, Mason, and Seitz [16] have treated the case $k = 1$. The result of this paper is a contribution to the case $k = 2$. Seitz [19] has also started work on the case $k \geq 2$. His results, however,

suggest that the case $k=2$ should be treated separately. At the moment of writing, his results assert (personal communications) that if the case $k=2$ and some cases with k small and $q=2$ are settled, then for arbitrary $k \geq 3$ it is possible to construct a subgroup G_0 with $L < G_0$ such that either G_0 is a Chevalley group or $G_0/Z(G_0) \cong L \times L$.

The method used in this paper is essentially a careful analysis of 2-local subgroups. Let $L \cong \text{Sp}(4, q)$ be a standard subgroup of G with $LO(G) \not\trianglelefteq G$ and let P_i , $i \in \{1, 2\}$, be two maximal parabolic subgroups of L containing a given Sylow 2-subgroup U of L . Then $U_i = O_2(P_i)$ is elementary abelian of order q^3 and $P'_i/U_i \cong \text{SL}(2, q)$. A major part of the proof is devoted to constructing a 2-subgroup V_i of G with $P_i \leq N_G(V_i)$ such that $U_i = V_i \cap P_i$ and $P'_i V_i/V_i$ is a standard subgroup of $N_G(V_i)/V_i$. Then using a previous classification theorem [16], we can determine the structure of the normal closure Q_i/V_i of $P'_i V_i/V_i$ in $N_G(V_i)/V_i$. It turns out that there are three cases to consider:

Case 1. $Q_1/V_1 \cong \text{SL}(2, q^2)$ or $\text{SL}(2, q) \times \text{SL}(2, q)$,

$Q_2/O(Q_2 \text{ mod } V_2) \cong \text{SL}(2, q)$.

Case 2. $Q_1/V_1 \cong \text{SL}(2, q^2)$ or $\text{SL}(2, q) \times \text{SL}(2, q)$,

$Q_2/Z(Q_2 \text{ mod } V_2) \cong \text{PSL}(3, q)$ or $\text{PSU}(3, q)$.

Case 3. $Q_1/V_1 \cong Q_2/V_2 \cong \text{SL}(2, q^2)$ or $\text{SL}(2, q) \times \text{SL}(2, q)$.

The rest of the paper is devoted to proving the following:

(A) $G_0 = \langle Q_1, Q_2 \rangle$ is a central extension of one of the groups listed in the theorem.

(B) G_0 is a normal subgroup of G .

If both Q_1/V_1 and Q_2/V_2 are central extensions of rank 1 Chevalley groups, then we can show that G_0 has a split BN-pair of rank 2, in which case we may appeal to a result of Fong and Seitz [7] to obtain (A) (see Lemma (1L) in Section 1). If $Q_1/V_1 \cong Q_2/V_2 \cong \text{SL}(2, q) \times \text{SL}(2, q)$, the same method applies. In the remaining cases, we have some trouble in proving (A). For example, in Case 1 we must show that $O(Q_2/V_2) = 1$. This problem is reduced to proving that $\hat{G} = \langle L^G \rangle$ is of “characteristic 2 type”, and to prove this requires a study of the centralizers of involutions of $\hat{G}$. In this analysis Goldschmidt’s “strong closure” theorem [11] and Gorenstein and Walter’s “balanced group” theorem [14] play important roles. If in Case 1 and 2 we have $Q_1/V_1 \cong \text{SL}(2, q) \times \text{SL}(2, q)$, we use Timmesfeld’s result [23] to prove $G_0 \cong \text{PSL}(4, q)$ or $\text{PSL}(5, q)$, although this may be avoidable by directly constructing a BN-pair. We prefer this approach because of its clarity. However, before using [23] it is again necessary to show that $\hat{G}$ is of characteristic 2 type.

The method used in proving (B) is a standard one. Unless $Q_1/V_1 \cong Q_2/V_2 \cong \text{SL}(2, q) \times \text{SL}(2, q)$, we show that $N_{\hat{G}}(G_0)$ is a strongly embedded sub-

group of $\hat{G}$. This again requires a study of the centralizers of involutions of $\hat{G}$. If $Q_1/V_1 \cong Q_2/V_2 \cong SL(2, q) \times SL(2, q)$, our approach is slightly different. In this case, we have $G_0 = G_1 \times G_2$ with $G_i \cong Sp(4, q)$. Our aim is to show that $G_1 \triangleleft \triangleleft C_G(x)$ for any involution x of $C_G(G_1)$, when we may use Aschbacher's result [1] to get $G_0 \triangleleft G$. Here again we must study the centralizers of involutions of G_1 and use the signalizer functor method of Gorenstein and Walter [14] (see Lemmas (1M) and (1N) in Section 1). However, this part of the analysis could be simplified considerably if we used the Unbalanced Group Theorem, the proof of which is near completion (see [20]).

We note that the above explanation gives only a conceptual outline of the proof and that the actual proof goes somewhat differently. For example, we divide the analysis according to the fusion pattern of an involution of $C_G(L)$.

I wish to thank Professor G. Seitz for communications which led me to this work, and I. Miyamoto for his helpful comments on Lemma (1B), which was a key to the proof. Some special case of the theorem was also worked out by H. Yamada and I am indebted to him for simplifications in Section 4.

Our notation is standard and taken from [13]. Possible exceptions are the use of the following: If X is a 2-group,

$\mathcal{E}^*(X)$ = set of maximal elementary abelian subgroups of X ,

$J_r(X)$ = subgroup of X generated by the abelian subgroups of maximal rank.

X^2 = subgroup of X generated by the squares of elements of X .

By $J(X)$ we denote the usual Thompson subgroup generated by the set $\mathcal{A}(X)$ of abelian subgroups of maximal order. For any finite group G ,

$O^2(G)$ = subgroup of G generated by the 2'-elements,

$O^{2'}(G)$ = subgroup of G generated by the 2-elements,

$E(G)$ = maximal normal semisimple subgroup of G ,

G^∞ = final term of the derived series of G ,

$I(G)$ = set of involutions of G .

If H, K are subgroups of a group G with $K \triangleleft H$,

$f(H \bmod K)$ = preimage in H of $f(H/K)$, where f is a function from groups to groups,

$C(H/K) = \{g \in G \mid [H, g] \leq K\}$.

If X, Y are subsets of a group,

$X * Y$ = product of X and Y such that $[X, Y] = 1$,

$X^Y = \{y^{-1}Xy \mid y \in Y\}$.

If G is a group, X is a subset of G , and H, K are subgroups of G such that $X^k \leq H$ for some $k \in K$, we write $X \leq_K H$ and if $X = \{x\}$, we write $x \in_K H$. By $V(2, q)$ we denote the vector space of 2-dimensional row vectors with

coefficients in F_q . Let V be an elementary abelian group of order q^2 on which $L = \text{SL}(2, q)$ acts. Then if $V \cong V(2, q)$ as L -modules, we say that V is a *natural module* for L . A E_{2^n} -subgroup is an elementary abelian subgroup of order 2^n . By “Goldschmidt group” we mean quasisimple groups that we call groups of type I and II in [11]. Let S be a Sylow 2-subgroup of a group G and a an element of S . Then an element $b \in a^G \cap S$ is an *extremal conjugate* of a in S if $|C_S(b)|$ is maximal. We also use the “bar” convention for homomorphic images. Thus if G is a group, N is a normal subgroup and $\bar{G}$ denotes the factor group G/N , then for any subset X of G , $\bar{X}$ will denote the image of X under the natural projection $G \rightarrow \bar{G}$. Conversely, when we refer to a subgroup or subset $\bar{X}$ of $\bar{G}$, it is understood that X is a preimage in G of $\bar{X}$. Similar convention will be used when a group G has a permutation representation on a set Ω . In this case we write X^Ω instead of $\bar{X}$.

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§ 1. Preliminaries

In this section we collect a number of preliminary lemmas to be used in later sections.

(1A) Let A be a normal elementary abelian 2-subgroup of a group H and suppose for some $t \in A^\#$, $C_H(t)$ acts transitively on $(A/\langle t \rangle)^\#$. Then one of the following holds:

- (1) $H = C_H(t)$.
- (2) $|H : C_H(t)| = |A|/2$ and H acts 2-transitively on t^H .
- (3) $|H : C_H(t)| = |A| - 1$.

PROOF. By hypothesis each coset $\neq \langle t \rangle$ of $\langle t \rangle$ in A contains the same number, n , of elements of t^H . Clearly, $0 \leq n \leq 2$ and the statement (1), (2) or (3) holds according as $n=0, 1$ or 2 .

(1B) Suppose (G, X) is a 2-transitive group, $x \in X$, $N \triangleleft G_x$ (the stabilizer of x), and $N^x \cong \text{SL}(2, q)^{V(2, q)}$. Then G has a regular normal subgroup.

PROOF. Let $y \in X - \{x\}$ and let B be the fixed point set of N_y . Then by a theorem of O’Nan [18] the sets B^g , $g \in G$, form a block design $\mathcal{D}$ on X with $\lambda=1$. Since $|B|=q$, parameter equalities imply that $\mathcal{D}$ is an affine plane of order q . By hypothesis G is transitive on the lines of $\mathcal{D}$, so that $\mathcal{D}$ is a translation plane and G contains the translation group by a theorem of Wagner [24]. This proves (1B).

(1C) Suppose $S=AB$ is a nonabelian 2-group, where A and B are distinct elementary abelian subgroups such that $C_S(a)=A$ for all $a \in A-B$. Then $Z(S)=A \cap B$ and $\mathcal{E}^*(S)=\{A, B\}$.

PROOF. As $C_S(b)=B$ for all $b \in B-A$, $Z(S)=A \cap B$. Let $E \in \mathcal{E}^*(S)$ and let $x=ab \in E-Z(S)$ with $a \in A$ and $b \in B$. Then $[a, b]=x^2=1$ and so, by symmetry, we may assume $x \in A-B$. Then $E \leq C_S(x)=A$ and hence $E=A$.

(1D) Let S be a group and let S_i , $i=1, 2, \dots, n$, be subgroups of S such that

- (i) $S=S_1S_2 \cdots S_n$,
- (ii) $S_iS_{i+1} \cdots S_n \triangleright S_{i+1} \cdots S_n$, $1 \leq i \leq n-1$, and
- (iii) $S_i \cap S_{i+1} \cdots S_n=1$, $1 \leq i \leq n-1$.

Furthermore, let u be an involution acting on S and suppose each S_i is u -invariant. Then the following holds:

- (1) $C_S(u)=C_{S_1}(u) \cdots C_{S_n}(u)$.
- (2) If $I(uS_i)=u^{S_i}$ for each i , then $I(uS)=u^S$.

PROOF. The assertions obviously hold when $n=1$. We proceed by induction on n . Set $T=S_2 \cdots S_n$. Let $x=ab \in S$ with $a \in S_1$ and $b \in T$, and suppose $x^u=x^{\pm 1}$. If $x^u=x$ then $a^{-1}a^u=bb^{-u} \in S_1 \cap T=1$, so $C_S(u)=C_{S_1}(u)C_T(u)$ and (1) follows. Suppose $x^u=x^{-1}$, so that $a^ub^u=b^{-1}a^{-1}$. We have $aa^u=(ab^{-1}a^{-1})b^{-u} \in S_1 \cap T=1$, so $ua \in I(uS_1)$, and by our assumption there is an element $g \in S_1$ such that $ua=u^g$. We also have $b^u=a^{-u}b^{-1}a^{-1}=ab^{-1}a^{-1}$, so $b^{u^g}=b^{ua}=b^{-1}$ and then $u^gb \in I(u^gT)$. T satisfies (i)–(iii) with regard to the factorization $T=S_2^g \cdots S_n^g$, and $I(u^gS_i^g)=(u^g)^{S_i^g}$, so by induction there is an element $h \in T$ such that $u^gb=u^{gh}$. Thus $ux=u^{gh} \in u^S$. This completes the proof.

The following is the well-known (generalized) ‘‘Thompson transfer lemma’’, and the proof is omitted.

(1E) Let S be a Sylow 2-subgroup of a group G and R a subgroup of

S with $S' \leq R$. Suppose x is an involution of $S-R$ such that $x^g \cap S \leq xR$. Then $x \notin O^2(G)$.

(1F) Let S be a Sylow p -subgroup of a group G , W a weakly closed subgroup of S (with respect to G) and A a subset of S such that $A \leq_c Z(W)$. Then there is an element $g \in G$ such that $A^g \leq Z(W)$ and $N_S(A)^g \leq S$. If A is a normal subgroup of S , then $A \leq Z(W)$.

PROOF. By hypothesis $A \leq Z(W^x)$ for some $x \in G$. By Sylow's theorem $\langle W^x, N_S(A)^n \rangle$ is a p -group for some $n \in N(A)$. Also, $\langle W^x, N_S(A)^n \rangle^y \leq S$ for some $y \in G$. Set $g = ny$, so that $N_S(A)^g \leq S$. The weak closure of W implies that $W^{xy} = W$, so $A^g = A^y \leq Z(W^{xy}) = Z(W)$. If $A \triangleleft S$, then $g \in N(S) \leq N(W)$, and hence $A \leq Z(W^{g^{-1}}) = Z(W)$.

Now we formulate an argument of Goldschmidt [11, Corollary 4].

(1G) Let S be a Sylow 2-subgroup of a group G and A a weakly closed elementary abelian subgroup of S . Suppose $a \in S-A$ is conjugate to an element of A . Set $X = C_A(a)$ and $X_0 = [A, a]$. Choose $g \in G$ such that

- (1) $a^g \in A$ and
- (2) $|A \cap A^{g^{-1}}|$ is maximal subject to (1).

Set $X_1 = A \cap A^{g^{-1}}$ and $X_2 = N_A(\langle X_1, a \rangle)$. Then the following holds:

- (i) $X_0 X_1 \leq X \leq X_2$, $A \cap X_2^g = X_1^g$, $X_2^g \not\leq A$.
- (ii) $|A/X| = |X_0| \leq |X_2/X_1|$, $|X_0 \cap X_1| = |X_2/X|$.

Furthermore we may choose g such that

- (3) $N_S(\langle X_1, a \rangle)^g \leq S$.

PROOF. By definition $\langle X_1, a \rangle^g \leq A$, so by (1F) there is an element $g_1 \in G$ such that $\langle X_1, a \rangle^{g_1} \leq A$ and $N_S(\langle X_1, a \rangle)^{g_1} \leq S$. Then $X_1 \leq A \cap A^{g_1^{-1}}$ so $X_1 = A \cap A^{g_1^{-1}}$ by maximality of $|X_1|$. This proves the last statement. As $\langle X_1, a \rangle$ is abelian, $X_1 \leq X \leq X_2$. As $X_1 = \langle X_1, a \rangle \cap A$ and $A \not\leq \langle X_1, a \rangle$, $X_1 < X_2$. We have $A \cap A^g = X_1^g \leq A \cap X_2^g \leq A \cap A^g$ so $X_1^g = A \cap X_2^g$ and $X_2^g \not\leq A$. Since A is elementary, commutation by a induces a homomorphism $A \rightarrow X$. Its image is X_0 and its kernel is X so $|A/X| = |X_0|$. Also, it maps X_2 onto $X_0 \cap X_1$. Thus $|X_0 \cap X_1| = |X_2/X|$ and

$$\begin{aligned} |X_0| &= |X_0 \cap X_1| \cdot |X_0/X_0 \cap X_1| \leq |X_2/X| \cdot |X/X_1| \\ &= |X_2/X_1|. \end{aligned}$$

This completes the proof.

When combined with the main theorem of [11], (1G) is very powerful, as is illustrated by the following result:

(1H) Let S be a Sylow 2-subgroup of a group G and suppose $S=AB$ with $\mathcal{E}^*(S)=\{A, B\}$. Then the following holds:

- (1) If $N(A) \not\leq N(A \cap B)$, then $N(A \cap B) \cap N(A)/A$ is a proper strongly embedded subgroup of $N(A)/A$.
- (2) If $|A| > |B|$, then $A \in \text{Syl}_2(\langle A^G \rangle)$.
- (3) If $|A|=|B|=|A \cap B|^2 \geq 16$ and if neither A nor B is strongly closed in S with respect to G , then $O^v(G)O(G)/O(G) \cong \text{PSL}(3, |A \cap B|)$.

PROOF. First, notice that $C_S(a)=A$ for all $a \in A-B$ and that A and B are normal in $N(S)$. Set $Z=A \cap B$. Then $Z(T)=Z$ for any subgroup T of S such that $A < T$ by (1C). Hence if $A < S \cap S^x$ for $x \in N(A)$, then $Z = Z(S \cap S^x) = Z^x$ and so $S^x \leq N(Z)$. This implies that $N(Z) \cap N(A)/A$ is a strongly embedded subgroup of $N(A)/A$ provided that $N(A) \not\leq N(Z)$. Now A and B are weakly closed in S with respect to G . Indeed, if $|A|=|B|$, this follows from Burnside's lemma. If $|A| < |B|$, then B is weakly closed and we can apply (1F). Suppose that $a \in S-A$ is conjugate to an element of A . Let $a^g \in A$ and choose notation as in (1G). By hypothesis $a \in B-A$ so $X=Z$. We may take g so that $N_S(\langle X_1, a \rangle)^g \leq S$. Then, as $\langle X_1, a \rangle \leq B$, $(BX_2)^g \leq S$ and so the weak closure of B implies that $g \in N(B)$. As $X_2^g \leq B$ by hypothesis, it follows that $X_2=Z$. Therefore

$$|S/B|=|A/Z|=|X_0| \leq |Z/X_1|=|X_2^g/X_1^g| \leq |S/A|.$$

Suppose $|A| > |B|$. Then the above shows that A is strongly closed in S . By the main theorem of [11], if $T=S \cap \langle A^G \rangle$ then $A=\Omega_1(T)$. As $S=AB$, we must have $T=A$.

It now remains to prove (3). Let $q=|Z|$. Then $q=|A/Z| \leq |Z/X_1|$ so $A \cap A^{q-1}=X_1=1$. This implies that $Z(G) \cap S=1$. Hence if $z \in Z^\#$, both A and B are strongly closed in S with respect to $C(z)$. Consequently, $Z=A \cap B$ is normal in $C(z) \cap N(B)$. Suppose $1 \neq z \in Z \cap Z^x$ for some $x \in N(B)$. Then $S^x \leq C(z) \cap N(B) \leq N(Z)$ so by Sylow's theorem $S^x=S^y$ for some $y \in N(Z)$, and then $Z^x=Z^y=Z$. This shows that Z is a T.I. set in $N(B)$. By the first paragraph $g \in N(B)-N(Z)$ so $|N(Z) \cap N(B)/B|$ is a strongly embedded subgroup of $N(B)/B$ by (1). As S/B is elementary of order q , a corollary of a theorem of Bender [4] shows that $|N(B):N(Z) \cap N(B)|=q+1$. As $|B|=q^2$, it follows that B is partitioned by $Z^{N(B)}$. We have shown that any involution of G is conjugate to an element of Z and that if $z \in Z^\#$ then A and B are strongly closed in S with respect to $C(z)$.

We show $SO(C(z)) < C(z)$. Set $H=C(z)$, $K=\langle Z^H \rangle$ and let bars denote images in $H/O(H)$. By [11] either $\bar{K}=\bar{Z}$ or $\bar{K}=O_2(\bar{K})\bar{K}_1 \cdots \bar{K}_n$ with $\bar{K}_i$ a Goldschmidt group, and $Z=\Omega_1(S \cap K)$. Suppose that the latter case holds and set $S_i=S \cap K_i$, $1 \leq i \leq n$. Then $\Omega_1(S_i)=Z \cap S_i \leq Z(S)$ so $\bar{K}_i$ is simple and

$\bar{S}$ normalizes $\bar{K}_i$. As $[S, S_i] \leq Z \cap S_i = \Omega_1(S_i)$, $\bar{S}$ induces inner automorphisms on $\bar{K}_i$ (see [11, § 3]), so $\bar{S} = \bar{T} \times \bar{U}$, where $\bar{T} = \bar{S} \cap E(\bar{K})$ and $\bar{U} = \bar{S} \cap C(E(\bar{K}))$. As $\Omega_1(\bar{S}) = \bar{S}$ and $\Omega_1(\bar{T}) = \bar{Z} \cap \bar{T}$, $\bar{T} \leq \bar{Z}$. Thus $U = (A \cap U)(B \cap U)$ and $\mathcal{E}^*(U) = \{A \cap U, B \cap U\}$. Let $a \in A \cap U - B$. Then $|[U, a]| = q$ as $C_U(a) = U \cap A$ has index q in U . But $[U, a]$ is contained in $Z \cap U$, whose order is less than q . This contradiction shows that $\bar{K} = \bar{Z}$. Consequently $\langle \bar{S}^H \rangle \leq C_{\bar{H}}(\bar{Z})$. As $\bar{A} \in \text{Syl}_2(\langle \bar{A}^H \rangle)$ and $\bar{B} \in \text{Syl}_2(\langle \bar{B}^H \rangle)$ by [11], it follows that $\langle \bar{A}^H \rangle \cap \langle \bar{B}^H \rangle = \bar{Z}$. Hence if $\bar{E} = E(\langle \bar{A}^H \rangle) \neq 1$ then $[\bar{B}, \bar{A} \cap \bar{E}] \leq \bar{E} \cap \bar{Z} = 1$, which is a contradiction. Thus $\langle \bar{A}^H \rangle = \bar{A}$ and $\langle \bar{B}^H \rangle = \bar{B}$. Therefore $\bar{S} \triangleleft \bar{H}$, as asserted.

We have shown that the centralizer of any involution of G is solvable of 2-length 1. By a result of [14], $\bar{G} = G/O(G)$ is a C -group [22]. As $O_2(N(\bar{A})) = \bar{A}$, $O_2(\bar{G}) \leq \bar{A}$ and also $O_2(\bar{G}) \leq \bar{B}$. Thus $O_2(\bar{G}) \leq \bar{Z}$ and, since $\bar{Z}$ is a T.I. set of $N(\bar{B})$, it follows that $O_2(\bar{G}) = 1$. Appealing to [22], we conclude that $O^*(\bar{G}) \cong \text{PSL}(3, q)$.

(1I) *Let G be a p -group acted on fixed-point-freely by a p' -group A and suppose $A = A_1 A_2$, $A_i \triangleleft A$, and $G = C_G(A_1) C_G(A_2)$. Then $G = C_G(A_1) \times C_G(A_2)$.*

PROOF. A_i acts fixed-point-freely on $C_G(A_{3-i})$ so $[G, A_i] = [C_G(A_{3-i}), A_i] = C_G(A_{3-i})$. Thus $C_G(A_i) \triangleleft G$. Since $C_G(A_1) \cap C_G(A_2) = 1$, we get the result.

(1J) *Let t be an involution of a group G , A a subgroup of G , and B a $\langle A, t \rangle$ -invariant subgroup of $[A, t]$ with $O^{2'}(B) = B$. Then if $[C_{O(G)}(t), B] = 1$, $[O(G), B] = 1$.*

PROOF. Let $F = F(O(G))$ and $U = C_F(B)$. Then $C_F(t) \leq U$, so $N_F(U)/U$ is inverted by t and hence centralized by $[A, t]$. Thus $B = O^{2'}(B)$ centralizes $N_F(U)$. As F is nilpotent, we conclude that $[F, B] = 1$. By the Feit-Thompson Theorem $O(G)$ is solvable and so $C_{O(G)}(F) \leq F$. Hence if D is any 2-subgroup of B , then $C_{O(G)D}(F) = Z(F) \times D$, so $D \triangleleft O(G)D$, and then $[O(G), D] \leq O(G) \cap D = 1$. This completes the proof.

(1K) *Suppose $L = \text{SL}(2, 2^n)$ acts nontrivially on an elementary abelian 2-group V of order 2^{2n} and $|C_V(P)| \geq 2^n$ for $P \in \text{Syl}_2(L)$. Then V is a natural module for L .*

PROOF. We choose the additive notation for V . By Lemma (4B) of [7] L is irreducible on V . Hence if $P \neq Q \in \text{Syl}_2(L)$, then $C_V(P) \cap C_V(Q) = \{0\}$ as $L = \langle P, Q \rangle$. Thus setting $U = C_V(P)$ and $W = C_V(Q)$, we have $V = U \oplus W$ and that V is partitioned by $U^\perp$. Let $H = N_L(P) \cap N_L(Q)$ and let $t \in I(N_L(H))$. Then t inverts H , and $P^t = Q$. Hence if $C_H(u) \neq 1$ for some $u \in U^*$, then $C_H(w) = C_H(u) \neq 1$ for some $w \in W^*$. However, $v = uw \in C_V(R)^*$ for some $R \in \text{Syl}_2(L) - \{P, Q\}$ by the above, and so $1 \neq C_H(u) \leq N_L(P) \cap N_L(Q) \cap N_L(R)$.

This is a contradiction because L is a Zassenhaus group on $Syl_2(L)$. Thus H acts regularly on both U and W , and hence on V/U also. Consequently, $[P, V] \leq U$.

Now we choose a basis $u = u_1, \dots, u_n, w = w_1, \dots, w_n$ of V such that $u_i \in U$ and $u_i^t = w_i, 1 \leq i \leq n$. With respect to this basis t is represented by the matrix $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$, where I and 0 denote the $n \times n$ identity matrix and zero matrix, respectively. Also, elements h , of H are represented by the matrices $\begin{pmatrix} X(h) & 0 \\ 0 & Y(h) \end{pmatrix}$. We may choose an element $a \in P$ such that $(at)^3 = 1$. Then, as $[P, V] \leq U$, we see that the matrix of a has the form $\begin{pmatrix} I & 0 \\ I & I \end{pmatrix}$.

Now since H is regular on $U^\#$, $X(h)$ is uniquely determined by its first row and the map $h \rightarrow X(h)$ is one to one. Thus the set of matrices $F^* = \{X(h) | h \in H\}$ is a cyclic group of order $2^n - 1$. Since t inverts H , $Y(h) = X(h)^{-1}$ and so a^h has the matrix form $\begin{pmatrix} I & 0 \\ X(h^2) & I \end{pmatrix}$. Also, if $h \neq h'$, there is $h'' \in H$ such that $a^h a^{h'} = a^{h''}$. These together imply that $X(h^2) + X(h'^2) = X(h''^2)$. Thus $F = F^* \cup \{0\}$ is closed under addition and so becomes a field of order 2^n . Furthermore, there is a natural isomorphism $f: L \rightarrow SL(2, F)$, as the matrices of elements of L has the form $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $A, B, C, D \in F$. Now suppose that $u^h + u^{h'} = u^{h''}$ with $h, h', h'' \in H$ and $h \neq h'$. This implies that the first rows of $X(h) + X(h')$ and $X(h'')$ are identical. Hence $X(h) + X(h') = X(h'')$. Likewise, $w^h + w^{h'} = w^{h''}$ implies $Y(h) + Y(h') = Y(h'')$. Hence there is an isomorphism $g: V \rightarrow V(2, F)$ such that $u^{hg} = (X(h), 0)$ and $w^{hg} = (0, Y(h))$. To complete the proof we have to show that $v^{xg} = v^{gx^f}$ for any $v \in V$ and $x \in L$. However, it is enough to consider the case where $v \in U^\# \cup W^\#$ and $x \in P^\#$ or $x = t$. Suppose $x \in P^\#$. Then $u^{hxg} = u^{hxh^{-1}hg} = u^{hg} = (X(h), 0) = u^{hg x^f}$. Next, if $x^f = \begin{pmatrix} I & 0 \\ X(k) & I \end{pmatrix}$, then we find that $w^{hx} = w^{htk} + w^h = u^{h^{-1}k} + w^h$. Thus $w^{hxg} = (X(h^{-1}k), Y(h)) = w^{hg x^f}$. Finally, $u^{htg} = w^{h^{-1}g} = (0, X(h)) = u^{hg t^f}$ and $w^{htg} = u^{h^{-1}g} = (Y(h), 0) = w^{hg t^f}$. Thus we have proved (1K).

(1L) Let S be a p -subgroup of a group G and let $M_i, i=1, 2$, be normal subgroups of S . Assume that for each i there is a subgroup L_i of $N_G(M_i)$ with $S < L_i$ such that

- (i) L_i/M_i is a perfect central extension of $PSL(2, p^n)$, $PSU(3, p^n)$ or $Sz(p^n)$ for some $n > 1$ with $Z(L_i/M_i)$ a p' -group,
- (ii) $S \in Syl_p(L_i)$, and
- (iii) $N_{L_i}(S) \leq N_G(L_{3-i})$.

Assume further that there is an involution $r_i \in L_i - N_{L_i}(S)$ such that

- (iv) $r_1 r_2$ has order m , m even > 2 ,

- (v) $S = M_i(S \cap S^{r_{3-i}} \cap S^{r_i r_{3-i}} \cap \dots \cap S^{\overbrace{r_{3-i} r_i \dots r_{3-i}}^{m-1}})$, and

$$(vi) \quad S \cap S^{\overbrace{r_1 r_2 \cdots r_1 r_2}^m} = 1.$$

Then $G_1 = \langle L_1, L_2 \rangle$ is a perfect central extension of

$$\begin{aligned} & \text{PSp}(4, p^n), \text{PSU}(4, p^n), \text{PSU}(5, p^n) \\ & G_2(p^n), {}^3D_4(p^n) \text{ or } {}^2F_4(p^n) \end{aligned}$$

for some n , and $Z(G_1)$ is a p' -group. Furthermore, S is a Sylow p -subgroup of G_1 and the Weyl group of the natural BN -pair of $G_1/Z(G_1)$ has order $2m$.

PROOF. By (iii) $B = N_{L_1}(S)N_{L_2}(S)$ is a subgroup and S is a (unique) Sylow p -subgroup of B . Let

$$W = \langle r_1, r_2 \rangle, \quad H = \bigcup_{w \in W} B^w, \quad N = WH.$$

We show that (B, N) is a BN -pair of G_1 such that $B \cap N = H$ is a complement for S in B and $N/H \cong W$.

By the structure of L_i/M_i , $L_i = \langle S, r_i \rangle$ so $G_1 = \langle B, N \rangle$. By the above remark $r_i B r_i \neq B$. Define

$$r = \overbrace{r_1 r_2 \cdots r_1 r_2}^m = \overbrace{r_2 r_1 \cdots r_2 r_1}^m,$$

and

$$S_0 = S, \quad S_1 = S^{r_1}, \quad S_2 = S^{r_1 r_2}, \quad \dots, \quad S_{m-1} = S^{\overbrace{r_1 r_2 \cdots r_1}^{m-1}}, \quad S_m = S^r.$$

Define B_k , $0 \leq k \leq m$, in a similar way using B . Furthermore define

$$H_1 = M_1, \quad H_2 = M_2^{r_1}, \quad H_3 = M_1^{r_2 r_1}, \quad \dots, \quad H_{m-1} = M_1^{\overbrace{r_2 r_1 \cdots r_1}^{m-2}}, \quad H_m = M_2^{\overbrace{r_1 r_2 \cdots r_1}^{m-1}}.$$

By the structure of L_i/M_i , $S \cap S^{r_i} = M_i$ so conjugating by suitable elements, we get $S_k \cap S_{k+1} = H_{k+1}$, $0 \leq k \leq m-1$. Define

$$D_k = \bigcap_{i=0}^k S_i = \bigcap_{i=1}^k H_i, \quad 1 \leq k \leq m.$$

By (v) we have

$$S = M_1(S \cap S^{r_2} \cap S^{r_1 r_2} \cap \dots \cap S^{\overbrace{r_1 r_2 \cdots r_2}^k})$$

for each even k , $2 \leq k \leq m-2$. Conjugating by $\overbrace{r_2 \cdots r_2 r_1}^k$, we get $S_k = H_{k+1} D_k$. Similarly, we have

$$S = M_2(S \cap S^{r_1} \cap \dots \cap S^{\overbrace{r_1 r_2 \cdots r_1}^k})$$

for each odd k , $1 \leq k \leq m-1$. Conjugating by $\overbrace{r_1 r_2 \cdots r_1}^k$, we again get $S_k = H_{k+1} D_k$. B normalizes L_i so S acts transitively on $\text{Syl}_p(L_i B) - \{S\}$. This implies that $B = S(B \cap B^{r_i})$. Conjugating by suitable elements, we get $B_k = S_k(B_k \cap B_{k+1})$, $0 \leq k \leq m-1$. Therefore $B_k = D_k H_{k+1}(B_k \cap B_{k+1}) = D_k(B_k \cap B_{k+1})$. As $D_k \leq B_0 \cap B_1 \cap \cdots \cap B_k$, it follows that $B_0 \cap B_1 \cap \cdots \cap B_k = D_k(B_0 \cap B_1 \cap \cdots \cap B_{k+1})$. Thus proceeding inductively, we get $B_0 = S_0(B_0 \cap B_1) = S_0(B_0 \cap B_1 \cap \cdots \cap B_m)$. As $B_0 \cap B_m$ is a p' -group by (vi), we conclude that

$$B \cap B^r = B \cap B^{r_1} \cap B^{r_2 r_1} \cap \cdots \cap B^{\overbrace{r_2 r_1 \cdots r_2}^m},$$

and that $B \cap B^r$ is a complement for S in B . The above argument applies with r_1 and r_2 interchanged. Therefore $H = B \cap B^r$ is a complement for S in B .

By the structure of L_i/M_i , H acts fixed-point-freely on S_k/H_{k+1} , $0 \leq k \leq m-1$. As $S_k = H_{k+1} D_k$ and $H_{k+1} \cap D_k = D_{k+1}$, H acts fixed-point-freely on D_k/D_{k+1} as well. As $D_m = 1$, it follows that $C_S(H) = 1$, and so $B \cap N = H$. The condition (v) in particular implies that $S \cap S^w \neq 1$ for any $w \in W - \{r\}$. Hence $S^r \cap S^w \neq 1$ for any $w \in W^\sharp$. As $S^r \cap S = 1$, it follows that $H \cap W = 1$, and so $N/H \cong W$.

It now remains to prove that $r_i B w \leq B r_i w B \cup B w B$ for any $i \in \{1, 2\}$ and $w \in W$. For definiteness let $i = 1$. As $\langle B, r_1 \rangle = B \cup B r_1 B$ and $W \leq N(H)$, it is sufficient to prove that $r_1 S w \leq S r_1 w S$ for any

$$w \in \{r_2, r_2 r_1, \dots, \overbrace{r_2 r_1 \cdots r_2}^{m-1}\}.$$

Let w be of length k , $1 \leq k \leq m-1$, and let $w = w_1 r_i$ with w_1 of length $k-1$ and $i \in \{1, 2\}$. Then by (v)

$$S \leq \begin{cases} M_1 M_2^{w_1^{-1}} & \text{if } k \text{ is odd,} \\ M_1 M_1^{w_1^{-1}} & \text{if } k \text{ is even.} \end{cases}$$

Hence if k is odd, then

$$\begin{aligned} r_1 S w &\leq r_1 (M_1 M_2^{w_1^{-1}}) w \\ &= r_1 M_1 w_1 M_2 r_2 \\ &= M_1 r_1 w_1 r_2 M_2 \\ &= M_1 r_1 w M_2. \end{aligned}$$

Similarly if k is even, we get $r_1 S w \leq M_1 r_1 w M_1$. Thus we have shown that (B, N) is a BN -pair with the properties mentioned before. As G_1 is generated by the conjugates of S , the desired conclusion follows from a theorem of Fong and Seitz [7].

(1.1) HYPOTHESIS. G is a finite group and $D=BB^t$ is a semisimple subgroup with (commuting) components B and B^t , where t is an involution such that $C(t) \leq H = N(D)$. Furthermore $C(D)$ has odd order.

Under Hypothesis (1.1) an involution $b \in B \cup B^t$ is *critical* if $C(b) \leq HO(C(b))$, and $\mathcal{E}$ will denote the set of noncyclic elementary abelian 2-subgroups of D all of whose nonidentity elements are critical. A simple group L is *balanced* if for any subgroup X of $\text{Aut}(L)$ containing $\text{Inn}(L)$ and any involution x of X , we have $O(C_x(x))=1$ (see [14, Definition 1]).

The following two results will be used in Section 10 to show that if $B \cong Sp(4, 2^n)$, $n \geq 2$, then D is a normal subgroup of G . It is possible to simplify the arguments if one uses the Unbalanced Group Theorem, the proof of which is now near completion (see [20]). Seitz [19] obtained a similar result.

(1M) Assume Hypothesis (1.1) with H maximal in G . Furthermore assume the following:

- (i) $\tilde{B} = B/Z(B)$ is balanced and of noncomponent type.
- (ii) $\mathcal{E}$ contains an element E such that $m(E) \geq 3$.

Then the following conditions hold:

- (1) If X is a Sylow 2-subgroup of B or $C(B)$, then $N(X) \leq H$.
- (2) If X is a 2-subgroup of $C(B)$ and BX/X is standard in $N(X)/X$, then $\langle B^{N(X)} \rangle$ is a semisimple group.

PROOF. Let $A = C_D(t)'$. Then $A = \{bb^t; b \in B\}$ is a quasisimple normal subgroup of $C(t)$ [14, Lemma 2.1] and so $[C_{O(G)}(t), A] = 1$. Hence if b is any 2-element of B , then $[O(G), bb^t] = 1$ by (1J). As $Z(D)$ has odd order, it follows that $C_D(O(G)) \not\leq Z(D)$ and consequently B or B^t is contained in $C_D(O(G))$. Thus $O(G) \leq C(D) \leq H$.

Let $b \in B$ be a critical involution, so that $C(b) = C_H(b)O(C(b))$. By a theorem of Gorenstein and Walter [15, Theorem 3.1] the 2-layer of $C_H(b)$ is contained in D , so by (i) B^t is the only 2-component of $C_H(b)$. Hence $(B^t O(C(b)))^\infty$ is the only 2-component of $C(b)$. Since $\tilde{B}$ is balanced, a result of [14], Lemma 2.3, shows that $O(C(a) \cap C(b)) \leq O(C(b))$ for any involution $a \in C(b)$.

