Finite groups with a standard subgroup isomorphic to $U_4(2^n)$

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(Received Feb. 10, 1977)

Let K be a subgroup of a finite group G . If K has even order while $K \cap K^g$ has odd order for any $g \in G - N_G(K)$, then K is said to be *tightly embedded* in G . Furthermore, a quasisimple subgroup L of G is said to be a *standard subgroup* if $K = C_G(L)$ is tightly embedded in G , $N_G(K) = N_G(L)$ and $[L, L^g] \neq 1$ for any $g \in G$. In this paper we prove the following theorem.

THEOREM. *Let G be a finite group and let L be a standard subgroup of G . Assume that $L/Z(L) \simeq U_4(q)$, where $q = 2^n > 2$. Furthermore assume that $C_G(L)$ has a cyclic Sylow 2-subgroup. Let X be the normal closure of L in G . Then one of the following holds:*

- (1) $LO(G) \triangleleft G$
- (2) $X \simeq L_4(q^2)$
- (3) $X \simeq U_4(q) \times U_4(q)$.

After the fundamental work of Aschbacher [2], much work has been done toward the classification of finite groups with a standard subgroup of known type (e.g. [5], [12], [16], [20], [21]). For details we refer the readers to [12]. As stated in the theorem above, the result of this paper classifies the groups with a standard subgroup isomorphic to $U_4(2^n)$ with $2^n > 2$ whose centralizer has a cyclic Sylow 2-subgroup.

In order to prove the theorem we may assume that $O(G) = 1$ (cf. Section 6). Then provided that $L \not\triangleleft G$, our aim is to obtain that $\langle L^G \rangle \simeq L \times L$ or $L_4(q^2)$. The method used in this paper is essentially a careful analysis of 2-local subgroups of G which heavily depends on the structure of 2-local subgroups of $U_4(q)$. Let A_1 and A_2 be the largest normal 2-subgroups of the maximal parabolic subgroups of L . Then a major part of the proof is devoted to constructing the 2-subgroups F_1 and F_2 such that $N_L(A_i)' < N(F_i)$ (cf. (3.5), (3.9)). Next, applying the previous classification theorem [16] to the groups $N_G(F_i)/F_i$, we construct the subgroups L_i of $N_G(F_i)$, $i = 1, 2$ (cf. (3.12)). Here two cases occur:

- (1) $L_1/F_1 \simeq L_2(q^2) \times L_2(q^2)$, $L_2/F_2 \simeq L_2(q^2)$;

$$(2) \quad L_1/F_1 \simeq L_2(q^2) \times L_2(q^2), \quad L_2/F_2 \simeq L_2(q) \times L_2(q).$$

Case (1) is treated in Section 4 and Case (2) in Section 5. We note that L_1 is obtained as a normal closure of $N_L(A_1)'$ in $N(F_1)$ while L_2 is not obtained in the same way and some fusion argument is needed to prove $L_2 \triangleleft N(F_2)$.

The author would like to express his hearty thanks to Dr. Kensaku Gomi. The general line of the proof is shown by him in his paper [12] on which the author greatly depends. Also the author is grateful to Dr. Hiromichi Yamada for providing a lemma appearing as (1.6) in the present paper which contributes to the simplification of the argument on 2-groups in Section 3.

Our notation is standard and we refer the readers to [13] except possibly the following. If X is a 2-group,

$\mathcal{E}(X)$ = the set of the maximal elementary abelian subgroups of X ,

$Jr(X)$ = the subgroup generated by the elementary abelian subgroup of X of maximal rank.

By $J(X)$ we denote the usual Thompson subgroup generated by the abelian subgroups of X of maximal order. For any finite group G ,

$E(G)$ = the maximal normal semisimple subgroup of G ,

$G^{(\infty)}$ = the final term of the derived series of G ,

$G^i = \langle g^i \mid g \in G \rangle$ for an integer i ,

$I(X)$ = the set of the involutions of any subset X of G .

If X and Y are subgroups of a group,

$X * Y$ = a central product of X and Y ,

$X \cdot Y$ = a semidirect product of X by Y .

For subsets X and Y of a group,

$$X^Y = \{y^{-1}xy \mid x \in X, y \in Y\}.$$

If X acts on a set Ω , then X^Ω denotes the image of the permutation representation of X on Ω . By Goldschmidt groups we mean quasisimple groups called groups of type I and II in [10].

§1. Preliminaries

(1.1) Let $\langle t \rangle \times A$ be a normal 2-subgroup of a group H and suppose $\langle t \rangle \times A$ is elementary abelian, $|H : C_H(t)| = \text{even}$ and $C_H(t) \cap N_H(A)$ acts transitively on $A^\#$ by conjugation. Let $\bar{H} = H / C_H(\langle t \rangle A)$. Furthermore suppose

that $\overline{C_H(t)}$ has a cyclic normal subgroup $\bar{J}$. Then $\overline{C_H(A)}$ is elementary abelian of order $|A|$ and $t^H t^{C(A)} = tA$.

PROOF. We will consider the action of H on t^H . Then $t^H = \{t\} \cup A^*$ or tA . In any case H is doubly transitive on t^H and contains a cyclic normal subgroup in its one-point-stabilizer. Hence by Theorem 3 of [1] either $\bar{H}$ has a regular normal subgroup on t^H or $\langle \bar{J}^H \rangle \simeq L_2(r)$ for some prime r . Suppose that the former holds and let M be the preimage of the regular normal subgroup and let S be a Sylow 2-subgroup of M . Then $Z(M) \cap \langle t \rangle A = Z(S) \cap \langle t \rangle A$. Hence $Z(M) \cap \langle t \rangle A \neq 1$. $C_H(t) \cap N_H(A)$ acts on $Z(M) \cap \langle t \rangle A$ and this action yields that $Z(M) \cap \langle t \rangle A = A$ and $t^H = tA$. Suppose that the latter holds. Then, since $\langle \bar{J}^H \rangle$ acts also on $(\langle t \rangle A)^* - t^H$, we get $\langle \bar{J}^H \rangle \simeq L_2(7)$ and $\langle t \rangle A$ has order 16. If $t^H = tA$, then $\langle \bar{J}^H \rangle$ is embedded in the parabolic subgroup of $GL_4(2)$ leaving the subspace A invariant. Then by a property of Levi decomposition (cf. [6]) $\langle \bar{J}^H \rangle$ has a fixed point on tA , which is a contradiction. If $t^H = \{t\} \cup A^*$, then a two-point-stabilizer of tA^* in $\langle \bar{J}^H \rangle$, which is a fours group, stabilizes a point on $\{t\} \cup A^*$. But a Sylow 2-subgroup of $L_2(7)$ acts regularly on $\{t\} \cup A^*$. This contradiction completes the proof.

(1.2) Let $q = 2^n$. Suppose that $q^4 - q^3 + q^2 - q + 1 = p^m$ for some prime p . Then $m = 1$.

PROOF. We prove by way of contradiction. First suppose that $m = 2m_1 + 1$, $m_1 \geq 1$. Then

$$q(q^3 - q^2 + q - 1) = (p - 1)(p^{2m_1} + p^{2m_1-1} + \cdots + 1).$$

Since $q = 2^n$, q divides $p - 1$. This yields that $m_1 = 1$. Put $p - 1 = qs$ for some integer s . Then

$$q^3 - q^2 + q - 1 = s((qs + 1)^2 + (qs + 1) + 1).$$

Hence

$$q(q^2 - q + 1) = s^3 q^2 + 3s^2 q + 3s + 1.$$

Then we have $q | 3s + 1$ and also $q \geq s^3 + 1$, which yield that $s = 1$. So $q = 2$ or 4. Then by direct calculation we can easily derive a contradiction in each case.

Next suppose that $m = 2m_1$. Then

$$q(q^3 - q^2 + q - 1) = (p^{m_1} + 1)(p^{m_1} - 1).$$

Hence $p^{m_1} - 1$ or $p^{m_1} + 1 = \frac{1}{2}qs$ for some odd integer s . Then

$$q(q^2 - q + 1) = \frac{1}{4}qs^2 + (1 \pm s).$$

Now by direct calculation we may assume that $q \geq 8$. Hence $s \pm 1 = \frac{1}{4}qr$ for some integer r . Then

$$(64 - r^2)q^2 + (-64 \pm 8r)q + 48 \mp 16r = 0.$$

In either case we have $r \leq 8$. Then checking for each r , $0 \leq r \leq 8$, we can easily obtain a contradiction.

(1.3) Let E be a normal 2-subgroup of a group H . Suppose that E is elementary abelian of order q^2 . Set $\bar{H} = H/C_H(E)$ and suppose that $\bar{H}$ has an abelian normal subgroup $\bar{J}$ which acts on E^* transitively. Let t be an involution in $H - C_H(E)$ and let $\bar{x}$ be an element of $\bar{J}$. Then $\bar{x}^t = \bar{x}^q$.

PROOF. We will define multiplication and addition $+$ in E and show that $(E, +, \cdot)$ is a field of q^2 elements. Name the identity element of E 0 and any one element of E^* 1. Define multiplication in E as follows. Let $a \cdot 0 = 0$ for all $a \in E$. For $b \in E$, $b \neq 0$, there exists a unique element $\bar{u}$ of $\bar{J}$ such that $1^{\bar{u}} = b$, since $\bar{J}$ is regular on E^* . Then let $a \cdot b = a^{\bar{u}}$ for $a \in E$. Thus $0 \cdot b = 0$. If $1^{\bar{u}} = b$ and $1^{\bar{v}} = c$ for $\bar{u}, \bar{v} \in \bar{J}$, then $1^{\bar{u}\bar{v}} = b^{\bar{v}}$, and $(a \cdot b) \cdot c = (a^{\bar{u}}) \cdot c = a^{\bar{u}\bar{v}} = a \cdot 1^{\bar{u}\bar{v}} = a \cdot b^{\bar{v}} = a \cdot (b \cdot c)$. Clearly $a \cdot 1 = a$, so 1 is a multiplicative identity of $(E, +, \cdot)$. If $1^{\bar{u}} = a$ and $1^{\bar{u}^{-1}} = d$ for $\bar{u} \in \bar{J}$, then $a \cdot d = a^{\bar{u}^{-1}} = 1^{\bar{u}\bar{u}^{-1}} = 1$. Also $b \cdot c = 1^{\bar{u}\bar{v}} = 1^{\bar{v}\bar{u}} = c \cdot b$. Hence $E - 0$ is an abelian group under the multiplication.

Next define addition as follows. $a + b = ab$ for $a, b \in E$. Then $0 + a = a + 0 = a$, since 0 is defined to be the identity of E . Let a, b and c be in E . If $c = 0$, then $(a + b) \cdot c = 0 = a \cdot c + b \cdot c$. Now assume that $c \neq 0$ and as above we put $1^{\bar{u}} = c$. Then we have

$$(a + b) \cdot c = (a + b)^{\bar{u}} = (ab)^{\bar{u}} = a^{\bar{u}}b^{\bar{u}} = (a \cdot c)(b \cdot c) = a \cdot c + b \cdot c.$$

Thus it is established that $(E, +, \cdot)$ is a finite field of q^2 elements.