Suppose $E \in \mathcal{E}$, so that $E \leq B$ or B^t . If $F \in \mathcal{E}$, then as $[B, B^t] = 1$, there is a sequence $E_0 = E, E_1, \dots, E_n = F$ of elements of $\mathcal{E}$ such that $[E_{i-1}, E_i] = 1$, $1 \leq i \leq n$. Define

$$X_E = \langle O(C(e)); e \in E^\# \rangle.$$

As we are assuming (ii), arguments of [14, Sections 4, 5] show that X_E has odd order and is independent of the choice of E . As $\mathcal{E}$ is H -invariant and

$(X_E)^h = X_{E^h}$ for $h \in H$, it follows that $H \leq N(X_E)$. If $X_E \triangleleft G$, then $X_E \leq O(G) \leq H$ by the first paragraph. If $X_E \not\triangleleft G$, then $N(X_E) = H$ by the maximality of H . Thus $X_E \leq H$ in any case.

Let $1 \neq b \in E \in \mathcal{E}$. Then $O(C(b)) \leq X_E \leq H$ and so $C(b) = C_H(b)O(C(b)) \leq H$. In particular, if $S \in \text{Syl}_2(B)$, then $C(S) \leq H$ and so B^i is the only 2-component of $C(S)$ as in the second paragraph, so $N(S) \leq N(B^i)$ and then $N(B) = B(N(S) \cap N(B)) \leq H$ by a Frattini argument. Therefore $N(S) \leq H$. Similarly, if X is a Sylow 2-subgroup of $C(B)$, then $N(X) \leq N(B) \leq H$.

Let X be a 2-subgroup of $C(B)$, $K = N(X)$, $\bar{K} = K/X$ and $L = \langle B^K \rangle$. Let $E \in \mathcal{E}$ with $E \leq B$. Then E acts on $O(C(X))$, so as $m(E) > 1$, $O(C(X)) = \langle O(C(X)) \cap C(e); e \in E^\# \rangle \leq H$. Thus $O(C(X))$ centralizes B . Assume that $\bar{B}$ is standard in $\bar{K}$. By a general result $\bar{L}$ is a 2-semisimple group and either $\bar{L}$ is 2-quasisimple or $\bar{L}$ is a product of two 2-components interchanged by an involution of $\bar{K}$. As $L \leq C(X)$, L must be 2-semisimple. Further, $O(L)$ centralizes B by the above, so $C_L(O(L))$ contains a 2-component of L , and it follows that $O(L) \leq Z(L)$. Thus L is semisimple.

We shall require the following theorem of Aschbacher [1].

THEOREM. *Let A be a quasisimple subgroup of a group G and assume that $A \triangleleft \triangleleft C(a)$ for any $a \in I(C(A))$ and that $[A, A^g] = 1$ for some $g \in G$. Then if $m(A) > 1$, $A \triangleleft \triangleleft G$.*

If A is a maximal component in the sense of [1], then this is a consequence of Theorem 9.7 of [1]. However, maximality of A was used only to show $A \triangleleft \triangleleft C(a)$ for any $a \in I(C(A))$, which we are now assuming.

(1N) *Assume Hypothesis (1.1) and the conditions (i), (ii) of (1M). Furthermore assume the following:*

- (iii) *The multiplier of $\bar{B}$ is a 2'-group.*
- (iv) *Let X be a nonidentity 2-subgroup of $C(B)$, K a subgroup of $N(X)$ containing X , $\bar{K} = K/X$, and $L = \langle B^K \rangle$. Then if $\bar{B}$ is standard in $\bar{K}$ with $\bar{B}O(\bar{K}) \triangleleft \bar{K}$ and $\bar{L}/Z(\bar{L}) \not\cong \bar{B} \times \bar{B}$, the multiplier of $\bar{L}/Z(\bar{L})$ is a 2'-group and $|N_L(\bar{B}) : \bar{B}|$ is odd, whereas $|\bar{L} : \bar{B}|$ is even.*

Then D is a normal subgroup of G .

PROOF. Suppose this is false and choose a counterexample G so that $|G : H|$ is minimal. If $H \leq M < G$, then $D \triangleleft M$ by the minimality of G and so $M = H$. Thus H is a maximal subgroup of G and (1M) applies.

We argue that $B \triangleleft \triangleleft N(X)$ for any nonidentity 2-subgroup X of $C(B)$. If X is a Sylow 2-subgroup of $C(B)$, this follows from (1M.1). Therefore we may assume that $X \notin \text{Syl}_2(C(B))$ and, by way of induction, that $B \triangleleft \triangleleft N(Y)$ for any 2-subgroup Y of $C(B)$ such that $|X| < |Y|$. Set $K = N(X)$

and let bars denote images in K/X . Then $C_K(\bar{B}) = \overline{C_K(B)}$ has even order and $\bar{B} \triangleleft \triangleleft C_K(\bar{x})$ for any $\bar{x} \in I(C_K(\bar{B}))$. If $[\bar{B}, \bar{B}^x] = 1$ for some $\bar{x} \in \bar{K}$, then $\bar{B} \triangleleft \triangleleft \bar{K}$ by the before-mentioned Aschbacher's theorem, and so $B \triangleleft \triangleleft K$. Suppose $[\bar{B}, \bar{B}^x] \neq 1$ for any $\bar{x} \in \bar{K}$. Then $\bar{B}$ is a standard subgroup of $\bar{K}$ and (iv) applies. Notice also that $L = \langle B^K \rangle$ is semisimple by (1M.2). Hence if $\bar{B}O(\bar{K}) \triangleleft \triangleleft \bar{K}$, then $\bar{B} \triangleleft \triangleleft \bar{K}$ and so $B \triangleleft \triangleleft K$. Therefore we may assume $\bar{B}O(\bar{K}) \not\triangleleft \triangleleft \bar{K}$. Then by (iii) and (iv), $Z(L)$ is a $2'$ -group and in particular $L \cap X = 1$. Hence $|L : B|$ is even and $|N_L(B) : B|$ is odd by (iv). (If $\bar{L}/Z(\bar{L}) \cong \tilde{B} \times \tilde{B}$, then $\bar{B}$ is a 'diagonal' subgroup of $\bar{L}$, and $N_L(\bar{B}) = \bar{B}Z(\bar{L})$; see [1, Lemma 2.5.3]). However, if $S \in \text{Syl}_2(B)$, then $N_L(S) \leq N_L(B)$ by (1M.1), a contradiction. Thus we have shown by induction that $B \triangleleft \triangleleft N(X)$ for any nonidentity 2-subgroup X of $C(B)$. We can therefore apply Aschbacher's theorem to conclude that $B \triangleleft \triangleleft G$. Since $C(D)$ has odd order, it follows that $D = E(G)$. The proof is complete.

§ 2. Properties of $\text{Sp}(4, q)$

In this section we fix notation for $L = \text{Sp}(4, q)$, where $q = 2^n \geq 4$, and set down some facts about its automorphisms.

With respect to a suitable basis of the underlying symplectic space we identify the elements of L with the 4×4 matrices x with entries in $GF(q)$ satisfying the condition

$$(2.1) \quad x'rx = r,$$

where x' denotes the transposed matrix of x , and $r = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}$. L has order $q^4(q^2-1)(q^4-1)$ (see for example [5, p. 4]). Denote by P the subgroup of L consisting of the matrices

$$(2.2) \quad (abcd) = \begin{bmatrix} 1 & & & \\ a & 1 & & \\ c & b & 1 & \\ d & c+ab & a & 1 \end{bmatrix},$$

where $a, b, c, d \in GF(q)$. Then $|P| = q^4$ and so P is a Sylow 2-subgroup of L .

We define

$$A_1 = \{(a0cd) ; a, c, d \in GF(q)\},$$

$$A_2 = \{(0bcd) ; b, c, d \in GF(q)\},$$

$$U_1 = \{(000d) ; d \in GF(q)\},$$

$$U_2 = \{(00c0) ; c \in GF(q)\}.$$

Denote by J the subgroup of L consisting of the matrices

$$h(x, y) = \begin{pmatrix} x & & & \\ & y & & \\ & & y^{-1} & \\ & & & x^{-1} \end{pmatrix},$$

where $x, y \in GF(q)$, $xy \neq 0$. Furthermore define

$$J_1 = \{h(1, y); y \in GF(q), y \neq 0\},$$

$$J_2 = \{h(x, x^{-1}); x \in GF(q), x \neq 0\}.$$

Denote by J_1 the group of matrices

$$\begin{pmatrix} s & & t \\ & 1 & \\ & & 1 \\ u & & v \end{pmatrix},$$

and denote by J_2 the group of matrices

$$\begin{pmatrix} s & & t \\ & s & t \\ u & & v \\ & u & v \end{pmatrix},$$

where

$$\begin{pmatrix} s & t \\ u & v \end{pmatrix} \in SL(2, q).$$

Finally define

$$r_1 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad r_2 = \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

Let $A = \text{Aut}(L)$. Then A is the semidirect product of L by a cyclic group $\langle g \rangle$ of order $2n$, where g is a *graph* automorphism defined by

$$(2.3) \quad \begin{aligned} r &\longrightarrow r, \\ (abcd) &\longrightarrow (b, a^2, d + ac + a^2b, c^2 + bd + abc), \end{aligned}$$

(see [5, Chapter 12]). Notice that $\langle g^2 \rangle$ is the group of *field* automorphisms of L .

The following is proved by direct computation.

- (2A) (1) $C_P(J_i) = U_i$ and $C_L(J_i) = J_i \times \hat{J}_i$, $1 \leq i \leq 2$.
 (2) J_i acts regularly on $(P/A_i)^*$, $(A_i/Z(P))^*$, $(Z(P)/U_i)^*$ and U_{3-i}^* , $1 \leq i \leq 2$.
 (3) J acts regularly on $Z(P) - U_1 \cup U_2$.
 (4) $N_L(P) = PJ = N_L(Z(P))$.
- (2B) (1) $P = A_1 A_2$, $\mathcal{E}^*(P) = \{A_1, A_2\}$, and $A_1 \cap A_2 = U_1 \times U_2 = Z(P) = P' = P^2$.
 (2) $Z(P) - U_1 \cup U_2$ is the set of square involutions of P .
 (3) P/U_i is isomorphic to a Sylow 2-subgroup of $SL(3, q)$, $1 \leq i \leq 2$.
 (4) P is indecomposable.

PROOF. (1) and (2) follow by direct computation. As $P^q = P$ and $U_1^q = U_2$, $P/U_1 \cong P/U_2$. The map

$$(abcd) \longrightarrow \begin{pmatrix} 1 & & \\ a & 1 & \\ c & b & 1 \end{pmatrix}$$

defines a homomorphism from P onto a Sylow 2-subgroup of $SL(3, q)$, and its kernel is U_1 . Thus (3) holds. Suppose that $P = P_1 \times \cdots \times P_n$, where P_i is indecomposable and $n \geq 2$. Since the square involutions of P generate $Z(P)$, each P_i is not elementary. Since $P' = P^2$, $P'_i = P_i^2$ and so $N_A(P)$ acts on $\{P_1^2, \dots, P_n^2\}$ by the Krull-Schmidt Theorem. Since there are precisely two $N_A(P)$ -conjugacy classes of elements of $(P^2)^*$, we must have $n = 2$. But then J normalizes P_1^2 and hence it can not act transitively on the square involutions of P , contrary to (2A.3). Thus P is indecomposable.

(2C) For $i \in \{1, 2\}$, the following conditions hold:

- (1) $N_L(A_i) = A_i(\hat{J}_i \times J_i)^{r_{3-i}}$, $C_L(U_i) = A_i \hat{J}_i^{r_{3-i}}$.
 (2) A_i/U_i is a natural module for $\hat{J}_i^{r_{3-i}} \cong SL(2, q)$.
 (3) If V is a normal subgroup of $C_L(U_i)$ contained in A_i , and $V \not\leq U_i$, then $V = A_i$.
 (4) $C_L(U_i)$ is perfect.
 (5) $N_L(A_i)$ acts transitively on $(A_i/U_i)^*$ and U_i^* .

PROOF. First, we note that g changes the subscripts i of the objects defined for L . Thus it is sufficient to consider the case $i = 1$. (1) can be checked by direct computation. (2) is a direct consequence of (1K). Now let V be a normal subgroup of $C_L(U_1)$ contained in A_1 and suppose $V \not\leq U_1$. Then $A_1 = VU_1$ by (2) and so $P = VA_2$. Since both V and A_2 are elementary, we have $P^2 \leq V \cap A_2$. Since $U_1 \leq P^2$, we conclude that $A_1 = V$, proving (3). Set $V = A_1 \cap C_L(U_1)'$. Then $C_L(U_1)$ centralizes A_1/V , and hence we must

have $V \not\leq U_1$. Thus $V = A_1$ by (3), and (4) follows. (5) is a consequence of (2) and (2A) since $J \leq N_L(A_1)$.

(2D) For $i \in \{1, 2\}$, the following conditions hold:

- (1) $N_A(A_i) = N_A(U_i) = N_L(A_i) \langle g^2 \rangle$.
- (2) $C_A(U_i) = C_L(U_i)$.
- (3) $C_A(C_L(U_i)/A_i) = A_i J_i^{r_3-i}$.
- (4) $C_A(P/A_i) = P J_i^{r_3-i}$, $C_A(Z(P)/U_i) = P J_{3-i}$, $C_A(A_i/U_i) = A_i$.

PROOF. Recall that $A = L \langle g \rangle$. Since g normalizes P , we have $N_A(P) = N_L(P) \langle g \rangle$. P is a Sylow 2-subgroup of $N_L(A_i)$. Hence using a Frattini argument, we may argue as follows:

$$\begin{aligned} N_A(A_1) &= N_L(A_1)(N_A(P) \cap N_A(A_1)) \\ &= N_L(A_1)(N_L(P) \langle g \rangle \cap N_A(A_1)) \\ &= N_L(A_1)(N_L(P) N_{\langle g \rangle}(A_1)) \\ &= N_L(A_1) \langle g^2 \rangle. \end{aligned}$$

Now, $O_2(C_L(U_1)) = A_1$ and $Z(N_L(A_1))' = U_1$ by (2C), so $N_A(U_1) = N_A(A_1) = C_L(U_1) J_2 \langle g^2 \rangle$ by the above. Thus $C_A(U_1) = C_L(U_1) C_{J_2 \langle g^2 \rangle}(U_1)$. We have

$$(000d)^{h(x, x^{-1})g^{2k}} = (000(x^2d)^{2k}).$$

Hence it follows easily that $C_{J_2 \langle g^2 \rangle}(U_1) = 1$. Therefore $C_A(U_1) = C_L(U_1)$. Analogous reasoning shows that $C_A(P/A_1) = P J_1^{r_3}$. Thus $C_A(C_L(U_1)/A_1) = A_1 J_1^{r_3}$. As $N_A(Z(P)) = N_A(P)$ by (2A.4), $N_A(Z(P)) \cap N_A(U_1) = N_L(P) \langle g^2 \rangle \leq N_A(U_2)$, so $C_A(Z(P)/U_1) \leq C_A(U_2) = C_L(U_2)$. Thus $C_A(Z(P)/U_1) = P J_2$. Hence $C_A(A_1/U_1) = C_{P J_2}(A_1/U_1) = A_1$.

(2E) Every involution of L is L -conjugate to an element of $Z(P)$. Let f be an involution of $\langle g \rangle$. Then the following holds:

- (1) $I(N_A(P) - L) = f^{N_A(P)}$. $I(A - L) = f^L$.
- (2) $C_L(f) = \begin{cases} \text{Sp}(4, q_0) & \text{if } q = q_0^2, \\ \text{Sz}(q) & \text{if } q \text{ is not square.} \end{cases}$
- (3) L is balanced (see Section 1 for the definition).

PROOF. The first assertion is a direct consequence of (2B.1) and (2C.5). If $q = q_0^2$, then f is a field automorphism and so

$$f : (abcd) \longrightarrow (a^{q_0} b^{q_0} c^{q_0} d^{q_0}).$$

If q is not a square, then by (2.3) we have

$$f : (abcd) \longrightarrow (b^\theta, a^{2\theta}, d^\theta + a^\theta c^\theta + a^{2\theta} b^\theta, c^{2\theta} + b^\theta d^\theta + a^\theta b^\theta c^\theta),$$

where $q=2^{2m+1}$ and $x^g=x^{2^m}$ for $x \in GF(q)$. In each case, we can check by direct computation that the number of elements of P that are inverted by f is equal to $|P:C_P(f)|=q^2$. This implies that $I(fP)=f^P$. (1) follows from this easily. (2) is well-known. It remains to prove (3). Since A/L is cyclic, it is enough to show that for any involution x of A , we have $O(C_A(x))=1$. If $x \in L$, we may assume $x \in Z(P)$. Then $C_L(x)=C_L(U_1), C_L(U_2)$ or P , so $O(C_L(x))=1$ and hence $O(C_A(x)) \leq C_A(C_L(x))$. As $C_A(P)=Z(P)$ by (2D), we must have $O(C_A(x))=1$. Assume $x \notin L$. Then we may assume $x=f$. Set $L_0=C_L(f)$, so that $C_A(f)=\langle g \rangle L_0$. It is clear that $C_A(f)/\langle f \rangle \cong \text{Aut}(L_0)$. Thus $O(C_A(f))=1$.

(2F) If $P \leq Q \in \text{Syl}_2(A)$, then $J_r(Q)=P$.

PROOF. Let X be an abelian subgroup of Q of rank at least $3n$, where $q=2^n$. We show $X \leq P$. Conjugating in A , we may assume $\Omega_1(X) \leq P \langle f \rangle$. If $\Omega_1(X) \not\leq P$, then by (2E.1) we may assume $f \in X$. But then $m(X) \leq 3n/2 + 1 < 3n$ by (2E.2), a contradiction. So $\Omega_1(X) \leq P$. Then A_1 or $A_2 \leq X$. Thus $X=A_1$ or A_2 by (2D.4). This completes the proof.

§ 3. Fusion of an involution of $C(L)$

In this section we begin the proof of the theorem stated in the introduction.

(3.1) NOTATION. Let G be a finite group with $O(G)=1$ and suppose L is a standard subgroup of G with $L \not\trianglelefteq G$ and $L/Z(L) \cong \text{Sp}(4, q)$, where $q=2^n \geq 4$. Furthermore assume that $C(L)$ has a cyclic Sylow 2-subgroup.

Results of Steinberg show (see [21]) that $Z(L)=1$ and so we may identify L with the group of 4×4 matrices x satisfying (2.1). The symbols used in Section 2 for various objects defined for $\text{Sp}(4, q)$ will retain their meaning for the balance of the paper. Thus P is a Sylow 2-subgroup of L consisting of the matrices (2.2).

Let t be an involution of $C(L)$ and set $C=C(t)$. Let Q be Sylow 2-subgroup of $LC_C(L)$ with $P \leq Q$. (We shall shortly show that $Q=\langle t \rangle P$.) Finally, for $i \in \{1, 2\}$ we define

$$B_i = \langle t \rangle A_i, \quad V_i = \langle t \rangle U_i.$$

(3A) The following conditions hold:

- (1) For some $R \in \text{Syl}_2(C)$, $Q=J_r(R)$.
- (2) $t^{N(Q)}=t^G \cap Z(Q) \neq \{t\}$.
- (3) $\langle t \rangle \in \text{Syl}_2(C(L))$.

- (4) $t^G \cap L = \emptyset$.
 (5) $N(Z(Q)) = N(Q)$.

PROOF. Let $T = C_Q(L)$ and $Q \leq R \in \text{Syl}_2(C)$. Then $Q = P \times T$ has rank $3n+1$, where $q=2^n$, and $Q = J_r(Q)$. If A is an abelian subgroup of R of rank at least $3n+1$, then $m(AT/T) \geq 3n$ and so $A \leq Q$ by (2F). This implies that $J_r(R) = Q$.

If $t^G \cap C = \{t\}$ then $t \in Z(G)$ by [9], since we are assuming $O(G)=1$. But then $G=C$ and $L \triangleleft G$ contrary to our assumption. Hence we may choose $t^g \in C - \{t\}$. Notice that $t \in C(t^g) \leq N(L^g)$. Set $K = C_{L^g}(t)^\infty$. Then $K \leq C^\infty = L$. If $t \notin L^g C(L^g)$, then $I(tL^g) \leq t^g$ by (2E.1) and so $I(tK) \leq t^g \cap tL$. Since K has even order by (2E.2), it follows that $t^g \cap tL \neq \{t\}$. If $t \in L^g C(L^g)$, then $t \neq t^{g^{-1}} \in LC_C(L)$. Thus in any case $t^g \cap \langle t \rangle L \neq \{t\}$. Since every involution of L is a central involution, we conclude that $t^G \cap Z(Q) \neq \{t\}$.

Let $s \in t^G \cap Z(Q)$. Then t is an extremal conjugate of s in a Sylow 2-subgroup of G containing R , and so there is an element $g \in G$ such that $s^g = t$ and $C_R(s)^g \leq R$. In particular $Q^g \leq R$, so $Q^g = Q$ by (1). Thus $t^G \cap Z(Q) = t^{N(Q)}$.

If $|T| > 2$, then t is the unique involution of $T^2 = Z(Q)^2$ and so $N(Q) \leq C$, contrary to (2). Thus $|T| = 2$ and (3) follows. Consequently $Q^2 = Z(P)$, so $t^G \cap Z(P) = \emptyset$ by (2), and (4) follows.

By a Frattini argument $C = LN_C(Q)$, so $N_C(Z(Q)) = N_L(Z(Q))N_C(Q) = N_C(Q)$, since $N_L(Z(Q)) = N_L(Z(P)) = N_L(P)$. It follows from (2) that $t^{N(Z(Q))} = t^{N(Q)}$. Thus $N(Z(Q)) = N(Q)$, and all parts of the lemma hold.

Before proceeding further we note the following two facts:

$$(3.2) \quad N(Q) \text{ normalizes } \{U_1, U_2\}.$$

This is because $U_1^\# \cup U_2^\#$ is the set of involutions of $Q^2 = Z(P)$ that have no square roots in Q .

$$(3.3) \quad N(Q) \text{ normalizes } \mathcal{E}^*(Q) = \{B_1, B_2\}.$$

(3B) Suppose X is a subgroup of $N(Q)$ such that $J \leq X$ and $X \not\leq C$. Then $t^X = tU_1, tU_2$ or $tZ(P)$.

PROOF. We may assume $C(Z(Q)) \leq X$. Set $\Omega = t^X$. Then $\Omega \leq tZ(P)$ by (3A). Since J acts transitively on each of the sets $U_1^\#, U_2^\#$, and $Z(P) - U_1 \cup U_2$, either (3B) holds or $\Omega = t(Z(P) - U_1^\# \cup U_2^\#)$, $t(U_1 \cup U_2)$, or $t(Z(P) - U_i^\#)$, $i \in \{1, 2\}$.

Assume that $\Omega = t(Z(P) - U_1^\# \cup U_2^\#)$. Then X^Ω is 2-transitive and $(X \cap C)^\Omega$ has a normal subgroup J^Ω , which is regular on $\Omega - \{t\}$. As $J^\Omega \cong J$ is not cyclic, X^Ω has no sharply doubly transitive normal subgroups, so a result

of Hering, Kantor, and Seitz [17] shows that X^q has a normal subgroup K^q with $K^q \leq X^q \leq \text{Aut}(K^q)$ such that K^q is isomorphic to $\text{PSL}(2, p^m)$, $\text{PSU}(3, p^m)$, $\text{Sz}(2^m)$, or a group of Ree type in their natural doubly transitive representations. In particular $|\Omega| - 1 = (q - 1)^2$ is a power of a prime, so $p = q - 1$ is a prime and $K^q \cong \text{PSL}(2, p^2)$. This shows that $|X/K|$ divides 4, so $J \leq K$ and $|K \cap C : JC_X(\Omega)| = (p^2 - 1)/2$. Here $C_X(\Omega) = C(Z(Q)) = PC_C(L)$ by (2D.4) and so $JC_X(\Omega) = N_L(P)C_C(L) = N_C(Q) \cap LC_C(L)$. As $N_C(Q)$ covers $C/LC_C(L)$, it follows that $(p^2 - 1)/2 \leq 2n$, where $q = 2^n$. The only possibility is $q = 4$, when $K \cap C = N_C(Q)$ and $|C/LC_C(L)| = 4$. Thus $K \cap C$ contains an element that interchanges U_1 and U_2 . But then $O^2(K^q) < K^q$ by (3.2), which is a contradiction.

Assume next that $\Omega = t(U_1 \cup U_2)$. Then $|\Omega| = 2q - 1$ is odd and so $N_X(U_1) = N_X(U_2)$ is transitive on Ω by (3.2). Choose $g \in N_X(U_1)$ such that $t^g \in tU_1^*$, and let $u \in U_2^*$. Then $(tu)^g \in \Omega \cap tU_1^*U_2^*$, a contradiction.

Finally assume that $\Omega = t(Z(P) - U_2^*)$, say. Then $\langle tZ(P) - \Omega \rangle = \langle tU_2^* \rangle = V_2$ since $q > 2$, so $X \leq N(V_2)$ and then $|t^{N(V_2)}| \geq |\Omega| = q^2 - q + 1 > q$. This, however, is impossible by (3A.4). Thus we have proved the lemma.

(3C) If $t^{N(Q)} = tU_1$ or tU_2 , then $t^G \cap C \leq LC(L)$.

PROOF. Suppose that $u = t^g \in C - LC(L)$. Then $C_L(u) \cong \text{Sz}(q)$ or $\text{Sp}(4, q_0)$, where $q = q_0^2$. We may assume that $C_P(u) \in \text{Syl}_2(C_L(u))$. Let z be a square involution of $C_P(u)$. Then, as $|C_L(u) : C_L(u)^\infty| \leq 2$, $z \in C_L(u)^\infty \leq L^g$. Also, our assumption and (3A) imply that $tz \notin t^G$. Hence we must have $t \in L^g C(L^g)$, so t centralizes a Sylow 2-subgroup Q_1 of $\langle u \rangle L^g$. Then $Q_1 = J_r(R)$ for some Sylow 2-subgroup R of C and so $Q_1 \leq \langle t \rangle L$ by (3A.1). Thus $Q_1^2 \leq L \cap L^g$. This, however, is a contradiction because $u \notin LC(L)$ implies $I(uQ_1^2) \leq u^G$, whereas $I(uQ_1^2) \not\leq u^G$ by our assumption and (3A).

(3D) Let $D_1 = O_2(N(V_1))$ and suppose $N(V_1) \not\leq C$. Then $N(V_1) = D_1 N_C(V_1)$, $D_1 \cap N_C(V_1) = B_1 \triangleleft N(V_1)$, and $D_1/B_1 \cong U_1$ as $N_C(V_1)$ -modules.

PROOF. By (2D.2), $C(V_1) = C_L(U_1)C_C(L)$ and so $O_2(C(V_1)) = B_1$. Hence $N(V_1) \leq N(B_1)$. Set $\Omega = t^{N(V_1)}$. Then (2C.5) and (3A) show that $\Omega = tU_1$ and $N(V_1)^q$ is 2-transitive. Let bars denote images in $N(V_1)/B_1$. Then $\overline{C(V_1)} = \overline{C_L(U_1)} \times \overline{O(C)}$ and $\overline{C_L(U_1)} \cong \text{SL}(2, q)$, so $\overline{C_L(U_1)} \triangleleft \overline{N(V_1)}$. Hence if $\overline{X} = \overline{C_L(U_1)} C(\overline{C_L(U_1)})$, then $C(V_1) \leq X \triangleleft N(V_1)$. Moreover, $X \not\leq N_C(V_1)$ as $|\text{Out}(\text{SL}(2, q))| < q$, so X^q is transitive. As $(X \cap C)^q = J_2^q$ is regular on $\Omega - \{t\}$ by (2D.3), X^q is a Frobenius group and therefore has a regular normal subgroup M^q . Clearly, $\overline{M} \triangleleft \overline{N(V_1)}$.

The definition of $\overline{X}$ and the structure of $\overline{C(V_1)}$ shows $\overline{M} = O_{2',2}(\overline{M}) \times \overline{C_L(U_1)}$. Moreover, $O(\overline{M}) = \overline{O(C)}$ so $J_2 C(V_1) \leq O(\overline{M}) C_X(O(\overline{M}))$ and, since X^q is primitive, we must have $\overline{X} = O(\overline{M}) C_X(O(\overline{M}))$. Hence $O_{2',2}(\overline{M}) = O_2(\overline{M}) \times$

$O(\overline{M})$, and therefore $\overline{N(V_1)}$ is a semidirect product of $O_2(\overline{M})$ by $\overline{N_c(V_1)}$. As $O_2(N_c(V_1)) = B_1$ by (2D), $O_2(\overline{M}) = \overline{D_1}$.

Now D_1 normalizes $U_1 = V_1 - \Omega$ and so $U_1 \cap Z(D_1) \neq 1$. Since $N_c(V_1)$ acts irreducibly on U_1 , it follows that $U_1 \leq Z(D_1)$. Thus commutation by t induces an isomorphism from D_1/B_1 onto U_1 , which commutes with the action of $N_c(V_1)$. The proof is complete.

(3E) For $i \in \{1, 2\}$, the following conditions hold:

- (1) $N(V_i) \leq N(B_i)$.
- (2) $N_c(V_i) = N_c(B_i)$.

PROOF. As remarked in (3D), $O_2(C(V_i)) = B_i$, so $N(V_i) \leq N(B_i)$. By (2C), $O_2(N_L(V_i)) = A_i$ and $Z(N_L(B_i)) = U_i$. Thus $N_c(V_i) = N_c(B_i)$.

§ 4. Case I: $t^{N(Q)} = tU_i$, $i = 1$ or 2

By (3A) and (3B) there are two cases to consider:

Case I: $t^{N(Q)} = tU_i$, $i = 1$ or 2 .

Case II: $t^{N(Q)} = tZ(P)$.

In this section we shall consider the first case. By symmetry we may assume

(4.1) HYPOTHESIS. $t^{N(Q)} = tU_1$.

(4A) The following conditions hold:

- (1) $t^G \cap B_1 = tU_1$.
- (2) $t^G \cap B_2 = \bigcup_{g \in N_L(A_2)} tU_1^g$
- (3) $N(V_1) = N(B_1)$.
- (4) $N(V_2) = N_c(B_2)$.
- (5) $N(Q) \leq N(B_i)$, $1 \leq i \leq 2$.

PROOF. By (4.1) and (3A), $t^G \cap Z(Q) = tU_1$. Since

$$A_i = \bigcup_{g \in N_L(A_i)} Z(P)^g, \quad 1 \leq i \leq 2,$$

by (2C.5), and $N_L(A_i) = N_L(U_i)$, (1) and (2) follow. In particular $t^G \cap B_1 \neq t^G \cap B_2$, so $N(Q) \leq N(B_i)$, $1 \leq i \leq 2$, by (3.3). Furthermore, $N(B_1) \leq N(V_1)$ as $\langle tU_1 \rangle = V_1$, and $N(V_2) \leq C$ as $t^G \cap V_2 = \{t\}$. Thus (3) and (4) hold by (3E).

(4.2) NOTATION. $D_i = O_2(N(B_i))$, $1 \leq i \leq 2$, $R = QD_1$, $T = QD_2$, $D = D_1 \cap D_2$.

Diagram 0 gives a picture of what we are trying to prove in (4B), (4D), and (4E) below. Sections 5 and 6 also contain some diagrams, but they

are not so satisfactory as Diagram 0 and are given there only to help the reader draw more detailed diagrams by himself. The reader is expected to consult these diagrams always. This dependence on the diagrams will be essential to the understanding of the involved configurations of subgroups which we encounter in this and the later sections, especially in Section 8.

(4B) *The following conditions hold:*

- (1) $N(B_1) = D_1 N_C(B_1)$, $D_1 \cap N_C(B_1) = B_1$, and $D_1/B_1 \cong U_1$ as $N_C(B_1)$ -modules.
- (2) $N(B_2) = D_2 N_C(B_2)$, $D_2 \cap N_C(B_2) = B_2$, and $D_2/B_2 \cong A_2/U_2$ as $N_C(B_2)$ -modules.
- (3) $N(B_i) \leq N(U_i)$, $1 \leq i \leq 2$.
- (4) $R \leq T$, and $\mathcal{E}^*(T/B_2) = \{R/B_2, D_2/B_2\}$

PROOF. By (4A), $N(B_1) = N(V_1) \not\leq C$, so (1) follows from (3D). Furthermore, $N(B_1) \leq N(U_1)$ as $U_1 = V_1 - t^G \cap B_1$ by (4A.1).

Let bars denote images in $N(U_2)/U_2$ and set $\Omega = \bar{t}^{N(\bar{B}_2)}$. Since $N(V_2) = N_C(B_2)$, $C(\bar{t}) \leq N(\bar{B}_2)$ and $C(\bar{t})$ acts transitively on $(\bar{B}_2/\langle \bar{t} \rangle)^*$. The above paragraph, (4A.5), and (3.2) show that $N(Q) \leq N(B_2) \cap N(U_2)$, whence $C(\bar{t}) < N(\bar{B}_2)$. Also, $\Omega \cap \bar{A}_2 = \emptyset$ by (3A). Thus it follows from (1A) that $\Omega = \bar{t}\bar{A}_2$ and that $N(\bar{B}_2)^o$ is 2-transitive. As a consequence we have $|N(B_2) \cap N(U_2) : N_C(B_2)| = q^2$. Since $|t^{N(B_2)}| = |t^G \cap B_2| = q^2$ by (4A), it follows that $N(B_2) \leq N(U_2)$. Now $\bar{C}_L(U_2)^o \cong \text{SL}(2, q)^{V(2, q)}$, so by (1B), $N(\bar{B}_2)^o$ has a regular normal subgroup $\bar{M}^o$. By (2D.4), $C(\Omega) = C(\bar{B}_2) = \bar{B}_2 \bar{O}(C)$, whence $O(\bar{M}) = O(C(\Omega)) = \bar{O}(C)$. Therefore $\bar{C}_L(U_2)C(\Omega) \leq O(\bar{M})C(O(\bar{M}))$. Since $\bar{C}_L(U_2)$ acts irreducibly on $\bar{M}/C(\Omega)$, it follows that $\bar{M} \leq O(\bar{M})C(O(\bar{M}))$. Thus $\bar{M} = O_2(\bar{M}) \times O(\bar{M})$, and arguing as in (3D), we obtain (2).

It remains to prove (4). (1) in particular shows that $[D_1, Q] \leq B_1 \leq Q$, so $R \leq N(Q) \leq N(B_2)$ and then $R \leq N(T)$ by the definition of T . Also, (2) and (2D.1) show that $N(T) \cap N(B_2)/T$ has a cyclic Sylow 2-subgroup. Since R/Q is elementary of order $q > 2$, we must have $|R/R \cap T| \leq 2$. Since J acts irreducibly on R/Q , we conclude that $R \leq T$. Let tildes denote images in $N(B_2)/B_2$. Then $\tilde{D}_2 \cong \bar{A}_2$ as $N_C(B_2)$ -modules. Hence it follows that $|C_{\tilde{D}_2}(\tilde{Q})| = q$ and $C_{\tilde{D}_2}(\tilde{x}) = C_{\tilde{D}_2}(\tilde{Q})$ for all $\tilde{x} \in \tilde{Q}^*$. So $C_{\tilde{D}_2}(\tilde{Q}) = \tilde{R} \cap \tilde{D}_2$ as $|R/Q| = q$, and (4) follows from (1C).

(4C) *The following conditions hold:*

- (1) $R^2 = Z(P) \leq Z(R)$.
- (2) $D_1^2 = U_1$, $Z(D_1) = A_1$.
- (3) $D_2^2 = A_2$, $Z(D_2) = U_2$, $Z_2(D_2) = A_2$.
- (4) $Z(T) = U_2$, $Z_2(T) = Z(P)$.
- (5) $N_T(X) = R$ for any nonidentity subgroup X of U_1 .

(6) $N_T(X)=R$ for any subgroup X of A_1 such that $Z(P)<X$.

PROOF. Let bars denote images in $N(U_2)/U_2$, and recall that $\bar{t}^{N(\bar{B}_2)}=\bar{t}^{D_2}=\bar{t}\bar{A}_2$ and $C(\bar{t})=\bar{N}_C(\bar{B}_2)$. By (4B), $\bar{R}^2$ and $\bar{D}_2^2\leq\bar{B}_2$. Clearly $\bar{t}$ has no square roots in $\bar{Q}$, hence in $\bar{T}$ also. It follows that $\bar{R}^2$ and $\bar{D}_2^2\leq\bar{A}_2$. So $R^2\leq B_1\cap A_2=Z(P)$ and then $R^2=Z(P)$ as $P^2=Z(P)$. As a consequence we have $D_1^2\leq Z(P)$. But $D_1^2\neq 1$ because $C_{D_1}(t)=B_1$. Hence $D_1^2=U_1$ by (2C.5). By (4A.5) and (4B.3), $U_i\triangleleft R$, $1\leq i\leq 2$. So $Z(R)\cap U_i\neq 1$ and then $Z(P)=U_1U_2\leq Z(R)$ by (2A.2). Consequently $U_1<Z(D_1)\cap A_1$. Thus $Z(D_1)=A_1$ by (2C.5).

Now $\bar{A}_2=\bar{B}_2-\bar{t}^{\bar{D}_2}\triangleleft\bar{D}_2$, so $\bar{A}_2\cap Z(\bar{D}_2)\neq 1$. Further $Z(\bar{D}_2)\leq C_{\bar{D}_2}(\bar{t})\leq\bar{B}_2$. Thus (2C.5) forces $Z(\bar{D}_2)=\bar{A}_2$. Similarly we have $\bar{D}_2^2=\bar{A}_2$. Since $U_2\triangleleft D_2$, a similar argument shows that $Z(D_2)=U_2$ or A_2 . Suppose that $Z(D_2)=A_2$. Then, since $D_2^2\leq A_2$ and $C_{D_2}(t)=B_2$, $[D_2, t]$ is a $N_C(B_2)$ -invariant subgroup of A_2 of order q^2 , which is impossible by (2C.3). Therefore $Z(D_2)=U_2$ and consequently $Z_2(D_2)=A_2$. Since $\bar{D}_2^2=\bar{A}_2$, a similar argument shows that either $D_2^2=A_2$ or $A_2=D_2^2\times U_2$. The latter is again impossible.

Now since $T=D_2P$ and $Z(\bar{T})<Z(\bar{Q})=\bar{Z}(\bar{Q})$, we have $Z(\bar{T})=\bar{Z}(\bar{P})$, whence $Z(T)\leq Z(D_2)=U_2$. Hence $Z(T)=U_2$ and $Z_2(T)=Z(P)$.

Let tildes denote images in $N(B_2)/B_2$. The $N_C(B_2)$ -isomorphism $\tilde{D}_2\rightarrow\bar{A}_2$ maps $C_{\tilde{D}_2}(\tilde{Q})=\tilde{R}\cap\tilde{D}_2$ onto $Z(\bar{P})$. This shows that J_1 acts regularly on T/R . Hence if $X\leq U_1$, then $N_T(X)=R$ or T . Since $Z(T)=U_2$, (5) follows. If $Z(P)<X\leq A_1$, then by (4B.4) $N_T(X)\leq N_T(XB_2)=R$. Thus (6) holds.

(4D) D has a J -invariant subgroup M of order q^3 such that $D=MZ(Q)$ and $M\cap Z(Q)=Z(P)$. Furthermore the following holds:

- (1) $M=C_M(J_1)\times C_M(J_2)$.
- (2) $C_M(J_1)$ has order q^2 and is elementary abelian or homocyclic of exponent 4.
- (3) $C_M(J_1)\leq\langle\hat{J}_1^{C(J_1)}\rangle$, $C_M(J_2)=U_2$.

PROOF. Since $T=B_1D_2$ and $B_1\cap D_2=Z(Q)$, D has order $2q^3$. Furthermore, $D^2\leq Z(P)$ by (4C). Thus M exists. We have $D_1=MB_1$ and $B_1\cap M=Z(P)$, so $M/Z(P)\cong U_1$ as J -modules by (4B). It follows that $M=C_M(J_1)Z(P)=C_M(J_1)U_2$. This is a central product by (4C). Thus $M=C_M(J_1)\times U_2$. As J_2 acts fixed-point-freely on $C_M(J_1)$, we also have $C_M(J_2)=U_2$.

Since J_2 is regular on $C_M(J_1)\cap C(t)=U_1$, Lemma (2.4) of [16] shows that (2) holds.

Set $J_0=J_1^{r_2}$, so that $J_0\leq J\cap\hat{J}_1$. J_0 acts regularly on U_1 , and hence on $C_M(J_1)/U_1$. Therefore $C_M(J_1)=[C_M(J_1), J_0]\leq\langle\hat{J}_1^{C(J_1)}\rangle$.

(4.3) NOTATION. $M_i=MM^{r_i}$, $1\leq i\leq 2$.

(4E) For $i\in\{1, 2\}$, the following conditions hold:

- (1) $M \cap M^{r_i} = C_M(J_i)$.
- (2) M_i is a subgroup of D_i such that $D_i = M_i B_i$ and $M_i \cap B_i = A_i$.
- (3) $N(B_i) = N(D_i) \leq N(M_i)$.
- (4) $M_i \triangleleft T$.
- (5) $M_1 \cap M_2 = M$.
- (6) M_1 is elementary, and $M_2^2 = Z(M_2) = U_2$.

PROOF. Set $\hat{M} = MA_1$. Then $\hat{M}$ is a maximal subgroup of D_1 and is abelian by (4C) and (4D). Since $Z(D_1) = A_1$, it follows that $\hat{M}$ is characteristic in D_1 . J_1 acts fixed-point-freely on $A_1/Z(P)$, and hence on $\hat{M}/M$, so $C_{\hat{M}}(J_1) = C_M(J_1)$. Thus $C_M(J_1) \leq M^{r_1} \leq \hat{M}$. Also, $\hat{M} = MZ(P)^{r_1} \leq MM^{r_1}$. Thus $\hat{M} = MM^{r_1}$, and comparing orders, we conclude that $M \cap M^{r_1} = C_M(J_1)$.

Now let $M_0 = MA_2$, so that $R \cap D_2 = \langle t \rangle M_0$. Let bars denote images in $N(U_2)/U_2$ and tildes denote images in $N(B_2)/B_2$. The $N_c(B_2)$ -isomorphism $\tilde{D}_2 \rightarrow \bar{A}_2$ maps $\tilde{R} \cap \tilde{D}_2$ onto $Z(\bar{P})$ and $\tilde{R}^{r_2} \cap \tilde{D}_2$ onto $Z(\bar{P})^{r_2}$. Since $Z(\bar{P}) \cap Z(\bar{P})^{r_2} = 1$, it follows that $(\tilde{R} \cap \tilde{D}_2) \cap (\tilde{R}^{r_2} \cap \tilde{D}_2) = 1$. Thus $M_0 \cap M_0^{r_2} = A_2$ and $M \cap M^{r_2} = U_2$. Recall that $D_2^2 = A_2$ and $A_2 = Z(P)Z(P)^{r_2}$. Thus $M_2 = M_0 M_0^{r_2}$ is a maximal subgroup of D_2 , and $D_2 = \langle t \rangle M_2$ as $D_2 = (R \cap D_2)(R^{r_2} \cap D_2)$. It is clear that $M_1 \cap M_2 = M$.

Now set $E = MP$. Then E is a maximal subgroup of R with $R = \langle t \rangle E$, and $E \geq M_i$, $0 \leq i \leq 1$. By (4C), $\bar{M}$ centralizes $\bar{A}_i$, $1 \leq i \leq 2$. So $\bar{E} = \bar{M} * \bar{P}$ and, since $\bar{P}$ is of type $SL(3, q)$, $\bar{M}_i \in \mathcal{A}(\bar{E})$, $0 \leq i \leq 1$. Let $\bar{X}$ be an abelian subgroup of $\bar{R}$ of order at least q^3 and suppose $\bar{X} \not\leq \bar{E}$. Let $\bar{Y}$ be a maximal abelian subgroup of $\bar{E}$ containing $\bar{X} \cap \bar{E}$. Then $\bar{M} \leq \bar{Y}$ and $|\bar{Y}/\bar{M}| = q$, so $|\bar{X} \cap \bar{M}| \geq q^2/2 > q$. We also have $\bar{X} \cap \bar{M} \leq Z(\bar{R})$ as $\bar{X} \not\leq \bar{E}$. This, however, is a contradiction because $C_{\bar{M}}(\bar{t}) = \bar{M} \cap \bar{Q} = Z(\bar{P})$ by (4A.4). Thus we must have $\bar{X} \leq \bar{E}$. Hence it follows that $\bar{E} = J(\bar{R})$, and consequently $E \triangleleft T$ as $R \triangleleft T$ by (4B.4).

Suppose that $x \in T - N(M_1)$. Then $\bar{M}_1^x$ is a member of $\mathcal{A}(\bar{E})$ different from $\bar{M}_1$. Structure of $\bar{P}$ then shows that $\bar{E} = \bar{M}_1 \bar{M}_1^x$; that is $E = M_1 M_1^x$. Since M_1 is abelian, it follows that $M \leq Z(E)$, and in particular $[A_2, M] = 1$. But then A_2 centralizes $D_2 = \langle t \rangle MM^{r_2}$, which contradicts (4C). Thus $M_1 \triangleleft T$.

Now $M_1 = A_1 C_M(J_1)$. Assume that M_1 is not elementary. Then $C_M(J_1)$ is homocyclic by (4D). But then $A_1 = \Omega_1(M_1) \triangleleft T$ contrary to (4C.6). Thus M_1 is elementary.

Now $\bar{M} = Z(\bar{E}) \triangleleft \bar{T}$, so $Z_2(\bar{T}) \cap \bar{M} > Z(\bar{P})$ as $Z(\bar{T}) = Z(\bar{P})$. Since J acts irreducibly on $\bar{M}/Z(\bar{P})$, it follows that $\bar{M} \leq Z_2(\bar{T})$. Thus

$$[\bar{M}, \bar{M}^{r_2}] \leq Z(\bar{T}) \cap Z(\bar{T}^{r_2}) = Z(\bar{P}) \cap Z(\bar{P}^{r_2}) = 1.$$

Therefore $\bar{M}_2$ is elementary abelian. Further, since $Z(\bar{D}_2) = \bar{A}_2$, it follows that $\bar{M}_2$ is characteristic in $\bar{D}_2$. In particular $M_2 \triangleleft T$. Also, $M_2^2 \triangleleft N_c(B_2)$,

so $M_2^2 = U_2$ by (2C.5) as $[A_2, M] \neq 1$. Next $Z(M_2) \leq C_{M_2}(U_1) = R \cap M_2 = M_0$ by (4C.5), so $Z(M_2) \leq M_0 \cap M_0^{r_2} = A_2$. Furthermore $U_2 \leq Z(M_2) \triangleleft N_C(B_2)$. Thus $Z(M_2) = U_2$ as $[A_2, M] \neq 1$.

It remains to prove (3). As $D_1 = B_1 M_1$ and $C_{D_1}(t) = B_1$, $\mathcal{E}^*(D_1) = \{B_1, M_1\}$ by (1C). Thus (3) holds if $i=1$. Similarly, we have $\mathcal{E}^*(D_2/U_2) = \{B_2/U_2, M_2/U_2\}$ as $N_{D_2}(V_2) = B_2$ by (4A.4). Further $N(D_2) \leq N(U_2)$ as $Z(D_2) = U_2$. Thus (3) holds for $i=2$ as well.

(4.4) NOTATION. $S = M_1 M_2$, $E = MP$.

Thus S is a maximal subgroup of T such that $T = \langle t \rangle S$.

(4F) The following conditions hold.

- (1) $\mathcal{E}^*(S/U_2) = \{M_1/U_2, M_2/U_2\}$.
- (2) $\mathcal{E}^*(E) = \{M_1, A_2\}$.
- (3) M_1 is weakly closed in S with respect to G .
- (4) $Z(S) = U_2$, $Z_2(S) = M$, $Z_3(S) = S$.
- (5) $N(S) = N(M_1) \cap N(M_2)$.

PROOF. Let bars denote images in $N(U_2)/U_2$ and recall that $\bar{E} = \bar{M} * \bar{P}$. Set $M_0 = M A_2$. Then, since $\bar{M}$ is elementary, we have $\mathcal{E}^*(\bar{E}) = \{\bar{M}_0, \bar{M}_1\}$. Also, $\mathcal{E}^*(S/A_2) = \{E/A_2, M_2/A_2\}$ by (4B.4). Thus (1) follows.

Now $C_{M_2}(u) = M_0$ for any $u \in U_1^*$ by (4C.5). By symmetry $C_{M_0}(v) = M_0^{r_2} \cap M_0 = A_2$ for any $v \in (U_1^{r_2})^*$. As $A_2 = Z(P)U_1^{r_2}$, (1C) implies that $\mathcal{E}^*(M_0) = \{M, A_2\}$. Thus $\mathcal{E}^*(E) = \{M_1, A_2\}$.

Suppose $X = M_1^g \leq S$. Then either $X = M_1$ or $X \leq M_2$ by (1). Assume the latter. Then $X \cap M_0 = M$ or A_2 by (2). But then $X \leq C_{M_2}(U_1) = M_0$ by (4C.5), which is a contradiction proving (3).

Now $Z(\bar{S}) = \bar{M}$ and $Z_2(\bar{S}) = \bar{S}$ by (1), so $Z(S) \leq Z(M_2) = U_2$. Clearly we also have $U_2 \leq Z(S)$. Thus (4) holds. In particular, $N(S) \leq N(U_2)$, so (5) easily follows from (4.4) and (1).

(4.5) NOTATION. For $i \in \{1, 2\}$, denote by L_i the normal closure of $C_L(U_i)M_i$ in $N(M_i)$.

(4G) L_1/M_1 is isomorphic to $SL(2, q^2)$ or $SL(2, q) \times SL(2, q)$, and if $L_1/M_1 \cong SL(2, q) \times SL(2, q)$, then t interchanges two components of L_1/M_1 . We have $C(L_1/M_1) = O(N(M_1) \bmod M_1)$ and $S \in \text{Syl}_2(L_1)$.

PROOF. Let bars denote images in $N(M_1)/M_1$. Then, since $D_1 = \langle t \rangle M_1$, (4E.3) implies that $C(\bar{t}) = \overline{N(D_1)} = \overline{N(B_1)} = \overline{N_C(B_1)}$. Thus setting $\bar{K}_1 = \overline{C_L(U_1)}$, we have $\bar{K}_1 \triangleleft C(\bar{t})$. Furthermore, $\bar{K}_1 \cong SL(2, q)$, and $C(\bar{K}_1) \cap C(\bar{t}) = \langle \bar{t} \rangle \bar{J}_1^* \bar{O}(C)$ by (2D.3). Thus $\bar{K}_1$ is standard in $\overline{N(M_1)}$, and we may appeal to [16].

Notice that $S \leq N(M_1)$ and $\bar{J}_1 \leq \bar{K}_1$. Since $\bar{J}_1$ acts irreducibly on $\bar{S}/\bar{E}$ and $\bar{E}/\bar{M}_1$ (see the proof of (4C.5)), $\bar{S} = [\bar{S}, \bar{J}_1] \leq \langle \bar{K}_1^{N(\bar{M}_1)} \rangle$. In particular, the 2-rank of $\bar{L}_1$ is at least $2n$. Since $\bar{t} \in N(\bar{S})$, the only possibility is the one described in (4G).

(4H) If $L_1/M_1 \cong \text{SL}(2, q^2)$ and $S \neq S_1 \in \text{Syl}_2(L_1)$, then $Z(S) \cap Z(S_1) = 1$.

PROOF. By the structure of $\text{SL}(2, q^2)$, S_1 is S -conjugate to S^{r_1} . Hence it is sufficient to show $Z(S) \cap Z(S^{r_1}) = 1$. This, however, is clear because $Z(S) = U_2$.

(4I) Assume $L_1/M_1 = K/M_1 \times K^t/M_1$ with $K/M_1 \cong \text{SL}(2, q)$, and set $V = S \cap K$. Then the following holds:

- (1) $M_1 = Z(V) \times Z(V^{r_1})$.
- (2) $M = Z(V)Z(V^t)$, $Z(V) \cap Z(V^t) = U_2 = Z(V) \cap Z(P)$.
- (3) K^t acts transitively on $Z(V)^\#$.
- (4) If $S \neq S_1 \in \text{Syl}_2(L_1)$, then $Z(S) \cap Z(S_1) = 1$.
- (5) If $u \in U_2^\#$, then $u^{N(M_1)} \cap M = Z(V)^\# \cup Z(V^t)^\#$.

PROOF. By (4F.1), $Z(V) \leq Z(V \bmod U_2) = M$. Since $S = VV^t$, we have $Z(V) \cap Z(V^t) = Z(S) = U_2$. If $Z(V) = U_2$, then $Z(V^{r_1}) \cap Z(V^t) = U_2^{r_1} \cap U_2 = 1$, a contradiction because V^t normalizes V^{r_1} . Hence $Z(V) \neq Z(V^t)$, which shows that K^t/M_1 acts nontrivially on $Z(V)$. So Lemma (4B) of [7] yields that $|Z(V)| \geq q^2$. Suppose that $X = Z(V) \cap Z(V^{r_1}) \neq 1$. Then, since V^t normalizes X , we have $U_2 \cap X = Z(V^t) \cap X > 1$ and so $U_2 \cap X^t \neq 1$. Furthermore, $X \cap X^t = U_2 \cap U_2^{r_1} = 1$, so $U_2 \cap X < U_2$. This is a contradiction because J normalizes X and acts irreducibly on U_2 . Therefore, $Z(V) \cap Z(V^{r_1}) = 1$. (1) and (2) now follow easily. Also, (3) is a consequence of (1K).

Now let $S \neq S_1 \in \text{Syl}_2(L_1)$. Then by the structure of $\text{SL}(2, q) \times \text{SL}(2, q)$, S_1 is S -conjugate to S^{r_1} , VV^{tr_1} or $V^tV^{r_1} = (VV^{tr_1})^t$. As before $Z(S) \cap Z(S^{r_1}) = 1$. We also have $Z(VV^{tr_1}) = Z(V) \cap Z(V^{tr_1})$, so by (1) $Z(S) \cap Z(VV^{tr_1}) = Z(V) \cap Z(V^t) \cap Z(V^{tr_1}) = 1$. Thus (4) follows as in (4H).

It remains to prove (5). In view of (3), it is enough to show that for any $S_1 \in \text{Syl}_2(L_1)$, we have $M \cap Z(S_1) \leq Z(V) \cup Z(V^t)$. If $S_1 = S, S^{r_1}$ or VV^{tr_1} , this is easy to check. Since $M = Z_2(S)$, the assertion follows as in (4H).

(4.6) NOTATION.. $G_1 = \langle L_1, L_2 \rangle$.

(4J) Let x be an involution of T . Then the following conditions hold:

- (1) If $x \in Z(P) - U_2$, then $x \in_S U_1$.
- (2) If $x \in M_1$, then $x \in_{L_1} U_1 \cup U_2$.
- (3) If $x \in M_2$, then $x \in_{L_2} M$.
- (4) If $x \in S$, then $x \in_{\sigma_1} U_1 \cup U_2$.

(5) If $x \in tS$, then $t \in_{L_1} tZ(P)$.

(6) $t^g \cap T = t^g \cap B_2 = t^S$.

PROOF. For any nonidentity subgroup X of U_1 , $N_S(X) = E$ by (4C.5). In particular $N_S(U_1) = E$. Let $s \in S - E$ and set $X = U_1 \cap U_1^s$. Then, as $s^2 \in E$, $s \in N(X)$ and so we must have $X = 1$. This implies that U_1 is a T.I. set of S . Thus by an easy counting argument, we have

$$Z(P) - U_2 = \bigcup_{s \in S} U_1^s.$$

This proves (1). Analogous reasoning using (4C.6) yields

$$M_1 - M = \bigcup_{s \in S} A_1^s.$$

Next, $M^{r_1} \cap U_2 = C_M(J_1) \cap U_2 = 1$, so, as $Z(S) = U_2$, we have

$$\bigcap_{g \in L_1} M^g = 1.$$

Let $x \in M_1$. Then the last equation shows $x \in_{L_1} M_1 - M$. So by the second equation $x \in_{L_1} A_1$, and then $x \in_{L_1} Z(P)$ by (2C.2) as $C_L(U_1) \leq L_1$. Thus by the first equation $x \in_{L_1} U_1 \cup U_2$. Next, let $y \in M_2$. Then $y^g \in E \cap M_2$ for some $g \in L_2$ as $C_L(U_2)$ acts transitively on $(M_2/A_2)^*$. Then $y^g \in M \cup A_2$ by (4F.2). If $y^g \in A_2$, then $y \in_{L_2} Z(P)$ by (2C.2). Thus we have proved (2) and (3). Since $I(S) \leq M_1 \cup M_2$ by (4F.1), (4) follows.

Suppose $x \in tS$. It follows from (4G) that involutions of $T/M_1 - S/M_1$ are all S/M_1 -conjugate. So $x \in_S D_1$ and then $x \in_S tA_1$ as $\mathcal{E}^*(D_1) = \{B_1, M_1\}$. As $C_L(U_1)$ acts transitively on $(A_1/U_1)^*$, we conclude that $x \in_{L_1} tZ(P)$.

It remains to prove (6). It is enough to show $t^g \cap T \leq t^S$. Let $x \in t^g \cap T$. Then, since $m(C) = 3n + 1$, where $q = 2^n$, (4) forces $x \in tS$. So $x \in_S tA_1$ by the above, and then $x \in_S tU_1$ by (4A.1). As $tU_1 = t^{M_1}$, the assertion holds.

(4K) Let U be a Sylow 2-subgroup of $N(S)$ containing T and suppose uS is an involution of $N_U(T)/S - T/S$. Then $uS \cap C$ contains an involution v such that $I(uS) = v^S$ and $|C_S(v)| = q^3$.

PROOF. By (4J.6), $t^{N_U(T)} = t^S$ so $N_U(T) = C_U(t)S$. Clearly we also have $C_S(t) = P$. Hence by the structure of $\text{Aut}(L)$, $uS \cap C_U(t)$ contains an involution v . We show that v has the desired properties.

Since v normalizes $A_i = C_{M_i}(t)$, v acts on L as a nontrivial field automorphism. Notice that $N_C(P) \leq N(Q) \leq N(S) \cap N(T)$. Hence conjugating in $N_C(P)$, we may assume that the action of v on L is a canonical one, so that v normalizes J_i and centralizes r_i , $1 \leq i \leq 2$. Then v normalizes $C_M(J_i)$ and $C_M(J_i)^{r_i^{q-1}}$. Using (4D.1) and the definition of S and M_i , we have

$$S = C_M(J_1)^{r_2} C_M(J_2)^{r_1} C_M(J_1) C_M(J_2).$$

Furthermore, we see that S satisfies the hypothesis of (1D) with respect to this decomposition.

Since U/S is not cyclic, L_1/M_1 must be the product of two copies of $\text{SL}(2, q)$ interchanged by t . Since v commutes with t and acts nontrivially on S/M_1 , v interchanges these two components or acts on each of them as a nontrivial field automorphism. Also, v acts on $C_L(U_2)M_2/M_2$ as a nontrivial field automorphism. It follows that $|C_{S/M_i}(v)| = |S/M_i|^{1/2}$ is equal to the number of elements of S/M_i inverted by v . Since S is a semidirect product of M_i by $C_M(J_i)^{r_3-t}$, each factor in the above decomposition of S is $\langle v \rangle$ -isomorphic to S/M_1 or S/M_2 . Thus we can apply (1) and (2) of (1D) to conclude that $I(vS) = v^S$ and $|C_S(v)| = q^3$.

(4L) *We have*

$$L_2/O(L_2 \bmod M_2) \cong \text{SL}(2, q) \quad \text{and} \quad D_2 \in \text{Syl}_2(C_{N(M_2)}(L_2/O(L_2 \bmod M_2))).$$

PROOF. Let bars denote images in $N(M_2)/M_2$. Arguing as in (4G), we see that $\bar{K}_2 = \overline{C_L(U_2)}$ is a standard subgroup of $\bar{N}(\bar{M}_2)$ and that $\langle \bar{t} \rangle \in \text{Syl}_2(C(\bar{K}_2))$. Thus it is enough to show that $\bar{t}$ is an isolated involution. (4J. 6) shows that $\bar{t}$ is strongly closed in $\bar{T}$. Let $\bar{V}$ be a Sylow 2-subgroup of $C(\bar{t})$ containing $\bar{S}$, so that $\bar{T} \triangleleft \bar{V} \leq N(\bar{S})$. If $\bar{t}$ is conjugate to an element of $\bar{V} - \bar{T}$, then by (4K) t is conjugate to an element of $C - LC(L)$ against (3C). Therefore $\bar{t}$ is strongly closed in $\bar{V}$, as asserted.

(4.7) NOTATION. $G_0 = \langle L^G \rangle$.

(4M) *We have* $G_1 \leq G_0$ *and* $S \in \text{Syl}_2(G_0)$.

PROOF. By (4D. 3) $C_M(J_i) \leq G_0$, $1 \leq i \leq 2$, and so the decomposition of S used in the proof of (4K) yields $S \leq G_0$. As L_i 's are generated by their Sylow 2-subgroups, we have $G_1 \leq G_0$.

It is clear that $G_0 \leq O^2(G)$. Thus it is enough to show $S \in \text{Syl}_2(O^2(G))$. Let U be a Sylow 2-subgroup of $N(S)$ containing T . Then $T \triangleleft U \in \text{Syl}_2(N(M_2))$ and U/S is abelian by (4L). Indeed, U/T is cyclic. If $t^g \in U - T$, then t is conjugate to an element of $C - LC(L)$ by (4K), which is impossible by (3C). Therefore

$$t^G \cap U = \bigcup_{g \in N_L(A_2)} tU_1^g = t^S$$

by (4A. 2) and (4J. 6). This shows $\langle t^G \cap U \rangle = B_2$ so $N(U) \leq N(B_2) \leq N(M_2)$ by (4E. 3). Thus $U \in \text{Syl}_2(G)$. Furthermore, (1E) shows $t \notin O^2(G)$. Hence if U/S is cyclic, we have $S \in \text{Syl}_2(O^2(G))$.

Therefore we may assume that U/S is not cyclic. Let uS be an involution of $U/S - T/S$. By (4K) we may take u to be an involution, and moreover, $|C_U(u)| \leq |U/S| q^3$ for all such u . We argue that $u^g \cap T = \emptyset$. Suppose by way of contradiction that $u^g \in T$. By (4J) we may take $u^g \in Z(Q)$, so $|C_S(u^g)| \geq q^4 \geq |U/S| q^3$ as $|U/S| \leq 2n$, where $q = 2^n$. Therefore we may take u^g to be an extremal conjugate of u in U , when we may take g so that $C_U(u)^g \leq U$. By (4K) we may take from the outset $u \in C - LC(L)$, so it follows that $t^g \in U$. Since $t^g \cap U = t^S$, we may finally take $g \in C$. But then $u^g \in C_T(t) = Q \leq LC(L)$, whereas $u \in C - LC(L)$. This contradiction shows that $u^g \cap T = \emptyset$. Since U/T is cyclic, it follows as before that $S \in \text{Syl}_2(O^2(G))$. The proof is complete.

Now we turn to a study of the centralizers of the involutions of G_0 .

(4N) *We have $N_{G_0}(U_2) = N_{G_0}(M_2)O(N_{G_0}(U_2))$.*

PROOF. Let bars denote images in $N(U_2)/O(N(U_2) \bmod U_2)$, and let X be the normal closure of M_2 in $N_{G_0}(U_2)$. Then $M_2 \in \text{Syl}_2(X)$ by (1H), (4F.1), and (4M), so either $\bar{X} = \bar{M}_2$ or $\bar{X} = O_2(\bar{X})\bar{X}_1 \cdots \bar{X}_k$ with $\bar{X}_i$ a simple Goldschmidt group by [11]. Assume that the latter case holds. Since $\bar{t}$ normalizes $\bar{X}$, we have

$$1 \neq C(\bar{t}) \cap \bar{M}_2 \cap \bar{X}_1 \bar{X}_1^t = \bar{A}_2 \cap \bar{X}_1 \bar{X}_1^t.$$

Also, $\overline{C_L(U_2)}$ normalizes $\{\bar{A}_2 \cap \bar{X}_i \bar{X}_i^t \mid 1 \leq i \leq k\}$ and acts transitively on $\bar{A}_2^*$. If $\bar{X}_1 \bar{X}_1^t \neq \bar{X}_i \bar{X}_i^t$, then elements of $\bar{X}_1 \bar{X}_1^t$ are not $\overline{C_L(U_2)}$ -conjugate to an element of $\bar{X}_1 \bar{X}_1^t \bar{X}_i \bar{X}_i^t$. Hence we must have $E(\bar{X}) = \bar{X}_1 \bar{X}_1^t$, and moreover $\bar{A}_2 \leq \bar{X}_1 \bar{X}_1^t$. Since $\overline{C_L(U_2)}$ is perfect and normalizes $\bar{M}_2 \cap \bar{X}_i$, it follows that $\overline{C_L(U_2)}$ centralizes each $\bar{X}_i$. But then $\overline{C_L(U_2)}$ centralizes $\bar{A}_2$, a contradiction. Therefore, $\bar{M}_2 \triangleleft \overline{N_{G_0}(U_2)}$. Since $U_2 = Z(M_2)$, it follows that $M_2 O(N_{G_0}(U_2)) \triangleleft N_{G_0}(U_2)$. The proof is complete.

For $i \in \{1, 2\}$, we choose $u_i \in U_i^*$ and set $C_i = C_{G_0}(u_i)$.

(4O) *We have $C_2 = N_{C_2}(U_2)O(C_2)$ and C_2 is 2-constrained.*

PROOF. Suppose that an element u of U_2 is C_2 -conjugate to an element v of $S - U_2$. By (4F.1) and (4J.3) we may take $v \in M_1 - U_2$ as $L_2 \leq C_2$. Then since M_1 is weakly closed in S , v is $N_{C_2}(M_1)$ -conjugate to u . But $N_{C_2}(M_1) \leq N(S) \leq N(U_2)$ by (4H) and (4I.4). This contradiction shows that U_2 is strongly closed in S with respect to C_2 . Notice that $S \in \text{Syl}_2(C_2)$.

Set $X = \langle U_2^{C_2} \rangle$ and let bars denote images in $C_2/O(C_2)$. By [11] either $\bar{X} = \bar{U}_2$ or $\bar{X} = O_2(\bar{X})\bar{X}_1 \cdots \bar{X}_k$ with $\bar{X}_i$ a quasisimple Goldschmidt group. Further, $U_2 = \Omega_1(S \cap X)$. If $U_2 < S \cap X$, (4F.4) forces $U_2 < M \cap X \leq \Omega_1(S \cap X)$,

which is a contradiction. So $U_2 \in \text{Syl}_2(X)$. Hence if $\bar{X} \neq \bar{U}_2$, then $\overline{C_L(U_2)}$ normalizes each $\bar{X}_i$, and then as in (4N) $\overline{C_L(U_2)}$ centralizes $\bar{X}_i$. But then $\bar{U}_2 \cap \bar{X}_i \leq Z(\bar{X}_i)$, a contradiction. Thus $\bar{X} = \bar{U}_2$, which implies that $C_2 = N_{C_2}(U_2)O(C_2)$. Now $C_S(M_2) \leq M_2 \triangleleft S$. Therefore $N_{G_0}(M_2)$ is 2-constrained, and so also is $N_{G_0}(U_2)$ by (4N). Thus C_2 is 2-constrained.

(4P) *We have $C_1 = N_{C_1}(M_1)O(C_1)$ and C_1 is 2-constrained.*

PROOF. As $C_L(U_1)M_1$ is a maximal subgroup of L_1 (see [1, Lemma 2.5.3]), we have $C_1 \cap L_1 = C_L(U_1)M_1$. So $E \in \text{Syl}_2(C_1 \cap N(M_1))$ as $|N_{G_0}(M_1) : L_1|$ is odd. Since M_1 is weakly closed in S , we conclude that $E \in \text{Syl}_2(C_1)$. Now set $X = \langle M_1^{G_1} \rangle$ and let bars denote images in $C_1/O(C_1)$. Using (1H) and (4F.2), we have that $M_1 \in \text{Syl}_2(X)$ and that as usual either $\bar{X} = \bar{M}_1$ or $\bar{X} = O_2(\bar{X})\bar{X}_1 \cdots \bar{X}_k$ with $\bar{X}_i$ a simple Goldschmidt group.

Suppose that the latter case holds with $k \geq 2$ and set $\bar{Y}_i = \bar{M}_1 \cap \bar{X}_i$. Let $\bar{u} \in \bar{U}_2^\#$ and write $\bar{u} = \bar{v}\bar{u}_1 \cdots \bar{u}_k$ with $\bar{v} \in O_2(\bar{X})$ and $\bar{u}_i \in \bar{Y}_i$, $1 \leq i \leq k$. By (4O) $C_{\bar{X}}(\bar{u})$ is 2-constrained and so we must have $\bar{u}_i \neq 1$ for all i . This implies that the map $\bar{u} \rightarrow \bar{u}_i$ is one to one. Therefore $|\bar{Y}_i| \geq q$ for all i . As elements of $\bar{Y}_i^\#$ are all conjugate in $N_{\bar{X}_i}(\bar{Y}_i)$, M_1 contains at least $|\bar{Y}_1^\#| \cdots |\bar{Y}_k^\#|$ elements that are G_0 -conjugate to u . By (4F.3) and (4M), $u^{G_0} \cap M_1 = u^{N_{G_0}(M_1)}$. So by (4H) and (4I.4), $|u^{G_0} \cap M_1| \leq (q+1)^2(q-1)$. Thus $(q-1)^k \leq (q+1)^2(q-1)$. This forces $k \leq 3$. Since $C_L(U_1)$ is perfect, it follows as in (4N) that $\overline{C_L(U_1)}$ centralizes each $\bar{X}_i$, if it existed. But then we have

$$1 \neq C(\bar{t}) \cap \bar{M}_1 \cap \bar{X}_1 \bar{X}_1^t = \bar{A}_1 \cap \bar{X}_1 \bar{X}_1^t \leq Z(\bar{X}_1 \bar{X}_1^t),$$

a contradiction. Thus $\bar{X} = \bar{M}_1$, which implies that $C_1 = N_{C_1}(M_1)O(C_1)$. That C_1 is 2-constrained follows as in (4O).

(4Q) *The following conditions hold:*

- (1) $C_i \leq N(M_i)$, $1 \leq i \leq 2$.
- (2) $L_2/M_2 \cong \text{SL}(2, q)$.

PROOF. By (4J.4) and (4M), involutions of G_0 are conjugate to an element of $U_1 \cup U_2$, so by (4O) and (4P), the centralizer of every involution of G_0 is 2-constrained. Furthermore, $M_1 \in \mathcal{SCN}_3(S)$. So by [14] every 2-local subgroup of G_0 is 2-constrained and has a trivial core. (1) now follows from (4N), (4O), and (4P). Also, $C_{G_0}(M_2) \leq M_2$ as $M_2 = O_2(N_{G_0}(M_2))$ by (4L).

Suppose that $M_2 < X = S \cap S^x$ for some $x \in L_2$. We may take $x \in Y = \langle S, S^x \rangle$. Since S/M_2 is abelian, $Y \leq N(X)$ and so Y normalizes $Z(X \bmod U_2) = M$ (see (4F.1)). So by (4F.4), Y stabilizes the series $1 \leq U_2 \leq M \leq M_2$, and then $O^2(Y) \leq C_{G_0}(M_2) \leq M_2$. Thus $S = S^x$. This implies that L_2/M_2 is a T.I. group. (2) now follows from (4L).

We have now arrived at the goal of this section.

(4R) If $L_1/M_1 \cong \text{SL}(2, q^2)$, then $G_0 \cong \text{PSU}(4, q)$.

PROOF. We show that (1L) can be used here. It will suffice to prove

$$(1) \quad S = M_i(S \cap S^{r_3-i} \cap S^{r_i r_3-i} \cap S^{r_3-i r_i r_3-i}), \quad 1 \leq i \leq 2.$$

$$(2) \quad S \cap S^{r_1 r_2 r_1 r_2} = 1.$$

Other conditions required in (1L) are easily verified by (4E), (4F.5), (4G), and (4Q.2).

First, consider (1). We have $S^{r_2} \cap S \cap S^{r_1} = M_2 \cap M_1 = M$, so conjugating by r_1 , we have $S \cap S^{r_1} \cap S^{r_2 r_1} = M^{r_1}$. Combining these and using (4E.1), we have

$$S^{r_2} \cap S \cap S^{r_1} \cap S^{r_2 r_1} = M \cap M^{r_1} = C_M(J_1).$$

Conjugating by r_2 , we conclude that

$$S \cap S^{r_2} \cap S^{r_1 r_2} \cap S^{r_2 r_1 r_2} = C_M(J_1)^{r_2}.$$

Using (4D), (4.3), and (4.4), we may also argue as follows:

$$\begin{aligned} S &= M_1 M_2 = M_1 (M M^{r_2}) = M_1 M^{r_2} \\ &= M_1 (C_M(J_2) C_M(J_1))^{r_2} = M_1 C_M(J_1)^{r_2}. \end{aligned}$$

Thus (1) holds for $i=1$, and by symmetry of arguments it holds for $i=2$ as well. To prove (2), we notice that $C_S(t) = P$. Since r_i 's centralize t , we have

$$S \cap S^{r_1 r_2 r_1 r_2} \cap C(t) = P \cap P^{r_1 r_2 r_1 r_2} = 1.$$

Thus (2) holds, and (1L) applies. We conclude that $G_1/Z(G_1) \cong \text{PSU}(4, q)$.

It remains to prove $G_0 = G_1$. First, $N(S) \leq N(L_i)$, $1 \leq i \leq 2$, by (4F.5), so $N(M_i) = L_i N(S) \leq N(G_1)$ by a Frattini argument. Thus $C_i \leq N(G_1)$ by (4Q.1), and then $C_{G_0}(x) \leq N(G_1)$ for any $x \in I(S)$ by (4J.4). Since $S \in \text{Syl}_2(G_0)$, it follows that $N_{G_0}(G_1)$ is a strongly embedded subgroup of G_0 . As G_0 has two conjugacy classes of involutions, we must have $G_1 \triangleleft G_0$. As G_0 is simple, we conclude that $G_0 = G_1$, as asserted.

(4S) If $L_1/M_1 \cong \text{SL}(2, q) \times \text{SL}(2, q)$, then $G_0 \cong \text{PSL}(4, q)$.

PROOF. Let $L_1/M_1 = K/M_1 \times K^t/M_1$ with $K/M_1 \cong \text{SL}(2, q)$ and define

$$V = S \cap K, \quad A = Z(V)Z(V^{r_2}).$$

Recall from (4I) that $Z(V)$ is a E_{q^2} -subgroup of M containing $U_2 = M_2^2$. Since $M \cap M^{r_2} = U_2$, it follows that A is a subgroup of M_2 of order q^3 .

Let $x \in M - U_2$. We show that the following holds:

- (1) If $x \in Z(V)$, then $C_S(x) = V$.
- (2) If $x \notin Z(V) \cup Z(V^t)$, then there is an element $g \in L_1$ such that $x^g \in U_1$ and $C_S(x)^g \leq E$.