Suppose that $h \in H$ such that $0^{\bar{h}} = 0$ and $1^{\bar{h}} = 1$. Let $1^{\bar{u}} = a$ and let $1^{\bar{v}} = b$. Then

$$\begin{aligned} (a + b)^{\bar{h}} &= (ab)^{\bar{h}} = a^{\bar{h}}b^{\bar{h}} = a^{\bar{h}} + b^{\bar{h}} \\ (a \cdot b)^{\bar{h}} &= a^{\bar{v}\bar{h}} = 1^{\bar{u}\bar{v}\bar{h}} = 1^{\bar{h} - 1\bar{u}\bar{v}\bar{h}} = 1^{\bar{h} - 1\bar{u}\bar{h}} \cdot 1^{\bar{h} - 1\bar{v}\bar{h}} = 1^{\bar{u}\bar{h}} \cdot 1^{\bar{v}\bar{h}} = a^{\bar{h}} \cdot b^{\bar{h}}. \end{aligned}$$

Thus $h \in \text{Aut}((E, +, \cdot))$. So $a^{\bar{h}} = \underbrace{a \cdot a \cdots a}_{2^i}$ for any $a \in E$ and for some integer

i . Put $j = 2^i$. For any $h \in H$ there exists $\bar{u} \in \bar{J}$ such that $0^{\bar{u}\bar{h}} = 0$ and $1^{\bar{u}\bar{h}} = 1$. Then $1^{\bar{x}\bar{u}\bar{h}} = 1^{\bar{x}^j}$ for any $\bar{x} \in \bar{J}$. On the other hand $1^{\bar{x}\bar{u}\bar{h}} = 1^{(\bar{u}\bar{h})^{-1}\bar{x}\bar{u}\bar{h}} = 1^{\bar{h} - 1\bar{x}\bar{h}} = 1^{\bar{x}\bar{h}}$. Thus $1^{\bar{x}\bar{h}} = 1^{\bar{x}^j}$ and so $\bar{x}^{\bar{h}} = \bar{x}^j$. Hence in particular if $t \in H - C_H(E)$ is an involution, then $\bar{x}^t = \bar{x}^q$.

(1.4) Let V be an elementary abelian group of order 2^{2m} for some integer m . Let f be an element of order 2^i which acts on V . Suppose that $|C_V(f^{2^{i-1}})| = 2^m$. Then any element x in $f^j V$ whose order is equal to $|f^j|$ is conjugate to f^j in $V\langle f \rangle$.

PROOF. If $|x|=2$, then the assertion is easily checked. Suppose that $|x|>2$. Take $v \in V$ such that $x=f^j v$. Then by induction we may assume that there exists an element u in V such that $x^{2u}=f^{2j}$. This implies that $[v, f^j]=[u, f^{2j}]=[u, f^j, f^j]$. Hence $v[u, f^j] \in C_V(f^j)$. Since $C_V(f^j) \leq C_V(f^{2^{i-1}}) = [V, f^{2^{i-1}}] \leq [V, f^j]$ by the assumption, we have $v[u, f^j]=[w, f^j]$ for some $w \in V$. Then $v=[uw, f^j]$ and consequently $x^{uw}=f^j$.

The following is the well known (generalized) "Thompson transfer lemma" and the proof is omitted.

(1.5) Let S be a Sylow 2-Subgroup of a group G and let R be a subgroup of S such that $S' \leq R$. Suppose that x is an involution of $S-R$ such that $x^G \cap S \subseteq xR$. Then $x \notin O^2(G)$.

The following is due to H. Yamada [20] (2.4).

(1.6) Let X be a p' -group acting on a p -group A . Suppose that $C_A(X) \trianglelefteq A$. Then $A = C_A(X) * [A, X]$.

§2. Properties of $U_i(q)$

In this section we fix the notation for various subgroups of $L \simeq U_i(q)$, $q=2^n > 2$, and obtain some basic properties about them and the automorphism group of L , which are required in the following sections.

We identify the elements of L with 4×4 matrices x with entries in $GF(q^2)$ satisfying the conditions

$$x^* r x = r \quad \text{and} \quad \det(x) = 1,$$

where $r = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}$ and x^* denotes the adjoint matrix of x conjugated by the

involutive automorphism of $GE(q^2)$. L has order $q^6(q^2-1)(q^3+1)(q^4-1)$ (see for example [7, p. 8]). For $a \in GF(q^2)$ set $\bar{a} = a^q$ and denote by $GE(q^2)^\times$ the multiplicative subgroup of $GF(q^2)$. Denote by P the subgroup of L consisting of the matrices

$$\begin{pmatrix} 1 & & & \\ a & 1 & & \\ c & b & 1 & \\ d & \bar{c} + \bar{a}b & \bar{a} & 1 \end{pmatrix},$$

where $b \in GF(q)$ and $d + \bar{d} + a\bar{c} + \bar{a}c = 0$. Then P is a Sylow 2-subgroup of L .

We define the following subgroups of P :

$$A_1 = \left\{ \begin{pmatrix} 1 & & & \\ & 1 & & \\ c & b & 1 & \\ d & \bar{c} & & 1 \end{pmatrix} \right\} \quad \text{and} \quad A_2 = \left\{ \begin{pmatrix} 1 & & & \\ a & 1 & & \\ c & & 1 & \\ d & \bar{c} & \bar{a} & 1 \end{pmatrix} \right\}.$$

Then A_1 is elementary abelian. Moreover we have

$$Z(P) = A'_2 = \left\{ \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ d & & & 1 \end{pmatrix} \right\} \quad \text{and} \quad Z_2(P) = A_1 \cap A_2 = \left\{ \begin{pmatrix} 1 & & & \\ & 1 & & \\ c & & 1 & \\ d & \bar{c} & & 1 \end{pmatrix} \right\}.$$

Denote by I the subgroup of L consisting of the matrices

$$\begin{pmatrix} a & & & \\ & b & & \\ & & \bar{b}^{-1} & \\ & & & \bar{a}^{-1} \end{pmatrix},$$

where $ab\bar{a}^{-1}\bar{b}^{-1} = 1$. Furthermore define

$$I_1 = \left\{ \begin{pmatrix} a & & & \\ & a & & \\ & & a^{-1} & \\ & & & a^{-1} \end{pmatrix} \middle| a \in GF(q)^\times \right\}$$

and

$$I_2 = \left\{ \begin{pmatrix} a^2 & & & \\ & a^{q-1} & & \\ & & a^{q-1} & \\ & & & \bar{a}^{-2} \end{pmatrix} \middle| a \in GF(q^2)^\times \right\},$$

which are subgroups of I .

Denote by N_1 the subgroup consisting of the matrices

$$\begin{pmatrix} s & t & & \\ u & v & & \\ & & \bar{s} & \bar{t} \\ & & \bar{u} & \bar{v} \end{pmatrix},$$

where $\begin{pmatrix} s & t \\ u & v \end{pmatrix} \in SL_2(q^2)$. Denote by N_2 the subgroup consisting of the matrices

$$\begin{pmatrix} 1 & & & \\ & s & t & \\ & u & v & \\ & & & 1 \end{pmatrix},$$

where $\begin{pmatrix} s & t \\ u & v \end{pmatrix} \in SL_2(q)$. Finally define

$$r_1 = \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{pmatrix}, \quad r_2 = \begin{pmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{pmatrix},$$

$$z_1 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ 1 & & & 1 \end{pmatrix}, \quad z_2 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ 1 & & 1 & \\ & 1 & & 1 \end{pmatrix}.$$

Let $A = \text{Aut}(L)$. Then A is a semidirect product of L by a cyclic subgroup generated by a field automorphism f of order $2n$ which acts on L canonically. Let $u \in I(\langle f \rangle)$ and let $\tilde{P} \in \text{Syl}_2(A)$ such that $\langle u, p \rangle \leq \tilde{P}$.

Most of the results of this section are obtained by direct calculation and the proofs are omitted or given briefly.

$$(2.1) \quad (1) \quad N_L(P) = PI.$$

$$(2) \quad Z(P) = C_P(I_2^{r_1}), Z_2(P) = Z(P) \cdot C_P(I_1^{r_2}), A_i = Z_2(P) \cdot C_P(I_{3-i}), P = A_i \cdot C_P(I_i),$$

$i = 1, 2.$

$$(3) \quad I \text{ acts transitively on } C_P(I_i)^* \text{ and on } C_P(I_i^{3-i})^*, i = 1, 2$$

$$(4) \quad N_L(A_i) = A_i \cdot (I_i \times N_i), i = 1, 2.$$

$$(5) \quad N_L(Z(P)) = N_L(A_2) \text{ and } C_L(Z(P)) = A_2 \cdot (I_2^{q-1} \times N_2).$$

(2.2) (1) N_1 acts on A_1 irreducibly and $I_1 \times N_1$ has just two orbits $z_1^{I_1 N_1}$ and $z_2^{I_1 N_1}$ on A^* whose lengths are $(q-1)(q^2+1)$ and $(q-1)q(q^2+1)$ respectively. In particular $A_1 N_1$ is perfect. Also it holds that $|z_1^{N_1}| = (q-1)(q^2+1)$ and $|z_2^{N_1}| = q(q^2+1)$.

(2) $A_2 N_2$ is perfect. Set $\overline{N_L(Z(P))} = N_L(Z(P))/Z(P)$. Let $x_1 \in I(A_2 - Z(P))$ and let $x_2 \in A_2^* - I(A_2)$. Then $|\bar{x}_1^{I_2 N_2}| = (q+1)(q^2-1)$ and $|\bar{x}_2^{I_2 N_2}| = (q-1)q(q^2-1)$.

(3) $C_L(z_1) = C_L(Z(P))$ and $C_{N_1}(z_2) = C_{N_1}(u) \simeq SL_2(q)$. Moreover $C_L(z_2)' = (A_1 \cap C(u))C_{N_1}(u)$ is perfect.

PROOF. By Lemma 4B of [8] A_1 is an irreducible N_1 -module. By calculation we have

$$C_{N_1}(z_1) = \left\{ \begin{bmatrix} s & 0 & & \\ u & v & & \\ & & \bar{s} & 0 \\ & & \bar{u} & \bar{v} \end{bmatrix} \left| \begin{pmatrix} s & 0 \\ u & v \end{pmatrix} \in SL_2(q^2) \right. \right\}$$

and

$$C_{N_1}(z_2) = C_{N_1}(u) = \left\{ \begin{bmatrix} s & t & & \\ u & v & & \\ & & s & t \\ & & u & v \end{bmatrix} \left| \begin{pmatrix} s & t \\ u & v \end{pmatrix} \in SL_2(q) \right. \right\}.$$

Also

$$C_{I_1 N_1}(z_1) = \left\{ \begin{bmatrix} s & 0 & & \\ u & v & & \\ & & \bar{s} & 0 \\ & & \bar{u} & \bar{v} \end{bmatrix} \begin{bmatrix} 1 & & & \\ & a & & \\ & & a^{-1} & \\ & & & 1 \end{bmatrix} \left| \begin{pmatrix} s & 0 \\ u & v \end{pmatrix} \in SL_2(q^2), a \in GF(q)^\times \right. \right\}$$

and

$$C_{I_1 N_1}(z_2) = \left\{ \begin{bmatrix} s & t & & \\ u & v & & \\ & & s & t \\ & & u & v \end{bmatrix} \left| \begin{pmatrix} s & t \\ u & v \end{pmatrix} \in SL_2(q) \right. \right\}.$$

Hence $|z_1^{I_1 N_1}| = (q-1)(q^2+1)$ and $|z_2^{I_1 N_1}| = (q-1)q(q^2+1)$. Since $(q-1)(q^2+1) + (q-1)q(q^2+1) = q^4 - 1$, (1) holds.