Suppose $x \in Z(V)$. Then $x \in M - Z(P)$ by (4I.2) and so $C_E(x) = M_1$ by (4F.2). Since $S = VE$, it follows that $C_S(x) = V$. Suppose $x \notin Z(V) \cup Z(V^t)$. Then by (4I.5) and (4J.2), there is an element $g \in L_1$ such that $x^g \in U_1$. Since $E \in \text{Syl}_2(C_{L_1}(x^g))$ (see the proof of (4P)), we may take g so that $C_S(x)^g \leq E$.

Now let $x \in M \cap V^{r_2} - U_2$, so that $Z(V^{r_2}) \leq C_S(x)$. By (1) and (2) above $Z(V^{r_2}) \leq V$ or $Z(V^{r_2}) \leq V^t$ or there is an element $g \in L_1$ such that $x^g \in U_1$ and $Z(V^{r_2})^g \leq E$. Suppose that the last case holds. Then $Z(V^{r_2})^g \leq A_2$ by (4F.2). As $C_L(U_2)$ acts transitively on $(A_2/U_2)^\#$, it follows that an element of U_2 is G_0 -conjugate to an element of $Z(P) - U_2$. But this is impossible by (4F.3) and (4I.5). Thus the last case does not hold. We define

$$\hat{A} = \begin{cases} Z(V)Z(V^{r_2}) & \text{if } Z(V^{r_2}) \leq V, \\ Z(V)Z(V^{r_2^t}) & \text{if } Z(V^{r_2}) \leq V^t. \end{cases}$$

Thus $\hat{A}$ is elementary of order q^3 . Let $x \in Z(V^{r_1})^\#$. If $M_1 < C_V(x)$, then $K = \langle V^{r_1}, C_V(x) \rangle$ centralizes x against (4I.1). Therefore $C_V(x) = M_1$ for all $x \in Z(V^{r_1})^\#$. Since $V = M_1 \hat{A}$ and $M_1 = Z(V) \times Z(V^{r_1})$, (1C) implies that $\mathcal{E}^*(V) = \{M_1, \hat{A}\}$. Consequently, $\hat{A} \triangleleft S$ as $V \triangleleft S$. By symmetry $\{M_1^{r_2}, \hat{A}\} = \mathcal{E}^*(V^{r_2})$ or $\mathcal{E}^*(V^{r_2^t})$, and $\hat{A} \triangleleft S^{r_2}$. Therefore $\hat{A} \triangleleft \langle S, S^{r_2} \rangle = L_2$ (see (4Q)). Since $Z(V) \leq \hat{A}$ and $r_2 \in L_2$, it follows that $Z(V^{r_2}) \leq \hat{A}$, and consequently $\hat{A} = A$. Thus $\mathcal{E}^*(V) = \{M_1, A\}$ and $A \triangleleft L_2$. Furthermore, (1K) shows that L_2 acts transitively on $(A/U_2)^\#$ as $A \cap Z_2(S) = Z(V)$. Thus elements of $A^\#$ are all G_0 -conjugate by (4I.3).

We argue that A is weakly closed in S with respect to G_0 . Since $A \not\leq M_1$, (1F) implies that A is not G_0 -conjugate to a subgroup of M_1 . In view of (4F.1), it will thus suffice to show that A is weakly closed in M_2 with respect to G_0 . For this purpose we determine $\mathcal{E}^*(M_2)$ explicitly.

Let $X \in \mathcal{E}^*(M_2)$ and let $x \in X - U_2$. By (4J) there is an element $g \in L_2$ such that $x^g \in M - U_2$. If $x^g \in Z(V)$, then $X^g \leq C_{M_2}(x^g) = V \cap M_2$ by (1) and so $X^g = M$ or A . Similarly, if $x^g \in Z(V^t)$, then $X^g = M$ or A^t . Suppose $x^g \notin Z(V) \cup Z(V^t)$. Then by (2) there is an element $h \in L_1$ such that $x^{gh} \in U_1$ and $X^{gh} \leq E$. If $X^{gh} \leq M_1$, then $X^g \leq M_1 \cap M_2 = M$ and then $X^g = M$. If $X^{gh} \not\leq M_1$, then $X^{gh} \leq A_2$ by (4F.2). Moreover, $M_1 < S^h \cap E$, so that the structure of L_1/M_1 forces $S^h = S$. Thus $gh \in N_{G_0}(M_2)$ by (4F.5), and therefore $X^{gh} = A_2$. We have proved that $\mathcal{E}^*(M_2)$ consists of A, A^t, M^g , and A_2^g as g ranges over $N_{G_0}(M_2)$. Now the elements of $A^\#$ are all G_0 -conjugate, while neither $M^\#$ nor $A_2^\#$ fuses completely as U_2 is strongly closed in $Z(P)$ with

respect to G_0 . Thus $M \not\sim A \not\sim A_2$ and therefore A is weakly closed in S with respect to G_0 . Consequently, elements of $A^\#$ are all conjugate in $N_{G_0}(A)$.

Now suppose $A^g \leq C_2$ for some $g \in G_0$. Then $A^g \leq L_2$ by (4Q). Take $h \in L_2$ so that $A^{gh} \leq S$. By the weak closure of A , we have $A^{gh} = A$ and then $A^g = A$. This and the above remark imply that for all $a \in A^\#$, A is weakly closed in $C_{G_0}(a)$ with respect to G_0 . Since G_0 is simple, we also have $G_0 = \langle A^{G_0} \rangle$. Thus we can appeal to a result of Timmesfeld [23] to conclude that G_0 is isomorphic to $PSL(4, q)$.

§ 5. Case II: $t^{N(Q)} = tZ(P)$

From now on we shall assume the following hypothesis:

(5.1) HYPOTHESIS: $t^{N(Q)} = tZ(P)$.

In this section we study the structure of $N(B_i)$, $1 \leq i \leq 2$.

(5.2) NOTATION. $D_i = O_2(N(V_i))$, $E_i = O_2(N(B_i))$, $1 \leq i \leq 2$.

(5A) For $i \in \{1, 2\}$, the following conditions hold:

- (1) $N(V_i) = D_i N_C(V_i)$, $D_i \cap N_C(V_i) = B_i \triangleleft N(V_i)$, and $D_i/B_i \cong U_i$ as $N_C(V_i)$ -modules.
- (2) $N(B_i) = E_i N(V_i)$, $E_i \cap N(V_i) = D_i \triangleleft N(B_i)$, and $E_i/D_i \cong A_i/U_i$ as $N_C(V_i)$ -modules.
- (3) $N(B_i) \leq N(U_i) \cap N(A_i)$.

PROOF. Suppose for definiteness that $i=1$. First, $|N(Q) : N_C(Q)| = q^2$ by (5.1). As $N(B_1) \cap N(Q)$ has index at most 2 in $N(Q)$ by (3.3), it follows that

$$|N(B_1) \cap N(Q) : N_C(B_1) \cap N_C(Q)| \geq q^2/2 > q.$$

Applying (3B) with $X = N(B_1) \cap N(Q)$, we conclude that $t^{N(B_1) \cap N(Q)} = tZ(P)$; that is $|N(B_1) \cap N(Q) : N_C(B_1) \cap N_C(Q)| = q^2$. As $N(B_1) \cap N(Q) \cap N(U_1)$ has index at most 2 in $N(B_1) \cap N(Q)$ by (3.2), and $N_C(B_1) \cap N_C(Q) \leq N(U_1)$ by (3E), analogous reasoning yields that $N(B_1) \cap N(Q) \leq N(U_1)$. Notice also that $N(V_1) \leq N(B_1) \cap N(U_1)$ as $t^g \cap V_1 = tU_1$ and $O_2(C(V_1)) = B_1$ (see (3A) and (3E)).

Now set $X = N(B_1) \cap N(U_1)$ and $Y = N(B_1) \cap N(Q)$. Let bars denote images in $N(U_1)/U_1$. Then $\bar{t} \in \bar{B}_1 \triangleleft \bar{X}$. Recall that $N_C(V_1) = N_C(B_1)$ by (3E). Hence if $N(V_1) \leq C$, then $N_Y(V_1) = N_C(B_1) \cap N_C(Q)$ has index q^2 in Y by the first paragraph, which implies that $|\bar{Y} : C_Y(\bar{t})| = q^2$. Since $\bar{t}^X \cap \bar{A}_1 = \emptyset$ by (3A), it follows that $\bar{t}^Y = \bar{t}\bar{A}_1$. But then, since $\bar{Q} \leq C(\bar{t})$ and $\bar{Y} \leq N(\bar{Q})$, $\bar{Q}$ acts trivially on $\bar{t}\bar{A}_1$, and consequently $[\bar{Q}, \bar{B}_1] = 1$. Since this is false, we must have $N(V_1) \not\leq C$. (1) now follows from (3D).

Consequently, $|N(Q) \cap N(V_1) : N_c(Q) \cap N_c(V_1)| = q$, which shows that $Y \not\leq N(V_1)$. Thus $C_X(\bar{t}) < \bar{X}$. Since $N_c(V_1)$ acts transitively on $(\bar{B}_1 / \langle \bar{t} \rangle)^\#$, (1A) yields that $\bar{t}^x = \bar{t} \bar{A}_1$ and that $\bar{X}$ acts 2-transitively on $\Omega = \bar{t}^x$. In particular, $|X : N(V_1)| = q^2$ and so $|X : N_c(B_1)| = q^3$. Since $t^g \cap A_1 = \emptyset$, it follows that $N(B_1) \leq N(U_1)$, $t^{N(B_1)} = tA_1$, and $A_1 \triangleleft N(B_1)$. So $\bar{A}_1 \cap Z(\bar{D}_1) \neq 1$ and, since $N_c(V_1)$ acts irreducibly on $\bar{A}_1$, it follows that $\bar{A}_1 \leq Z(\bar{D}_1)$. Therefore $\bar{D}_1 \leq C(\bar{B}_1) = C(\Omega)$. Definition of D_1 shows $\bar{D}_1 = O_2(C(\bar{B}_1)) \triangleleft N(\bar{B}_1)$. Furthermore, $C(\Omega) = \bar{D}_1 \bar{O}(C)$ by (2D.4). Thus arguing as in (4B.2), we obtain (2).

(5.3) NOTATION. $Q_i = QD_i$, $R_i = Q_i E_i$, $1 \leq i \leq 2$, $H = R_1 \cap R_2$.

(5B) For $i \in \{1, 2\}$, the following conditions hold:

- (1) $H = N_{R_i}(Q) = N_{R_i}(Q_i) = N_{R_i}(A_{3-i})$.
- (2) $\mathcal{C}^*(R_i/D_i) = \{E_i/D_i, H/D_i\}$.
- (3) $H/Q = Q_1/Q \times Q_2/Q$.

PROOF. By the last statement of (5A.1), $[D_i, Q] \leq B_i \leq Q$ so $Q_i = QD_i \leq N_{R_i}(Q)$. Hence $Q_2 \leq N(B_2) \cap N(Q) \leq N(B_1)$ by (3.3), and consequently Q_2 normalizes $R_1 = QE_1$. Since Q_2/Q is elementary and $N(R_1) \cap N(B_1)/R_1$ has cyclic Sylow 2-subgroups by (5A), we must have $|Q_2 : R_1 \cap Q_2| \leq 2$. Since J acts irreducibly on Q_2/Q , we conclude that $Q_2 \leq R_1$. By symmetry $Q_1 \leq R_2$ so $\langle Q_1, Q_2 \rangle \leq H$. Further, $Q_1 \cap Q_2 \leq N(V_1) \cap N(V_2) \leq C$ as $V_1 \cap V_2 = \langle t \rangle$. Hence $Q_1 \cap Q_2 = Q$. Consequently, $|H : Q| \geq q^2$. Next, $B_1 \cap B_2 = Z(Q)$ so $H \leq N(Q)$ by (3A.5). Hence $H \leq N_{R_i}(Q) \leq N_{R_i}(Q_i)$ as $Q_i = QD_i$. Now (5A.2) in particular implies that $|N_{R_i}(Q_i) : Q_i| = q$, whence $|N_{R_i}(Q_i) : Q| = q^2$. Thus $H = N_{R_i}(Q) = N_{R_i}(Q_i)$ and $H/Q = Q_1/Q \times Q_2/Q$. Now $A_2 \triangleleft H$ by (5A.3). On the other hand, $N_{R_1}(A_2) \leq N_{R_1}(Q) = H$ as $Q = B_1 A_2$. Thus $N_{R_1}(A_2) = H$. Finally, the proof of (2) is similar to that of (4B.4) and is omitted.

(5.4) NOTATION. $H_i = H \cap E_i$, $1 \leq i \leq 2$, $E = E_1 \cap E_2$.

Diagram 1 shows the configuration of subgroups defined in (5.2), (5.3), and (5.4).

(5C) The following conditions hold:

- (1) $H^2 = Z(H) = Z(P)$.
- (2) $E_i^2 = A_i$, $1 \leq i \leq 2$.
- (3) $D_i^2 = U_i$, $Z(D_i) = A_i$, $1 \leq i \leq 2$.

PROOF. By (5B.1) and (5.4), $H/D_i = H_i/D_i \times Q_i/D_i$ so $H^2 \leq H_i \cap Q_i = D_i$, $1 \leq i \leq 2$. As $D_1 \cap D_2 = D_1 \cap D_2 \cap Q = B_1 \cap B_2 = Z(Q)$ by (5B.3), we have $H^2 \leq Z(Q)$. Also, $Z(P) = P^2 \leq H^2$, so $H^2 = Z(P)$ or $Z(Q)$. Next, (5B.2) in particular implies that $H \triangleleft R_i$, so $H^2 \triangleleft R_i$. Since $N_{R_i}(Z(Q)) = N_{R_i}(Q) = H$ by (3A.5) and

(5B.1), we conclude that $H^2 = Z(P)$. Now $Z(H) \leq C_H(t) = Q$ so $Z(H) < Z(Q)$. On the other hand, since $U_i \triangleleft H$ by (5A.3), $U_i \cap Z(H) \neq 1$. Since J acts irreducibly on U_i , it follows that $Z(P) = U_1 U_2 \leq Z(H)$. Thus $Z(H) = Z(P)$.

The $N_c(V_i)$ -isomorphism $E_i/D_i \rightarrow A_i/U_i$ maps H_i/D_i onto $Z(P)/U_i$, so E_i/D_i is a union of conjugates of H_i/D_i . Since $H_i^2 \leq A_i \triangleright N(B_i)$, it follows that $E_i^2 \leq A_i$. In particular, $[E_i, t] \leq E_i^2 \leq A_i$. As $C_{E_i}(t) = B_i$, we also have $|[E_i, t]| \geq q^3$. Thus $E_i^2 = A_i$.

Finally, the proof of (3) is similar to that of (4C.2) and is omitted.

(5D) *E has a J-invariant subgroup F of order q^4 such that $E = FZ(Q)$ and $F \cap Z(Q) = Z(P)$. Furthermore the following conditions hold:*

- (1) $F = C_F(J_1) \times C_F(J_2)$.
- (2) $C_F(J_i)$ has order q^2 and is elementary or homocyclic of exponent 4, $1 \leq i \leq 2$.
- (3) $C_F(J_i) \leq \langle \hat{J}_i^{C(J_i)} \rangle$, $1 \leq i \leq 2$.

Let $F_i = F \cap D_i$, $1 \leq i \leq 2$. Then

- (4) $F_i = C_F(J_i) \times U_{3-i}$.

PROOF. That F exists is proved as in (4D). The $N_c(V_i)$ -isomorphism $E_i/D_i \rightarrow A_i/U_i$ maps H_i/D_i onto $Z(P)/U_i$ so J_i has no nontrivial fixed points on H_i/D_i . Hence it follows that $C_F(J_i) = C_{F_i}(J_i)$. Now $D_i = F_i B_i$ and $F_i \cap B_i = Z(P)$. Hence (2), (3), and (4) follow exactly as in (4D). Next, as $F_1 \cap F_2 = Z(P)$, we have $F = F_1 F_2 = C_F(J_1) C_F(J_2)$. Furthermore, $C_F(J_1) \cap C_F(J_2) = C_{Z(P)}(J) = 1$. Thus (1) follows from (1I).

(5.5) NOTATION. $K_i = A_i F_i$, $1 \leq i \leq 2$, $K = K_1 K_2$.

Thus for $i \in \{1, 2\}$, K_i is a subgroup of D_i such that $D_i = K_i B_i$ and $K_i \cap B_i = A_i$. Also, K is a subgroup of H such that $H = KQ$ and $K \cap Q = P$. Notice also that $K = FP$ and $F \cap P = Z(P)$.

(5E) *For $i \in \{1, 2\}$, the following conditions hold:*

- (1) $R_i^2 \leq FK_i$.
- (2) $K_i = F_i F_i^{r_i}$ and K_i is the unique abelian maximal subgroup of D_i .
- (3) $F \cap F^{r_i} = C_F(J_i)$.
- (4) FF^{r_i} is a $N_L(A_i)$ -invariant subgroup of E_i such that $E_i = FF^{r_i} D_i$ and $FF^{r_i} \cap D_i = K_i$.

PROOF. Suppose for definiteness that $i=1$, and set $r=r_1$ and $s=r_2$. Now H^2 and $E_1^2 \leq FK_1$ by (5C), so $FK_1 \triangleleft R_1 = HE_1$. Let bars denote images in R_1/FK_1 . J_1^s acts irreducibly on $A_1/Z(P)$ and centralizes $A_2/Z(P)$, so J_1^s acts irreducibly on E_1/H_1 and centralizes K/FK_1 by (5A.2). Hence if $k \in K - FK_1$, then $C_{\bar{E}}(\bar{k}) = \bar{H}_1$ or $\bar{E}_1$. However, $|\bar{R}_1 : C_{\bar{R}_1}(\bar{k})| \leq 2$ as $\bar{R}_1' \leq \bar{H}_1$. Thus

$C_{\bar{E}_1}(\bar{k}) = \bar{E}_1$ for $\bar{k} \in \bar{K}$, which implies that $\bar{R}_1 = \bar{E}_1 \times \bar{K}$. Therefore $R_1^2 \leq FK_1$.

Now arguing as in the first part of (4E), we obtain (2) and, moreover, $F_1 \cap F_1^r = C_F(J_1)$. In particular, $K_1 \leq FF^r$ so $FF^r = FK_1(FK_1)^r$. Since $E_1^2 \leq K_1$, it follows that FF^r is a subgroup of E_1 . Next, (5A.2) in particular implies that $H_1 \cap H_1^r = D_1$. Since $t \notin FK_1$, this shows $FK_1 \cap (FK_1)^r = K_1$. Thus FF^r is a maximal subgroup of E_1 . Moreover, $F \cap F^r = F \cap F^r \cap K_1 = F_1 \cap F_1^r = C_F(J_1)$. As $H_1 = FK_1 D_1$ and $E_1 = H_1 H_1^r$, we also have $E_1 = FF^r D_1$ and $FF^r \cap D_1 = K_1$. Finally, (1) in particular implies that P normalizes FF^r . Thus $N_L(A_1) = \langle P, J, r \rangle$ normalizes FF^r . The proof is complete.

(5F) For $i \in \{1, 2\}$, the following conditions hold:

- (1) If $[A_i, F_{3-i}] = 1$, then $Z(E_i) = A_i$ and $Z(R_i) = Z(P)$.
- (2) If $[A_i, F_{3-i}] \neq 1$ then $Z(E_i) = Z(R_i) = U_i$. Furthermore, $C_{A_i}(x) = Z(P)$ for any $x \in F_{3-i} - Z(P)$.
- (3) FK_i/U_i is abelian.

PROOF. For definiteness, let $i=1$. First, assume that $[A_1, F_2] = 1$. Then, since $[A_1, F_1] = 1 = [F_1, F_2]$ by (5D) and (5E), $FK_1 = A_1 F_1 F_2$ is abelian, so A_1 centralizes $E_1 = FF^r D_1$. Since $Z(E_1) \leq C_{E_1}(t) = B_1$, it follows that $Z(E_1) = A_1$. Furthermore, $Z(P)$ centralizes $R_1 = E_1 P$. Since $Z(R_1) \leq C_{R_1}(t) = Q$, we also have $Z(R_1) = Z(P)$.

Next, assume that $[A_1, F_2] \neq 1$ and let $x \in F_2 - Z(P)$. Then $Z(P) \leq C_{A_1}(x) \leq A_1$ and $C_{A_1}(x)$ is J_2 -invariant as J_2 centralizes $F_2/Z(P)$ by (5D). Since J_2 acts irreducibly on $A_1/Z(P)$, it follows that $C_{A_1}(x) = Z(P)$ or A_1 . Suppose that $C_{A_1}(x) = A_1$. Then $Z(P) < C_{F_2}(A_1) \leq F_2$. Since J acts irreducibly on $F_2/Z(P)$, we must have $C_{F_2}(A_1) = F_2$, contrary to our assumption. Thus $C_{A_1}(x) = Z(P)$. Hence commutation by x induces a J_2 -isomorphism $A_1/Z(P) \rightarrow [A_1, x]$, so $[A_1, x]$ is a subgroup of $Z(P)$ of order q on which J_2 acts regularly. The only possibility is $[A_1, x] = U_1$ (see (2A)). Thus $[A_1, F_2] = U_1$. As before $Z(E_1) \leq B_1$, so $Z(E_1) \leq A_1$ as $t^{E_1} = tA_1$. So $Z(E_1) \leq Z(P)$ by the above. Also, since $U_1 \triangleleft E_1$ by (5A.3), $Z(E_1) \cap U_1 \neq 1$. Thus $Z(E_1) = U_1$ by (2C.5). As before $Z(R_1) \leq Z(Q) \leq E_1$, so $Z(R_1) \leq Z(E_1) = U_1$. Since $R_1 = PE_1$, we conclude that $Z(R_1) = U_1$.

Now we have shown that $[A_1, F_2] \leq U_1$ in any case. Thus $FK_1/U_1 = A_1 F_1 F_2 / U_1$ is abelian.

(5G) Either $[A_1, F_2] = 1$ or $[A_2, F_1] = 1$.

PROOF. Suppose that $[A_1, F_2] \neq 1 \neq [A_2, F_1]$. We shall eventually derive a contradiction. We need some notation:

$$\begin{aligned} X_1(a) &= (a000), & X_2(a) &= (0a00), \\ Z_1(a) &= (000a), & Z_2(a) &= (00a0). \end{aligned}$$

Now commutation by t induces an isomorphism $F_i/Z(P) \rightarrow U_i$ so we may choose $Y_i(a) \in F_i$ such that

$$(1) \quad [Y_i(a), t] = Z_i(a).$$

Then $Y_i(a)$ is unique modulo $Z(P)$ and consequently

$$(2) \quad Y_i(a)Y_i(b) \equiv Y_i(a+b) \pmod{Z(P)}.$$

Now conjugating (1) with $i=1$ by $h(x, y)$, we have

$$[Y_1(a)^{h(x,y)}, t] = Z_1(x^2a).$$

Therefore

$$(3) \quad Y_1(a)^{h(x,y)} \equiv Y_1(x^2a) \pmod{Z(P)}.$$

Notice that $[F_1, A_2] \leq U_2$ by (5F.3). Define the map $F: GF(q) \times GF(q) \rightarrow GF(q)$ by

$$[Y_1(a), X_2(b)] = Z_2(F(a, b)).$$

Conjugating this by $h(x, y)$ and using (3), we have

$$[Y_1(x^2a), X_2(y^2b)] = Z_2(xyF(a, b)).$$

This implies that $F(x^2a, y^2b) = xyF(a, b)$. Thus setting $f = F(1, 1)$ and changing notation, we have $F(a^2, b^2) = fab$, if $ab \neq 0$. Since $ab = 0$ implies $F(a, b) = 0$, this holds for all (a, b) . Thus we conclude that

$$(4) \quad [Y_1(a), X_2(b)] = Z_2(f(ab)^{1/2})$$

for some nonzero element f of $GF(q)$. Analogous computation starting with (1) with $i=2$ shows that

$$(5) \quad [Y_2(a), X_1(b)] = Z_1(gab)$$

for some nonzero element g of $GF(q)$. Also, matrix multiplication shows

$$(6) \quad [X_2(b), X_1(a)] = Z_1(ba^2)Z_2(ba).$$

Now set

$$[abcd] = X_1(a)X_2(b)Y_1(c)Y_2(d).$$

Then using commutator relations (4), (5) and (6) together with the fact that F and K_i are abelian, we obtain

$$\begin{aligned}
(7) \quad & [abcd][\tilde{a}\tilde{b}\tilde{c}\tilde{d}] \\
& = X_1(a+\tilde{a})X_2(b+\tilde{b})Y_1(c)\tilde{Y}_1(\tilde{c})Y_2(d)\tilde{Y}_2(\tilde{d}) \\
& \quad \times Z_1(b\tilde{a}^2+gd\tilde{a})Z_2(b\tilde{a}+f(c\tilde{b})^{1/2}).
\end{aligned}$$

Using (7), we can determine the centralizer of $[abcd]$ in K . The result is

$$C_K([abcd]) = \left\{ [\tilde{a}\tilde{b}\tilde{c}\tilde{d}]z \left| \begin{array}{l} z \in Z(P), \\ b\tilde{a}^2+gd\tilde{a}=\tilde{b}a^2+g\tilde{d}a, \\ b^2\tilde{a}^2+f^2c\tilde{b}=\tilde{b}^2a^2+f^2\tilde{c}b \end{array} \right. \right\}.$$

For simplicity we shall write this as follows:

$$C_K([abcd]) = \left\{ \begin{array}{l} b\tilde{a}^2+gd\tilde{a}=\tilde{b}a^2+g\tilde{d}a, \\ b^2\tilde{a}^2+f^2c\tilde{b}=\tilde{b}^2a^2+f^2\tilde{c}b \end{array} \right\}.$$

From this we obtain

$$(8) \quad C_K([00cd]) = \left\{ \begin{array}{l} \tilde{a}=0 \\ \tilde{b}=0 \end{array} \right\}, \quad cd \neq 0,$$

$$(9) \quad C_K([a0cd]) = \left\{ \begin{array}{l} d\tilde{a}=\tilde{d}a \\ \tilde{b}=0 \end{array} \right\} \cup \left\{ \begin{array}{l} gd\tilde{a}=\tilde{b}a^2+g\tilde{d}a \\ f^2c=\tilde{b}a \end{array} \right\},$$

$$(10) \quad C_K([0bcd]) = \left\{ \begin{array}{l} \tilde{a}=0 \\ c\tilde{b}=\tilde{c}b \end{array} \right\} \cup \left\{ \begin{array}{l} b\tilde{a}+gd=0 \\ b^2\tilde{a}^2+f^2c\tilde{b}=f^2\tilde{c}b \end{array} \right\}.$$

For each $j \in GF(q)$, we define

$$\begin{aligned}
A_{1j} &= \left\{ [a0c(ja)]z \left| \begin{array}{l} a, c \in GF(q), \\ z \in Z(P) \end{array} \right. \right\}, \\
A_{2j} &= \left\{ [0b(jb)d]z \left| \begin{array}{l} b, d \in GF(q), \\ z \in Z(P) \end{array} \right. \right\}.
\end{aligned}$$

Thus $A_{i0} = K_i$, $1 \leq i \leq 2$. We show

$$(11) \quad \mathcal{A}(K) = \{F\} \cup \{A_{ij} \mid 1 \leq i \leq 2, j \in GF(q)\}.$$

First, (2) and (7) show that A_{ij} is, in fact, a subgroup of K . Furthermore, A_{ij} has order q^4 and is abelian by (9) and (10). Let $F \neq A \in \mathcal{A}(K)$. If $A \leq FK_1$, then A contains an element $x = [a0cd]z$ with $a \neq 0$ and $z \in Z(P)$. Setting $j = d/a$, we obtain from (9) that $A \leq C_K(x) = A_{1j}$. Thus $A = A_{1j}$. Similarly if $A \leq FK_2$, then $A = A_{2j}$ for some $j \in GF(q)$. Suppose that $A \not\leq FK_i$, $1 \leq i \leq 2$. Since $A \neq F$, (8) shows $A \cap F = Z(P)$. So $(A \cap FK_1) \cap (A \cap FK_2) = Z(P)$ and then $|A \cap FK_1 : Z(P)| \leq |K : FK_2| = q$. Also, $|A : A \cap FK_1|$

$\leq |K:FK_1|=q$. Since $|A| \geq q^4$, it follows that $|A \cap FK_i|=q^3$, $1 \leq i \leq 2$. Let $x=[a0cd]z \in A \cap FK_1 - Z(P)$ and let $\tilde{x}=[0\tilde{b}\tilde{c}\tilde{d}]\tilde{z} \in A \cap FK_2 - Z(P)$, where $z, \tilde{z} \in Z(P)$. Then $a\tilde{b} \neq 0$, and (9) shows $f^2c = \tilde{b}a$. This implies that $\tilde{b}$ is uniquely determined. But then (7) shows that the product of any two elements of $A \cap FK_2 - Z(P)$ is contained in $A \cap F = Z(P)$. This a contradiction because $|A \cap FK_2|=q^3$. Thus we have proved (11).

We can now complete the proof of (5G). From (7) we obtain

$$(12) \quad \begin{aligned} [a0c(ja)]^2 &= Y_1(c)^2 Y_2(ja)^2 Z_1(gja^2), \\ [0b(jb)d]^2 &= Y_1(jb)^2 Y_2(d)^2 Z_2(fj^{1/2}b). \end{aligned}$$

Assume that $F_i^2=1$, $1 \leq i \leq 2$. Then $F^2=K_1^2=K_2^2=1$ and $A_{ij}^2=U_i$ for $j \neq 0$ by (12), so that $N(K)$ normalizes $\{U_1, U_2\}$ by (11). But since R_1 normalizes K by (5E.1), and $Z(R_1)=U_1$, this yields a contradiction. Assume that $F_1^2 \neq 1 \neq F_2^2$. Then (5D.2) and (12) show that $K_i^2=U_i$, $1 \leq i \leq 2$, and $A^2=Z(P)$ for any other member A of $\mathcal{A}(K)$. This again yields a contradiction. So assume $F_1^2=1 \neq F_2^2$ say. Then $K_1^2=1$, $F^2=A_{2j}^2=U_2$, and $[a0c(ja)]^2=Y_2(ja)^2 Z_1(gja^2)$ by (12). Let $Z_1(x_1)Z_2(x_2) \in Z(P) - U_1 \cup U_2$, so that $x_1x_2 \neq 0$. There is an element $x \in GF(q)$ such that $Y_2(x)^2=Z_2(x_2)$. Set $a=x_1/gx$ and $j=gx^2/x_1$. Then

$$[a00(ja)]^2=Y_2(x)^2 Z_1(x_1)=Z_2(x_2)Z_1(x_1).$$

Thus elements of $Z(P) - U_1^*$ have a square root in some member of $\mathcal{A}(K)$ while elements of U_1^* have none. This forces $U_1 \triangleleft N(K)$, which again yields a contradiction. The proof of (5G) is complete.

§ 6. The case $[A_i, F_{3-i}] \neq 1$, $i=1$ or 2

We shall retain the notation of Section 5. In (5G) we have shown that either $[A_1, F_2]=1$ or $[A_2, F_1]=1$. In this and the next two sections we shall consider the case where $[A_i, F_{3-i}] \neq 1$ for some $i \in \{1, 2\}$. By symmetry we may assume

$$(6.1) \quad \text{HYPOTHESIS. } [A_1, F_2] \neq 1 = [A_2, F_1].$$

(6A) *The following conditions hold:*

- (1) $Z(E_1)=Z(R_1)=U_1$.
- (2) $Z(E_2)=A_2$, $Z(R_2)=Z(P)$.
- (3) $C_{A_1}(x)=Z(P)$ for all $x \in F_2 - Z(P)$.
- (4) FK_2 is abelian.
- (5) $N(Q) \leq N(B_i) \cap N(H) \cap N(U_i)$, $1 \leq i \leq 2$.

PROOF. Since we are assuming (6.1), statements (1)–(3) are the restatements of (5F). Also, (4) is a consequence of (5D) and (5E). Now $N(Q)$

normalizes $\{B_1, B_2\}$ by (3.3). Since $E_i = O_2(N(B_i))$, and $E_1 \not\sim E_2$ by (1) and (2), it follows that $N(Q) \leq N(B_i)$, $1 \leq i \leq 2$. Furthermore, $N(Q)$ normalizes $R_i = QE_i$, so $N(Q)$ normalizes $H = R_1 \cap R_2$. That $N(Q)$ normalizes U_1 follows from (1).

(6.2) NOTATION. $\hat{N} = N(H) \cap N(U_1)$.

Diagram 2 will help the reader understand the involved configuration which we shall consider in (6B) and (6C) below.

(6B) *The following conditions hold:*

- (1) $C_F(J_1)$ is homocyclic of exponent 4.
- (2) $C_F(J_2)$ is elementary abelian.
- (3) *The following groups are normal in $\hat{N} : K, F, F_2, FK_1, FK_2, K_2$.*
- (4) $\hat{N} = N_{\hat{N}}(H_i)R_{3-i}$ and $N_{R_{3-i}}(H_i) = H$, $1 \leq i \leq 2$.
- (5) $N(R_i) \cap N(B_i)$ is a subgroup of $N_{\hat{N}}(H_i)$ of index q , $1 \leq i \leq 2$.

PROOF. First, notice that $R_i \leq \hat{N}$, $1 \leq i \leq 2$, by (5B.2) and (6A). Let bars denote images in $N(U_1)/U_1$. Then $\bar{F}\bar{K}_i$ is abelian, $1 \leq i \leq 2$, by (5F.3) and (6A.4). Now $\bar{P} = \bar{A}_1\bar{A}_2$ is of type $SL(3, q)$, so $\bar{K} = \bar{F}*\bar{P}$ and $Z(\bar{K}) = \bar{F}$. In addition, $\bar{F}\bar{K}_i \in \mathcal{A}(\bar{K})$, $1 \leq i \leq 2$. Let $\bar{A}$ be an abelian subgroup of $\bar{H}$ with $|\bar{A}| \geq q^4$ and suppose that $\bar{A} \not\leq \bar{K}$. Let $\bar{B}$ be a maximal abelian subgroup of $\bar{K}$ containing $\bar{A} \cap \bar{K}$. Then $\bar{F} \leq \bar{B}$ and $|\bar{B} : \bar{F}| = q$ by the above, whence $|\bar{A} \cap \bar{F}| > q^2$. However, $\bar{A} \cap \bar{F} \leq Z(\bar{H}) \cap \bar{F} \leq \bar{F}_1$ as $\bar{H} = \bar{A}\bar{K}$ and $N_F(V_1) = F \cap Q_1 = F_1$ by (5A). Hence we must have $\bar{A} \leq \bar{K}$. This implies that $J(\bar{H}) = \bar{K}$. Thus $K \triangleleft \hat{N}$ and, since $\bar{F} = Z(\bar{K})$, $F \triangleleft \hat{N}$.

Now suppose that $x \in R_1 - N(FK_2)$. Then $\bar{F}\bar{K}_2^x$ is a member of $\mathcal{A}(\bar{K})$ different from $\bar{F}\bar{K}_2$, so $K = FK_2(FK_2)^x$ and $F = FK_2 \cap FK_2^x$ by the structure of $\bar{K}$. But then, since FK_2 is abelian, $F \leq Z(K)$ against (6.1). Therefore $FK_2 \triangleleft R_1$. Next, suppose that $C_F(J_2)$ is not elementary, so that $C_F(J_2)$ is homocyclic of exponent 4 by (5D). If $C_F(J_1)$ is elementary, then $U_2 = (FK_2)^2$ is normal in R_1 by the above, against (6A.1), so $C_F(J_1)$ is homocyclic of exponent 4 by (5D). But then $A_2 = \Omega_1(FK_2)$ is normal in R_1 . Since this is false by (5B.1), $C_F(J_2)$ must be elementary abelian. Consequently, $\bar{F} = \overline{C_F(J_1)C_F(J_2)}$ is elementary, and then $\mathcal{E}^*(\bar{K}) = \{\bar{F}\bar{K}_1, \bar{F}\bar{K}_2\}$. Since $FK_1 \not\sim FK_2$, this shows that $FK_i \triangleleft \hat{N}$, $1 \leq i \leq 2$.

Next, we prove (4). Consider first the case $i=1$. Since $N_{H_1}(V_1) = D_1$ by (5A), (1C) shows that $\mathcal{E}^*(\bar{H}_1) = \{\bar{D}_1, \bar{F}\bar{K}_1\}$, whence $N(\bar{H}_1) \leq N(\bar{D}_1)$. Also, $N(D_1) \leq N(A_1)$ since $Z(D_1) = A_1$. Thus

$$(6) \quad N_{\hat{N}}(H_1) \leq N_{\hat{N}}(D_1) \leq N_{\hat{N}}(A_1).$$

Thus $N_{R_2}(H_1) = H$ by (5B.1) and, in particular, $N_{\hat{N}}(H_1) < \hat{N}$. Also, J , which

is contained in $N_{\hat{N}}(H_1)$, acts transitively on $(H/H_1)^*$. Since FK_1 and $K \triangleleft \hat{N}$, (1A) applied to $\hat{N}/FK_1$ and its elementary abelian normal subgroup H/FK_1 now yields that $|\hat{N} : N_{\hat{N}}(H_1)| = q$. Thus $\hat{N} = N_{\hat{N}}(H_1)R_2$, and (4) holds for $i=1$. It follows easily from (6A.2) that $Z(H_2) = A_2$. So $N_{\hat{N}}(H_2) \leq N_{\hat{N}}(A_2)$, and arguing as above, we see that (4) holds for $i=2$ as well.

Now $N(B_i) = R_i N_C(B_i)$ and $R_i \cap C = Q$ by (5A). Moreover, $N(Q) \leq N(B_i)$ by (6A.5). Hence $N(R_i) \cap N(B_i) = R_i N_C(Q)$, and then $N(R_i) \cap N(B_i) \leq N_{\hat{N}}(H_i)$ by (6A.5). Furthermore, $|N(R_i) \cap N(B_i)| = |N(Q)|q$ since $N_{R_i}(Q) = H$ by (5B). Also, (4) in particular implies that $|\hat{N} : N_{\hat{N}}(H_i)| = q$. Thus

$$|N_{\hat{N}}(H_1) : N(R_1) \cap N(B_1)| = |N_{\hat{N}}(H_2) : N(R_2) \cap N(B_2)|.$$

We must show that this index is equal to q . First, we have

$$(7) \quad N_{\hat{N}}(B_i) = N(R_i) \cap N(B_i)$$

since $R_i = HE_i$. Recalling (6) above, we apply (1A) to $N_{\hat{N}}(H_1)/A_1$ and its elementary abelian normal subgroup D_1/A_1 . Then, since $N_{\hat{N}}(B_1) = N(R_1) \cap N(B_1)$ and $K_1 = D_1 \cap K \triangleleft N_{\hat{N}}(H_1)$, we have $|N_{\hat{N}}(H_1) : N(R_1) \cap N(B_1)| = 1$ or q . Suppose that $N_{\hat{N}}(H_1) = N(R_1) \cap N(B_1)$. Then, in particular, $\hat{C} = C_{\hat{N}}(H/FK_1) \leq N(R_1) \cap N(B_1) = R_1 N_C(Q)$. Also, $R_1 \leq \hat{C}$ by (5E.1), so $\hat{C} = R_1 C_C(P/A_1)$ since $H = PFK_1$ and $P \cap FK_1 = A_1$. Since $C_C(P/A_1) = QJ_1^{r_2}O(C)$ by (2D.4), this shows that R_1 is the unique Sylow 2-subgroup of $\hat{C}$. Thus $R_1 \triangleleft \hat{N}$ and, by symmetry, $R_2 \triangleleft \hat{N}$. If F_1 is elementary, then (1C) and (6A.3) imply that $\mathcal{E}^*(FK_1) = \{F, K_1\}$, and so $K_1 \triangleleft \hat{N}$. But then we have $H_1 \triangleleft \hat{N}$, as (5B.2) and (5C) show that $H_1 = Z(R_1 \text{ mod } K_1)$. Since this contradicts (4), F_1 is not elementary. But then $K_2 = \mathcal{Q}_1(FK_2) \triangleleft \hat{N}$, and this yields the same contradiction. Thus we have proved (5).

Now (5), (6), and (7) show that $N_{\hat{N}}(B_1) < N_{\hat{N}}(D_1)$. If $C_F(J_1)$ is elementary, then so also is $K_1 = A_1 C_F(J_1)$, and then $\mathcal{E}^*(D_1) = \{B_1, K_1\}$ as $C_{D_1}(t) = B_1$. But this forces $N(D_1) = N(B_1)$, a contradiction. Thus $C_F(J_1)$ is homocyclic of exponent 4 by (5D). Consequently, $K_2 = \mathcal{Q}_1(FK_2) \triangleleft \hat{N}$ and $F_2 = F \cap K_2 \triangleleft \hat{N}$. Thus all parts of the lemma hold.

(6.3) NOTATION. $N_1 = O_2(N(D_1))$, $N_2 = O_2(N(E_2) \cap N(K_2))$, $N_0 = N_1 \cap N_2$, $T_1 = R_1 N_1$, $T_0 = R_2 N_2$, $T_2 = N_{T_0}(R_2)$, $T_3 = T_1 \cap T_2$.

(6C) The following conditions hold:

- (1) $N(D_1) = N_1 N(B_1)$, $N_1 \cap N(B_1) = E_1 \triangleleft N(D_1)$, and $N_1/E_1 \cong U_1$ as $N_C(V_1)$ -modules.
- (2) $N(D_2) = N(B_2)$. $N(E_2) \cap N(K_2) = N_2 N(B_2)$, $N_2 \cap N(B_2) = E_2$, and $N_2/E_2 \cong A_2/U_2$ as $N_C(V_2)$ -modules. Furthermore, $FF^{r_2} = Z(N_2 \text{ mod } K_2) = N_2^3 K_2$.

- (3) $N_{\hat{N}}(H_i) = T_i(N(R_i) \cap N(B_i)), T_i \cap N(R_i) \cap N(B_i) = R_i, 1 \leq i \leq 2.$
- (4) $T_3 = N_{T_i}(H_{3-i}), 1 \leq i \leq 2.$
- (5) $T_i \triangleleft \hat{N}, 1 \leq i \leq 3.$
- (6) $T_3^2 \leq F.$

PROOF. Let bars denote images in $N(A_1)/A_1$, and notice that $N(D_1) \leq N(A_1)$ by (5C). As remarked in the last paragraph of the proof of (6B), $N(B_1) < N(D_1)$. This implies that $C(\bar{t}) < N(\bar{D}_1)$. In addition, $N(\bar{D}_1) \leq N(\bar{K}_1)$ by (5E.2). Using arguments parallel to those used to prove (3D), we can now prove (1). We leave the detail to the reader. As a consequence we have $R_1 \triangleleft T_1$, so $T_1 \leq N_{\hat{N}}(H_1)$ by (5B.2) and (6A.1). Comparing orders and using (6B.5), we see that (3) holds for $i=1$.

Now $K_2 = A_2 C_F(J_2)$ is elementary by (5C) and (6B.2). Also, $C_{D_2}(t) = B_2$, so $\mathcal{E}^*(D_2) = \{B_2, K_2\}$ by (1C). Thus $N(D_2) = N(B_2) \leq N(E_2) \cap N(K_2)$. In particular, $N_{\hat{N}}(D_2) = N_{\hat{N}}(B_2) = N(R_2) \cap N(B_2)$, so $R_2 \leq C_{\hat{N}}(H_2/K_2) \leq N(R_2) \cap N(B_2) = R_2 N_C(Q)$. Thus $C_{\hat{N}}(H_2/K_2) \leq R_2 C_C(H_2/K_2)$. Since the $N_C(V_2)$ -isomorphism $E_2/D_2 \rightarrow A_2/U_2$ maps H_2/D_2 onto $Z(P)/U_2$, $C_C(H_2/K_2) = QJ_1O(C)$ by (2D.4). Hence it follows that R_2 is the unique Sylow 2-subgroup of $C_{\hat{N}}(H_2/K_2)$. Since $K_2 \triangleleft \hat{N}$, this forces $R_2 \triangleleft N_{\hat{N}}(H_2)$. Thus

$$N_{\hat{N}}(H_2) \leq N_{N(E_2) \cap N(K_2)}(R_2),$$

since (5B.2) and (5C) imply that $\mathcal{E}^*(R_2/K_2) = \{E_2/K_2, H/K_2\}$. Since $N(R_2) \cap N(B_2)$ contains a Sylow 2-subgroup of $N(D_2) = N(B_2)$, (6B.5) now shows that $|N(E_2) \cap N(K_2) : N(D_2)|$ is divisible by q . Hence if bars denote images in $N(K_2)/K_2$, $C(\bar{t})$ has even index in $N(\bar{E}_2)$. Using arguments parallel to those used to prove (4B.2), we can now prove that $N(\bar{E}_2) = \bar{N}_2 N(\bar{B}_2)$, $\bar{N}_2 \cap N(\bar{B}_2) = \bar{E}_2$, and $\bar{N}_2^2 \leq \bar{E}_2$. Then $[\bar{N}_2, \bar{t}] < \bar{E}_2$, while $[\bar{N}_2, \bar{t}]$ contains q^2 elements $\bar{t}\bar{t}^x$ as $\bar{x}$ runs through $\bar{N}_2$. Thus $[\bar{N}_2, \bar{t}]$ is a maximal subgroup of $\bar{E}_2$, and $\bar{E}_2 = \langle \bar{t} \rangle \times [\bar{N}_2, \bar{t}]$. Since $N_L(A_2)$ acts irreducibly on $\bar{F}\bar{F}^{r_2}$ by (5E.4), we must have $\bar{F}\bar{F}^{r_2} = [\bar{N}_2, \bar{t}]$. In particular, $\bar{F}\bar{F}^{r_2} \triangleleft \bar{N}_2$, so $\bar{F}\bar{F}^{r_2} \cap Z(\bar{N}_2) \neq 1$ and then $\bar{F}\bar{F}^{r_2} = Z(\bar{N}_2)$. Now arguing as in (4B.2), we obtain the second assertion of (2). That $\bar{N}_2^2 = Z(\bar{N}_2)$ follows from the fact that $N_L(A_2)$ acts transitively on $(N_2/E_2)^\#$.

Now (2) in particular implies that

$$N_{N(E_2) \cap N(K_2)}(R_2) = T_2(N(R_2) \cap N(B_2))$$

and that $|T_2 : R_2| = q$. Thus comparing orders, we have that (3) holds for $i=2$ as well. Consequently, $T_2 \triangleleft N_{\hat{N}}(H_2)$ and $N_{\hat{N}}(H_2)/T_2$ has a cyclic Sylow 2-subgroup. Let $X = N_{T_1}(H_2)$. Then $T_1 = R_1 X$ and $R_1 \cap X = H$ by (6B.4), so J acts irreducibly on X/H . Hence $|X : X \cap T_2| \leq 2$ and then $X \leq T_2$ by the above remark. Thus $X = T_3$, and also $T_3 = N_{T_2}(H_1)$ by (6B.4).

Since $K \triangleleft T_1$ and J acts irreducibly on K/FK_1 , T_1 centralizes H/FK_1 . Hence as before T_1 is the unique Sylow 2-subgroup of $C_{\hat{N}}(H/FK_1)$, whence $T_1 \triangleleft \hat{N}$. Likewise $T_2 \triangleleft \hat{N}$, and so $T_3 = T_1 \cap T_2 \triangleleft \hat{N}$.

It remains to prove (6). First, $T_1/E_1 = R_1/E_1 \times N_1/E_1$ by (1), whence $T_1^2 \leq E_1$. Likewise $T_2^2 \leq E_2$, so $T_3^2 \leq E_1 \cap E_2 = E$. Suppose that $T_3^2 F = E$. Then, since T_3 and $F \triangleleft \hat{N}$, $Z(Q) = B_1 \cap E \triangleleft E_1$ and then $t^{E_1} \leq tZ(P)$ by (3A). However, $|t^{E_1}| = q^3$ since $C_{E_1}(t) = B_1$. Therefore, $T_3^2 \leq F$ as asserted.

(6.4) NOTATION. $\hat{T} = T_1 T_2$.

(6D) The following conditions hold:

- (1) $\hat{T}$ is a normal subgroup of $\hat{N}$.
- (2) $\hat{T} = R_i T_{3-i}$, $R_i \cap T_{3-i} = H$, $1 \leq i \leq 2$.
- (3) $T_0 \cap \hat{T} = T_2$.
- (4) $E_i \cap N_{3-i} = E$, $1 \leq i \leq 2$.

PROOF. The first assertion follows from (6C.5). Also, (2) follows easily from (6B.4) and (6C.4). Now $\hat{T} \cap T_0 \leq N(R_2) \cap N(K_2)$ as $R_2 = HE_2$. As remarked before $Z(R_2 \text{ mod } K_2) = H_2$. Thus by (6C.4), $T_0 \cap \hat{T} = N_{\hat{T}}(H_2) = N_{T_1}(H_2)T_2 = T_2$. Consequently, $T_0 \cap T_1 = T_3$. Using (6B.4) and the fact that $R_2 \cap N_2 = E_2$, we easily find that $E_1 \cap T_3 = H_1$ and $H_1 \cap N_2 = E$. Thus $E_1 \cap N_2 = E_1 \cap T_3 \cap N_2 = H_1 \cap N_2 = E$. Likewise, $E_2 \cap N_1 = E$. Thus all parts of the lemma hold.

(6E) There exists a J -invariant subgroup M of N_0 of order q^5 such that $N_0 = ME$ and $M \cap E = F$. Furthermore, the following conditions hold:

- (1) $M = C_M(J_1) \times C_M(J_2)$.
- (2) $\langle \hat{J}_1^{C(J_1)} \rangle$ is isomorphic modulo center to $PSU(3, q)$ or $PSL(3, q)$.
- (3) $\langle \hat{J}_2^{C(J_2)} \rangle$ is isomorphic to $SL(2, q^2)$ or $SL(2, q) \times SL(2, q)$.
- (4) $C_M(J_i) \in \text{Syl}_2(\langle \hat{J}_i^{C(J_i)} \rangle)$, $1 \leq i \leq 2$.

PROOF. By (6C.6), $N_0^3 \leq F$. Further, $|N_0| = 2q^5$ by (6D.4). That M exists now follows as in (4D). Furthermore, J_1 centralizes M/F and J acts fixed-point-freely on M/F . Thus $M = C_M(J_1)F = C_M(J_1)C_F(J_2)$ and $C_M(J) \leq C_F(J) = 1$ by (5D), and (1) follows from (1I). Clearly we also have $C_M(J_2) = C_F(J_2)$. As in (4D), we have $C_M(J_i) \leq X_i = \langle \hat{J}_i^{C(J_i)} \rangle$. Since $C_L(J_i) = \hat{J}_i \times J_i$ by (2A.1), $\hat{J}_i \triangleleft C \cap C(J_i)$. Next, since $U_i \leq \hat{J}_i$, and since $C(U_i) \cap C \leq LC_C(L)$ by (2D.2), we have $C(\hat{J}_i) \cap C = C_L(\hat{J}_i)C_C(L)$ and so $C(\hat{J}_i) \cap C \cap C(J_i) = C_L(\hat{J}_i J_i)C_C(L) = J_i C_C(L) = J_i O(C) \langle t \rangle$. Therefore $\hat{J}_i$ is a standard subgroup of $C(J_i)$ and $\langle t \rangle \in \text{Syl}_2(C(\hat{J}_i) \cap C(J_i))$. Moreover, t is not an isolated involution in $C(J_i)$ since $C_M(J_i) \not\leq C$. Since $C_F(J_i)$ is homocyclic by (6B.1), Lemma (2.4) of [16] shows that $C_M(J_i)$ is isomorphic to a Sylow 2-subgroup of $PSL(3, q)$ or $PSU(3, q)$, or $C_M(J_i)$ is homocyclic. The main theorem of

[16] now shows that (2) holds. Also, since $C_M(J_2) = C_F(J_2)$ is t -invariant and elementary of order q^2 by (6B.2), (3) follows. By order consideration $C_M(J_i) \in \text{Syl}_2(X_i)$ for $1 \leq i \leq 2$.

(6.5) NOTATION. $M_i = MM^{r_i}$, $1 \leq i \leq 2$.

(6F) For $i \in \{1, 2\}$, the following conditions hold:

(1) $M \cap M^{r_i} = C_M(J_i)$.

(2) M_i is a subgroup of N_i such that $N_i = M_i E_i$ and $M_i \cap E_i = FF^{r_i}$.

REMARK. Diagram 3 shows the configuration of subgroups defined in (5D), (5.5), (6E) and (6.5). Complexes of these groups are also shown. Here, MK and $MK_i = M(FK_i)$ are subgroups since K and FK_i are normal subgroups of $\hat{N}$. We shall see that all other complexes are also subgroups.

PROOF. Since F_i/U_1 is elementary by (6B), so is $FK_i/U_1 = A_1 F_1 F_2/U_1$. By (5E.4), $N_L(A_1)$ acts transitively on $(FF^{r_1}/K_1)^*$. Hence FF^{r_1}/U_1 is elementary abelian and since $N_{E_1}(V_1) = D_1$, it follows that $\mathcal{E}^*(E_1/U_1) = \{FF^{r_1}/U_1, D_1/U_1\}$. Thus $FF^{r_1} \triangleleft N(E_1)$. Now (6C.1) implies that $C_L(U_1)$ stabilizes the series $FF^{r_1} \leq E_1 \leq N_1$, so by (2C.4), $C_L(U_1)$ centralizes N_1/FF^{r_1} . In particular, r_1 normalizes $\hat{M} = MFF^{r_1}$, so $M^{r_1} \leq \hat{M}$. Thus $M_1 = \hat{M}$ is a subgroup. We have $MK_1 = M_1 \cap T_3 \triangleleft M_1$ and $M = MK_1 \cap N_2 \triangleleft MK_1$. Furthermore, we see from Diagram 3 that $M_1/MK_1 \cong MK_1/M \cong A_1/Z(P)$ as J -modules. So J_1 acts fixed-point-freely on each of the factors of the series $M < MK_1 < M_1$. Thus $C_M(J_1) = C_{M_1}(J_1)$ is normalized by r_1 and so $C_M(J_1) \leq M^{r_1}$. By order consideration, we conclude that $M \cap M^{r_1} = C_M(J_1)$. Thus the assertions hold for $i=1$.

Since $N_2^2 \leq FF^{r_2}$ by (6C.2), $M_2 = M(FF^{r_2})M^{r_2}$ is a subgroup of N_2 . Thus arguing just as in (5E.4), we see that the assertions hold for $i=2$ as well.

(6G) T_0 and $\hat{T}$ normalize each other.

PROOF. First, $A_i = E_i^2 \triangleleft T_3$. Next, F and $K_i \triangleleft T_3$ by (5E.2) and (6B.3), whence $F_i = F \cap K_i \triangleleft T_3$. Since J acts irreducibly on each of the groups $A_i/Z(P)$ and $F_i/Z(P)$, it follows that $K = A_1 A_2 F_1 F_2 \leq Z(T_3 \bmod Z(P))$. Since $M^2 = U_1$ by (6E), we conclude that $MK/Z(P)$ is elementary abelian. Likewise, MF^{r_2}/K_2 is an elementary abelian group since $FF^{r_2} = Z(N_2 \bmod K_2)$. Now set

$$X = N(T_2) \cap N(K_2) \cap N(MK) \cap N(MF^{r_2}),$$

and let bars denote images in X/K_2 . Then $\overline{MK} \overline{F}^{r_2} = \overline{MK}_2 * \overline{K} \overline{F}^{r_2}$ by the above, and $\overline{K} \overline{F}^{r_2} \cong R_2/D_2 \cong P/U_2$ by (5A). Hence

$$\mathcal{E}^*(\overline{MKF}^{r_2}) = \{\overline{MK}, \overline{MF}^{r_2}\}.$$

Since $N_{T_2}(D_2) = R_2$ by (6C.2), arguments of the third paragraph of the proof of (4E) now show that

$$J(\overline{T}_2) = \overline{MKF}^{r_2}.$$

Therefore $N(K_2) \cap N(T_2)$ normalizes MKF^{r_2} and $\{MK, MF^{r_2}\}$. By (6B) and (6C), $\hat{T} \leq N(K_2) \cap NT_2$, so $\hat{T}$ normalizes $MK = T_3 \cap MKF^{r_2}$ and MF^{r_2} , whence $\hat{T} \leq X$. An analogue of (5B.2) shows that

$$\mathcal{E}^*(T_0/E_2) = \{T_2/E_2, N_2/E_2\},$$

whence $T_0 \leq N(K_2) \cap T(T_2)$. Hence we also have $T_0 \leq X$. Since $N_{T_3}(D_2) = T_3 \cap R_2 = H$, $\mathcal{E}^*(T_3/K_2) = \{H/K_2, MK/K_2\}$ and so $N_X(T_3) \leq N_X(H)$. Since $H \cap MF^{r_2} = FK_2$ and $(FK_2)^2 = U_1$ by (6B), it follows that $N_X(T_3) \leq \hat{N} = N(H) \cap N(U_1)$. Set $Y = C_X(T_2/MK)$. Then $Y \leq \hat{N}$ by the above, and $\hat{T} \leq Y$ because $\hat{T}/T_3$ is elementary. It easily follows from (6B.4), (6C.3), and (6C.4) that $\hat{N} = \hat{T}(N(R_2) \cap N(B_2))$. Thus $Y \leq \hat{T}(N(R_2) \cap N(B_2) \cap C(R_2/H))$, and as before $\hat{T}$ is the unique Sylow 2-subgroup of Y . Thus $\hat{T} \triangleleft X$. Consequently, $N_{\hat{T}}(T_0) > T_2$. From Diagram 2 we see that J acts irreducibly on $\hat{T}/T_2$. Hence $N_{\hat{T}}(T_0) = \hat{T}$. The proof is complete.

(6.6) NOTATION. $T = \hat{T}T_0$.

(6H) The following conditions hold:

- (1) $M_1^2 = Z(M_1) = U_1$, $M_2^2 = Z(M_2) = K_2$.
- (2) $M_i \triangleleft T$, $1 \leq i \leq 2$.
- (3) $\mathcal{E}^*(M_1K_2/U_1) = \{M_1/U_1, MK_2/U_1\}$.
- (4) $Z(M_1K_2) = U_1$.

PROOF. By (6E), $M_1 = C_M(J_1) * \langle F_2, F_2^{r_1} \rangle$ and $C_M(J_1)^2 = U_1$. As FF^{r_1}/U_1 is elementary (see the first paragraph of the proof of (6F)), $M_1^2 = U_1$.

In the proof of (6G) we have shown that $MKF^{r_2} \triangleleft T_0$ and $(MKF^{r_2})^2 \leq FK_2$. Since $N_2^2 \leq FF^{r_2}$ by (6C.2), this shows that $T_0^2 \leq MF^{r_2}$. In particular, $T_0 \leq N(M_2)$ so $N_L(A_2) = \langle P, J, r_2 \rangle$ normalizes M_2 . Moreover, $N_L(A_2)$ acts transitively on $(M_2/FF^{r_2})^*$ by (6C.2), and MF^{r_2}/K_2 is elementary. Thus $M_2^2 \leq K_2$, and since $N_{N_2}(D_2) = E_2$ by (6C.2), $\mathcal{E}^*(N_2/K_2) = \{M_2/K_2, E_2/K_2\}$. Recall from the proof of (6G) that $J(T_2/K_2) = MKF^{r_2}/K_2$, $\mathcal{E}^*(T_0/E_2) = \{N_2/E_2, T_2/E_2\}$, and $\mathcal{E}^*(MKF^{r_2}/K_2) = \{MK/K_2, MF^{r_2}/K_2\}$. Thus M_2/K_2 is the only E_{q^4} -subgroup of T_0/K_2 and, since $T \leq N(T_0) \cap N(K_2)$, it follows that $M_2 \triangleleft T$.

By (6E) $U_1 = M^2 \leq M_2^2 \cap A_2$ and, since $M_2^2 \cap A_2$ is $N_L(A_2)$ -invariant, (2C.3) forces $A_2 \leq M_2^2$. Since $M_2^2 \triangleleft T$ by the above and $N_T(A_2) = T_0$ by (5B.1) and (5C.2), $A_2 \neq M_2^2$. Since J acts irreducibly on K_2/A_2 , we conclude that $M_2^2 =$

K_2 . Now $Z(T) \leq N_T(A_2) = T_0$, so $Z(T) \leq C_{T_0}(t) = Q$. Thus $Z(T) \leq Z(R_1) = U_1$ and, since J acts irreducibly on U_1 , $Z(T) = U_1$. In particular, $Z(M_2) \not\leq U_2$ so $A_2 \leq Z(M_2)$ as above. Consequently, $MK/U_1 = (M/U_1) * (P/U_1)$ and, since $MK \triangleleft T$ (see (6G)), $M = Z(MK \bmod U_1) \triangleleft T$. In particular, $M \triangleleft M_2$ so $C_M(J_2) = M \cap M^{r_2} \triangleleft M_2$ and

$$M_2/C_M(J_2) = M/C_M(J_2) \times M^{r_2}/C_M(J_2).$$

Thus $Z(M_2) \leq Z(M_2 \bmod C_M(J_2)) = K_2$ by (6E), and then $Z(M_2) = K_2$ as before.

As remarked in (6G), $(MK)^2 \leq Z(P)$, so

$$M_1K_2/K_1 = (MK_1/K_1) * (F^{r_1}K/K_1)$$

and, since $F^{r_1}K/K_1 \cong R_1/D_1 \cong P/U_1$, $\mathcal{E}^*(M_1K_2/K_1) = \{M_1/K_1, MK/K_1\}$. Likewise, $\mathcal{E}^*(MK/U_1) = \{MK_1/U_1, MK_2/U_1\}$ by the above paragraph. Hence (3) follows.

Now set $S = M_1M_2$. Then $T = S\langle t \rangle$, so $M_1K_2 = T_1 \cap S \triangleleft \hat{T}$ and then $M_1 \triangleleft \hat{T}$ by (3). So $\hat{T} \cap S/M = M_1/M \times MF^{r_2}/M$ and, since $MF^{r_2}/M \cong C_F(J_1)$ is homocyclic, $M_1K_2 = \Omega_1(\hat{T} \cap S \bmod M)$. Thus $M_1K_2 \triangleleft T$, and then $M_1 \triangleleft T$ by (3).

By (6A.3) $F_2^{r_1} \cap C(Z(P)) = Z(P)^{r_1}$, so $C_{M_1}(Z(P)) = MK_1$. Likewise, $C_{MK_1}(F_2) = M$. Thus $Z(M_1) \leq M$ and so $Z(M_1) \leq M \cap M^{r_1} = C_M(J_1)$. Hence $Z(M_1) = U_1$ by (6E). This and (3) imply (4).

(6.7) NOTATION. Let $S = M_1M_2$ and let L_i , $1 \leq i \leq 2$, denote the normal closure of $C_L(U_i)M_i$ in $N(M_i)$.

(6I) The following conditions hold:

- (1) $L_1/Z(L_1 \bmod M_1)$ is isomorphic to $\text{PSU}(3, q)$ or $\text{PSL}(3, q)$.
- (2) L_2/M_2 is isomorphic to $\text{SL}(2, q^2)$ or $\text{SL}(2, q) \times \text{SL}(2, q)$, and if $L_2/M_2 \cong \text{SL}(2, q) \times \text{SL}(2, q)$, then t interchanges two components of L_2/M_2 .
- (3) For $i \in \{1, 2\}$, we have $S \in \text{Syl}_2(L_i)$ and $C(L_i/M_i) = O(N(M_i) \bmod M_i)$.

PROOF. Since $M_1^2 = U_1$, and since $N_{N_1}(V_1) = D_1$ by (5A), it follows that $\mathcal{E}^*(N_1/U_1) = \{M_1/U_1, D_1/U_1\}$. Consequently $Z(N_1) = U_1$, so $N(N_1) = N(D_1) \leq N(M_1)$. Also, since K_2 is a characteristic subgroup of M_2 and $\mathcal{E}^*(N_2/K_2) = \{M_2/K_2, E_2/K_2\}$, we have $N(N_2) \cap N(M_2) = N(N_2) \cap N(K_2) = N(E_2) \cap N(K_2)$. Also, notice that $\langle C_M(J_1)^{r_2}, C_M(J_1)^{r_2r_1} \rangle$ is isomorphic modulo center to $\text{PSU}(3, q)$ or $\text{PSL}(3, q)$ by (6E) and is embedded in $N(M_1)/M_1$. Thus arguing as in (4G), we get the result.

(6J) The following conditions hold:

- (1) $Z(S) = U_1$, $Z_2(S) = F_2$, $Z_3(S) = MK_2 = Z(S \bmod K_2)$.
- (2) $C_S(F_2) = M_2 = C_T(K_2)$.
- (3) $N(M_1K_2) \leq N(S)$.

PROOF. By (6I) $C_{L_i}(M_i/M_i^2) \leq Z(L_i \text{ mod } M_i)$ so $C_S(M_i/M_i^2) = M_i$, $1 \leq i \leq 2$. In particular, $Z(S) \leq M_1$ and hence $Z(S) = U_1$ by (6H). Likewise $Z_2(S) \leq M_1 \cap M_2 = M$. Notice that $[M, M^{r_2}] \leq M \cap M^{r_2} = C_M(J_2)$. Since $C_M(J_2) \cap U_1 = 1$ and M_1/U_1 is abelian, we have $Z_2(S) = C_M(M^{r_2})$ and therefore $Z_2(S) = C_M(M^{r_2b})$ for any $b \in B = PJ$. Let $x \in P - P^{r_2}$. Then $r_2 x r_2 \in N_L(A_2) - B$ so $r_2 x r_2 \in B r_2 B$. Also, $x r_2 \in B r_2 B$. Hence $M_2 = M_2^{r_2 x r_2} = (M M^{r_2})^{r_2 x r_2} = \langle M^{r_2 b} \mid b \in B \rangle$. Thus $Z_2(S) = M \cap K_2 = F_2$ by (6H.1). Since $[M_1, K_2] \leq M_1 \cap K_2 = F_2$, $K_2 \leq Z_3(S)$. Also, since $[M, M^{r_2}] \leq C_M(J_2) \leq F_2$, $M \leq Z_3(S)$. Thus $M K_2 \leq Z_3(S)$. Conversely, $Z_3(S) \leq C_S(M_2/M_2^2) \leq M_2$ and $Z_3(S) \leq Z(S \text{ mod } M_1) = M_1 K_2$ by (6I). Thus $Z_3(S) = M_2 \cap M_1 K_2 = M K_2$. Let bars denote images in S/K_2 . Then $\mathcal{E}^*(\bar{K}_1 \bar{M}_2) = \{\bar{M} \bar{K}, \bar{M}_2\}$ because $\mathcal{E}^*(T_0/E_2) = \{N_2/E_2, T_2/E_2\}$ and $\mathcal{E}^*(\bar{M} \bar{K} \bar{F}^{r_2}) = \{\bar{M} \bar{K}, \bar{M} \bar{F}^{r_2}\}$ (see the proof of (6G)). Since $Z(\bar{S}) \leq C_{\bar{S}}(\bar{M}_2) = \bar{M}_2$, we conclude that $Z(\bar{S}) \leq \bar{M} \bar{K} \cap \bar{M}_2 = \bar{M} \bar{K}_2$. Thus we have proved (1).

In the last paragraph of the proof of (6H) we have shown that $C_{M_1}(F_2) = M$. Thus $C_S(F_2) = M_2$. Suppose that $C_T(K_2) > M_2$. Then $C_T(K_2)/M_2 - S/M_2$ contains an involution. However, (6I) implies that involutions of $T/M_2 - S/M_2$ are conjugate to $t M_2$ in T/M_2 , and consequently $C_T(K_2) = N_2$. Since this is false, $C_T(K_2) = M_2$.

By (6H) $N(M_1 K_2) \leq N(M_1)$. Hence if $x \in N(M_1 K_2)$, then $S^x \in \text{Syl}_2(L_1)$ and $Z(S/M_1) = M_1 K_2/M_1 = Z(S^x/M_1)$. So the structure of L_1/M_1 forces $S^x = S$. Thus (3) holds.

(6K) The following conditions hold:

- (1) $Z(P) - U_1 = \bigcup_{x \in S} U_2^x$.
- (2) $K_2 - F_2 = \bigcup_{x \in S} A_2^x$.
- (3) $I(tS) = t^S = t^G \cap T$.

PROOF. All these follow as in (4J). We only sketch a proof. Since $Z(R_1) = U_1$, $N_{R_1}(X) = H$ for any nonidentity subgroup X of U_2 . So U_2 is a T.I. set of R_1 , and (1) follows. Likewise, (5B.2) implies that $N_{R_1}(X) = H$ for any subgroup X of A_2 such that $Z(P) < X$. Hence (2) follows. Now (6I) implies that involutions of $T/M_1 - S/M_1$ are all conjugate in T/M_1 . Hence if $x \in I(tS)$, then $x \in {}_T N_1$ and, since $\mathcal{E}^*(N_1/U_1) = \{M_1/U_1, D_1/U_1\}$, $x \in {}_T D_1$. Since $t K_1 = t^{M_1}$, we conclude that $I(tS) = t^S$. Suppose that $x \in t^G \cap S$. Then by (6I) we may take $x \in M_1 K_2$, in which case $x \in M_1 \cup M_2$ by (6H.3). If $x \in M_1$, then $|x^{M_1}| \leq |M_1'| = q$ so $|C_{M_1}(x)| \leq q^q < |C|_2$, a contradiction. If $x \in M_2$, then $K_2 \leq {}_G C$, whereas $m(C) = 3n + 1$, where $q = 2^n$. This is a contradiction. Thus $t^G \cap T \leq tS$, and (3) follows.

(6.8) NOTATION. $G_0 = \langle L^G \rangle$, $G_1 = \langle L_1, L_2 \rangle$.

(6L) We have $G_1 \leq G_0$.

PROOF. This follows as in (4M).

§ 7. Identification of G_0 (1)

We shall retain the notation of Sections 5 and 6. In this and the next section we shall identify G_0 with $\text{PSU}(5, q)$ and $\text{PSL}(5, q)$. Under Hypothesis (6.1) we have shown that L_2/M_2 is isomorphic to $\text{SL}(2, q^2)$ or $\text{SL}(2, q) \times \text{SL}(2, q)$. In this section we shall assume the following hypothesis:

(7.1) HYPOTHESIS. $L_2/M_2 \cong \text{SL}(2, q^2)$.

(7A) *The following conditions hold:*

- (1) $L_1/Z(L_1 \bmod M_1) \cong \text{PSU}(3, q)$.
- (2) $\Omega_1(M_2) = K_2$.
- (3) $\mathcal{E}^*(S/U_1) = \{M_1/U_1, MK_2/U_1\}$, $\mathcal{E}^*(S/K_2) = \{M_1K_2/K_2, M_2/K_2\}$.
- (4) $N(S) = N(M_1) \cap N(M_2)$.
- (5) *If $S \neq S_1 \in \text{Syl}_2(L_2)$, then $Z(S) \cap (S_1) = 1$.*

PROOF. Let bars denote images in $N(K_2)/K_2$. Then $Z(\bar{S}) = \bar{M}$ has order q^2 by (6J) and, as we are assuming (7.1), $N_{L_2}(\bar{S})$ acts transitively on $\bar{M}^\#$ by (1K). Suppose that (1) is false. Then $C_M(J_1) \cong S/M_1$ is of type $\text{SL}(3, q)$ by (6I), and so $MK_2 = C_M(J_1) * K_2$ has precisely two maximal elementary abelian subgroups, which contain K_2 and are normal in S as $MK_2 = Z_3(S)$ and $K_2 \geq Z_2(S)$. But then $N_{L_2}(S)$ cannot act irreducibly on MK_2/K_2 , which is a contradiction proving (1). In particular, $\Omega_1(S \bmod M_1) = M_1K_2$ so $\Omega_1(M_2) \leq MK_2 \cap (MK_2)^{r_2} = K_2$, which proves (2). Likewise $\Omega_1(S/U_1) \leq M_1K_2/U_1$, and so the first assertion of (3) follows from (6H.3). Next, since both $\bar{M}_1$ and $\bar{M}_2$ are elementary abelian, (1C) and (1K) imply that $\mathcal{E}^*(\bar{S}) = \{\bar{M}_1, \bar{M}_2\}$, proving the second assertion of (3). Now $N(S) \leq N(K_2)$ as $Z_3(S) = MK_2$ and $Z(MK_2) = K_2$. So $N(S) \leq N(M_2)$ by (3). Since $U_1 = Z(S)$, we also have $N(S) \leq N(M_1)$. Thus (4) holds. Finally, the proof of (5) is similar to that of (4H) and is omitted.

The proof of the following result is similar to that of (4R) and is omitted.

(7B) $G_1/Z(G_1) \cong \text{PSU}(5, q)$ with $|Z(G_1)|$ odd.

The rest of the section is devoted to proving $G_0 = G_1$.

(7C) *The following conditions hold.*

- (1) $S \in \text{Syl}_2(G_0)$.
- (2) M_2 is weakly closed in S with respect to G .

PROOF. First, recall that $G_1 \leq G_0$, so $S \leq O^2(G)$. Let U be a Sylow 2-subgroup of $N(S)$ containing T . We argue that M_2 is weakly closed in U with

respect to G . Suppose $M_2^g \leq U$. Then, as U/S is cyclic by (6I) and (7A.1), $K_2^g \leq T$ and then $K_2^g \leq S$ by (6K.3). Since M_2^g/K_2^g is elementary, it follows that $M_2^g \leq T$, and therefore $K_2^g = (M_2^g)^2 \leq K_1 M_2$ as $T/K_1 M_2$ is elementary. So by (7A.3) $K_2^g \leq MK_1$ or MK_2 . Since $\Omega_1(M_2) = K_2$, it follows that $F_2 \leq K_2^g$, so $K_2^g = K_2$ and $M_2^g \leq C_T(K_2) = M_2$ by (6J.2). Thus $M_2^g = M_2$, as claimed. Consequently, $N(U) \leq N(M_2)$ and, as $U \in \text{Syl}_2(N(M_2))$ by (7A.4), $U \in \text{Syl}_2(G)$. Since U/S is cyclic and $t^G \cap T = I(tS)$, (1E) implies that $t \notin O^2(G)$. Therefore $S \in \text{Syl}_2(G_0)$.

(7D) We have $N_{G_0}(U_1) = N_{G_0}(M_1)O(N_{G_0}(U_1))$.

PROOF. First, $N(M_1) \leq N(U_1)$ by (6H.1). Let bars denote images in $N(U_1)/U_1$ and set $X = N_{G_0}(U_1)$. Then (7A.3) shows that $\bar{M}_1$ is a weakly closed subgroup of $\bar{S}$ with respect to $\bar{X}$. Suppose that an element $\bar{a}$ of $\bar{S} - \bar{M}_1$ is $\bar{X}$ -conjugate to an element to $\bar{M}_1$. Choose $\bar{g} \in \bar{X}$ so that $\bar{g}$ satisfies the conditions (1), (2) of (1G) and set $\bar{X}_1 = \bar{M}_1 \cap \bar{M}_1^{g^{-1}}$ and $\bar{X}_2 = N_{\bar{M}_1}(\langle \bar{X}_1, \bar{a} \rangle)$. By (1G), $\bar{X}_2 \not\leq \bar{M}_1$ and we may assume $\bar{X}_2 \leq \bar{S}$. Then $\langle \bar{X}_2, \bar{a} \rangle \leq \bar{M} \bar{K}_2$ and $C_{\bar{M}_1}(\bar{a}) = \bar{M}$ by (7A.3), so $|\bar{X}_2 : \bar{X}_2 \cap \bar{M}_1| \leq q < q^2 = |\bar{M}_1 / C_{\bar{M}_1}(\bar{a})|$. Since this contradicts (1G), we conclude that $\bar{M}_1$ is strongly closed in $\bar{S}$ with respect to $\bar{X}$.