By calculation $P' = Z_2(P)$. Since $r_2 \in N_2$ and since $A_2 = Z_2(P)Z_2(P)^{r_2}$, we have $A_2 \leq (A_2 N_2)'$. Which implies that $A_2 N_2$ is perfect. Similarly we have

$$C_{I_2 N_2}(\bar{z}_2) = \left\{ \begin{pmatrix} a & & & \\ & a^{-1} & & \\ & u & a & \\ & & & a^{-1} \end{pmatrix} \mid a \in GF(q)^\times, u \in GF(q) \right\}.$$

Hence $|\bar{z}_2^{I_2 N_2}| = (q+1)(q^2-1)$. Let $x = \begin{pmatrix} 1 & & & \\ c & 1 & & \\ d & \bar{c} & 1 & \\ & & 1 & 1 \end{pmatrix}$, $c \neq \bar{c}$. Then

$$C_{I_2 N_2}(\bar{x}) = \left\{ \begin{pmatrix} 1 & & & \\ & s & t & \\ & u & v & \\ & & & 1 \end{pmatrix} \begin{pmatrix} a^2 & & & \\ & a^{q-1} & & \\ & & a^{q-1} & \\ & & & \bar{a}^{-2} \end{pmatrix} \begin{pmatrix} s = a^{-q+3} + tc \\ u = a^{-q+3}c + a^{q-3}c + tc^2, \\ v = a^{q-3} + tc \end{pmatrix} \right. \\ \left. \left(\begin{pmatrix} s & t \\ u & v \end{pmatrix} \in SL_2(q), a \in GF(q^2)^\times \right) \right\}.$$

Hence in particular $a^{q-3} + a^{-q+3} \in GF(q)$, so $a^{q-3} + a^{-q+3} = a^{1-3q} + a^{-1+3q}$. This implies that $a^{q-1}(a^q + a^{-1})^2(a^{-(q+1)} + 1)^4 = 0$. Hence $a^{q+1} = 1$. Then, since $t = (a^4 + a^{-4})/(c + \bar{c})$, $|\bar{x}^{I_2 N_2}| = (q-1)q(q+1)(q^2-1)/(q+1) = (q-1)q(q^2-1)$. Now $(q+1)(q^2-1) + (q-1)q(q^2-1) = q^4 - 1 = |\bar{A}_2^*|$. Thus (2) holds. (3) is obtained by direct calculation.

- (2.3) (1) $\mathcal{C}(P \bmod Z(P)) = \{A_1, A_2\}$.
 (2) $Jr(\bar{P}) = A_1$, $Jr(\bar{P} \bmod A_1) = P$ and $Jr(P \bmod Z(P)) = A_2$.
 (3) $|C_{A_1}(x)| \leq q^2$ for $x \in P - A_1$.
 (4) $|C_{A_1}(x)| \leq q^3$ for $x \in \bar{P} - P$.
 (5) If B is a maximal abelian subgroup of P different from A_1 , then $|B| \leq q^3$.

PROOF. Set $\bar{P} = P/Z(P)$. Then $\bar{P} = \bar{A}_1 \bar{A}_2$, where $\bar{A}_1$ and $\bar{A}_2$ are elementary abelian, and $C_{\bar{P}}(\bar{x}) = \bar{A}_1$ for any $\bar{x} \in \bar{A}_1 - \bar{A}_2$. This implies (1). For (5) assume

that $B < A_2$. Set $(a, c, d) = \begin{pmatrix} 1 & & & \\ a & 1 & & \\ c & & 1 & \\ d & \bar{c} & \bar{a} & 1 \end{pmatrix}$. Let $x = (x_1, x_2, x_3) \in A_2 - Z(P)$. Then

$C_{A_2}(x) = \{(a, c, d) | a\bar{x}_2 + \bar{a}x_2 = c\bar{x}_1 + \bar{c}x_1\}$. Hence $|C_{A_2}(x)| = q^4$. Moreover we have $C_{A_2}(x) = C_{A_2}(x')$ for any element $x' = (kx_1, kx_2, x'_3)$, where $k \in GF(q)$. Suppose that $y = (y_1, y_2, y_3) \in C_{A_2}(x) - Z(P)$ and that $y \neq (kx_1, kx_2, x'_3)$ for any $k \in GF(q)$. Since $|C_{A_2}(x)| = q^4$ and $|\langle (kx_1, kx_2, x'_3), Z(P) | k \in GF(q) \rangle| = q^2$, such y really exists.

Then $C_{A_2}(x, y) = \{(a, c, d) | a\bar{x}_2 + \bar{a}x_2 = c\bar{x}_1 + \bar{c}x_1, a\bar{y}_2 + \bar{a}y_2 = c\bar{y}_1 + \bar{c}y_1\}$. Hence $|C_{A_2}(x, y)| \leq q^3$. Suppose that $x \in P - (A_1 \cup A_2)$. Then it is easily obtained that $|C_P(x)| = q^3$. Thus (5) holds by (1). (3) and (4) follow from direct calculation. Now (3), (4) and (5) give that $Jr(\tilde{P}) = A_1$, since $q \geq 4$. By (1) we have $Jr(P \bmod Z(P)) = A_2$. We get $Jr(\tilde{P} \bmod A_1) = P$ easily.

The following is an immediate consequence of (2.2) and (2.3.2).

(2.4) *L has just two conjugacy classes of involutions which are represented by z_1 and z_2 .*

(2.5) (1) $C_A(I) = (I_1 \times N_1) \langle u \rangle$ if $q > 4$, and $C_A(I) = (I_1 \times N_1) \langle fr \rangle$ if $q = 4$.
 $C_A(I_2) = I_2 N_2$.

(2) $C_A(A_1) = C_A(Z_2(P)) = A_1$, $C_A(Z(P)) = A_2 \cdot (I_2^{q-1} \times N_2) \langle u \rangle$.

(3) $C_A(A_1 N_1 / A_1) = A_1 I_1$, $C_A(A_2 N_2 / A_2) = A_2 I_2 \langle u \rangle$.

(4) $C_A(A_2 I_2^{q-1} / A_2) = C_L(A_2 I_2^{q-1} / A_2)$.

(5) $C_A(P / A_1) = P I_1$, $C_A(A_2 / Z(P)) = A_2$.

PROOF. (1) is obtained by direct calculation. Similarly we have $C_L(A_1) = C_L(Z_2(P)) = A_1$. Hence by the Frattini argument $C_A(Z_2(P)) = A_1(N_A(A_1) \cap C(Z_2(P))) = A_1((I_1 N_1 \langle f \rangle \cap C(Z_2(P))) = A_1$. By (2.1.5) $C_A(Z(P)) = A_2 I_2^{q-1} N_2(N_A(P) \cap C(Z(P))) = A_2 I_2^{q-1} N_2(I \langle f \rangle \cap C(Z(P))) = A_2 I_2^{q-1} N_2 \langle u \rangle$. Thus (2) holds. By (2.1.4) $C_A(A_1 N_1 / A_1) = A_1(I_1 N_1 \langle f \rangle \cap C(I_1)) = A_1 I_1$. Similarly $C_A(A_2 I_2 / A_2) = A_2 I_2 \langle u \rangle$, and $C_A(A_2 I_2^{q-1} / A_2) = A_2(I_2 N_2 \langle f \rangle \cap C(I_2^{q-1})) = A_2 I_2 N_2$, since any nontrivial element of $\langle f \rangle$ does not centralize $q+1$ elements of $GF(q^2)^\times$. This gives (4). (5) follows from (2.1.1) and (2.1.4).

The following is (19.8) (ii) of [4].

(2.6) *Let $x \in I(A - L)$. Then x is conjugate to u or uz_1 . Furthermore $C_L(u) \simeq PSp_4(q)$ and $C_L(uz_1)$ is isomorphic to the centralizer of a transvection in $PSp_4(q)$.*

(2.7) *Any complement of A_1 in $A_1 N_1$ is conjugate to N_1 in $A_1 N_1$.*

PROOF. Let M be a complement. Set $i(a) = \begin{pmatrix} a & & & \\ & a^{-1} & & \\ & & \bar{a} & \\ & & & \bar{a}^{-1} \end{pmatrix} \in L$ and

set $I' = \{i(a) | a \in GF(q^2)^\times\}$. Then $N_{A_1 N_1}(P) = P I'$. So we may assume that $I' \leq M$ by Schur-Zassenhauss' theorem. Set $\overline{A_1 N_1} = A_1 N_1 / A_1$. Since $\bar{I}' \langle \bar{r}_1 \rangle = N_{\overline{A_1 N_1}}(I') \geq \overline{N_{A_1 N_1}(I')}$, it follows that $N_{A_1 N_1}(I') = N_{A_1}(I') I' \langle r_1 \rangle = C_{A_1}(I') I' \langle r_1 \rangle = I' \langle r_1 \rangle$. Therefore $N_{N_1}(I') = N_M(I') = I' \langle r_1 \rangle$. Then $M = \langle P \cap M, r_1 \rangle$ by the structure of $M \simeq SL_2(q^2)$. Hence in order to prove the assertion we will

show that $P \cap N_1 = P \cap M$. Let $x = \begin{pmatrix} 1 & & & \\ c & 1 & & \\ f & d & 1 & \\ h & g & e & 1 \end{pmatrix} \in P \cap M$. Then, since $P =$

$A_1 \cdot (P \cap M)$ and $(P \cap M)^{\#} = x^{I'}$ by the structure of M , we have $c \neq 0$. Hence $(P \cap M)^2 = 1$, which yields $d = 0$ and so $g = \bar{f}$. Since for any $a \in GF(q^2)^{\times}$ there exists an element b in $GF(q^2)^{\times}$ such that $xx^{i(a)} = x^{i(b)}$, we have

$$\begin{cases} c + a^2c = b^2c, \\ f + a\bar{a}^{-1}f = b\bar{b}^{-1}f, \\ h + a\bar{a}h + \bar{a}^2\bar{c}f + a^{-1}\bar{a}c\bar{f} = b\bar{b}h. \end{cases}$$

Then provided that $f \neq 0$, we have $\bar{a}^2 = a$ from the first two equalities. This yields $f = 0$. Also we have $h = 0$, which implies that $x \in P \cap N_1$. Thus $(P \cap N_1)^{\#} = x^{I'} = (P \cap M)^{\#}$ and the proof is complete.

(2.8) Suppose that $x \in \tilde{P} - P$ and set $\bar{x} = x\tilde{P}/P$. If $|x| = |\bar{x}|$, then $m(C_P(x)) = 3m$, where $m = 2n/|x|$.