Now set $\bar{Y} = \langle \bar{M}_1^x \rangle$ and let tildes denote images in $\langle \bar{t} \rangle \bar{X} / O(\bar{X})$. Then, as $S \in \text{Syl}_2(X)$ by (7C), $\tilde{Y} = O_2(\tilde{Y}) \tilde{Y}_1 \cdots \tilde{Y}_k$ with $\tilde{Y}_i$ a quasisimple Goldschmidt group [11]. Moreover, $\tilde{M}_1 \in \text{Syl}_2(\tilde{Y})$ because $Z(\tilde{S} \text{ mod } \tilde{M}_1) = \tilde{M}_1 \tilde{K}_2$. Since $\tilde{L}_1$ acts irreducibly on $\tilde{M}_1$ by Lemma (4D) of [7], either $\tilde{Y} = O_2(\tilde{Y})$ or $\tilde{Y} = E(\tilde{Y})$. Suppose that $\tilde{Y} = E(\tilde{Y})$. Then, as $N_{\tilde{M}_1}(V_1) = K_1$, we have

$$1 \neq C(\bar{t}) \cap \tilde{M}_1 \cap \tilde{Y}_1 \tilde{Y}_1^t = \tilde{K}_1 \cap \tilde{Y}_1 \tilde{Y}_1^t.$$

Now $A_2 \leq C(C_F(J_1))$, and so $C_L(U_1) = \langle A_2, A_2^{r_1} \rangle \leq C(C_F(J_1))$. Hence if $\widetilde{C_F(J_1)} \cap \tilde{Y}_1 \tilde{Y}_1^t \neq 1$, $\widetilde{C_L(U_1)} \leq N(\tilde{Y}_1 \tilde{Y}_1^t)$. Assume that $\widetilde{C_F(J_1)} \cap \tilde{Y}_1 \tilde{Y}_1^t = 1$. Then a non-trivial element $\tilde{x}$ of $\tilde{K}_1 \cap \tilde{Y}_1 \tilde{Y}_1^t$ is written as $\tilde{x} = \tilde{a} \tilde{b}$ with $\tilde{a} \in \tilde{A}_1^*$ and $\tilde{b} \in \widetilde{C_F(J_1)}$. Let $\tilde{g} \in \widetilde{C_L(U_1)}$. Then $\tilde{x}^g = \tilde{a}^g \tilde{b}^g \in \tilde{K}_1 \cap \tilde{Y}_1^g \tilde{Y}_1^{gt}$. Hence if $\tilde{Y}_1^g \tilde{Y}_1^{gt} \neq \tilde{Y}_1 \tilde{Y}_1^t$, then $\tilde{a}^g \neq \tilde{a}$ and so there is an element $\tilde{h} \in \widetilde{C_L(U_1)}$ such that $\tilde{a}^h = \tilde{a} \tilde{a}^g$. Set $\tilde{y} = \tilde{a} \tilde{a}^g$. Then $\tilde{b} = \tilde{a}^h \tilde{b} \tilde{a} \tilde{a}^g = \tilde{x}^h \tilde{y}$, $\tilde{x}^h \in \tilde{K}_1 \cap \tilde{Y}_1^h \tilde{Y}_1^{ht}$, and $\tilde{y} = \tilde{a} \tilde{b} \tilde{a}^g \tilde{b} \in (\tilde{K}_1 \cap \tilde{Y}_1 \tilde{Y}_1^t)^* (\tilde{K}_1 \cap \tilde{Y}_1^g \tilde{Y}_1^{gt})^*$. Since $\widetilde{C_L(U_1)} \leq C(\tilde{b})$, this is possible only if $\widetilde{C_L(U_1)}$ normalizes $\{\tilde{Y}_1 \tilde{Y}_1^t, \tilde{Y}_1^g \tilde{Y}_1^{gt}, \tilde{Y}_1^h \tilde{Y}_1^{ht}\}$ and, since $\widetilde{C_L(U_1)}$ is perfect, it follows that $\widetilde{C_L(U_1)} \leq N(\tilde{Y}_1 \tilde{Y}_1^t)$. Thus in any case $\widetilde{C_L(U_1)}$ normalizes $\tilde{Y}_1 \tilde{Y}_1^t$, and so normalizes $\tilde{Y}_1$. Then as usual $\widetilde{C_L(U_1)} \leq C(\tilde{Y})$ and, since $\tilde{Y}_1$ was an arbitrary component of $\tilde{Y}$, we conclude that $\widetilde{C_L(U_1)} \leq C(\tilde{Y})$. This, however, is a contradiction because $\widetilde{C_L(U_1)} \not\leq C(\tilde{M}_1)$. Therefore $\tilde{Y} = \tilde{M}_1$, from which (7D) follows as in (4N).

(7E) We have $N_{G_0}(K_2) = N_{G_0}(M_2)O(N_{G_0}(K_2))$.

PROOF. First, $N(M_2) \leq N(K_2)$ by (6H.1). Let bars denote images in $N(K_2)/K_2$ and set $Y = N_{G_0}(K_2)$. We know that $\bar{S} = \bar{M}_1 \bar{M}_2 \in \text{Syl}_2(\bar{Y})$ and $\mathcal{E}^*(\bar{S}) = \{\bar{M}_1, \bar{M}_2\}$. Hence both $\bar{M}_1$ and $\bar{M}_2$ are weakly closed in $\bar{S}$ with respect to $\bar{Y}$. Suppose that an element $\bar{a}$ of $\bar{S} - \bar{M}_2$ is $\bar{Y}$ -conjugate to an element of $\bar{M}_2$. Then there is an element $\bar{g} \in \bar{Y}$ such that $\bar{a}^g \in \bar{M}_2$ and $C_{\bar{S}}(\bar{a})^g \leq \bar{S}$ by (1F). Since $\bar{a} \in \bar{M}_1$, we have $\bar{g} \in N(\bar{M}_1)$, and consequently $g \in N(S)$ by (6J.3). But then $g \in N(M_2)$ by (7A.4), which is a contradiction because $a^g \in M_2$. Therefore $\bar{M}_2$ is strongly closed in $\bar{S}$ with respect to $\bar{Y}$. Let tildes denote images in $\langle \bar{t} \rangle \bar{Y} / O(\bar{Y})$ and let $\tilde{X} = \langle \tilde{M}_2^{\bar{Y}} \rangle$. Then as usual $\tilde{X} = O_2(\tilde{X}) \tilde{X}_1 \cdots \tilde{X}_k$ with $\tilde{X}_i$ a quasisimple Goldschmidt group and $\tilde{M}_2 = \Omega_1(\tilde{S} \cap \tilde{X})$. Since $\tilde{M}_1$ is elementary, $\tilde{M}_2 = \tilde{S} \cap \tilde{X}$. Suppose that $k \geq 1$ and set $\tilde{Y}_i = \tilde{M}_2 \cap \tilde{X}_i$. Then $1 \neq C(\tilde{t}) \cap \tilde{Y}_i \tilde{Y}_i^t \leq \tilde{F} \tilde{F}^{r_2}$ as $N(D_2) = N(B_2)$. Since $C_L(U_1)$ acts transitively on $(\tilde{F} \tilde{F}^{r_2})^{\#}$, we must have $E(\tilde{X}) = \tilde{X}_1 \tilde{X}_1^t$ and thus as usual $\tilde{L}_2$ centralizes $\tilde{X}_1$. This is a contradiction because $\tilde{L}_2$ has no nontrivial fixed points on $\tilde{M}_2$. Therefore $\tilde{X} = O_2(\tilde{X})$ and, since $K_2 = Z(M_2)$, (7E) follows.

Now we choose $u_i \in U_i^{\#}$, $1 \leq i \leq 2$, and set $C_i = C_{G_0}(u_i)$.

(7F) We have $C_1 = N_{C_1}(U_1)O(C_1)$ and C_1 is 2-constrained.

PROOF. Suppose that an element u of U_1 is C_1 -conjugate to an element v of $S - U_1$. Then $v \in M_1 \cup K_2$ by (7A). Now $G_1/Z(G_1) \cong \text{PSU}(5, q)$. By the structure of $\text{PSU}(5, q)$, involutions of M_1 are L_1 -conjugate to an element of K_2 and, as $L_1 \leq N_{C_1}(U_1)$, we may assume that $v \in K_2$. Then u is $N_{C_1}(M_2)$ -conjugate to v by (7C). However, $N_{C_1}(M_2) \leq N(S) \leq N(U_1)$ by (7A.5), whereas $v \notin U_1$. This contradiction shows that U_1 is strongly closed in S with respect to C_1 . Define $X = \langle U_1^{C_1} \rangle$ and let bars denote images in $C_1/O(C_1)$. Then, as $S \in \text{Syl}_2(C_1)$, either $\bar{X} = \bar{U}_1$ or $\bar{X} = O_2(\bar{X}) \bar{X}_1 \cdots \bar{X}_k$ with $\bar{X}_i$ a quasisimple Goldschmidt group [11]. Furthermore, $U_1 \in \text{Syl}_2(X)$ because $Z_2(S) = F_2$ is elementary. Thus arguing as in (4O), we get the result.

(7G) We have $C_2 = N_{C_2}(K_2)O(C_2)$ and C_2 is 2-constrained.

PROOF. As in (4P), $K_1 M_2 \in \text{Syl}_2(C_2)$. Furthermore, an argument of (7C) implies that K_2 is weakly closed in $K_1 M_2$ with respect to C_2 . Suppose that an element a of $K_1 M_2 - K_2$ is C_2 -conjugate to an element of K_2 and choose an element $g \in C_2$ so that g satisfies the conditions (1)–(3) of (1G). Then $a \in M K_1 - M_2$ and $C_{M_2}(a) \leq M$ by (7A.2) and (7A.3), so $Z(P) \leq C_{K_2}(a) \leq F_2$. However, since $C_{K_1 M_2}(x) = M_2$ for $x \in F_2 - Z(P)$ by (6A.3), we must have $C_{K_2}(a) = Z(P)$. Hence if $X_1 = K_2 \cap K_2^{g^{-1}}$ and $X_2 = N_{K_2}(\langle X_1, a \rangle)$, then $|X_2/X_1| \geq q^2$ by (1G). But also $X_1^g = X_2^g \cap K_2 = X_2^g \cap M_2$ and therefore $|X_2/X_1| \leq q$. This contradiction shows that K_2 is strongly closed in $K_1 M_2$ with respect to C_2 .

Set $X = \langle K_2^{C_2} \rangle$ and let bars denote images in $C_2/O(C_2)$. As usual

$\bar{X} = O_2(\bar{X})\bar{X}_1 \cdots \bar{X}_k$ with $\bar{X}_i$ a quasisimple Goldschmidt group and $\bar{K}_2 = \Omega_1(\bar{K}_1\bar{M}_2 \cap \bar{X})$. Suppose that $k \geq 2$ and set $\bar{Y}_i = \bar{K}_1\bar{M}_2 \cap \bar{X}_i$, $1 \leq i \leq k$. (7F) implies that for any $\bar{u} \in \bar{U}_1^\#$, $C_{\bar{X}}(\bar{u})$ is 2-constrained. Hence if we write $\bar{u} = \bar{v}\bar{v}_1 \cdots \bar{v}_k$ with $\bar{v} \in O_2(\bar{X})$ and $\bar{v}_i \in \bar{Y}_i$, then $\bar{v}_i \in \Omega_1(\bar{Y}_i) - Z(\bar{X}_i)$, and so the map $\bar{u} \rightarrow \bar{v}_i Z(\bar{X}_i)$ is one to one. Thus $|\Omega_1(\bar{Y}_i)/Z(\bar{X}_i)| \geq q$ and then $|\bar{u}^x \cap \bar{K}_2| \geq (q-1)^k$. But as $N_{G_0}(M_2)$ controls the fusion of elements of K_2 by (7C), (7A.5) forces $|u^{G_0} \cap K_2| \leq (q-1)(q^2+1) < (q-1)^4$. Hence it follows that $k \leq 3$. Thus as usual $\bar{C}_L(U_2)$ centralizes each $\bar{X}_i$. But (6A.3) and (6H.3) imply that $C_{K_2}(A_1) = Z(P)$. So $\Omega_1(\bar{Y}_i) \leq C_{K_2}(C_L(U_2)) = \bar{U}_2$ and therefore $\Omega_1(\bar{Y}_i) \leq Z(\bar{X}_i)$, which is a contradiction. Thus $\bar{K}_2 = O_2(\bar{X})$, and (7G) follows as in (4P).

We have arrived at the goal of this section.

(7H) We have $G_0 \cong \text{PSU}(5, q)$.

PROOF. Structure of $\text{PSU}(5, q)$ and (7C.1) show that involutions of G_0 are G_1 -conjugate to an element of $U_1 \cup U_2$. Thus we may use (7D)–(7G) and arguments of (4R) to prove $G_0 = G_1$.

§ 8. Identification of G_0 (2)

In this section we complete our analysis under Hypothesis (6.1). In view of (6I) and (7H) we shall assume the following hypothesis:

(8.1) HYPOTHESIS. $L_2/N_2 = N/M_2 \times N^t/M_2$, $N/M_2 \cong \text{SL}(2, q)$.

Furthermore, we let $V = S \cap N$. For simplicity of notation we let $r = r_1$ and $s = r_2$.

The proof of the following result is similar to that of (4I) and is omitted.

(8A) The following conditions hold:

- (1) $K_2 = Z(V) \times Z(V^s)$.
- (2) $F_2 = Z(V)Z(V^t)$, $Z(V) \cap Z(V^t) = U_1 = Z(V) \cap Z(P)$.
- (3) N^t acts transitively on $Z(V)^\#$.
- (4) If $S \neq S_1 \in \text{Syl}_2(L_2)$, then $Z(S) \cap Z(S_1) = 1$.
- (5) If $u \in U_1^\#$, then $u^{N(M_2)} \cap F_2 = Z(V)^\# \cup Z(V^t)^\#$.

(8B) Let bars denote images in $N(K_2)/K_2$ and set $\bar{B} = Z(\bar{N})$. Then the following conditions hold:

- (1) $\bar{L}_2 = \bar{N}' \times \bar{N}^t$.
- (2) $O_2(\bar{N}') = \bar{B}^t$, and $\bar{B}^t$ is a natural module for $\bar{N}'/\bar{B}^t \cong \text{SL}(2, q)$.
- (3) $\bar{M}_2 = Z(\bar{V})Z(\bar{V}^t)$, $Z(\bar{V}) \cap Z(\bar{V}^t) = Z(\bar{S})$.

$$(4) \quad \mathcal{E}^*(\bar{S}) = \{\bar{M}_1, \bar{M}_2, \bar{M}_1\bar{B} \cap \bar{V}, \bar{M}_1\bar{B}^t \cap \bar{V}^t\}.$$

$$(5) \quad N(S) = N(M_1) \cap N(M_2).$$

PROOF. First of all $C_{L_2}(\bar{M}_2) = \bar{M}_2$, whence $\bar{B}Z(\bar{S}) \leq Z(\bar{V}) \leq \bar{M}_2$. Since $S = VV^t$, it follows that $Z(\bar{S}) = Z(\bar{V}) \cap Z(\bar{V}^t)$. Also, since $N = \langle V, V^s \rangle$, $Z(\bar{V}) \cap Z(\bar{V}^s) = \bar{B}$. Next $Z(\bar{S}) = \bar{M}$ by (6J) and so we have

$$Z(\bar{S}) \cap Z(\bar{S}^s) = 1.$$

Combining these, we conclude that $\bar{B} \cap \bar{B}^t = 1$. Now since V normalizes V^{st} , $Z(\bar{V}) \cap Z(\bar{V}^{st}) \neq 1$. Hence the above equation forces $Z(\bar{S}) < Z(\bar{V})$. Then by order consideration $\bar{B} = Z(\bar{V}) \cap Z(\bar{V}^s) \neq 1$. Thus $\bar{N}^t$ acts nontrivially on $\bar{B}$, and so Lemma (4B) of [7] implies that $|\bar{B}| \geq q^2$. Thus $\bar{M}_2 = \bar{B} \times \bar{B}^t$ and consequently, $\bar{M}_2 = Z(\bar{V})Z(\bar{V}^t)$. Hence (3) holds.

Since $\bar{S} = \bar{M}_1\bar{M}_2$, and $\bar{M}_1$ is elementary, there is a complement $\bar{X}$ of $\bar{M}_2$ in $\bar{L}_2$ by Gaschütz's theorem. Let $\bar{Y} = \bar{N} \cap \bar{X}$. Then $\bar{Y} \cong \text{SL}(2, q)$ and $\bar{Y}$ acts irreducibly on $\bar{B}^t$ by Lemma (4B) of [7]. Hence $\bar{Y}\bar{B}^t$ is perfect and, since $\bar{N} = \bar{Y}\bar{B}^t \times \bar{B}$, $\bar{Y}\bar{B}^t = \bar{N}'$. Thus (1) follows. Also, (1K) implies (2). In particular, a Sylow 2-subgroup of $\bar{N}'$ has precisely two maximal elementary abelian subgroups, which are of order q^2 . Thus $\mathcal{E}^*(\bar{S})$ consists of four E_{q^4} -subgroups. Certainly, $\bar{M}_1$ and $\bar{M}_2$ are E_{q^4} -subgroups of $\bar{S}$. Since $\bar{M} = Z(\bar{S})$, $\bar{M}_1 \cap \bar{B} = \bar{M} \cap \bar{B}$ has order q . So $\bar{M}_1\bar{B} \cap \bar{V} = (\bar{M}_1 \cap \bar{V})\bar{B}$ has order q^4 and is elementary as $\bar{B} \leq Z(\bar{V})$. Thus (4) holds.

As in (7A), $N(S) \leq N(K_2)$ and so $N(S)$ normalizes $\mathcal{E}^*(\bar{S})$. We know that $Z(M_1K_2) = U_1 < Z(V) \leq Z(M_1B \cap V)$, and $Z(M_1B \cap V) \neq K_2$ as $C_S(K_2) = M_2$ by (6J). Thus no two elements of $\mathcal{E}^*(\bar{S})$ are conjugate, and (5) follows as in (7A).

(8.2) NOTATION. Let B be the preimage of $\bar{B}$ defined in (8B) and let W be the preimage of $\bar{S} \cap \bar{N}'$. Here bars denote images in $N(K_2)/K_2$.

(8C) *The following conditions hold:*

$$(1) \quad \mathcal{E}^*(M_2) = \{B, B^t\}.$$

$$(2) \quad L_1/Z(L_1 \text{ mod } M_1) \cong \text{PSL}(3, q).$$

$$(3) \quad B \cap C_M(J_1) = B^r \cap C_M(J_1).$$

PROOF. Suppose that $B^2 \neq 1$. Then $B^2 \cap Z(V) \neq 1$ so $Z(V) \leq B^2$ by (8A.3). Likewise $Z(V^s) \leq B^2$ so $B^2 = K_2$ by (8A.1). By definition $[N, B] \leq K_2$, so we conclude that $[O^2(N), B] = 1$. But then, as $N = O^2(N)M_2$, $K^2 \leq Z(N)$, which is a contradiction. Therefore, $B^2 = 1$. Let $x \in B - K_2$. Then, since N centralizes B/K_2 , $C_{B^t}(x)$ is N -invariant. Since N acts irreducibly on B^t/K_2 , $C_{B^t}(x) = K_2$ or B^t . However, since $M_2 = BB^t$ and $x \notin Z(M_2) = K_2$, $C_{B^t}(x) = K_2$. Thus (1) holds by (1C). Now $M_1K_2 \cap B = MK_2 \cap B$. As remarked in the second paragraph of the proof of (8B), $MK_2 \cap B$ has order q^5 , so $M_1B/M_1 = M_1K_2B/M_1$

is a E_{q^2} -subgroup of S/M_1 and consequently S/M_1 is not of type $PSU(3, q)$. Thus (2) follows from (6I). Finally, (3) follows from the fact that $C_L(U_1) = \langle A_2, A_2^r \rangle$ centralizes $C_M(J_1)$.

(8.3) NOTATION. $D = B \cap C_M(J_1) = B^r \cap C_M(J_1)$, $\hat{F} = F_2 F_2^r$, $Z = Z(V)Z(V^r)$.

(8D) The following conditions hold:

- (1) $C_M(J_1)$ is isomorphic to a Sylow 2-subgroup of $SL(3, q)$, and $\mathcal{E}^*(C_M(J_1)) = \{D, D^t\}$.
- (2) $M_1 = C_M(J_1) * \hat{F}$.
- (3) $[Z(V), Z(V^r)] = 1$ and hence Z is an elementary abelian group of order q^3 .
- (4) $C_L(U_1)$ normalizes Z and acts transitively on $(Z/U_1)^*$.
- (5) $\mathcal{E}^*(\hat{F}) = \{Z, Z^t, A_1^g, F_2^g \mid g \in N_{G_1}(\hat{F})\}$.
- (6) If X and Y are distinct elements of $\mathcal{E}^*(\hat{F})$, then $|X \cap Y| \leq q^2$.

PROOF. By a remark in the proof of (8C), $|MK_2 \cap B| = q^5$ and so, as $MK_2 = C_M(J_1)K_2$ and $C_M(J_1) \cap K_2 = U_1$, $|D| = q^2$. Further, $D \cap D^t \leq C_M(J_1) \cap K_2 = U_1$. Thus (1) follows from (6E). (2) is immediate from the fact that $[C_M(J_1), F_2] = 1$.

Set $R = \hat{F}K_2$, $V_0 = V \cap R$, and $X_i = \langle R, R^{r_i} \rangle$, $1 \leq i \leq 2$. We know that $Y_2 = E(C(J_2^r))$ is isomorphic to $SL(2, q^2)$ or $SL(2, q) \times SL(2, q)$ and that $C_F(J_2)^r \in \text{Syl}_2(Y_2)$. Since $J_2^{rs} = J_2^r$ and $C_F(J_2)^r \cap C_F(J_2)^{rs} = 1$, $Y_2 = \langle C_F(J_2)^r, C_F(J_2)^{rs} \rangle \leq X_2$. Also, Y_2 covers L_2/M_2 which is isomorphic to $SL(2, q) \times SL(2, q)$. Thus $Y_2 \cong SL(2, q) \times SL(2, q)$ and L_2 is a semidirect product of M_2 by Y_2 . As $X_2 = Y_2 K_2$, it follows that $X_2/K_2 \cong SL(2, q) \times SL(2, q)$. Next, since $M_1 K_2/F_2$ is abelian, $\hat{F} \triangleleft R$ and so $\hat{F} \triangleleft X_1$. Moreover, since $C_L(U_1)$ covers $X_1/\hat{F}$, $X_1/\hat{F} \cong SL(2, q)$ and X_1 acts transitively on $(\hat{F}/A_1)^*$ by (1K). Since $\mathcal{E}^*(A_1 F_2) = \{A_1, F_2\}$ by (6A.3), it follows that involutions of $\hat{F}$ are X_1 -conjugate to an element of F_2 .

Now we shall use the arguments of (4S). First, using arguments of the second paragraph of (4S), we can prove the following:

- (a) If $x \in Z(V) - U_1$, then $C_R(x) = V_0$.
- (b) If $x \in F_2 - Z(V) \cup Z(V^t)$, then there is an element $g \in X_2$ such that $x^g \in U_2$ and $C_R(x)^g \leq A_1 K_2$.

Indeed, (a) follows from (6A.3), and (b) from (6K). Next, let $x \in V^r \cap F_2 - U_1$. Then as in the third paragraph of (4S), $[Z(V), Z(V^r)] = 1$ or $[Z(V^t), Z(V^r)] = 1$ or there is an element $g \in X_2$ such that $x^g \in U_2$ and $Z(V^r)^g \leq A_1 K_2$. Suppose that the last case holds. It follows from (6H.3) that $\mathcal{E}^*(A_1 K_2/U_1) = \{A_1 F_2/U_1, K_2/U_1\}$. Also, $\mathcal{E}^*(A_1 F_2) = \{A_1, F_2\}$. So $\mathcal{E}^*(A_1 K_2) = \{A_1, K_2\}$, and therefore $U_1^q \leq Z(V^r)^g \cap K_2 \leq A_1 \cap K_2 = Z(P)$. Thus $U_1^q = U_1$ by (8A.5), and then $g \in N(S)$ by (8A.4). Now $S \cap S^r = M_1$ and, by the structure of

$\text{PSL}(3, q)$, S acts transitively on the Sylow 2-subgroups $\hat{S}$ of L_1 such that $S \cap \hat{S} = M_1$. This implies that $N(S) = (N(S) \cap N(S^r))S$, so we may write $g = hkh$ with $h \in N(S) \cap N(S^r)$ and $k \in S$. Then $Z(V^r)^h \leq A_1^{k-1} \leq MK_1$, whereas $Z(V^r)^h = Z(V^r)$ or $Z(V^{rt})$ as $N(S^r) \leq N(M_2^r)$. Since this is impossible, either $[Z(V), Z(V^r)] = 1$ or $[Z(V), Z(V^{rt})] = 1$. Then arguing as in (4S), we get (3) and (4). Furthermore, we have $\mathcal{E}^*(V_0) = \{K_2, Z\}$. Also, arguments of the fifth paragraph of (4S) imply (5).

It remains to prove (6). Suppose that x is an element of $X \cap Y - U_1$. Conjugating by an element of X_1 , we may assume that $x \in F_2$. If $x \in Z(V)$, then $X, Y \leq V_0$ by (a), so $\{X, Y\} = \{Z, F_2\}$ and $X \cap Y = Z(V)$ has order q^2 . Likewise if $x \in Z(V^t)$, then $X \cap Y = Z(V^t)$. Suppose $x \notin Z(V) \cup Z(V^t)$. Then by (b) there is an element $g \in X_2$ such that $x^g \in U_2$ and $\langle X, Y \rangle^g \leq A_1 K_2$. If $X^g \leq K_2$, then $X = \hat{F} \cap K_2 = F_2$. If $X^g \not\leq K_2$, then $X^g = A_1$ since $\mathcal{E}^*(A_1 K_2) = \{A_1, K_2\}$. Hence we may suppose that $X^g \leq K_2$ and $Y^g \not\leq K_2$. Then $X = F_2$ and $Y^g = A_1$, so $X^g \cap Y^g \leq K_2 \cap A_1 = Z(P)$. This completes the proof of (6).

(8E) *Let U be a Sylow 2-subgroup of $N(S)$ containing T . Then the following conditions hold:*

- (1) *If $B^g \leq U$, then $B^g = B$ or B^t .*
- (2) *M_2 is weakly closed in U with respect to G .*
- (3) *$U \in \text{Syl}_2(G)$.*
- (4) *G -fusion of elements of K_2 is controlled by $N(M_2)$.*

PROOF. Since $M_2 = BB^t$, (2) follows from (1). If (2) holds, then $N(U) = N(M_2) \cap N(U) \leq N(S)$ as $U \cap L_2 = S$, and (3) follows. Also, (4) is a consequence of (2) and (3) as $K_2 = Z(M_2)$. Thus it is enough to prove (1).

Suppose that $B^g \leq U$ and $B^g \not\leq S$. Then an element of $B^g - S$ acts on L_1/M_1 as a field, graph or graph-field automorphism, and consequently $|B^g M_1 \cap S/M_1| \leq q$. Since $B^g M_1 \cap T = B^g M_1 \cap S$ by (6K.3) and since U/T is cyclic, it follows that $|B^g M_1/M_1| \leq 2q$. Set $X = B^g \cap M_1$ and $Y = B^g \cap \hat{F}$. Then $|X| \geq q^5/2$ and $|Y| \geq q^3/2$ as $|M_1 : \hat{F}| = q^2$. Let $Y \leq Y_1 \in \mathcal{E}^*(\hat{F})$. Let $x \in X \cap D\hat{F}$ and write $x = de$ with $d \in D$ and $e \in \hat{F}$. Then $1 = [x, Y] = [e, Y]$ as $[D, \hat{F}] = 1$. So let $\langle e, Y \rangle \leq Y_2 \in \mathcal{E}^*(\hat{F})$. Then, as $Y \leq Y_1 \cap Y_2$, (8D.6) forces $Y_1 = Y_2$ and so $e \in Y_1$. This shows that $X \cap D\hat{F} \leq DY_1$. Since $|X \cap D\hat{F}| \geq q^4/2 > q^3 = |Y_1|$, there is an element $x \in X \cap D\hat{F} - Y_1$. Write $x = de$ with $d \in D - U_1$ and $e \in Y_1$. Likewise we may choose an element $x' \in X$ such that $x' = d'e'$ with $d' \in D^t - U_1$ and $e' \in Y_1$. But then $1 = [x, x'] = [d, d']$, which is a contradiction because $\mathcal{E}^*(C_M(J_1)) = \{D, D^t\}$.

Assume, therefore, that $B^g \leq S$. Then by (8B.4), $B^g \leq M_1 K_2, M_2, M_1 B \cap V$ or $M_1 B^t \cap V^t$. Notice that $B^g \cap M_1 K_2 \leq M_1, B \cap MK_2 \leq M_1, B \cap MK_2$ or $B^t \cap MK_2$ by (6H.3) and (8C.1). Furthermore, arguments of the above paragraph

imply that $|B^g \cap M_1| < q^5/2$. Hence if $B^g \not\leq M_2$, then $B^g \cap M_1 K_2 = B \cap M K_2$ or $B^t \cap M K_2$. But then $B^g \leq C_S(K_2) = M_2$ by (6J.2), a contradiction. Thus $B^g \leq M_2$, and then $B^g = B$ or B^t by (8C). The proof is complete.

(8.4) NOTATION. $M_1 B / M_1$ is a E_{q^2} -subgroup of S / M_1 and so $N_{L_1}(M_1 B) / M_1$ is a maximal parabolic subgroup of L_1 / M_1 . Since $S \cap S^r = M_1$, there is a Sylow 2-subgroup S^w of $N_{L_1}(M_1 B)$ such that $S^w \cap S^r > M_1$. Then $S^w \cap S^{wr} / M_1$ is a E_{q^2} -subgroup of S^w / M_1 different from $M_1 B / M_1$, as otherwise $S^w \cap S^{wr} = M_1$ and $M_1 B^r / M_1$ is L_1 / M_1 -conjugate to $M_1 B^t / M_1$, which is not the case. We may take w such that $w^2 \in M_1$.

(8F) *The following conditions hold:*

- (1) $N(M_1 B) \leq N(B)$.
- (2) $N_{L_1}(M_1 B) \leq N(M_1 B \cap V) \cap N(Z(V))$.

Let $X = O^{2'}(N_{L_1}(M_1 B))$ and let bars denote images in $N(B)/B$. Then

- (3) $\bar{X} = \bar{X}' \times \bar{W}^t$, $O_2(\bar{X}) = \bar{M}_1 \cap \bar{V}$, $O_2(\bar{X}')$ is a natural module for $\bar{X}' / O_2(\bar{X}) \cong \text{SL}(2, q)$, and $\bar{V} \in \text{Syl}_2(\bar{X})$. (See (8.2) for the definition of $\bar{W}$.)

PROOF. Since $B^t \not\leq M_1 B$, (1) is a consequence of (8E). Set $\bar{W}^* = Z(\bar{S}) \cap Z(\bar{S}^w)$. From (8B) we deduce that $Z(\bar{S}) = \bar{M}_1 \cap \bar{V}^t$. Hence $|\bar{M}_1 / \bar{W}^*| \leq q^2$. Since $X = \langle S, S^w \rangle$, we also have $\bar{W}^* \leq Z(\bar{X})$. If $\bar{W}^* \cap \bar{M}_2 \neq 1$, then the center of the preimage of $\bar{W}^* \cap \bar{M}_2$ is K_2 by (8C.1), and so $K_2 \triangleleft X$. But then $M_1 K_2 \triangleleft X$, which is a contradiction because X acts irreducibly on $M_1 B / M_1$. Thus $\bar{W}^* \cap \bar{M}_2 = 1$ and, since $Z(\bar{S}) \cap \bar{M}_2 \neq 1$, $Z(\bar{S}) \neq Z(\bar{S}^w)$. This implies that $\bar{X}$ acts nontrivially on $\bar{M}_1 / \bar{W}^*$. Hence $|\bar{M}_1 / \bar{W}^*| = q^2$ by Lemma (4B) of [7] and, moreover, $\bar{M}_1 / \bar{W}^*$ is a natural module for $\bar{X} / \bar{M}_1 \cong \text{SL}(2, q)$ by (1K). Moreover, $\bar{V}^t = \bar{W}^* \times \bar{M}_2$, and then $\bar{S} = \bar{W}^* \times \bar{V}$. So by Gaschütz's theorem, $\bar{W}^*$ has a complement in $\bar{X}$. Since $\bar{X} / \bar{W}^*$ is perfect, we conclude that $\bar{X} = \bar{X}' \times \bar{W}^*$.

Let $\bar{Y}$ be a complement for $\bar{S}$ in $N_{\bar{X}}(\bar{S})$. Then $\bar{Y}$ acts fixed-point-freely on $\bar{S} / \bar{W}^*$, and moreover $\bar{Y}$ normalizes $\bar{V}$ by (8B.5). Hence $\bar{V} = [\bar{S}, \bar{Y}] = \bar{S} \cap \bar{X}'$, and so $\bar{M}_1 \cap \bar{V} = O_2(\bar{X}')$. Thus $M_1 B \cap V \triangleleft N_{L_1}(M_1 B)$. To see $Z(V) \triangleleft N_{L_1}(M_1 B)$, it is sufficient to show that $Z(M_1 B \cap V) = Z(V)$. Let $\hat{Z} = Z(M_1 B \cap V)$. Then $\hat{Z} \leq C_S(K_2) \leq M_2$, so $\hat{Z} \leq Z(M_1 B \cap M_2) = K_2$ by (8C.1). So $\hat{Z} \leq M$ by (6H.3), and then $\hat{Z} \leq F_2$. Since $F_2 = Z(V)Z(V^t)$ and $C_S(x) = V^t$ for $x \in Z(V^t) - Z(V)$, we conclude that $\hat{Z} = Z(V)$, as required.

To prove (3), it is sufficient to show $\bar{W}^* = \bar{W}^t$. We know that $\bar{N}^t = \bar{M}_2 \times \bar{N}^t$. Let $\bar{Y}$ be a complement for $\bar{V}^t$ in $N_{\bar{N}^t}(\bar{V}^t)$. Since $\mathcal{E}^*(\bar{S}) = \{\bar{V}^t, \bar{M}_1\}$, $\bar{Y}$ normalizes $\bar{M}_1$. Also, $\bar{Y} \leq N(\bar{S}) \leq N(\bar{M}_1)$, so $\bar{Y}$ normalizes the centrer $\bar{W}^*$ of $\bar{X}$. Furthermore, $\bar{Y}$ centralizes $\bar{M}_2$ and acts fixed-point-freely on $\bar{V}^t / \bar{M}_2$. Since $\bar{V}^t = \bar{W}^* \times \bar{M}_2 = \bar{W}^t \times \bar{M}_2$, it follows that $\bar{W}^* = [\bar{V}^t, \bar{Y}] = \bar{W}^t$. The proof is complete.

(8.5) NOTATION. $A = D^t Z$.

(8G) The following conditions hold:

- (1) A is the only E_{q^4} -subgroup of $D^t \hat{F}$ all of whose nonidentity elements are G_0 -conjugate. A is a normal subgroup of L_1 .
- (2) $C_S(D) \leq M_1 B$.