PROOF. Applying (1.4) to P/A_1 , we may assume that $x = f^i a$ for some $a \in A_1$ and some integer i , where f is the canonical field automorphism of L . Then $C_{A_1}(x) = C_{A_1}(f^i)$ and $m(C_{A_1}(x)) = 3m$. Moreover $m(C_{Z_2(P)}(x)) = 2m$. Since $P/A_1 \simeq A_2/Z_2(P)$ as $N_A(P)$ -modules, we have $|C_{P/A_1}(x)| = |C_{A_2/Z_2(P)}(x)|$, so $m(C_{A_2/Z_2(P)}(x)) = m$. Therefore $m(C_{A_2}(x)) \leq 3m$. Now (2.3.1) implies that any elementary abelian subgroup of P is contained in either A_1 or A_2 . Thus we have $m(C_P(x)) = 3m$.

§ 3. Fusion of the involution of $C_c(L)$

In this section we begin the proof of the theorem stated in the introduction under the following hypothesis.

HYPOTHESIS. Let G be a finite group and let L be a standard subgroup of G such that $L/Z(L) \simeq U_i(q)$, where $q = 2^n \geq 4$. Suppose that $O(G) = 1$ and $L \ntrianglelefteq G$. Furthermore suppose that $C(L)$ has a cyclic Sylow 2-subgroup.

It is known (cf. [15] [18]) that the Schur multiplier of $U_i(q)$, $q \geq 4$, is trivial. So we may identify L with the group of 4×4 matrices x 's defined in the beginning of Section 2. The symbols used in Section 2 for various objects of $U_i(q)$ and $\text{Aut}(U_i(q))$ will retain their meaning. Thus P is a Sylow 2-subgroup of L .

Let t be an involution of $C(L)$ and set $C = C(t)$. Let Q be a Sylow 2-subgroup of $LC_c(L)$ containing P and let T be a Sylow 2-subgroup of C containing Q . Finally, for $i = 1, 2$, we define $B_i = \langle t \rangle A_i$.

(3.1) *The following statements hold.*

- (1) $Jr(T) = B_1$.
- (2) Set $C_T(B_1) = \tilde{B}_1$. Then $t^{N(\tilde{B}_1)} = t^G \cap \tilde{B}_1 \neq \{t\}$.
- (3) $\tilde{B}_1 = B_1$ and $\langle t \rangle \in \text{Syl}_2(C(L))$.
- (4) $t^G \cap L = \phi$.
- (5) $Q = Jr(T \bmod B_1)$ and $B_2 = Jr(Q \bmod Z(Q))$.
- (6) $t^{N(B_1)} = tA_1$.

PROOF. Let $T_0 = C_T(L)$. Then $Q = T_0 \times P$ and $Jr(Q) = B_1$ by (2.3.2). If B is an abelian subgroup of T of rank at least $4n+1$, then $m(BT_0/T_0) \geq 4n$, since T_0 is assumed to be cyclic by the hypothesis. So again by (2.3.2) $B \leq Q$. This proves (1). If $t^G \cap C = \{t\}$, then $t \in Z(G)$ by [9], since we are assuming $O(G) = 1$. But then $C = G$ and $L \triangleleft G$, contrary to our hypothesis. So we may take $t^G \in C - LC(L)$. Then by (2.6) $t^{g^h} = u, uz_1, tu$ or $tu z_1$ for some

$$h \in C. \text{ Let } x = \begin{pmatrix} 1 & & & \\ a & 1 & & \\ c & (c+\bar{c})/a & 1 & \\ d & c & a & 1 \end{pmatrix} \in L. \text{ Then } t^{g^h x} = t^{g^h} \begin{pmatrix} 1 & & & \\ 0 & 1 & & \\ c+\bar{c} & 0 & 1 & \\ 0 & c+\bar{c} & 0 & 1 \end{pmatrix}.$$

Since $L^{g^h} \geq C_L(t^{g^h})^{(\infty)}$, we have $\begin{pmatrix} 1 & & & \\ 0 & 1 & & \\ c+\bar{c} & 0 & 1 & \\ 0 & c+\bar{c} & 0 & 1 \end{pmatrix} \in L^{g^h}$. Hence $t^{g^h x} \in L^{g^h} C(L^{g^h})$

$-\{t^{g^h}\}$. So we may assume that $t^G \in LC(L)$. Then by (2.4) $t^{g^k} = z_1, z_2, z_1 t$ or $z_2 t$ for some $k \in C$. Thus $t^G \cap B_1 \neq \{t\}$. Let $s \in t^G \cap B_1 - \{t\}$ and set $s^{g'} = t$. Since $T \in \text{Syl}_2(C)$, we can choose g' such that $C_T(s)^{g'} \leq T$. In particular $\tilde{B}_1^{g'} < T$. Since $\tilde{B}_1 = T_0 \times A_1 \geq B_1$, $\tilde{B}_1$ is weakly closed in T . Hence $\tilde{B}_1^{g'} = \tilde{B}_1$ and $g' \in N(\tilde{B}_1)$. Thus (2) holds. If $|T_0| \neq 2$, then $\langle t \rangle$ is a characteristic subgroup of $\tilde{B}_1$. But this contradicts (2). So (3) holds. Now (4) follows from (2.2.3), and (5) follows from (3) and (2.3.2).

By (2), (4) and (2.2.1) we have $t^{N(B_1)} \subseteq tA_1$, and $|t^{N(B_1)}| = q^3 - q^2 + q, q^4 - q^3 + q^2 - q + 1$ or q^4 . Set $\Omega = t^{N(B_1)}$. If either of the former two cases holds, then $N(B_1)^\Omega$ is doubly transitive and the one-point-stabilizer $N_C(B_1)^\Omega$ has a cyclic normal subgroup I_1^Ω . Notice that $N_C(B_1)^\Omega$ is not solvable. Hence by theorem 3 of [1] together with (1.2) neither of these cases hold. Thus $t^{N(B_1)} = tA_1$.

- (3.2) (1) $N(Z(Q)) = O_2(N(Z(Q)))N_C(Z(Q))$ and $|O_2(N(Z(Q)))| = 2q^2$.
- (2) $N(Q) = O_2(N(Z(Q)))N_C(Q)$.
- (3) $N(B_2) = N(Z(Q))$.

PROOF. Set $\Omega = t^G \cap Z(Q)$. Then $\Omega = tZ(P)$ by (3.1.4) and (3.1.6). (3.1.6) also implies that T is not a Sylow 2-subgroup of G . Hence by (3.1.5) $t^{N(Z(Q))} \neq \{t\}$. By (2.1.3) $N(Z(Q))^\Omega$ is doubly transitive and the one-point-stabilizer

$N_c(Z(Q))^q$ has a cyclic normal subgroup I_2^q . Thus by (1.1) $N(Z(Q))^q$ has a regular normal subgroup. Let M be the preimage of the regular normal subgroup and let X be a Sylow 2-subgroup of M . Then $M^q = X^q$, so $[X^q, I_2^q] = X^q$. Now $C(Z(Q)) = B_2 \cdot (I_2^{q-1} \times N_2) \cdot O(C)\langle u_0 \rangle$ by (2.5.2), where $u_0 = 1$ or u according as $Q = T$ or not. Since $C(I_2^{q-1}N_2O(C)) \geq I_2$, we have $[X, I_2] \leq C(B_2I_2^{q-1}N_2O(C)/B_2)$ by three subgroup lemma. Let $X_1 \in \text{Syl}_2([X, I_2]B_2)$. Since u acts on I_2^{q-1} nontrivially, it follows that $u \notin [X, I_2]B_2$. Thus we have $M = X_1 \cdot (I_2^{q-1} \times N_2)O(C)\langle u_0 \rangle$ and (1) holds.

Since Q is characteristic in T by (3.1.5), $N(Q)^q$ is transitive as well as $N(Z(Q))^q$. So $|N(Q):N_c(Q)| = |N(Z(Q)):N_c(Z(Q))| = q$. Since $N(Q) \leq N(Z(Q))$, (2) follows. Since $Z(B_2) = Z(Q)$, we have (3) similarly.

(3.3) *There exists a 2-subgroup E of $C(P)$ of order q^2 such that $O_2(N(Z(Q))) = EB_2$, $E \cap B_2 = Z(P)$ and $E/Z(P) \simeq Z(P)$ as $N_c(Z(Q))$ -modules.*

PROOF. Set $M = O_2(N(Z(Q)))$. Then $Z(M) \cap Z(P) \neq 1$, since $Z(Q) \triangleleft M$ and $t^M = Z(Q) - Z(P)$ by (3.2). Hence $Z(P) \leq Z(M)$ by (2.1.3). Now $M \cap C = B_2$ and $Z(B_2) = Z(Q)$, so $Z(M) \leq Z(Q)$. Thus $Z(P) = Z(M)$ and $t \in Z_2(M)$. Then the map $[\cdot, t]$ from M to $Z(P)$ is an $N_c(Z(P))$ -homomorphism. Clearly the kernel is B_2 and the image is $Z(P)$. Hence $M = B_2C_M(I_2^{r_1})$, $B_2 \cap C_M(I_2^{r_1}) = Z(Q)$ and $C_M(I_2^{r_1})/Z(Q) \simeq Z(P)$ as I -modules. Set $E = (C_M(I_2^{r_1})I)'$ and we will show that E has the desired property.

$Z(P) \leq C_M(I_2^{r_1})^2 \leq Z(Q)$ by (2.1.3). If the second equality holds, then t is a square involution in $C_M(I_2^{r_1})$, since $t^M = Z(Q) - Z(P)$. But this contradicts the fact that $C_M(t) = B_2$. Hence $C_M(I_2^{r_1})/Z(P)$ is elementary, $t \notin E$ and $C_M(I_2^{r_1}) = E\langle t \rangle$. Then E is of order q^2 . Moreover $O_2(N(Z(Q))) = EB_2$ and $E \cap B_2 = Z(P)$. By (2.1.3) and (3.2.2) $Z_2(Q) \leq Z(M \bmod Z(Q))$. So $EZ(Q) \triangleleft EZ_2(Q)$. Since $Z_2(P) = Z(P)[Z_2(P), I_2^r]$, we have $[EZ(Q), Z_2(P)] = 1$ by (1.6). By definition E is r_2 -invariant. Hence $[EZ(Q), A_2] = 1$, since $A_2 = Z_2(P)Z_2(P)^{r_2}$. Then $EZ(Q) = C_M(A_2) \cap N(Z(Q))$ and $E = [EZ(Q), I_2] = [EZ(Q), MI_2] \triangleleft N(Z(Q))$. Hence $E/Z(P) \simeq Z(P)$ as $N_c(Z(Q))$ -modules. Moreover by (1.6) we have $MQ = EZ(Q) * [MQ, I_2^{r_1}]$. Since $P = [P, I_2^{r_1}]$ by (2.1.2), $MQ = EZ(Q) * P$ and the proof is complete.

(3.4) *Set $D_1 = O_2(N(B_1))$. Then $N(B_1) = D_1N_c(B_1)$ and $|D_1| = 2q^3$. Furthermore $Z(D_1) = A_1$ and $D_1/B_1 \simeq A_1$ as $N_c(B_1)$ -modules.*

PROOF. Set $t^{N(B_1)} = \Omega$. Then by (2.2.1) and (3.1.6) $N(B_1)^q$ is primitive. Since $A_1 = B_1 - \Omega$, we have $A_1 \triangleleft N(B_1)$. Hence $(C(A_1) \cap N(B_1))^q$ is a nontrivial normal subgroup of $N(B_1)^q$, since it contains E^q . Hence it is transitive. Since $C(A_1) \cap N_c(B_1) = B_1O(C)$ by (2.5.2), $(C(A_1) \cap N(B_1))^q$ is a regular normal subgroup of $N(B_1)^q$. Thus $N(B_1) = (C(A_1) \cap N(B_1))N_c(B_1)$ and $(C(A_1) \cap N(B_1))^q$

$\simeq A_1$ as $N_C(B_1)$ -modules by the $[\cdot, t]$ -homomorphism. Hence N_1 acts on $(C(A_1) \cap N(B_1))^q$ irreducibly by (2.2.1). Since $N_1 \leq C(B_1 O(C)/B_1)$, we obtain that $C(A_1) \cap N(B_1)$ is 2-closed just as in (3.2). Therefore $D_1 = O_2(C(A_1) \cap N(B_1))$, $t^{D_1} = tA_1$ and $|D_1| = 2q^8$. Now it is clear that $D_1/B_1 \simeq (C(A_1) \cap N(B_1))^q$ as $N_C(B_1)$ -modules, which implies the last assertion.

(3.5) Set $F_1 = (D_1 I_1)'$, $P_1 = F_1 P$, $Q_1 = F_1 Q$, $T_1 = F_1 T$ and $F = EC_{F_1}(I_1^{r_2})$. Then the following statements hold.

- (1) $D_1 = F_1 \langle t \rangle$, $F_1 \triangleleft N(B_1)$ and $F_1/A_1 \simeq A_1$ as $N_C(B_1)$ -modules.
- (2) $Z(P_1) = E$, $Z_2(P_1) = F$ and $Z_3(P_1) = P_1$.
- (3) F_1 is elementary or homocyclic.
- (4) $T_1 \in \text{Syl}_2(N(B_1))$, $J(T_1) = F_1$ and $Jr(T_1 \bmod F_1) = Q_1$.
- (5) $N_{P_1}(P) = EP$, $N_{P_1}(EP) = FP$ and $N_{P_1}(FP) = P_1$.
- (6) $E = C_{P_1}(I_2^{r_1})$, $F = E \cdot C_{P_1}(I_1^{r_2})$, $F_1 = F \cdot C_{P_1}(I_2)$, $P_1 = F_1 \cdot C_{P_1}(I_1)$.

PROOF. By (3.4) $C_{D_1}(I_1) = \langle t \rangle$. Hence in order to prove (1) it is enough to show that $t \notin F_1$. Since D_1 is not elementary, $D_1^2 \neq 1$. So by (2.2.1) $A_1 \leq D_1^2 \leq B_1$. Suppose $D_1^2 = B_1$. Then t has a square root in D_1 , since $t^{D_1} = tA_1$. But this yields that $C_{D_1}(t) \neq B_1$, which is a contradiction. Thus D_1/A_1 is elementary. So if we set $\bar{N}(B_1) = N(B_1)/A_1$, then $\bar{D}_1 = \langle \bar{t} \rangle \times [\bar{D}_1, \bar{I}_1]$ and (1) follows.

By (2.1.2) $A_1 = C_P(I_2^{q-1}) \times C_P(I_1^{r_2})$. Hence $F_1 = C_{F_1}(I_2^{q-1}) C_{F_1}(I_1^{r_2})$ and $C_{F_1}(I_2^{q-1}) \cap C_{F_1}(I_1^{r_2}) = 1$. Since $[I_2^{q-1}, I_1^{r_2}] = 1$, by (1I) of [12] we have $F_1 = C_{F_1}(I_2^{q-1}) \times C_{F_1}(I_1^{r_2})$. Since $C_P(I_2^{q-1}) = C_P(I_2^{r_1}) \times C_P(I_2)$, $C_{F_1}(I_2^{q-1}) = C_{F_1}(I_2^{r_1}) \times C_{F_1}(I_2)$. Consequently we have $F_1 = C_{F_1}(I_2^{r_1}) \times C_{F_1}(I_2) \times C_{F_1}(I_1^{r_2})$ and in particular $E = C_{F_1}(I_2^{r_1}) \triangleleft F_1$. Now by (2.1.5) and (2.2.3) $Z(P) \cap Z(P)^x = 1$ for $x \in N_C(A_1) - N(Z(P))$, so it follows that $[E, E^x] = 1$. By the structure of $U_4(q)$ it is easily seen that $A_1 = \langle Z(P)^x | x \in N_C(A_1) - N(Z(P)) \rangle$. Hence $F_1 = \langle E^x | x \in N_C(A_1) - N(Z(P)) \rangle$ and we have $E \leq Z(F_1)$, which implies that F_1 is abelian. Then, since $F_1^2 = 1$ or A_1 by (1), F_1 is elementary or homocyclic respectively.

$N_{P_1}(P) = N_{F_1}(P)P$. By (1) $N_{F_1}(P) = \{x \in F_1 | [x, P] \leq A_1\} = \{x \in F_1 | [x, t] \in Z(P)\} = EA_1$. Hence $N_{P_1}(P) = EP$ and $C_{P_1}(P) = E$. Similarly $N_{F_1}(EP) = \{x \in F_1 | [x, t] \in Z_2(P)\}$. Since $|A_1 EC_{F_1}(I_1^{r_2})/A_1| = q^3$, $|[EC_{F_1}(I_1^{r_2}), t]| = q^3$. Also $[EC_{F_1}(I_1^{r_2}), t] = [E, t][C_{F_1}(I_1^{r_2}), t] \leq Z(P)C_{A_1}(I_1^{r_2}) = Z_2(P)$ by (2.1.2), since $[E, t] = Z(P)$. Thus we have $N_{P_1}(EP) = FP$. Then $N_{P_1}(FP) = N_{F_1}(FP)P = F_1P = P_1$. Since $E = C_{P_1}(I_2^{r_1})$ and $E = Z(EP)$, we have by (1.6) $E = Z(FP)$. Hence inductively we have $E = Z(P_1)$. By (2.3.2) $Jr(EP \bmod E) = EA_2$. Hence $N_{P_1}(EA_2) \leq N_{P_1}(EA_2 A_1) = N_{P_1}(EP)$, which gives $N_{P_1}(EA_2) = N_{P_1}(EP)$. Since $P_1 = F_1P = F_1EA_2$, $F_1 \cap EA_2 = EZ_2(P)$. Thus $FP/EZ_2(P)$ is abelian and so F is normal in P_1 . Then applying (1.6), we have $F/E \leq Z(P_1/E)$. Since $N_{P_1}(EP) = FP$, $Z(P_1/E) \leq FP/E$. Suppose $F/E < Z(P_1/E)$. Then $EZ_2(P)/E < EP/E \cap Z(P_1/E)$, which is absurd.

Hence $F = Z_2(P_1)$. By (1I) of [12] $P_1/F = C_{P_1/F}(I_1) \times C_{P_1/F}(I_2)$. Note that $C_{P_1/F}(I_1) \simeq C_{P/Z_2(P)}(I_1)$ and $C_{P_1}(I_2) < F_1$. So this gives $Z_3(P_1) = P_1$. Thus we get (2), (3) and (5). Also (6) is proved implicitly.

It remains to show (4). It is clear that $T_1 \in \text{Syl}_2(N(B_1))$ by (3.4). (2.3.3) gives that $|C_{F_1}(x)| \leq q^4$ for $x \in P_1 - F_1$. So $|C_{P_1}(x)| \leq q^6$ for $x \in P_1 - F_1$. Hence $J(P_1) = F_1$. Let B be an abelian subgroup of T_1 different from F_1 . The above argument shows that $B \cap P_1 \leq F_1$ or else $|B \cap P_1| \leq q^6$. If $B \not\leq P_1$ and $B \leq Q_1$, then $B \cap Z(P_1) = Z(P)$. Hence $|(B \cap P_1)Z(P)| = |B \cap P_1| |Z(P)| / |B \cap Z(P_1)|$ and so $|B| \leq 2nq^6 < q^8$, since $q \geq 4$. If $B \not\leq Q_1$ and $B \cap P_1 \not\leq F_1$, then $|B| \leq 2nq^6 < q^8$ as above. Finally if $B \not\leq Q_1$ and $B \cap P_1 \leq F_1$, then by (2.3.4) $|C_{F_1}(x)| \leq q^6$ for $x \in T_1 - Q$ and this yields that $|B| \leq 2nq^6$. Thus we have $J(T_1) = F_1$. Then (2.3.2) gives $Jr(T_1 \text{ mod } F_1) = Q_1$, since $T_1/F_1 \simeq T/A_1$.

(3.6) F_1 is elementary.

PROOF. First we will show that $T_1 \notin \text{Syl}_2(G)$. By (3.3) $EB_2 = E\langle t \rangle * A_2$. Hence $J(EB_2) = EA_2$, since $q \geq 4$. Then $E = Z(J(EB_2))$. Set $N(EB_2) = N(EB_2)/E$ and set $\Omega = tA_2$. Since $Q_1/B_1 \simeq P_1/A_1$, $N_{Q_1}(EQ) = FQ$ by (3.5.5). Since $J(EQ \text{ mod } E) = EB_2$ by (2.3.2), we have $F \leq N_{Q_1}(EB_2)$. Suppose that $\bar{x}^f = \bar{x}$ for $\bar{x} \in \Omega$ and $f \in F$. Then by (3.5.2) $\bar{t}^f = \bar{t}$, which implies $f^q = 1^q$. Since $F = E \times C_F(I_1^{f^2})$ and each of these direct factors is t -invariant, we have $C_F(\Omega) = N_F(\langle t \rangle E) = EZ_2(P)$. Hence F^q is semiregular of order q^2 . By (2.2.2) and (3.2) $N_c(EB_2)^q$ has three orbits on Ω whose lengths are 1, $(q+1)(q^2-1)$ and $(q-1)q(q^2-1)$. Now the action of F^q yields that any orbit of $N(EB_2)^q$ has length divisible by q^2 . Hence the only possibility remains that $N(EB_2)^q$ is transitive. Then the index $|N(EB_2):N(Z(Q))|$ is divisible by q^4 , since $N(Z(Q))^q$ is contained in the stabilizer of $\bar{t}$. By (3.2.1) and (3.3) $ET \in \text{Syl}_2(N(Z(Q)))$. Thus the order of a Sylow 2-subgroup of $N(EB_2)$ is not less than $q^5|T|$, while $|T_1| = q^4|T|$. Therefore we have $T_1 \notin \text{Syl}_2(G)$.