Let x be an involution of S . Then the following holds:

- (3) If $x \in M_1$, then $x \in_{L_1} F_2$.
- (4) If $x \in M_2$, then $x \in_{L_2} M$.
- (5) $x \in_{G_0} U_1 \cup U_2$.
- (6) If $x \in U_1^*$, then

$$x^{G_0} \cap MK_2 \cap B = DZ(V^t)^\# \cup (\bigcup_{g \in N} Z(V^g)^\#),$$

and

$$x^{G_0} \cap C_M(J_1)A_1 = D^\# \cup D^{t\#}.$$

PROOF. First, A is elementary or order q^4 by (8D). By the structure of $\text{PSL}(3, q)$, $\langle S^w, S^{wr} \rangle = \langle B, B^r, M_1 \rangle$ and so we have

$$(a) \quad D \leq Z(\langle S^w, S^{wr} \rangle \bmod U_1).$$

In particular, $D \leq Z_2(S^w)$. Also, $Z(V) = Z(V)^w \leq Z_2(S^w)$ by (8F.2). So $F_2^w = Z_2(S^w) = DZ(V)$ and equality holds in (a). Now let $u \in U_1^*$. By (6E) involutions of $C_M(J_1)$ are all G_0 -conjugate and, in particular, $D^\# \leq u^{G_0} \cap F_2^w$. However, (8A.5) and (8E.4) imply that

$$(b) \quad u^{G_0} \cap F_2 = Z(V)^\# \cup Z(V^t)^\#.$$

Hence

$$(c) \quad u^{G_0} \cap DZ(V) = Z(V)^\# \cup D^\#,$$

and $D = Z(V^{tw})$. Since $w^2 \in M_1$, it follows that

$$(d) \quad D^{wt} = Z(V).$$

Since $D\hat{F} = C_{M_1}(D)$ by (8D), we have

$$(e) \quad (D\hat{F})^{wt} = C_{M_1}(Z(V)) = M_1 \cap V.$$

Hence (8D.5) shows that

$$(f) \quad \mathcal{E}^*(M_1 \cap V) = \{(DZ)^{wt}, (DZ^t)^{wt}, (DA_1^g)^{wt}, (DF_2^g)^{wt} \mid g \in N_{G_1}(\hat{F})\}.$$

In particular, $(DZ)^{wt} \leq M_1 \cap V$. Also, since $DZ \leq M_1 \cap V$ and $M_1 \cap V = M_1 \cap (M_1 B \cap V) \triangleleft N_{L_1}(M_1 B)$ by (8F.2), $(DZ)^{wt} \leq M_1 \cap V^t$. Thus $(DZ)^{wt} \leq M$. However, $(DZ)^{wt} \not\leq B^t$ as $DZ \not\leq B = B^w$ (see (8F.1)). Therefore

$$(g) \quad (DZ)^{wt} = M \cap B$$

by (8C.1). Now $C_L(U_1)$ centralizes D and acts transitively on $(Z/U_1)^\#$ by (8D.4). Hence it follows from (c) that $u^{G_0} \cap DZ = Z^\# \cup D^\#$. Conjugating by wt , we obtain from (d) and (g) that $u^{G_0} \cap M \cap B = (Z^{wt})^\# \cup Z(V)^\#$ and since $D^\# \cup Z(V^t)^\# \leq u^{G_0} \cap M \cap B$ by (b), it follows that $Z^{wt} = DZ(V^t)$. Thus

$$(h) \quad u^{G_0} \cap M \cap B = DZ(V^t)^\# \cup Z(V)^\#.$$

In particular, elements of $D^t Z(V)^\#$ are all G_0 -conjugate and since $C_L(U_1)$ centralizes D^t and acts transitively on $(Z/U_1)^\#$, it follows that elements of $A^\#$ are all G_0 -conjugate. Thus we deduce from (b), (c), and (f) that $A = (DZ^t)^{wt}$ and that A is the only element of $\mathcal{E}^*(M_1 \cap V)$ all of whose non-identity elements are G_0 -conjugate. Thus $A = A^{twt}$ and so by (e) A is the only element of $\mathcal{E}^*(D^t \hat{F})$ all of whose non-identity elements are G_0 -conjugate. Furthermore, $A \triangleleft S$ as $M_1 \cap V \triangleleft S$. So $A = A^r \triangleleft S^r$, and then $A \triangleleft L_1$ as $L_1 = \langle S, S^r \rangle$ by the structure of $PSL(3, q)$.

Now let bars denote images in L_1/U_1 and let $\bar{d} \in \bar{D}^\#$. Then $\langle \bar{S}^w, \bar{S}^{wr} \rangle \leq C_{L_1}(\bar{d}) < \bar{L}_1$ by (a), so $O^2(C_{L_1}(\bar{d})) = \langle \bar{S}^w, \bar{S}^{wr} \rangle$ by the structure of $PSL(3, q)$. As a consequence we have $C_{\bar{S}}(\bar{D}) = \bar{M}_1 \bar{B}$, whence $C_S(D) \leq M_1 B$, proving (2). As remarked earlier, equality holds in (a), so it also follows that $\bar{D}$ is a T.I. set of $\bar{L}_1$ and, since $|\bar{L}_1 : N_{L_1}(\bar{D})| = q^2 + q + 1$, $\bar{A}^t$ is partitioned by $\bar{L}_1$ -conjugates of $\bar{D}$. Hence if $x \in M_1$, then $x \in_{L_1} DA = C_M(J_1)Z$ and so $x \in_{L_1} C_M(J_1)Z(V)$ since $C_L(U_1)$ centralizes $C_M(J_1)$ and acts transitively on $(Z/U_1)^\#$. Hence if x is an involution, then $x \in_{L_1} DZ(V)$ or $D^t Z(V)$, and so by (d) either $x \in_{L_1} Z(V)^t Z(V) = F_2$ or $x \in_{L_1} A$. If $x \in A$, then $x \in_{L_1} D^t$ and so $x \in_{L_1} Z(V)$ by (d). Hence (3) holds.

By definition of B , $N \leq N(MK_2 \cap B)$. Next, $DZ(V^t) = A^t \cap B \triangleleft M_2$, so $[D, M_2] \leq DZ(V^t) \cap K_2 = Z(V^t)$. So $M \cap B = DF_2 \leq Z(V \bmod Z(V^t))$. Let $s \equiv vv^t \bmod M_2$, where $v \in N$. Then $V^s = V^v$. Thus setting $X = (M \cap B) \cap (M \cap B)^v$, we have $[N, X] \leq Z(V^t)$. Since $K_2 = Z(V^t)Z(V^{st})$ and N acts transitively on $Z(V^{st})^\#$ by (8A), it follows that $X \cap K_2 = Z(V^t)$ and, since $|MK_2 \cap B : X| \leq q^2$, $MK_2 \cap B = XK_2$. Thus N acts transitively on $(MK_2 \cap B/X)^\#$. Since N^t is transitive on $(B/K_2)^\#$ by (8B), we conclude that elements of B are L_2 -conjugate to an element of M . Since $\mathcal{E}^*(M_2) = \{B, B^t\}$, (4) follows.

Let Y be a complement for V in $N_N(V)$. Then $Y \leq N(S) \cap N(V^t) \leq N(M_1 \cap V^t)$, and so $Y \leq N(A^t)$ since A^t is the only element of $\mathcal{E}^*(M_1 \cap V^t)$ all of whose nonidentity elements are G_0 -conjugate. Thus Y normalizes $A^t \cap B = DZ(V^t)$. Since Y has no nontrivial fixed points on $K_2/Z(V^t)$ by (1K), it follows that

$$X/Z(V^t) = C_{MK_2 \cap B/Z(V^t)}(Y) = DZ(V^t)/Z(V^t),$$

which implies that $DZ(V^t) \triangleleft N$. Now let $x \in u^{G_0} \cap MK_2 \cap B$. Then $x^g \in M \cap B$ for some $g \in N$, and so $x^g \in DZ(V^t) \cup Z(V)$ by (h). This proves the first equality of (6). It also follows from (h) that $u^{G_0} \cap DZ(P) = D^*$. Hence $u^{G_0} \cap C_M(J_1)Z(P) = D^* \cup D^{t*}$ and, since $C_L(U_1)$ centralizes $C_M(J_1)$ and acts transitively on $(A_1/U_1)^*$, we have $u^{G_0} \cap C_M(J_1)A_1 = D^* \cup D^{t*}$, proving the second equation of (6).

Finally, let x be an involution of S . Then $x^g \in M_1K_2$ for some $g \in L_1$, and then $x^g \in M_1 \cup M_2$ by (6H.3). Thus x is G_0 -conjugate to an element of F_2 by (3) and (4). Using (6K) and arguments parallel to those used to prove (4J.4), we obtain (5).

(8H) S is a Sylow 2-subgroup of G_0 .

PROOF. Our arguments here are similar to those used in (4K) and (4M). We only sketch a proof. It is sufficient to show $S \in \text{Syl}_2(O^2(G))$. Let U be a Sylow 2-subgroup of $N(S)$ containing T . Then $U \in \text{Syl}_2(G)$ by (8E). Notice that U/S is abelian and U/T is cyclic because t acts on L_1/M_1 as a graph automorphism. Since $t^g \cap T \leq tS$ by (6K), we may assume that $T < U$. Then L_1/M_1 admits an involutory field automorphism and so q is a square. By (6K) $U = C_U(t)S$, so $C_U(t) - Q$ contains an involution u , which acts on L as a field automorphism. Then u acts on L_1/M_1 as a nontrivial field or graph-field automorphism. Also, either u interchanges two components of L_2/M_2 or acts on each of them as a nontrivial field automorphism. Thus arguing as in (4K), we conclude that $|C_S(u)| = q^5$ and $I(uS) = u^S$. Now we may assume that $C_U(t) \in \text{Syl}_2(C)$ since $N(Q) \leq N(S)$ by (6A.5). Since $\langle t \rangle$ has a complement $N_U(N)$ in U , $C_U(t) = \langle t \rangle \times X$ for some 2-group X embedded in $\text{Aut}(L)$. Hence $C_U(u) \cap C_U(t)$ covers $C_U(t)/P$, and so $C_U(u)$ covers U/S . Thus

$$|C|_2 = |U/S| q^4 < |U/S| q^5 = |C_U(u)|.$$

Thus $t \not\sim u$, and so by (6K) involutions of tS and uS are not conjugate. Since all these hold with u interchanged by ut and since involutions of U/S are represented by t, u , and ut , it follows that $t^g \cap U \leq tS$. Next, involutions of S are conjugate to an element of $Z(P)$ by (8G). Hence arguing as in the last part of the proof of (4M), we conclude that $u^g \cap T = \emptyset$. Thus $t, u \notin O^2(G)$. This completes the proof.

(8I) A is weakly closed in S with respect to G_0 .

PROOF. First, $A \triangleleft L_1$ by (8G). Clearly, $C_{L_1}(A) \leq M_1$, so $C_{L_1}(A) \leq C_S(Z(V)) \cap C_S(Z(V))^r = V \cap V^r = C_M(J_1)Z$. So $C_{L_1}(A) = C_M(J_1)Z \cap C(D^t) = D^tZ = A$. In particular, $A \in \mathcal{SCN}(S)$, so by (8H) A is a Sylow 2-subgroup of $C_{G_0}(A)$. Hence A can not be conjugate to a subgroup of M_2 .

Suppose that $A^g \leq M_1 B \cap V$ and set $\hat{A} = A^g \cap M_1 K_2$. Then $Z(V) \leq \hat{A}$ by the above. We show $\hat{A} \leq M_1$. If $Z(V) < \hat{A} \cap K$, then $\hat{A} \cap Z(V^s)$ contains a nonidentity element, say, z by (8A. 1). But then as $V^s \leq C_N(z) < N$, $A^g \leq C_V(z) = V \cap V^s = M_2$, which is impossible. So $\hat{A} \cap K_2 = Z(V)$, and consequently $\hat{A} \not\leq K_2$. If $\hat{A} \leq MK_2 \cap B$, then, as $\hat{A}^*$ fuses completely in G_0 , (8G. 6) implies that $\hat{A} \cap (DZ(V^t) - K_2)$ contains an element x . But then elements of $x(Z(V) - U_1)$ can not fuse to x , a contradiction. If $\hat{A} \leq MK_2 \cap B^t$, then the same lemma implies that $\hat{A} = D^t Z(V)$. But then $A^g \leq C_{M_1 B}(D^t) \leq M_1 B \cap M_1 B^t = M_1 K_2$ by (8G. 2), a contradiction. Thus $\hat{A} \not\leq MK_2$ and so $\hat{A} \leq M_1$ by (6H. 3). Next, let $x \in N_{L_1}(M_1 B)$. Then $A^{g^{x^{-1}}} \leq M_1 B \cap V$ by (8F), so $A^{g^{x^{-1}}} \cap M_1 K_2 \leq M_1$ by the above. By the structure to $PSL(3, q)$, $M_1 B$ is a union of $M_1 K_2^x$ as x runs through $N_{L_1}(M_1 B)$. Thus we conclude that $A^g \leq M_1$. We have shown that if $A^g \leq M_1 B \cap V$, then $A^g \leq M_1$. Thus (8B. 4) implies that if $A^g \leq S$, then $A^g \leq M_1$.

Assume that $A^g \leq M_1$ for $g \in G_0$. Then $|A^g \cap C_M(J_1)A_1| \geq q^2$, and so $D \leq A^g$ or $D^t \leq A^g$ by (8G. 6). If $D \leq A^g$, then $A^g \leq C_{M_1}(D) = D\hat{F}$ so $A^g = A^t$ by (8G. 1). Likewise if $D^t \leq A^g$, then $A^g = A$. We have shown that if $A^g \leq S$ for $g \in G_0$, then $A^g = A$ or A^t . Since both A and A^t are normal in S , and since $S \in \text{Syl}_2(G_0)$, (8I) follows from Burnside's lemma.

(8J) *Let bars denote images in $N(B)/B$. Then $\bar{W}^t$ is strongly closed in $\bar{S}$ with respect to $\bar{N}_{G_0}(B)$.*

PROOF. From (8B) we deduce that $\bar{S} = \bar{V}^t \bar{M}_1$, $\mathcal{E}^*(\bar{S}) = \{\bar{V}^t, \bar{M}_1\}$, and $\bar{W}^t = \bar{V}^t \cap \bar{M}_1$. Since $S \in \text{Syl}_2(N_{G_0}(B))$ by (8H), it is enough to show that $N_{G_0}(V^t)$ and $N_{G_0}(M_1 B)$ normalizes W^t . By (8E. 2) $N(V^t) \leq N(M_2)$, so $N(V^t) \leq N(N^t)$. Since $W^t = V^t \cap N^{t'}$, $N(V^t) \leq N(W^t)$. Next, $N_{G_0}(M_1 B) \leq N(M_1)$ by (8I), so $N_{G_0}(M_1 B) \leq N(N_{L_1}(M_1 B))$. Since $\bar{W}^t = Z(O^{2'}(\bar{N}_{L_1}(M_1 B)))$ by (8F. 3), $N_{G_0}(M_1 B) \leq N(W^t)$.

(8K) *We have $N_{G_0}(U_1) = N_{G_0}(M_1)O(N_{G_0}(U_1))$.*

PROOF. We argue that A is strongly closed in S with respect to $N_{G_0}(U_1)$. Suppose by way of contradiction that an element a of A is $N_{G_0}(U_1)$ -conjugate to an element b of $S - A$. Notice that $L_1 \leq N_{G_0}(U_1) \cap N(A)$. Hence we may assume that $b \in M_1 K_2$. Then $b \in M_1 \cup K_2$ by (6H. 3) and (8G. 6). So by (8G. 3) we may assume that $b \in K_2$. It also follows from (8G. 3) that elements of A are L_1 -conjugate to an element of $A \cap F_2 = Z(V)$. So we may also assume that $a \in K_2$. Then, since M_2 is weakly closed in S , a is $N_{G_0}(U_1) \cap N(M_2)$ -conjugate to b . However, this is impossible since $N_{G_0}(U_1) \cap N(M_2) \leq N_{G_0}(S) \leq N(A)$ by (8A. 4) and (8I). Thus A is strongly closed.

Let bars denote images in $N_{G_0}(U_1)/O(N_{G_0}(U_1) \bmod U_1)$, and let $\bar{X}$ denote

the normal closure of $\bar{A}$. As remarked above, elements of A are L_1 -conjugate to an element of $Z(V)$. Since J acts transitively on $(Z(V)/U_1)^\#$, it follows that $\bar{L}_1 J$ acts transitively on $\bar{A}^\#$. Hence either $\bar{X} = \bar{A}$ or $\bar{X}$ is a simple Goldschmidt group by [11]. In the latter case $[\bar{L}_1^\infty, \bar{X}] = 1$ as before, and then $[\bar{L}_1, \bar{A}] = 1$ as $\bar{L}_1 = \bar{M}_1 \bar{L}_1^\infty$. Since this is false, we must have $\bar{A} \triangleleft \bar{N}_{G_0}(U_1)$. By symmetry $\bar{A}^t \triangleleft \bar{N}_{G_0}(U_1)$, so $\bar{M}_1 \triangleleft \bar{N}_{G_0}(U_1)$ and the assertion follows.

(8L) We have $N_{G_0}(K_2) = N_{G_0}(M_2)O(N_{G_0}(K_2))$.

PROOF. Let bars denote images in $N(K_2)/K_2$. We shall show that $\bar{M}_2$ is a strongly closed subgroup of $\bar{S}$ with respect to $\bar{N}_{G_0}(K_2)$. Let $\bar{a} \in \bar{S} - \bar{M}_2$ and suppose by way of contradiction that $\bar{a}^g \in \bar{M}_2$ for some $\bar{g}$. Choose $\bar{g}$ so that $\bar{X}_1 = \bar{M}_2 \cap \bar{M}_2^{\bar{g}^{-1}}$ has maximal order. Let $\bar{X} = C_{\bar{M}_2}(\bar{a})$, $\bar{X}_0 = [\bar{M}_2, \bar{a}]$, and $\bar{X}_2 = N_{\bar{M}_2}(\langle \bar{X}_1, \bar{a} \rangle)$. Since $\bar{M}_2$ is a weakly closed subgroup of $\bar{S}$ by (8E), (1G) applies here. Thus $|\bar{M}_2/\bar{X}| = |\bar{X}_0| \leq |\bar{X}_2/\bar{X}_1|$ and $\bar{X}_2^g \cap \bar{M}_2 = \bar{X}_1^g$. Moreover, we may assume that $N_{\bar{S}}(\langle \bar{X}_1, \bar{a} \rangle)^g \leq \bar{S}$. Let $\langle \bar{X}, \bar{a} \rangle \leq \bar{Y} \in \mathcal{E}^*(\bar{S})$. Then $\bar{Y}^g \leq \bar{S}$. As remarked in (8B), each element of $\mathcal{E}^*(\bar{S})$ is weakly closed in $\bar{S}$ and so $\bar{g} \in N(\bar{Y})$. Suppose that $\bar{Y} = \bar{M}_1$. Then $\bar{a}^g \in \bar{M}_2 \cap \bar{Y} = \bar{M}$. But as $N(M_1 K_2) \leq N(M K_2)$ by (6H), this yields a contradiction. So $\bar{Y} \neq \bar{M}_1$, and therefore $\bar{Y} = \bar{M}_1 \bar{B} \cap \bar{V}$ or $\bar{M}_1 \bar{B}^t \cap \bar{V}^t$ by (8B). Assume $\bar{Y} = \bar{M}_1 \bar{B} \cap \bar{V}$, say. Then $\bar{X}_2 \geq \bar{X} = \bar{Y} \cap \bar{M}_2 \geq \bar{B}$ and $\bar{g} \in N(\bar{Y}) \leq N(\bar{B})$ by (8E.1). So $\bar{B} \leq \bar{X}_2^g$. Since $\bar{M}_2$ and $\bar{Y}$ are the only elements of $\mathcal{E}^*(\bar{S})$ that contain $\bar{B}$, it follows that $\bar{X}_2^g \leq \bar{Y}$. Therefore $\bar{X}_2 = \bar{Y} \cap \bar{M}_2 = Z(\bar{V})$. Let tildes denote images in $N_{G_0}(B)/B$, and recall from (8J) that $\tilde{W}^t$ is a strongly closed subgroup of $\tilde{S}$. By a theorem of Glauberman [10], there is an element $\tilde{h} \in N(\tilde{W}^t) \cap N(\tilde{Y})$ such that $\tilde{X}_2^{\tilde{g}^{-1}} = \tilde{X}_2^{\tilde{h}}$. Since $\tilde{Y} \tilde{W}^t \tilde{M}_1 \tilde{B}$, $\tilde{h} \in N(\tilde{M}_1 \tilde{B})$. However, $N_{G_0}(M_1 B) \leq N(M_1)$ by (8I), and then $N_{G_0}(M_1 B) \cap N(K_2) \leq N(M_1 K_2) \leq N(S)$ by (6J.3). Thus $\tilde{h} \in N(\tilde{M}_2)$ by (8B.5), and so $\tilde{h}$ normalizes $\tilde{V} = \tilde{Y} \tilde{M}_2$. This is a contradiction because $\tilde{h}$ moves $\tilde{X}_2 = Z(\tilde{V})$. Thus $\bar{M}_2$ is strongly closed in $\bar{S}$ with respect to $\bar{N}_{G_0}(K_2)$ and arguing as in (7E), we have the result.

For each $i \in \{1, 2\}$, we choose an element $u_i \in U_i^\#$ and set $C_i = C_{G_0}(u_i)$.

(8M) We have $C_1 = N_{C_1}(U_1)O(C_1)$ and C_1 is 2-constrained.

PROOF. Since $L_1 \leq C_1$, arguments of (8K) imply that U_1 is strongly closed in S with respect to C_1 . The result then follows as in (7F).

(8N) We have $C_2 = N_{C_2}(K_2)O(C_2)$ and C_2 is 2-constrained.

PROOF. As in (4P), $K_1 M_2 \in \text{Syl}_2(C_2)$. We argue that B is strongly closed in $X = K_1 M_2$ with respect to C_2 . Suppose this is false, and let $b \in X - B$ be C_2 -conjugate to an element of B . Since $S \in \text{Syl}_2(G_0)$, (8E) shows that B is a weakly closed subgroup of S with respect to G_0 . Choose $g \in C_2$ so that

g satisfies conditions (1), (2) of (1G). Let $X_1 = B \cap B^{g^{-1}}$, $X_2 = N_B(\langle X_1, b \rangle)$, $X_3 = C_B(b)$, and $X_0 = [B, b]$. Then by (1G), $X_2^g \cap B = X_1^g$, $X_2^g \not\leq B$, and $X_0 X_1 \leq X_3 \leq X_2$. Furthermore, we may assume that $N_X(\langle X_1, b \rangle)^g \leq X$. In particular, $X_2^g \leq X$ and so $X_2^g \leq M_2$ or MK_1 by (8B.4) and (6H.3). Hence by (1G),

$$|B/X_3| \leq |X_2/X_1| = |X_2^g/X_1^g| \leq q^2.$$

Also, $b \in M_2$ or MK_1 . Suppose that $b \notin M_2$. Then (8B.4) and (6H.3) imply that $C_X(b) = C_{MK_1}(b)$, and so $X_3 \leq C_{M \cap B}(b) \leq M \cap B \leq B$. Since $|B/X_3| \leq q^2$ and $|B : M \cap B| = q^2$, it follows that $X_3 = M \cap B$. But then $F_2 \leq X_3$ and $b \in C_S(F_2) = M_2$ by (6J), a contradiction. Thus $b \in M_2$ and so $b \in B^t$ and $X_3 = K_2$ by (8C). In particular $|B/X_3| = q^2$, so $|X_2/X_1| = q^2$. Also, $B^t \leq N_X(\langle X_1, b \rangle)$. So $B^{tg} \leq X$ and since B^t is also weakly closed in S with respect to G_0 , $g \in N(B^t)$. Suppose that $X_2^g \leq MK_1$. Then $K_2^g \leq X_2^g \cap B^t \leq M \cap B^t$ and so $K_2^g = M \cap B^t$. But then (1F) forces $M \cap B^t = K_2$, a contradiction. Hence $X_2^g \leq M_2$ and since $X_2^g \not\leq B$, $X_2^g \leq B^t$. Thus $X_2 = B \cap B^t = K_2$. Moreover, (1G.ii) implies that $K_2 = X_0 \times X_1$ and $|X_1| = q^2$. Since $X_1^g \leq B \cap B^t = K_2$, $X_1 \leq K_2^{g^{-1}}$. So $X_1 = K_2 \cap K_2^{g^{-1}}$ and $B^t = K_2 K_2^{g^{-1}}$.

Now let $\hat{b}$ be an arbitrary element of $K_2^{g^{-1}} - X_1$ and choose $\hat{g} \in C_2$ so that $\hat{b}^{\hat{g}} \in B$ and $\hat{X}_1 = B \cap B^{\hat{g}^{-1}}$ has maximal order. Then the above paragraph imply that $|\hat{X}_1| = q^2$ and $[B, \hat{b}] \cap \hat{X}_1 = 1$. Since $\hat{b}^{\hat{g}} \in B$ and $B \cap B^{\hat{g}^{-1}} = X_1$ has order q^2 , the above applies with $\hat{g}$ interchanged by g . Since $B^t = K_2 K_2^{g^{-1}}$, we conclude that $[B, a] \cap X_1 = 1$ for any $a \in B^t - K_2$. Let $a \in D^t - D$. Then $U_1 = [D, a] \leq [B, a]$ and so $[B, a] \cap Z(V) \neq 1$. Since $[N^t, B^t] \leq K_2$, N^t normalizes $[B, a]$. Since N^t acts irreducibly on $Z(V)$, and $[B, a] = q^2$ by (8C.1), we conclude that $[B, a] = Z(V)$. We know that N acts transitively on $(B^t/K_2)^*$. Hence

$$\{[B, a] \mid a \in B^t - K_2\} = \{Z(V)^x \mid x \in N\}.$$

Hence if $u \in U_1^*$, then $u^{G_0} \cap X_1 = \emptyset$ by (8G.6).

Now set $Y = \langle M_2, M_2^{g^{-1}} \rangle$ and notice that $Y \leq C(X_1)$. Suppose that $M_2 \notin \text{Syl}_2(N_Y(M_2))$. Then, since $X \in \text{Syl}_2(N_{C_2}(M_2))$, there is an element $h \in N_{C_2}(M_2)$ such that $M_2 < C_X(X_1^h)$. We set $\hat{X} = C_X(X_1^h)$ and argue that $Z(\hat{X}) = Z(P)$. Indeed, $Z(\hat{X}) \leq C_X(M_2) = K_2$. Next, since $A_1 \cap \hat{X} > Z(P)$, $Z(\hat{X}) \leq K_2 \cap M = F_2$ by (6H.3) and then $Z(\hat{X}) \leq Z(P)$ by (6A.3). Thus $Z(\hat{X}) = Z(P)$, as asserted. But now $X_1^h = Z(P)$, which is a contradiction because $u^{G_0} \cap X_1 = \emptyset$ for $u \in U_1^*$. Thus $M_2 \in \text{Syl}_2(Y)$ and we may assume $g \in Y$. Furthermore $Y \not\leq N(K_2)$ since $K_2 \neq K_2^{g^{-1}}$. It now follows from (1H) that Y/B^t has a strongly embedded subgroup. We shall show that this case does not occur.

Let bars denote images in $N(B)/B$. Then by (8J) $\bar{W}^t$ is a strongly closed subgroup of $\bar{S}$ with respect to $\bar{N}_{G_0}(B)$. Since $Z(\bar{S} \bmod \bar{W}^t) = \bar{M} \bar{W}^t$ is elementary, [11] implies that $\bar{W}^t$ is a Sylow 2-subgroup of its normal closure

in $\overline{N_{G_0}(B)}$. Also, $\overline{W}^t$ is a Sylow 2-subgroup of $\overline{N}^t$, which is isomorphic to $\text{SL}(2, q)$. In particular, nonidentity elements of $\overline{W}^t$ are all conjugate. Hence [11] implies that $\overline{Y}_0 = \overline{N}^t O(\overline{N_{G_0}(B)})$ is normal in $\overline{N_{G_0}(B)}$. Set $\overline{Y}_1 = C(\overline{Y}_0 / O(\overline{Y}_0))^\infty$ and $\overline{Y}_2 = O'(\overline{N_{L_1}(\overline{M}_1 B)})$. Then $\overline{Y}_2$ centralizes a Sylow 2-subgroup $\overline{W}^t$ of $\overline{N}^t$ by (8F). So $\overline{Y}_2 = \overline{Y}_2^\infty \leq \overline{Y}_1$. Also, $\overline{N} \leq \overline{Y}_1$ since $[\overline{N}, \overline{N}^t] = 1$ and $\overline{N}$ is perfect. Now $\overline{V} \in \text{Syl}_2(\overline{Y}_1)$ and $\mathcal{C}^*(\overline{V}) = \{\overline{M}_2, \overline{M}_1 B \cap \overline{V}\}$. Since $\overline{M}_2$ is a natural module for $\overline{N}$ by (8B), $\overline{N}$ acts transitively on $\overline{M}_2^\#$ and so $\overline{M}_1 B \cap \overline{V}$ is not strongly closed in $\overline{V}$. Likewise $\overline{M}_2$ is not strongly closed (use (8F)). Thus $\overline{Y}_1 / O(\overline{Y}_1) \cong \text{PSL}(3, q)$ by (1H.3) and so if $x \in N_{G_0}(B)$, then $\langle \overline{M}_2, \overline{M}_2^x \rangle / B$ cannot have a strongly embedded subgroup.

We have shown that B is strongly closed in X with respect to C_2 , and by symmetry so also is B^t . Thus K_2 is strongly closed and arguing as in (7G), we can complete the proof.

Using (8G)–(8N) and arguments in the last part of the proof of (4S), we can now prove the following:

(8O) G_0 is isomorphic to $\text{PSL}(5, q)$.

§ 9. The case $[A_1, F_2] = 1 = [A_2, F_1]$

We shall retain the notation of Section 5. From now on we may assume (5.1) and

(9.1) HYPOTHESIS. $[A_1, F_2] = 1 = [A_2, F_1]$.

(9.2) NOTATION. $M_i = FF^{r_i}$, $1 \leq i \leq 2$.

(9A) For $i \in \{1, 2\}$, M_i is elementary abelian.

PROOF. Since we are assuming (9.1), (5F) shows that $Z(E_i) = A_i$ and $Z(R_i) = Z(P)$, $1 \leq i \leq 2$. Thus $FK_i = FA_i$ is abelian by (5D). So $K = F * P$, and then it follows from the structure of P that $\mathcal{A}(K) = \{FK_1, FK_2\}$. Let A be an abelian subgroup of H of order at least q^5 and suppose that $A \not\leq K$. Choose a maximal abelian subgroup B of K containing $A \cap K$. Then $|A \cap K : A \cap F| \leq |B : F| \leq q$ and so $|A \cap F| \geq q^4/2$. But $A \cap F \leq Z(H) = Z(P)$ by (5C). Hence we must have $A \leq K$. Thus $\mathcal{A}(H) = \mathcal{A}(K)$ and $J(H) = K$. Consequently, $N(H) \leq N(K)$. Since $R_i \leq N(H)$ by (5B.2), R_i acts on $\mathcal{A}(H)$. Also, $R_i \leq N(FK_i)$ by (5E.1). Therefore $FK_{3-i} \triangleleft R_i$, $1 \leq i \leq 2$.

Suppose that $C_F(J_2)$ is homocyclic of exponent 4. Then $\Omega_1(FK_2/U_1) = F_1A_2/U_1$ by (5C) and (5D). Since U_1 and FK_2 are normal in R_1 , it follows that $F_1A_2 \triangleleft R_1$. But then $Q_1 = D_1(F_1A_2) \triangleleft R_1$ contrary to (5B.1). Thus $C_F(J_2)$ is elementary by (5D), and by symmetry so also is $C_F(J_1)$. Thus $FK_i =$

$A_i C_F(J_1) C_F(J_2)$ is elementary. As $N_L(A_i)$ acts transitively on $(M_i/K_i)^*$ by (5E.4), we conclude that M_i is elementary.

(9B) We have $R_i \leq N(R_{3-i})$, $1 \leq i \leq 2$.

PROOF. Set $X = N(H) \cap N(FK_1) \cap N(FK_2)$ and recall from the proof of (9A) that $R_i \leq X$, $1 \leq i \leq 2$. Since FK_1 is elementary and $C_{H_1}(t) = B_1$, (1C) implies that $\mathcal{E}^*(H_1) = \{B_1, FK_1\}$, and so $N(H_1) \leq N(B_1)$. In particular, $Y = C_X(H/FK_1) \leq N(B_1)$. Since $E_1 \triangleleft N(B_1)$ and $R_1 = E_1 H$, we have

$$Y \leq N(R_1) \cap N(B_1) = R_1(N_c(Q) \cap N_c(B_1)).$$

Furthermore, $R_1 \leq Y$ by (5E.1). Thus $R_1 \leq Y \leq R_1 C_c(P/A_1)$. Since $C_c(P/A_1) = QJ_1^2 O(C)$ by (2D.4), this implies that R_1 is the unique Sylow 2-subgroup of Y . Therefore $R_1 \triangleleft X$ and, by symmetry, $R_2 \triangleleft X$. Thus (9B) holds.

(9.3) NOTATION. $T = R_1 R_2$.

(9C) We have $M_i \triangleleft T$, $1 \leq i \leq 2$.

PROOF. Recall from (5B) that $\mathcal{E}^*(R_i/D_i) = \{E_i/D_i, H/D_i\}$. Since $N_{E_i}(Z(Q)) = N_{E_i}(Q) = H_i$ by (3A) and (5B), we also have $\mathcal{E}^*(E_i/Z(P)) = \{M_i/Z(P), H_i/Z(P)\}$. Thus $\mathcal{E}^*(R_i/Z(P)) = \{M_i/Z(P), H/Z(P)\}$. Since R_i and $Z(P) = Z(R_i)$ are normal in T by (9B), so is M_i .

(9.4) NOTATION. Set $S = M_1 M_2$. For $i \in \{1, 2\}$, define N_i to be the normal closure of $C_L(U_i)M_i$ in $N(M_i)$.

(9D) For $i \in \{1, 2\}$, N_i/M_i is isomorphic to $SL(2, q^2)$ or $SL(2, q) \times SL(2, q)$. If $N_i/M_i \cong SL(2, q) \times SL(2, q)$, then t interchanges two components of N_i/M_i . We have $C(N_i/M_i) = O(N(M_i) \bmod M_i)$ and $S \in \text{Syl}_2(N_i)$.

PROOF. First, we note that S is a maximal subgroup of T such that $T = \langle t \rangle S$. Let bars denote images in $N(M_i)/M_i$. Since $C_{E_i}(t) = B_i$ and M_i is elementary, (1C) implies that $\mathcal{E}^*(E_i) = \{\bar{M}_i, \bar{B}_i\}$. Hence we have $N(E_i) = N(B_i) \leq N(M_i)$. This implies that $C(\bar{t}) = \overline{N(B_i)}$. Thus arguing as in (4G), we get the result.

(9E) We have $Z(S) = M_1 \cap M_2 = F$ and $Z_2(S) = S$.

PROOF. By (9D) $C_{N_i}(M_i) = M_i$ or N_i . But clearly the latter case does not hold. So $C_{N_i}(M_i) = M_i$. In particular, $Z(S) \leq M_1 \cap M_2$. Clearly we also have $M_1 \cap M_2 \leq Z(S)$, so $Z(S) = M_1 \cap M_2 = F$. Also, $Z_2(S) = S$ by (9A) and (9C).

(9F) If $N_i/M_i \cong SL(2, q^2)$, $1 \leq i \leq 2$, then $\langle L^G \rangle \cong Sp(4, q^2)$.

PROOF. Let $x \in F^{r_1} - F$. Then $S^{r_1} \leq C_{N_1}(x) < N_1$ by (9E), so the structure of $\text{SL}(2, q^2)$ shows that $C_{N_1}(x) \leq N_{N_1}(S^{r_1})$. So $C_S(x) = M_1$ as $S \cap S^{r_1} = M_1$. Since $M_1 = FF^{r_1}$, (1C) implies that $\mathcal{E}^*(S) = \{M_1, M_2\}$. Consequently, $N_{N_i}(S) \leq N(M_{3-i}) \leq N(N_{3-i})$, $1 \leq i \leq 2$. Now arguing as in (4R), we find that $G_1 = \langle N_1, N_2 \rangle$ is a central extension of $\text{Sp}(4, q^2)$ with $|Z(G_1)|$ odd and $S \in \text{Syl}_2(G_1)$.

Now $L = \langle C_L(U_1), C_L(U_2) \rangle \leq G_1$ so $C(G_1) \triangleleft C(L) \cap N(G_1)$. Since $\langle t \rangle \in \text{Syl}_2(C(L) \cap N(G_1))$ by (3A) and $t \notin C(G_1)$, $C(G_1)$ has odd order. Hence if $S \leq U \in \text{Syl}_2(N(G_1))$, then $S = J_r(U)$ by (2F). Thus $N(U)$ normalizes $\mathcal{E}^*(S) = \{M_1, M_2\}$. Then, since $N_i = E(N(M_i) \bmod M_i)$ by (9D), we have $N(U) \leq N(G_1)$. Thus $U \in \text{Syl}_2(G)$. Also, U/S is cyclic as the outer automorphism group of $\text{Sp}(4, q^2)$ is cyclic. Furthermore, $t^G \cap S = \emptyset$ as $m(C) = m(B_i) < m(M_i)$ by (3A). Hence (1E) implies that $S \in \text{Syl}_2(O^{2,2'}(G))$. Since $S \leq G_1 \leq O^{2,2'}(G)$, it follows that $O^{2,2'}(G)$ is simple (recall that we are assuming $O(G) = 1$). Thus appealing to [8], we conclude that $\langle L^G \rangle = O^{2,2'}(G) \cong \text{Sp}(4, q^2)$. The proof is complete.

In view of (9F) we may assume from now on that N_i/M_i is isomorphic to $\text{SL}(2, q) \times \text{SL}(2, q)$ for some $i \in \{1, 2\}$. By symmetry we shall assume

(9.5) HYPOTHESIS. $N_1/M_1 = L_1/M_1 \times L'_1/M_1$, $L_1/M_1 \cong \text{SL}(2, q)$.

(9.6) NOTATION. $S_1 = S \cap L_1$, $G_0 = \langle N_1, N_2 \rangle$.

(9G) We have $G_0 = G_1 \times G'_1$, where $G_1 \cong \text{Sp}(4, q)$.