Suppose that F is homocyclic. Then $F_1^3 = A_1$ and $D_1 = C_{T_1}(A_1)$ char T_1 by (2.5.2) and (3.5.4). If $txtx = 1$ for some $x \in F_1 - A_1$, then $txtx = 1$ for any $x \in F_1 - A_1$, since $F_1 = \langle x^{N_c(B_1)} \rangle$ by (2.2.1). Hence if $txtx \neq 1$ for some $x \in F_1 - A_1$, then $txtx \neq 1$ for any $x \in F_1 - A_1$. In this case we get $Jr(D_1) = B_1$. Then B_1 char D_1 char T_1 , which is a contradiction, since $T_1 \in \text{Syl}_2(N(B_1))$ and $T_1 \notin \text{Syl}_2(G)$. Therefore we have $txtx = 1$ for any $x \in F_1 - A_1$. Now the first paragraph shows that $N(EB_2)$ is transitive on tA_2 . So t is $N(EB_2)$ -conjugate to txa for some $x \in E$ and $a \in A_2 - I(A_2)$. But $txatxa = txtxa^2 = a^2 \neq 1$, a contradiction. Thus by (3.5.3) F_1 is elementary.

(3.7) Set $O_2(N(Q)) = S$. Then $N(Q_1) = SN_c(Q)$ and $S/Q_1 \simeq P/A_1$ as $N_c(Q)$ -modules. Furthermore set $R = (SI)'$. Then $S = R\langle t \rangle$.

PROOF. Set $\Omega = (Q_1/F_1) - (P_1/F_1)$. Since $T_1 \notin \text{Syl}_2(G)$, we can apply (1.1) to obtain that $N(Q_1)^q$ has a regular normal subgroup. By (3.6) $\mathcal{E}(D_1) = \{B_1, F_1\}$. So $N(D_1) = N(B_1) = D_1 N_c(A_1)$. Then, since $Q_1 = D_1 Q$, we have $N(D_1) \cap N(Q_1) = Q_1 N_c(Q)$. Thus $C(\Omega) = C(Q_1/F_1) = Q_1 C_c(Q/A_1) = Q_1 I_1 O(C)$ by (2.5.5). Therefore as in (3.2) we can obtain that the preimage of the regular normal subgroup is 2-closed and that $N(Q_1) = SN_c(Q)$. By the $[\cdot, t]$ -homomorphism we have $S/Q_1 \simeq P_1/F_1$. Hence $S/Q_1 \simeq P/A_1$ as $N_c(Q)$ -modules. Since $N(D_1) = N(B_1)$, the last assertion is obtained as in (3.5.1).

(3.8) Set $D_2 = C_S(EA_2/E) \cap N(EB_2)$. Then $D_2 = O_2(N(EB_2))$, $N(EB_2) = D_2 N_c(B_2)$ and $D_2/EB_2 \simeq A_2/Z(P)$ as $N_c(B_2)$ -modules.

PROOF. By (3.7) $R \cap Q_1 = P_1$, so $E, F \triangleleft R$. Hence $FC_R(I_1)/E = (EC_R(I_1^{r_1})/E) \times (EC_R(I_1)/E)$ by (1I) of [12]. Then $EA_2 \leq Z(FC_R(I_1) \bmod E)$, since $FC_R(I_1) \cap P_1 = FC_{P_1}(I_1) = FC_P(I_1) > EA_2$ and since I acts on $C_P(I_1)^*$ transitively by (2.1.3). Hence by (3.5) and (3.7) $FC_S(I_1) = C_S(EA_2/E)$. Now $EB_2/E = (EZ_2(P)/E) \times (EC_{Q_1}(I_1)/E)$. Moreover $EZ_2(P) \cap EC_R(I_1^{r_2}) = F$ and R normalizes Q_1 . These facts yield that $EB_2 \triangleleft FC_S(I_1)$. Notice that $|C_S(I_1)F : EB_2| = q^4$. By (3.6) E is elementary. Hence $\mathcal{E}(\langle t \rangle E) = \{Z(Q), E\}$ and so $N(\langle t \rangle E) = N(Z(Q)) = EN_c(B_2)$ by (3.3). Then $EN_c(B_2) \cap C(EA_2/E) = EB_2 O(C)$ by (2.5.5). Therefore if we set Ω as in the proof of (3.6), then it follows that $(C_S(I_1)F)^q$ is a regular normal subgroup of $N(EB_2)^q$. Since $I_2 \times N_2 < C(O(C))$ and it acts on $(C_S(I_1)F)^q$ irreducibly by (2.2.2), it is obtained that the preimage of $(C_S(I_1)F)^q$ is 2-closed as in the proof of (3.2.1). Hence $D_2 = C_S(I_1)F$ and $N(EB_2) = D_2 N_c(B_2)$. Then by the $[\cdot, t]$ -homomorphism we have $D_2/EB_2 \simeq FA_2/E$ as $N_c(B_2)$ -modules. Since $EA_2/E \simeq A_2/Z(P)$, the last assertion follows.

(3.9) (1) $Z(R) = E$, $Z_2(R) = F$, $Z_3(R) = R$, $F_1 \text{ char } R$.

(2) Set $F_2 = D_2 \cap R$. Then $F_2 \triangleleft N(EB_2)$, F_2/E is elementary and $F_2/EA_2 \simeq EA_2/E \simeq A_2/Z(P)$ as $N_c(B_2)$ -modules.

(3) $E = C_R(I_2^{r_1})$, $F = E \cdot C_R(I_1^{r_2})$, $F_i = F \cdot C_R(I_{3-i})$, $R = F_i \cdot C_R(I_i)$, $i = 1, 2$.

PROOF. By (3.5.6) and (3.7) $E = C_R(I_2^{r_1})$ and $F = E \cdot C_R(I_1^{r_1})$. By (1I) of [12] $R/F = (FC_R(I_1)/F) \times (FC_R(I_2)/F)$. Hence $F_i = F \cdot C_R(I_{3-i})$ and $R = F_i \cdot C_R(I_i)$, $i = 1, 2$. Thus (3) holds.

By (1.6) $E \leq Z(R)$. Suppose that $E < Z(R)$. Then by (3) $Z(R) \not\leq P_1$ and $[Z(R), I_1] \leq P_1$. Hence $Z(R) \cap C(I_1) \neq 1$. Since $Z(R) \cap C(I_1)$ is t -invariant, we have $Z(R) \cap C(I_1) \cap C \neq 1$, which contradicts (2.1.2). Similarly we have $Z_2(R) = F$. By (3.7) and (3.8) $F_2 = (D_2 I_2)'$, so $F_2 \triangleleft N(EB_2)$, since $N(EB_2) = D_2 N_c(B_2) \triangleleft D_2 I_2$. Then $(F_2/E)^2 = (C_R(I_1)E/E)^2 \times (F/E)^2 = (C_R(I_1)E/E)^2 < EA_2/E$. Since $I_2 \times N_2$ acts on EA_2/E irreducibly, we have $F_2^2 \leq E$. Thus (2) holds and at the same time we have $Z_3(R) = R$. By (3.6) $C_R(F) \geq F_1$. If $C_R(F) > F_1$, then by (3)

$C_R(F) \cap C_R(I_1) \neq 1$. Hence $C \cap C_R(F) \cap C_R(I_1) \neq 1$. This yields that $C_P(Z_2(P)) \cap C_P(I_1) \neq 1$, which contradicts (2.1.2) and (2.5.2). Thus $C_R(F) = F_1$ and (1) holds.

(3.10) $\mathcal{E}(D_1) = \{F_1, B_1\}$ and $\mathcal{E}(D_2 \bmod Z_2(D_2)) = \{F_2, EB_2\}$. In particular $N(D_1) = N(B_1)$ and $N(D_2) = N(EB_2)$.

PROOF. By (3.6) $\mathcal{E}(D_1) = \{F_1, B_1\}$. $Z(D_2) \leq Z(C_{D_2}(t)) = Z(B_2) = Z(Q)$. Since $t \notin Z(D_2)$, $Z(P) = Z(D_2)$ by (2.1.3). Then $E \leq Z_2(D_2)$, since $E \triangleleft D_2$ and since I acts on $E/Z(P)$ irreducibly by (3.3). If $E < Z_2(D_2)$, then $EA_2 = Z_2(D_2)$, since by (3.8) $N_C(B_2)$ acts irreducibly on $EA_2/E = Z(D_2/E)$. This yields that $A_2F/Z_2(P) \simeq A_1A_2F/A_1 \simeq FP/A_1$, which is not abelian by (3.5.5). This is a contradiction. Thus $E = Z_2(D_2)$ and by (3.9.2) we have $\mathcal{E}(D_2 \bmod Z_2(D_2)) = \{F_2, EB_2\}$.

$$(3.11) \quad (1) \quad E^{r_1} \cap F = 1.$$

$$(2) \quad C_R(I_1^{r_2})^{r_2} \cap F = 1.$$

PROOF. By (3.9.3) $E^{r_1} = C_R(I_2^{r_1})^{r_1} = C_{F_1}(I_2^{r_1})^{r_1} = C_{F_1}(I_2)$, since F_1 is r_1 -invariant. Then (1) follows from (3.9.3) again. We can obtain (2) similarly.

(3.12) Set $M_1 = \langle N_1^{N(F_1) \cap C(I_1)} \rangle$ and set $M_2 = \langle N^{N(F_2) \cap C(I_2)} \rangle$. Then the following statements hold.

$$(1) \quad M_1 \simeq SL_2(q^2) \times SL_2(q^2) \text{ or } SL_2(q^4).$$

$$(2) \quad M_2 \simeq SL_2(q) \times SL_2(q) \text{ or } SL_2(q^2).$$

Moreover in either case, t interchanges the components of M_i if the former holds, and t acts on M_i as a field automorphism if the latter holds.

PROOF. By (2.5.1) and (3.5.1) N_1 is a standard subgroup of $N(F_1) \cap C(I_1)$ and $\langle t \rangle$ is a Sylow 2-subgroup of $N(F_1) \cap C(I_1) \cap C(N_1)$. By (3.9) t is not a central involution in $N(F_1) \cap C(I_1)$ and $C_R(I_1)$ is elementary. We also have $[C_R(I_1), I \cap N_1] = C_R(I_1)$. Hence we can appeal to the result of [16] to obtain (1). (2) is obtained similarly.

DEFINITION. $L_1 = F_1 \cdot M_1$ and $L_2 = F_2 \cdot M_2$.

$$(3.13) \quad (1) \quad L_1 \triangleleft N(F_1).$$

$$(2) \quad E = Z(L_2).$$

$$(3) \quad L_1 \text{ and } L_2/E \text{ are perfect.}$$

PROOF. Set $\overline{N(F_1)} = N(F_1)/F_1$. By (2.5.3) and (3.10) $\overline{N_1}$ is a standard subgroup of $\overline{N(F_1)}$ and $\langle \bar{t} \rangle$ is a Sylow 2-subgroup of $C_{\overline{N(F_1)}}(\overline{N_1})$. Then as in (3.12) we have $\langle \overline{N_1}^{\overline{N(F_1)}} \rangle \simeq SL_2(q^2) \times SL_2(q^2)$ or $SL_2(q^4)$. Since $\overline{M_1} \leq \langle \overline{N_1}^{\overline{N(F_1)}} \rangle$, L_1 is the preimage of $\langle \overline{N_1}^{\overline{N(F_1)}} \rangle$, so $L_1 \triangleleft N(F_1)$.

Now $E \leq Z(F_2)$. Suppose that $E < Z(F_2)$. Then by (3.9.2) $Z(F_2)A_2 = F_2$.

Since $Z(F_2) \cap A_2 = Z(P)$ and $|F_2| = q^{10}$, it follows that $|Z(F_2)| = q^9$. By (3.9.3) $R = F_1 F_2$ and $F_1 \cap F_2 = F$. Therefore $Z(F_2) \cap F = E$. Then considering the order, we have $Z(F_2)F = F_2$, a contradiction. Hence $E = Z(F_2)$. By (2.1.5) and (3.3) $N_2 < C(E)$, since $N_2 = O^{2'}(N_2)$. So $C_{M_2}(E)$ is a nontrivial t -invariant normal subgroup of M_2 . Then the structure of M_2 yields that $C_{M_2}(E) = M_2$. Thus $E = Z(L_2)$.

By (2.2.1) and (3.5.1) $F_1 N_1$ is perfect. Hence $L_1 = F_1 M_1$ is perfect. Similarly L_2/E is perfect by (2.2.2) and (3.9.2).

- (3.14) (1) $R \in \text{Syl}_2(L_1) \cap \text{Syl}_2(L_2)$.
 (2) $N_{L_1}(R)$ and $N_{L_2}(R)$ normalize each other.
 (3) $N_{L_1}(R) \cap N_{L_2}(R) = R$.

PROOF. (1) is a direct consequence of (3.12). By (3.9.1) and (3.13.1) $N_{L_2}(R)$ normalizes $N_{L_1}(R)$. Set $J_1 = N_{M_1}(R) \cap N_{M_1}(R^r)$. Then $N_{M_1}(R) = (R \cap M_1) \cdot J_1$ and $I \cap M_1 < J_1$. Since $J_1 \simeq Z_{q^2-1} \times Z_{q^2-1}$ or Z_{q^4-1} and since $I = I_1 \times (I \cap N_1)$, we have $I_1 \times J_1 < C(I_2)$. By (3.9.3) $F_2 = (R I_2)^r$. Hence $J_1 < N(F_2) \cap C(I_2)$. This proves (2).

By the choice of J_1 , r_1 inverts the elements of J_1 . Set $N_{M_2}(R) = (R \cap M_2) \cdot J_2$. Then $(J_1 \cap R J_2)^{r_1} = J_1 \cap R J_2$. By (3.13.2) $E \leq C(J_1 \cap R J_2)$. By (3.11.1) $F_1 = F \cdot E^{r_1}$. Therefore $J_1 \cap R J_2$ acts on F_1/F trivially. Since $F_1/F \simeq R/F_2$ and since any non-identity element of J_2 acts on R/F_2 nontrivially, we have $J_1 \cap R J_2 = 1$. Since $R J_1 \cap R J_2 = R(J_1 \cap R J_2)$, (3) is obtained.

$$(3.15) \quad L_1/F_1 \simeq SL_2(q^2) \times SL_2(q^2).$$

PROOF. Assume that the assertion does not hold. Let J_1 be as in (3.14). Then $I \cap J_1 = I \cap M_1 = I \cap N_1$. Suppose that $L_2/F_2 \simeq SL_2(q^2)$. Then $J_2 \simeq Z_{q^2-1}$. By (3.14) $J_1 J_2 R/R \simeq Z_{q^4-1} \times Z_{q^2-1}$. So we may take J_2 such that $J_1 J_2 = J_1 \times J_2$. Then $(J_1 J_2)^{(q^2+1)(q-1)} = (J_1^{q^2+1} J_2)^{q-1} = (I \cap N_1)^{q-1} J_2^{q-1} \simeq Z_{q+1} \times Z_{q+1}$. Since $q = 2^n$, $J_1 J_2 / C_{J_1 J_2}(L_1/F_1)$ acts on L_1/F_1 as an inner automorphism. Therefore $C(L_1/F_1) \cap (I \cap N_1)^{q-1} J_2^{q-1} \neq 1$. By (2.1.2) and (3.3) $(I \cap N_1)^{q-1} < C(E)$. So by (3.11.1) $(I \cap N_1)^{q-1} < C(F_1/F)$, since I and N_1 are r_1 -invariant. Also by (3.13.2) $J_2^{q-1} < C(E)$ and $J_2^{q-1} \not\leq C(F_1/F)$, since $F_1/F \simeq R/F_2$. Consequently F_1 has a nontrivial L_1 -invariant subspace $C_{F_1}(C(L_1/F_1)) \cap (I \cap N_1)^{q-1} J_2^{q-1}$. But this can not hold, since L_1 acts on F_1 irreducibly by Lemma 4B of [8]. Suppose that $L_2/F_2 \simeq SL_2(q) \times SL_2(q)$. Then $J_2 \simeq Z_{q-1} \times Z_{q-1}$ and by (3.14) $C_{J_2}(L_1/F_1) \neq 1$. Then we can derive a contradiction in the same way as above.

DEFINITION. Set $L_1 = K_1 K_1^t$ and $U = R \cap K_1$, where $K_1 \triangleleft L_1$ and $K_1/F_1 \simeq SL_2(q^2)$.

$$(3.16) \quad (1) \quad |Z(U)| \geq q^4.$$

(2) If $|Z(U)| = q^4$, then $Z(U) < F$.

PROOF. Suppose that $|Z(U)| < q^4$. Then by Lemma 4B of [8] $K_1^t \leq C(Z(U))$. By symmetry $K_1 \leq C(Z(U^t))$. Then $K_1 K_1^t \leq C(Z(U) \cap Z(U^t))$, which is a contradiction, since $Z(U) \cap Z(U^t) = E$ and $N_1 \not\leq C(Z(P))$. Thus (1) holds.

Since $F = Z_2(R)$ and $E = Z(R) < Z(U)$ by (3.9.1), it follows that $E < Z(U) \cap F$. By (3.9.3) $I/I_1^{r_2}$ acts on F/E fixed-point-freely. Then, since $I = I_1^{r_2} \times I_2^{r_2}$, any $I_2^{r_2}$ -orbit on $(F/E)^*$ is of length $q^2 - 1$. This yields that $(Z(U) \cap F)/E$ is of order at least q^2 . Thus (2) holds.

For the following we refer the readers to (4I) of [12].

(3.17) Suppose $Z(U) \leq F$. Then the following statements hold.

(1) $F_1 = Z(U) \times Z(U^{r_1})$.

(2) $F = Z(U)Z(U^t)$.

(3) K_1^t acts on $Z(U)$ and on $Z(U^{r_1})$ as $SL_2(q^2)$ acts naturally on a 2-dimensional vector space over $GF(q^2)$.

(4) If $R \neq R' \in \text{Syl}_2(L_1)$, then $Z(R) \cap Z(R') = 1$.

(5) If $z \in E^*$, then $z^{N(F_1)} \cap F = Z(U)^* \cup Z(U^t)^*$.

PROOF. Suppose $Z(U) \cap Z(U^{r_1}) \neq 1$. Then, since K_1^t normalizes $Z(U) \cap Z(U^{r_1})$, we have $Z(U^t) \cap Z(U) \cap Z(U^{r_1}) \neq 1$. This implies that $1 \neq E \cap Z(U^{r_1}) \leq E \cap F^{r_1}$, which contradicts (3.11.1). Thus (3.16.1) gives (1). Since $Z(U) \cap Z(U^t) = E$, (2) follows. In particular $Z(U) \neq Z(U^t)$. Hence K_1^t acts on $Z(U)$ nontrivially. So we can appeal to (1K) of [12] to obtain (3). By the structure of $SL_2(q^2) \times SL_2(q^2)$, R' is R -conjugate to one of R^{r_1} , $UU^{r_1 t}$ and $U^{r_1}U^t$. Hence for the proof of (4) we may assume that R' is one of those. Then in any case we have $E \cap Z(U^{r_1})$, $E \cap Z(U^{r_1 t}) \leq E \cap F^{r_1} = 1$ by (3.11.1). Thus (4) holds. Since $F = Z_2(R)$, arguing as in (4), we see that it is enough for (5) to show that $Z(R') \cap F \subseteq Z(U) \cup Z(U^t)$ for R' mentioned in (4), and this obviously holds.

§ 4. Case for $L_4(q^2)$

In this section we assume the following and we will obtain that $\langle L^G \rangle \simeq L_4(q^2)$ under the hypothesis made at the beginning of Section 3.

HYPOTHESIS. $L_2/F_2 \simeq SL_2(q^2)$.

(4.1) (1) $\mathcal{C}(R \bmod E) = \{F_1, F_2\}$.

(2) $Z(U) < F$.

PROOF. If we show $I(R) = I(F_1) \cup I(F_2)$, then (3.6) and (3.9.2) will give (1). Suppose false and let $x \in I(R) - I(F_1) - I(F_2)$. Then by our hypothesis in this

section J_2 acts on $(R/F_2)^\#$ transitively. Since $F_1 \text{ char } R$ (see (3.9.1)), we have $x^y \notin I(R) - I(F_1) - I(F_2)$ for any $y \in J_1$. So we may take $x = fa_1 \notin F_1 \cup F_2$ for some $f \in F_2$ and $a_1 \in A_1 - Z_2(P)$. Then by (3.9.2) $[f, t] \in Z_2(P)E$. Hence $f \in FA_2$. Set $f = a_2 f'$ with $a_2 \in A_2$, $f' \in F$ and set $\bar{R} = R/E$. Then $\bar{1} = (\bar{a}_2 \bar{f}' \bar{a}_1)^2 = (\bar{a}_2 \bar{a}_1)^2$ by (3.9.1). But, since $EP/E \simeq P/Z(P)$, this yields that $a_2 \in A_1 \cap A_2$ and so $x \in F_1$, a contradiction. Thus (1) holds. Then (2) follows from (1), since $F = F_1 \cap F_2 < U$.

$$(4.2) \quad C_{J_2 L_1}(z_2) = C_{L_1}(z_2) \text{ and } C_{L_1}(z_2)/F_1 \simeq SL_2(q^2).$$

PROOF. By (2.2.3) $SL_2(q) \simeq C_{N_1}(z_2) \leq C_{M_1}(z_2)$. Since $C_{M_1}(z_2)$ is t -invariant, $C_{M_1}(z_2) \simeq SL_2(q)$, $SL_2(q) \times SL_2(q)$ or $SL_2(q^2)$ by the structure of M_1 . If $C_{M_1}(z_2) \simeq SL_2(q)$, then $|z_2^{L_1}| = |M_1 : C_{M_1}(z_2)| > |F_1|$. Hence this can not occur. So in any case $C_{M_1}(z_2)$ is maximal among the subgroups of M_1 which are t -invariant.