PROOF. First, $Z(N_1) = F \cap F^{r_1} = C_F(J_1)$ as $N_1 = \langle S, S^{r_1} \rangle$. Let bars denote images in $N(M_1)/Z(N_1)$. Denote by Z the preimage of $Z(\bar{N}_1)$. Then $[N_1, Z] \leq Z(N_1)$, and so $[N'_1, Z] = 1$ by the three-subgroup lemma. Since $N_1 = N'_1 M_1$, and M_1 is abelian, it follows that $Z = Z(N_1)$, whence $Z(\bar{N}_1) = 1$ and $Z(\bar{S}) = \bar{F}$. Thus, arguing as in (8B), we find that

$$(1) \quad \begin{aligned} \bar{N}_1 &= \bar{L}'_1 \times \bar{L}_1'^t, & O_2(\bar{L}_1) &= Z(\bar{L}_1), \\ \bar{M}_1 &= Z(\bar{S}_1)Z(\bar{S}_1'), & \text{and } Z(\bar{S}_1) \cap Z(\bar{S}_1') &= Z(\bar{S}). \end{aligned}$$

Furthermore, $\bar{L}'_1$ acts transitively on $Z(\bar{L}_1)^*$ and so $\bar{L}'_1$ is perfect. Now let X denote the preimage of $Z(\bar{L}_1)$. Then $[L_1, X] \leq Z(N_1)$ so $[L'_1, X] = 1$ by the three-subgroup lemma. Also, $L_1 = L'_1 X$ as $L_1/X \cong \bar{L}'_1$ is perfect. Thus L'_1 is perfect and $Z(\bar{L}_1) = Z(\bar{L}_1)'$. As a consequence we have

$$(2) \quad Z(\bar{S}_1) = \overline{Z(S_1)}$$

and

$$(3) \quad Z(S_1) = FZ(L_1),$$

since $Z(\bar{S}_1) = \overline{FZ(L_1)}$ by (1). Next, since $[L'_1, L_1'^t] \leq Z(N_1)$, another application of the three-subgroup lemma yields that

$$(4) \quad [L'_1, L'^t_1] = 1.$$

Now an analogue for $\bar{N}_1$ of (8B.4) shows that M_1 is the only E_{q^6} -subgroup of S that intersects with M_2 in $F = Z(S)$. Hence

$$(5) \quad N_{N_2}(S) \leq N(M_1).$$

Thus $Z(S_1)M_2$ is $N_{N_2}(S)$ -invariant, and moreover $|Z(S_1)M_2/M_2| = q$ by (2). In particular, $N_{N_2}(S)$ acts reducibly on S/M_2 , so by (9D) N_2/M_2 must be isomorphic to $SL(2, q) \times SL(2, q)$. Let

$$N_2/M_2 = L_2/M_2 \times L_2^t/M_2$$

with $L_2/M_2 \cong SL(2, q)$, and set $S_2 = S \cap L_2$. Then, by symmetry, the analogues of (1)–(5) with subscripts 1 and 2 interchanged hold. By the structure of $SL(2, q) \times SL(2, q)$, S_2/M_2 and S_2^t/M_2 are the only $N_{N_2}(S)$ -invariant E_q -subgroup of S/M_2 . So changing notation, we may assume that

$$(6) \quad Z(S_1)M_2 = S_2^t.$$

Then comparing orders, we have

$$(7) \quad S_1 \cap M_2 = Z(S_2^t).$$

Also, $Z(S_2^t) = FZ(L_2^t)$ by the analogue of (3), so $S_1 = M_1Z(S_2^t) = M_1Z(L_2^t)$. Since both M_1 and $Z(L_2^t)$ are elementary and normal in S , this implies that

$$(8) \quad S_1^2 \leq M_1 \cap Z(L_2^t).$$

Now set $P_i = S_i \cap L'_i$, $1 \leq i \leq 2$. Then S is a central product of P_1, P_1^t , and $Z(N_1)$ by (1) and (4), so $S^2 = P_1^2(P_1^t)^2$. Since $U_1 \leq P^2 \leq S^2$ and $\bar{N}_1 = \bar{L}'_1 \times \bar{L}_1^t$, it follows that

$$U_1 \leq (P_1^2 \cap Z(N_1))((P_1^t)^2 \cap Z(N_1)).$$

In particular, $P_1^2 \cap Z(N_1) \neq 1$. Also, $P_1^2 \cap (P_1^t)^2 \leq Z(N_2)$ by (8) and an analogue of (1), so $P_1^2 \cap (P_1^t)^2 \cap Z(N_1) = 1$. Thus $P_1^2 \cap U_1 = 1$. Since J acts irreducibly on $Z(N_1)/U_1$ (see the proof of (5D)), we conclude that

$$Z(N_1) = (P_1^2 \cap Z(N_1)) \times ((P_1^t)^2 \cap Z(N_1)).$$

Consequently, $Z(N_1) \leq L'_1 L_1^t$. Next, let tildes denote images in $N_1/P_1^2 \cap Z(N_1)$. Then $\widetilde{Z(\bar{N}_1)}$ has a complement in $\tilde{P}_1 \widetilde{Z(\bar{N}_1)}$, so by Gaschütz's theorem $\widetilde{Z(\bar{N}_1)}$ has a complement in $\tilde{L}'_1 \widetilde{Z(\bar{N}_1)}$. This implies that $L'_1 \cap Z(N_1) = P_1^2 \cap Z(N_1)$, so $L'_1 \cap L_1^t = 1$. Thus, by symmetry of the arguments,

$$(9) \quad N_i = L'_i \times L_1^t, \quad 1 \leq i \leq 2.$$

Consequently, $L'_i \cong C_{N_i}(t) = C_L(U_i)$, so that P_i is indecomposable and $P'_i = Z(P_i)$ by (2B).

We define

$$W_i = L'_i \cap M_i, \quad 1 \leq i \leq 2.$$

Thus for $i \in \{1, 2\}$ we have $W_i \triangleleft S$, $W_i \leq Z(S'_i)$, $M_i = W_i \times W_i^t$, and $P'_i = F \cap W_i$. Now $W_1 \cap W_2 \leq M_1 \cap M_2 \cap W_i = F \cap W_i = P'_i$, $1 \leq i \leq 2$, so $W_1 \cap W_2 = P'_1 \cap P'_2$. Next, since $S = P_1 \times P_1^t = P_2 \times P_2^t$, the Krull-Schmidt Theorem implies that $P'_2 = P'_1$ or P_1^t . Suppose that $P'_2 = P_1^t$. Then $W_1 \cap W_2 = 1$ by (9), and so $[W_1, W_2] = 1$. Also, since $W_1 \leq Z(S'_1) \leq S_2$ by (6) and $W_2^t \leq Z(S_2)$, $[W_1, W_2^t] = 1$. Hence W_1 centralizes $S = M_1 W_2 W_2^t$, which is a contradiction. Therefore

$$W_1 \cap W_2 = P'_1 = P'_2.$$

Now we define

$$R = W_1 W_2.$$

Then $|R| = q^4$ by the above. Thus

$$S = R \times R^t$$

since $S = RR^t$ and $|S| = q^8$.

Now $W_1 \leq M_1 \leq S_1$ and $W_2 \leq Z(S'_2) \leq S_1$ by (7), so $R \leq S_1$. Furthermore, $R \cap M_1 = W_1$ because $M_1 \cap W_2 = M_1 \cap M_2 \cap W_2 = F \cap W_2 = P'_2 = P'_1$. Let Y be a complement for S_1 in $N_{L_1}(S_1)$. Then $Y \leq N_{N_1}(S) \leq N(M_2)$ by an analogue of (5), and so Y normalizes $W_i = O_2(L_i)$, $1 \leq i \leq 2$, whence $Y \leq N(R)$. Since Y acts fixed-point-freely on S_1/M_1 and centralizes M_1/W_1 , we conclude that $R/W_1 = [S_1/W_1, Y] = P_1/W_1$. Thus, by symmetry

$$(10) \quad R = P_1 = P_2.$$

This shows that R is a Sylow 2-subgroup of both L'_1 and L'_2 .

Now we have $Z(R) = Z(P_1) = P'_1 = W_1 \cap W_2$. Since $M_1 \cap M_2 = Z(S) = Z(R) \times Z(R^t)$, it follows that

$$(11) \quad M_1 \cap M_2 = (W_1 \cap W_2)(W_1 \cap W_2)^t.$$

Hence

$$\begin{aligned} M_1 &= (M_1 \cap M_2)(M_1 \cap M_2^{r_1}) \\ &= (W_1 \cap W_2)(W_1 \cap W_2^{r_1})(W_1 \cap W_2)^t(W_1 \cap W_2^{r_1})^t. \end{aligned}$$

This implies that

$$(12) \quad W_1 = (W_1 \cap W_2)(W_1 \cap W_2^{r_1}).$$

Thus

$$(13) \quad R = W_2 W_1 = W_2 (W_1 \cap W_2^{r_1}).$$

Next, noticing that $S \cap S^{r_1} = M_1$, we deduce from (11):

$$(14) \quad \begin{aligned} M_2 \cap M_2^{r_1} &= (M_1 \cap M_2) \cap (M_1 \cap M_2^{r_1}) \\ &= (W_1 \cap W_2)(W_1 \cap W_2)^t \cap (W_1 \cap W_2^{r_1})(W_1 \cap W_2^{r_1})^t \\ &= (W_1 \cap W_2 \cap W_2^{r_1})(W_1 \cap W_2 \cap W_2^{r_1})^t \\ &= (W_2 \cap W_2^{r_1})(W_2 \cap W_2^{r_1})^t. \end{aligned}$$

By symmetry, this holds with subscripts interchanged. Hence

$$\begin{aligned} M_1 \cap M_2 &= (M_1 \cap M_1^{r_2})(M_2 \cap M_2^{r_1}) \\ &= (W_1 \cap W_1^{r_2})(W_2 \cap W_2^{r_1})(W_1 \cap W_1^{r_2})^t(W_2 \cap W_2^{r_1})^t. \end{aligned}$$

Thus

$$(15) \quad W_1 \cap W_2 = (W_1 \cap W_1^{r_2})(W_2 \cap W_2^{r_1}).$$

Now by (9) we may choose an involution $s_i \in L'_i$, $i \in \{1, 2\}$, such that $r_i = s_i s_i^t$. By the structure of $Sp(4, q)$ there is an element $a \in A_2 \cap A_2^{r_1 r_2}$ such that $r_1 \in (A_2^{r_1} \cap A_2^{r_1 r_2 r_1})^a$. From (14) we deduce

$$\begin{aligned} r_1 \in (M_2^{r_1} \cap M_2^{r_1 r_2 r_1})^a &= (M_2 \cap M_2^{r_1})^{r_2 r_1 a} \\ &= \{(W_2 \cap W_2^{r_1})(W_2 \cap W_2^{r_1})^t\}^{r_2 r_1 a} \\ &= (W_2^{r_1} \cap W_2^{r_1 r_2 r_1})^a \times (W_2^{r_1} \cap W_2^{r_1 r_2 r_1})^{a t}. \end{aligned}$$

Since $W_2 \leq P_1$ by (10), and $a \in N_1$, we also have $W_2^{r_1 a} \leq L'_1$. Thus $s_1 \in (W_2^{r_1} \cap W_2^{r_1 r_2 r_1})^a$. Now $[W_2^{r_1}, W_2^t] \leq [L'_1, L_1^t] = 1$ by (10). Also,

$$[W_2^{r_1 r_2 r_1}, W_2^{t r_1 r_2}] \leq [L_1'^{r_2 r_1 r_2}, L_1'^{t r_2 r_1 r_2}] = 1$$

since

$$W_2^{r_1 r_2 r_1} = W_2^{(r_2 r_1)^2} = W_2^{(r_1 r_2)^2} \leq L_1'^{(r_1 r_2)^2} = L_1'^{r_2 r_1 r_2}.$$

Furthermore, a normalizes W_2^t and $W_2^{t r_1 r_2}$ since $a \in M_2 \cap M_2^{r_1 r_2}$. Thus s_1 centralizes W_2^t and $W_2^{t r_1 r_2}$. Now by (13)

$$R^{t r_2} = \{W_2(W_1 \cap W_2^{r_1})\}^{t r_2} \leq W_2^t W_2^{t r_1 r_2}.$$

Hence $[s_1, R^{t r_2}] = 1$, and as $R^{r_1} = R^{s_1}$, we have

$$[R^{r_1}, R^{t r_2}] = [R, R^{t r_2}]^{s_1} \leq [L'_2, L_2^t]^{s_1} = 1.$$

Since $L'_i = \langle R, R^{r_i} \rangle$, we conclude that $[L'_1, L_2^t] = 1$.

Now we define

$$G_1 = \langle L'_1, L'_2 \rangle.$$

Then $[G_1, G'_1] = 1$ by the above. Therefore $G_0 = G_1 * G'_1$. It remains to show $G_1 \cong \text{Sp}(4, q)$. First, $1 = (r_1 r_2)^4 = (s_1 s_2)^4 (s'_1 s'_2)^4$, so $(s_1 s_2)^4 \in G_1 \cap G'_1 \leq Z(G_0)$. But $C(M_i) = M_i O(C(M_i))$ by (9D), whence it easily follows that $Z(G_0)$ has odd order. Hence we must have $(s_1 s_2)^4 = 1$. Next, $N_{L'_2}(R) \leq N_{N_2}(S) \leq N(M_1)$ by (5), so $N_{L'_2}(R) \leq N(L'_1)$. Now as in (4R) these together with the equations (12), (15), and their analogues with subscripts interchanged are enough to show $G_1 \cong \text{Sp}(4, q)$. Thus we have proved (9G).

To complete the proof of the main theorem, we must show that G_0 is a normal subgroup of G . However, we can not prove this directly, and it is necessary to assume that the main theorem holds in any proper section of G . This is because our approach here depends on (1N). Thus the rest of the section is devoted to proving that every involution of G_1 is critical.

(9.7) NOTATION. Let $R = S \cap G_1$, $\{W_1, W_2\} = \mathcal{E}^*(R)$ with $M_i = W_i W_i^t$, $1 \leq i \leq 2$, and $\hat{W}_i = Z(N_{G_1}(W_i))'$. Thus $W_1 \cap W_2 = Z(R) = \hat{W}_1 \times \hat{W}_2$. Let U be a Sylow 2-subgroup of $N(G_0)$ containing T , and let $V = N_U(G_1)$. Finally, set $W = C_V(G'_1)$.

(9H) *The following conditions hold:*

- (1) $C(G_0)$ has odd order.
- (2) $N(M_i) \leq N(G_0)$, $1 \leq i \leq 2$.
- (3) $N(S) \leq N(G_0)$.
- (4) $J_r(U) = S$.
- (5) $U \in \text{Syl}_2(G)$.
- (6) $N(G_0)$ controls the fusion of involutions of G_0 .
- (7) If X is a S -invariant subgroup of $N(G_0)$, then $N(G_0)$ contains a Sylow 2-subgroup of $N(X)$ containing S .

PROOF. For the proof of (1) see the second paragraph of the proof of (9F). By a Frattini argument $N(M_1) = N_1(N(S) \cap N(M_1))$. Since M_2 is the only E_{q^6} -subgroup of S that intersects with M_1 in $Z(S)$, $N(S) \cap N(M_1) \leq N(M_1) \cap N(M_2) \leq N(G_0)$. Thus $N(M_1) \leq N(G_0)$ and, by symmetry, $N(M_2) \leq N(G_0)$.

Let $x \in N(S)$. Then $\{M_1^x, M_2^x\}$ is a set of two E_{q^6} -subgroups of S that intersect in $Z(S)$. So $\{M_1^x, M_2^x\} = \{M_1, M_2\}$ or $\{W_1 W_2^t, W_2 W_1^t\}$. In the former case x normalizes $\{N_1, N_2\}$ and so $x \in N(G_0)$. In the latter case we may argue as follows:

$$\begin{aligned}
G_0 &= \langle N_{G_1}(W_1), N_{G_1}(W_2), N_{G_1}(W_1)^t, N_{G_1}(W_2)^t \rangle \\
&\leq \langle N(W_1 W_2^t), N(W_2 W_1^t) \rangle \\
&= \langle N(M_1), N(M_2) \rangle^x \\
&\leq N(G_0)^x.
\end{aligned}$$

Now $N(G_0)^\infty = G_0$ by (1). Thus $G_0 = N(G_0)^\infty = G_0^x$. This proves (3).

Let A be an abelian subgroup of U of rank at least $6n$, where $q=2^n$. Elements of $U-V$ interchange R and R^t . Hence if $A \not\leq V$, then $A \cap R = 1$ and so

$$m(A) \leq m(S/R) + m(SW/S) + m(V/SW) + m(U/V) \leq 3n + 3,$$

which is a contradiction. Therefore $A \leq V$, and it follows from (2F) that $A \leq S$. This proves (4).

Let X be a S -invariant subgroup of $N(G_0)$ and let Y be a Sylow 2-subgroup of $N(G_0) \cap N(X)$ containing S . (4) implies that S is a weakly closed subgroup of U with respect to G . So $N(Y) \cap N(X) \leq N(S) \cap N(X) \leq N(G_0) \cap N(X)$. Thus $Y \in \text{Syl}_2(N(X))$, and (7) holds. As a consequence we have (5). Moreover, $N(S)$ controls the fusion of elements of $Z(S)$. Since any involution of G_0 is G_0 -conjugate to an element of $Z(S)$, (6) follows.

(9I) If $t \in N(G_0)^g$, then $g \in N(G_0)$.

PROOF. First, since $m(C) < m(G_0)$, $t \notin G_0^g$. Suppose that $t \in N(G_1^q) \cap N(G_1^{tq})$. If t induces an inner automorphism on G_1^q , then $C_{G_1^q}(t)$ contains a Sylow 2-subgroup X of G_1^q . But then $\langle t, X \rangle \in \text{Syl}_2(\langle t, L \rangle)$ by (3A) and so an involution of G_1 is conjugate to an element of L , which is impossible by (9H.6). Thus t induces outer automorphisms on both G_1^q and G_1^{tq} . So by (2E), $C_{G_1^q}(t)^\infty$ and $C_{G_1^{tq}}(t)^\infty$ are isomorphic to $Sz(q)$, $Sp(4, q^{1/2})$ or A_6 when $q=4$, and moreover $C_{G_1^q}(t)^\infty \times C_{G_1^{tq}}(t)^\infty \leq L$. Since L has no such subgroups, we have $G_1^{qt} = G_1^{tq}$. Consequently $L = C_{G_1^q}(t)' = \{xx^t \mid x \in G_1^q\}$. Let $Y \in \text{Syl}_2(G_1^q)$. Then $\hat{P} = \{yy^t \mid y \in Y\}$ is a Sylow 2-subgroup of L . Now $N(Q)$ acts on $\{B_1, B_2\}$ and since M_i is the unique abelian maximal subgroup of $E_i = O_2(N(B_i))$, $N(Q)$ acts on $\{M_1, M_2\}$. Thus $N(Q) \leq N(S) \leq N(G_0)$ and so $N(\langle t, \hat{P} \rangle) \leq N(G_0)$. If $y \in I(Z(Y))$, then clearly $y \in N(\langle t, \hat{P} \rangle)$. So $L < N(G_0) \cap G_0^g$ and then Lemma 2.5.3 of [1] implies that $G_0^g \leq N(G_0)$. Thus $G_0^g = G_0$. The proof is complete.

If one uses the Unbalanced Group Theorem, one can stop here and appeal to a result of Seits [19]. However, using more fusion arguments, we can avoid the Unbalanced Group Theorem.

(9J) Assume that $S < V$. Then $N_v(T) - S$ contains an involution v such that $v^g \cap V = I(vS) = v^S$.

PROOF. First observe that $S < N_v(T)$. Since $I(tS) = t^s$, $N_v(T) = C_v(t)S$ and so $C_v(t) - P$ contains an involution v . Since $v \notin SW \cup SW^t$, v induces a nontrivial outer automorphism both on G_1 and on G_1^t , and hence $I(vS) = v^s$ by (1D) and (2E). Thus it remains to prove $v^g \cap V \leq vS$. Since V/SW is cyclic, involutions of V are contained in vS, SW or SW^t . Suppose $w = v^g \in SW$. Conjugating in $N(G_1) \cap N(G_1^t)$, we may assume that $|C_{wW^t}(w)| \geq |SW : S| q^6$. Then since $|C_U(v)| \leq |U : S| q^4$, it follows that $|C_U(v)| < |C_U(w)|$. Also, if $u \in I(tV)$, then $|C_U(u)| \leq |U : S| q^4 < |C_U(w)|$ since u interchanges R and R^t . Thus we may take w to be an extremal conjugate of v , in which case we may also assume that $C_U(v)^g \leq U$. In particular, $t^g \in U$ so $g \in N(G_0)$ by (9I). But then $v^g \notin SW$, a contradiction. Thus $v^g \cap V \leq vS$.

(9K) We have $N(\hat{W}_i \hat{W}_i^t) \leq N(G_0)$, $1 \leq i \leq 2$.

PROOF. Let $Y = N(\hat{W}_1 \hat{W}_1^t)$, $Y_0 = N(\hat{W}_1) \cap N(\hat{W}_1^t)$, and $Y_1 = O^2(Y)$. By (9H.6), $\hat{W}_1 \cup \hat{W}_1^t$ is strongly closed in $\hat{W}_1 \hat{W}_1^t$ and so Y acts on $\{\hat{W}_1, \hat{W}_1^t\}$. Thus $Y_1 \leq Y_0$ and $Y_1 \cap N(G_0) \leq N(G_1) \cap N(G_1^t)$. Also, $S \leq N_{G_1}(\hat{W}_1) N_{G_1}(\hat{W}_1^t)^t \leq Y_1$. Let $S \leq \hat{V} \in \text{Syl}_2(Y_1 \cap N(G_0))$. We may assume $\hat{V} \leq V$. By (9H), $N_{Y_1}(\hat{V}) \leq Y_1 \cap N(G_0)$ and so $\hat{V} \in \text{Syl}_2(Y_1)$. If $S < \hat{V}$, then by (9J) $N_{\hat{V}}(T) - S$ contains an involution v such that $v^g \cap \hat{V} \leq vS$. But then since V/S is abelian, (1E) forces $v \notin Y_1$, a contradiction. Thus $S \in \text{Syl}_2(Y_1)$.

Now any element of $\mathcal{E}^*(S)$ is normal in S . So two elements of $\mathcal{E}^*(S)$ are conjugate in Y_1 if and only if they are conjugate in $N_{Y_1}(S)$. Using (9H.3) and the fact that $W_1 = O_2(N_{G_1}(\hat{W}_1))$, we deduce that $N_{Y_1}(S) \leq N(W_1) \cap N(W_1^t)$. Thus $N_{Y_1}(S) \leq N(M_1)$ and so $N_{Y_1}(S) \leq N(M_2)$ as in the proof of (9H.2). Since $|\mathcal{E}^*(S)| = 4$, it follows that any element of $\mathcal{E}^*(S)$ is weakly closed in S with respect to Y_1 . We shall prove that M_1 is strongly closed in S with respect to Y_1 .

Let $a \in S - M_1$ and suppose that $a^g \in M_1$ for some $g \in Y_1$. Choose g so that $|M_1 \cap M_1^{g^{-1}}|$ is maximal and let $X_1 = M_1 \cap M_1^{g^{-1}}$, $X_2 = N_{M_1}(\langle X_1, a \rangle)$, and $X = C_{M_1}(a)$. By (1G) we may assume $N_S(\langle X_1, a \rangle)^g \leq S$. Let $\langle X, a \rangle \leq A \in \mathcal{E}^*(S)$. Then $(AX_2)^g \leq S$, so $g \in N_{Y_1}(A)$. If $A = M_2$, then $N_{Y_1}(A) \leq N_{Y_1}(G_0)$ by (9H.2). However, since $N_{Y_1}(G_0) \leq N(M_1)$ as before, this yields a contradiction. Thus $A \neq M_2$, and we may suppose that $A = W_1 W_2^t$, in which case $X = A \cap M_1 = W_1 Z(R^t)$. By (9H.6), $W_1 \cup W_2^t$ is strongly closed in A with respect to G and so $N(A)$ acts on $\{W_1, W_2^t\}$. Thus $N_{Y_1}(A) \leq N(W_1)$ and so $W_1 \leq X_1$. Also, $W_1 \leq X_2^g$ and since M_1 and A are the only element of $\mathcal{E}^*(S)$ that contain W_1 , it follows that $X_2^g \leq A$. Thus $X_2 = A \cap M_1$ and since $|X_2/X_1| \geq |M_1/X| = q$ by (1G) and since $W_1 \hat{W}_1^t \leq X_1$, it follows that $X_1 = W_1 \hat{W}_1^t$. Thus $X_1 \triangleleft N_{Y_1}(A)$ and so $AM_1 = C_S(X_1)$ is a Sylow 2-subgroup of $Z = C(X_1) \cap N_{Y_1}(A)$. Now $\mathcal{E}^*(AM_1) = \{A, M_1\}$ and $A \cap M_1 = X_2$ is not normal in Z , so (1H.1) implies that Z/A has

a strongly embedded subgroup. Let bars denote images in $N(A)/A$ and let $\bar{Z}_0 = O^*(\bar{Z})$. Then $\hat{Z}_0 = \bar{Z}_0/O(\bar{Z}_0)$ is isomorphic to $SL(2, q)$ by [4]. Next, $C_{G_1}(\hat{W}_1)$ is perfect and normalizes the Sylow 2-subgroup AM_1 of Z_0 . Hence $C_{G_1}(\hat{W}_1)$ centralizes $\hat{Z}_0$, which implies that $Z_0 = N_{Z_0}(RA)O(Z_0 \bmod A)$. However, since $\mathcal{E}^*(RA) = \{A, M_2\}$, $N_{Z_0}(RA) \leq N(M_2) \leq N(G_0)$ and so $N_{Z_0}(RA)^\infty \leq (C_{G_0}(X_1) \cap N_{G_0}(A))^\infty = 1$, a contradiction. Therefore M_1 is strongly closed in S with respect to Y_1 .

Let bars denote images in $Y_1/O(Y_1 \bmod \hat{W}_1\hat{W}_1^t)$ and set $X = \langle M_1^{\bar{Y}_1} \rangle$. Then by [11], $\bar{X} = O_2(\bar{X})\bar{X}_1 \cdots \bar{X}_k$ with $\bar{X}_i$ a Goldschmidt group and, moreover, $M_1 \in \text{Syl}_2(X)$ since $S = M_1M_2$. Now $N_1 = C_{G_1}(\hat{W}_1)C_{G_1}(\hat{W}_1)^t$ and so the structure of N_1 is known. In particular, $N_1 \leq Y_1$ and there are precisely three $\bar{N}_1$ -classes of elements of $\bar{M}_1^*$. Thus $k \leq 3$ and as before $\bar{N}_1$ centralizes $E(\bar{X})$. Since $Z(\bar{N}_1) = 1$, $E(\bar{X}) = 1$ and so $\bar{M}_1 \triangleleft \bar{Y}_1$, which shows that $M_1O(Y_1) \triangleleft Y_1$.

Now $N_C(V_1) \leq N(M_1) = N(M_1) \cap N(G_0) \leq Y$ and $\hat{W}_1\hat{W}_1^t \cap C = U_1$. Hence $C_{O(Y)}(t) \leq O(N_C(V_1)) = O(N_C(B_1)) \leq C(A_1)$ and since $[M_1, t] = A_1$, (1J) implies that $A_1 \leq C_{M_1}(O(Y))$. Clearly we also have $\hat{W}_1\hat{W}_1^t \leq C_{M_1}(O(Y))$. Since $\bar{W}_1$ and $\bar{W}_1^t$ are the only nontrivial $\bar{N}_1$ -invariant subgroups of $\bar{M}_1$, it follows that $[M_1, O(Y)] = 1$. Thus $M_1 \triangleleft Y_1$. Using (9H) and a Frattini argument, we conclude that $Y \leq N(G_0)$. The proof is complete.

(9L) *The following conditions hold:*

- (1) $N(Z(R)) \leq N(G_0)$.
- (2) *If $R^g \leq N(G_0)$, then $g \in N(G_0)$.*

PROOF. We first argue that $N(Z(S)) \leq N(G_0)$. By (9H.6), $Z(R) \cup Z(R^t)$ is strongly closed in $Z(S)$, so $N(Z(S))$ acts on $\{Z(R), Z(R^t)\}$. Likewise $\hat{W}_1 \cup \hat{W}_2$ is strongly closed in $Z(R)$. It therefore follows that $N(Z(S))$ acts on $\{\hat{W}_1\hat{W}_1^t, \hat{W}_2\hat{W}_2^t, \hat{W}_1\hat{W}_2^t, \hat{W}_2\hat{W}_1^t\}$. Hence if $g \in N(Z(S))$, then $\{\hat{W}_1\hat{W}_1^t, \hat{W}_2\hat{W}_2^t\}^g = \{\hat{W}_1\hat{W}_1^t, \hat{W}_2\hat{W}_2^t\}$ or $\{\hat{W}_1\hat{W}_2^t, \hat{W}_2\hat{W}_1^t\}$. In each case arguing as in the proof of (9H.3) and using (9K), we have $G_0^g = G_0$. Thus $N(Z(S)) \leq N(G_0)$.

Assume that $g \in N(Z(R)) - N(G_0)$. Then (9I) and (2.2) of [19] imply that there is an element $u \in t^g$ such that $G_0^g = G_0^u$. We have $G_1^u = G_1^g$ or G_1^{tg} . Assume the former and take an involution x of $\langle Z(R), Z(R)^u \rangle$ centralized by u . Since $Z(R)^u \leq G_1^{gu} = G_1$, x is a central involution of G_0 , and so by (9H.7) $C(x) \cap N(G_0)$ contains a Sylow 2-subgroup of $C(x)$. But this implies that t is conjugate to an element of $N(G_1) \cap N(G_1^t)$ against (9I). Hence we must have $G_1^u = G_1^{tg}$ and $G_1^t = G_1^{gu}$. But now $\hat{S} = RR^{gu}$ is a Sylow 2-subgroup of G_0 and $Z(\hat{S}) = Z(R)Z(R^u)$ is normalized by u , which is impossible as shown in the first paragraph. Therefore $N(Z(R)) \leq N(G_0)$.

Assume that $R^g \leq U$. Then since $m(U/S) \leq 3$, $W_i^g \cap S \neq 1$ and so $W_i^g \cap R \neq 1$ or $W_i^g \cap R^t \neq 1$ by (9H.6). Thus $W_i^g \leq V$, $1 \leq i \leq 2$. So $R^g \leq \Omega_1(V)$ and

since V/S is abelian, $Z(R^g) = (R^g)^2 \leq S$. Then by (9H.6) we may assume $Z(R^g) \leq R$, in which case (2E) implies that R^g induces inner automorphisms on G_1 , whence $R^g \leq SW^t$. By (9H.6), $W_i^g \cap S \leq R$ or R^t , $1 \leq i \leq 2$. However, $Z(R^g) \leq R$, so $W_i^g \cap S \leq R$. Also, since SW^t/S is cyclic, $|W_i^g \cap S| \geq q^3/2 > q^2$. Thus $W_1^g \cap S \leq W_i$ and $W_2^g \cap S \leq W_{3-i}$, where $i \in \{1, 2\}$, and so $Z(R^g) \leq W_1 \cap W_2 = Z(R)$. Therefore $g \in N(Z(R)) \leq N(G_0)$. This proves (2).

(9M) For any involution $a \in G_1$, we have $C(a) \leq N(G_0)O(C(a))$.

PROOF. We first argue that involutions of S are not conjugate to an element of $V-S$. Suppose by way of contradiction that $a \in I(V-S)$ and $b = a^g \in S$. Since $N_V(T)/S$ is cyclic, (9J) implies that $a \notin N_V(T)$. So $a \in SW^t$ or S^tW . Assume the former say. In this case conjugating by an element of G_1 , we may assume $a \in C(R)$ and so $R^g \leq C(b)$. Now by (9H.7), $N(G_0)$ contains a Sylow 2-subgroup of $C(b)$, so $R^{gc} \leq N(G_0)$ for some $c \in C(b)$. But then $gc \in N(G_0)$ by (9L) and $b = a^{gc} \in G_0$ whereas $a \notin G_0$. This is the desired contradiction.

Now let $a \in I(\hat{W}_1)$ and $X = C(a)$. By (9H.7), $N(G_0)$ contains a Sylow 2-subgroup $\hat{S}$ of X with $S \leq \hat{S}$. We show that W_1 is strongly closed in $\hat{S}$ with respect to X . For this end let $b \in \hat{S}$ and suppose $b^g \in W_1$ for some $g \in X$. Since $\hat{S} \leq N(G_1)$, the above paragraph implies that $b \in S$. Furthermore, since $ab \sim ab^g \in W_1$, (9H.6) forces $b \in R$. Thus $R^t \leq C_X(b) \cap C_X(b^g)$ and so $\langle R^t, R^{t^g-1c} \rangle$ is a 2-group for some $c \in C_X(b)$. Furthermore, $\langle R^t, R^{t^g-1c} \rangle^x \leq N(G_0)$ for some $x \in G$. It therefore follows from (9L) that $g^{-1}c \in N_X(G_0)$. Thus we may take $g \in N_X(G_0)$. However, $N_X(G_0) \leq N(W_1)$ since $O_2(C_{G_1}(a)) = W_1$. Thus W_1 is strongly closed, as asserted. Now a result of Goldschmidt [12] implies that $C(W_1)^\infty O(X) \triangleleft X$. Also, since $C(W_1) \leq N(G_0)$ by (9L.1), it follows that $C(W_1)^\infty = G_1^t$. Thus $G_1^t O(X) \triangleleft X$. Using a Frattini argument and (9L), we conclude that $X \leq N(G_0)O(X)$. Likewise if $a \in I(\hat{W}_2)$, then $C(a) \leq N(G_0)O(C(a))$.

Finally let $a \in Z(R) - \hat{W}_1 \cup \hat{W}_2$ and let $X = C(a)$. Arguing as before, we see that $N(G_0)$ contains a Sylow 2-subgroup $\hat{S}$ of X with $S \leq \hat{S}$ and that if $b \in Z(R)^*$ and $b^g \in \hat{S}$ for some $g \in X$, then we may take $g \in N_X(G_0)$. Since $N_X(G_0) \leq N(Z(R))$, it follows that $Z(R)$ is strongly closed in $\hat{S}$ with respect to X . Thus $X \leq N(G_0)O(X)$ just as before. This completes the proof of (9M).

§ 10. Conclusion of the proof of the theorem

So far we have been assuming $O(G) = 1$. In this section we drop this condition and assume instead that the main theorem holds in any

proper section of G satisfying the hypothesis of the theorem. Let X denote the normal closure of L . If $O(G) \neq 1$, let bars denote images in $G/O(G)$. Then $\bar{L}$ is a standard subgroup of $\bar{G}$ and $C_{\bar{G}}(\bar{L})$ has a cyclic Sylow 2-subgroup. By the induction hypothesis, the main theorem holds in $\bar{G}$ and so $\bar{X}$ is isomorphic to one of the groups listed in the theorem. In each case we easily find a t -invariant 2-subgroup A of X such that $1 \neq [A, t] \leq L$. Then (1J) implies that $[A, t, O(G)] = 1$ and, in particular, $C_X(O(X)) \not\leq O(X)$. Thus $O(X) = Z(X)$ and the main theorem holds in G . Therefore we may suppose that $O(G) = 1$. Furthermore, in view of (4R), (4S), (7H), (8O), and (9F) we may assume Hypothesis (9.5). Then (9G), (9H.1), (9I), and (9M) imply that G satisfies Hypothesis (1.1) with $B \cong Sp(4, q)$ and that every involution of B is critical. In particular, G satisfies (1M.ii). We know that B satisfies (1M.i) and (1N.iii). Finally, (1N.iv) holds by induction and by the results of Aschbacher [2] and Aschbacher-Seitz [3]. Thus we can apply (1N) to conclude that $X \cong Sp(4, q) \times Sp(4, q)$. This completes the proof of the theorem.

References

- [1] M. Aschbacher, On finite groups of component type, III. J. Math., **19** (1975), 87–115.
- [2] —, A characterization of certain Chevalley groups and its application to component type groups, to appear.
- [3] M. Aschbacher and G. Seitz, On groups with a standard component of known type, to appear.
- [4] H. Bender, Transitive Gruppen gerader Ordnung, in denen jede Involution genau einen Punkt festläßt, J. Algebra, **17** (1971), 527–554.
- [5] R. Carter, Simple groups of Lie type, John Wiley and Sons, London, 1972.
- [6] W. Feit and J. Thompson, Solvability of groups of odd order, Pacific J. Math., **13** (1963), 755–1029.
- [7] P. Fong and G. Seitz, Groups with a (B, N) -pair of rank 2 I, II, Invent. Math., **21** (1973), 1–57; **24** (1974), 191–240.
- [8] R. Gilman and D. Gorenstein, Finite groups with Sylow 2-subgroups of class two I, II, Trans. Amer. Math. Soc., **207** (1975), 1–126.
- [9] G. Glauberman, Central elements of core-free groups, J. Algebra, **4** (1966), 403–420.
- [10] G. Glauberman, A sufficient condition for p -stability, Proc. London Math. Soc., **25** (1972), 253–287.
- [11] D. Goldschmidt, 2-Fusion in finite groups, Ann. of Math., **99** (1974), 70–117.
- [12] D. Goldschmidt, Strongly closed 2-subgroups of finite groups, to appear.
- [13] D. Gorenstein, Finite groups, Harper and Row, New York, 1968.
- [14] D. Gorenstein and J. Walter, Centralizers of involutions in balanced groups, J. Algebra, **20** (1972), 284–319.
- [15] D. Gorenstein and J. Walter, Balance and generation in finite groups, J. Algebra, **33** (1975), 224–287.
- [16] R. Griess, D. Mason, and G. Seitz, Bender groups as standard subgroups, to appear.
- [17] C. Hering, W. Kantor and G. Seitz, Finite groups having a split BN -pair of

- rank 1, *J. Algebra*, **20** (1972), 435–475.
- [18] M. O’Nan, A characterization of $L_n(q)$ as a permutation group, *Math. Z.*, **127** (1972), 301–314.
- [19] G. Seitz, Chevalley groups as standard subgroups I, to appear.
- [20] R. Solomon, 2-Signalizers in groups of alternating type, to appear.
- [21] R. Steinberg, Lectures on Chevalley groups, Lecture Note, Yale University, 1967.
- [22] M. Suzuki, Finite groups in which the centralizer of any element of order 2 is 2-closed, *Ann. of Math.*, **82** (1965), 191–212.
- [23] F. Timmesfeld, Groups with weakly closed T.I. subgroups, *Math. Z.*, **143** (1975), 243–278.
- [24] A. Wagner, On finite affine line transitive planes, *Math. Z.*, **87** (1965), 1–11.

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Diagram 0

Diagram 1

Diagram 2

Diagram 3