Now by (3.14) $J_2 L_1/F_1 \simeq Z_{q^2-1} \times SL_2(q^2) \times SL_2(q^2)$. Set $\bar{J}_1 \bar{J}_2 = J_1 J_2 R/R$ and set $\bar{J} = C_{(\bar{J}_1 \cap \bar{K}_1) \bar{J}_2}(U/F_1)$. Then $\bar{J} \simeq Z_{q^2-1}$. Let $\langle \bar{x} \rangle = \bar{J}$ and let $\langle \bar{y} \rangle = \bar{J}_2$. Then $\bar{x} \bar{y}^i \in \bar{J}_1 \cap \bar{K}_1$ for some integer i . So $\bar{x}$ acts on U^t/F_1 as $\bar{y}^{-i}$ acts on it. Since t acts on L_2/F_2 as a field automorphism, we have $\bar{y}^t = \bar{y}^q$. Hence $\bar{x}^t$ centralizes U^t/F_1 and acts on U/F_1 as $\bar{y}^{-q^i}$ acts on it. Therefore $\bar{x} \bar{x}^{qt} \bar{y}^t$ centralizes R/F_1 . Set $\Omega = E^\#$ and we will show $\langle x x^{qt} y^i \rangle^q \simeq Z_{q^2-1}$. By (3.13.2) $(x x^{qt} y^i)^q = ((x y^i)(x^q y^{iq})^t)^q$. Since $\langle t \rangle N_{L_1}(R)$ normalizes E and since $N_{L_1}(R)^q$ is transitive, we can apply (1.3) to obtain $((x y^i)^t)^q = ((x y^i)^q)^t$. Hence $\langle x x^{qt} y^i \rangle^q = \langle x y^i \rangle^q$. Then, since $(J_1 \cap K_1)^q$ is faithful by (3.17.3), we have $\langle x x^{qt} y^i \rangle^q \simeq \langle \bar{x} \bar{y}^t \rangle$. So in order to see $\langle x x^{qt} y^i \rangle^q \simeq Z_{q^2-1}$ it is enough to show that $\bar{J} \cap \bar{J}_2 = \bar{1}$. Since J_2 is cyclic, $\bar{J} \cap \bar{J}_2$ is t -invariant. So $\bar{J} \cap \bar{J}_2 < C(R/F_1)$. Hence if $R \cdot J_0$ denotes the preimage of $\bar{J} \cap \bar{J}_2$, then $J_0 < C(L_1/F_1)$. By (3.13.2) $J_0 F_1 < C(E)$ and so by (3.17.3) and (4.1.2) we have $J_0 F_1 < C(F_1)$. In particular $J_0 \leq C(R/F_2)$, since $F_1/F \simeq R/F_2$ by (3.9). This implies $J_0 = 1$ by the structure of L_2/F_2 . Thus $\langle x x^{qt} y^i \rangle^q = C_{J_1 J_2}(R/F_1)^q \simeq Z_{q^2-1}$. In other words $C_{J_2 L_1}(L_1/F_1)^q \simeq Z_{q^2-1}$.

By (2.2.1) $\langle z_2^{N_1} \rangle = A_1$, and so $\langle z_2^{L_1} \rangle = \langle A_1^{L_1} \rangle = F_1$. Notice that $C_{J_2 L_1}(L_1/F_1)$ acts on the set of the orbits of L_1 on F_1 . Let $J' F_1$ be the subgroup of $C_{J_2 L_1}(L_1/F_1)$ which leaves the orbit $z_2^{L_1}$ invariant. Set $J'' = C_{J'}(z_2^{L_1})$. Then $[J'', F_1] = 1$, since $\langle z_2^{L_1} \rangle = F_1$. Hence the result of the previous paragraph yields $J'' = 1$ and so J' is faithful on $z_2^{L_1}$. So, if we set $\Delta = z_2^{L_1}$, then $(J' L_1)^\Delta = J'^\Delta \times K_1^\Delta \times K_1^{\Delta t}$. Since $|\Delta| = |J' L_1 : C_{J' L_1}(z_2)|$, $|C_{J' L_1}(z_2)| = |J'| |C_{L_1}(z_2)|$. Then by the result obtained in the first paragraph we have $C_{J' L_1}(z_2) = J' C_{L_1}(z_2)$. Since J'^Δ is normal and contained in a one-point-stabilizer of the transitive permutation group $(J' L_1)^\Delta$, we have $J'^\Delta = 1^\Delta$ and $C_{J_2 L_1}(z_2) = C_{L_1}(z_2)$. Therefore $|z_2^{J_2 L_1}| = (q^2 - 1)q^2(q^2 + 1)^2$ or $(q^2 - 1)q^2(q^2 + 1)$ according as $C_{M_1}(z_2) \simeq SL_2(q) \times SL_2(q)$ or $SL_2(q^2)$. But $(q^2 - 1)q^2(q^2 + 1)^2 > q^8$. Consequently $C_{J_2 L_1}(z_2)/F_1 \simeq SL_2(q^2)$ and the proof is complete.

(4.3) z_1 and z_2 are the representatives of $N(F_1)$ -conjugacy classes of the elements of F_1^* and the fusion is controled by $J_2 L_1$. Furthermore $|z_1^{N(F_1)}| = (q^2 - 1)(q^2 + 1)^2$ and $|z_2^{N(F_1)}| = (q^2 - 1)^2 q^2 (q^2 + 1)$.

PROOF. By (3.17.4) and (4.1.2) $|\bigcup_{x \in L_1} (E^x)^*| = (q^2 - 1)(q^2 + 1)$. By (4.2) we have

$$|z_1^{L_1}| = (q^2 - 1)(q^2 + 1)^2 \text{ and } |z_2^{J_2 L_1}| = (q^2 - 1)^2 q^2 (q^2 + 1).$$

Hence L_1 has one orbit of length $(q^2 - 1)(q^2 + 1)^2$ and $q^2 - 1$ orbits of length $(q^2 - 1)q^2(q^2 + 1)$ on F_1^* , and the orbits of length $(q^2 - 1)q^2(q^2 + 1)$ are permuted transitively by J_2 . Since $L_1 \triangleleft N(F_1)$, the assertion follows.

(4.4) Set $U \cap M_1 = U_0$. Then the following statements hold.

(1) $C_{F_1}(a) = Z(U)$ for $a \in U_0^*$ and $|C_{F_1}(b)| = q^4$ for $b \in U_0 U_0^t - U_0 - U_0^t$. In particular $C_{F_1}(a, b) = E$.

(2) $\mathcal{E}(C_R(a)) = \{Z(U)U_0, EU_0 U_0^t\}$ and $\mathcal{E}(C_R(b)) = \{V, EU_0 U_0^t\}$, where $V = C_R(C_{F_1}(b)) \cap C_R(b)$.

(3) $C_R(x, b) = V$ for any $x \in C_{F_1}(b) - E$. Also $\mathcal{E}(C_R(c)) = \{V, EU_0 U_0^t\}$ for any $c \in V^* \cap U_0 U_0^t$.

(4) If $x \in Z(U) - E$, then $C_R(x) = U$. If $x \in F - Z(U) - Z(U^t)$, then $|C_R(x)| = q^{10}$.

PROOF. Suppose $C_{F_1}(a) > Z(U)$. Then since $C_{F_1}(a) = C_{F_1}(a^x)$ for any $x \in K_1^t$, we have $C_{F_1}(a) = F_1$ by (3.17.1, 3) and (4.1.2). This yields that $C_R(a) = R$, since $R = F_1(U_0 \times U_0^t)$, which is against (3.9.1). Hence $C_{F_1}(a) = Z(U)$. By the structure of M_1 the elements of $U_0 U_0^t - U_0 - U_0^t$ are conjugate mutually in $N_{M_1}(R)$. So we may assume that $b \in C_{N_1}(z_2) < C_{M_1}(z)$. Now the proof of (4.2) shows that $C_{J_1 J_2}(L_1/F_1)$ acts on F_1^* fixed-point-freely. Therefore if we denote $X = C_{J_1 J_2}(L_1/F_1)$, then $\langle z_2^x \rangle E \leq C_{F_1}(b)$ and $|\langle z_2^x \rangle E| \geq q^4$. By (4.1.1) $C_{F_1}(b) \leq F$. Hence, if $|C_{F_1}(b)| > q^4$, then by (3.17.5) $Z(U) \cap C_{F_1}(b) > E$. Let $c \in Z(U) \cap C_{F_1}(b) - E$ and let $b = b_1 b_2$ with $b_1 \in K_1$ and $b_2 \in K_1^t$. Since $bc = cb$ and $b_1 c = c b_1$, we have $b_2 c = c b_2$, which contradicts the first assertion. Thus (1) holds. Also it is obtained that $C_{F_1}(b) \cap Z(U) = E$ and (4) follows immediately.

By (1) $C_R(a) = (Z(U)U_0)(EU_0 U_0^t)$, $Z(U)U_0 \cap EU_0 U_0^t = EU_0$ and $C_R(a, d) = EU_0 U_0^t$ for any $d \in EU_0 U_0^t - Z(U)U_0$. This implies that $\mathcal{E}(C_R(a)) = \{Z(U)U_0, EU_0 U_0^t\}$. Now we may take $b = z_2^{x^2}$. Then $C_{F_1}(b) = \langle z_2^x \rangle E$. Since $C_{F_1}(b) \cap Z(U) = E$, $C_{L_1}(x)/F_1 \simeq SL_2(q^2)$ for any $x \in C_{F_1}(b) - E$ by (3.17.5) and (4.2). Hence we have $C_R(x) \in \text{Syl}_2(C_{L_1}(x))$ by the structure of M_1 . Then $C_{F_1}(y) = \langle x^x \rangle E = \langle z_2^x \rangle E$ for any $y \in C(x) \cap U_0 U_0^t$, since $C(x) \cap (M_1 \cap K_1) \cap U_0 U_0^t = C(x) \cap (M_1 \cap K_1^t) \cap U_0 U_0^t = 1$ by the proof of (4.2). Hence $V = C_R(C_{F_1}(b)) \cap C_R(b)$ is abelian. So in order to prove (2) and (3) we only show that $\mathcal{E}(C_R(b)) = \{V, EU_0 U_0^t\}$. Let $x \in I(C_R(b))$ and let $x = x_1 x_2 x_3$, where $x_1 \in C_{F_1}(b)$, $x_2 \in U_0$ and

$x_3 \in U_0^t$. Furthermore we can choose x'_3 in U_0^t such that $x_2x'_3 \in C(x_1)$. Then $1 = (x_1x_2x_3)^2 = (x_1x_2x'_3x'_3x_3)^2 = (x_1x'_3x_3)^2$, which yields that $x'_3x_3 = 1$ or $x_1 \in E$. This gives the claim.

$$(4.5) \quad (1) \quad I(F_2) = \bigcup_{x \in L_2} (F^\#)^x.$$

(2) If $x \in I(R)$, then x is either conjugate to z_1 or to z_2 in $L_1 \cup L_2$.

(3) $t^a \cap RT = t^R = I(tR)$.

(4) $|C_{F_1}(u)| = q^4$ and $|C_{F_1}(ut)| = q^6$, where the involution u defined in Section 2 is chosen to normalize each component of M_1 .

PROOF. By (3.9.3) and (3.11.2) $F \cap F^{r_2} = E$. Hence $F \cap F^x = E$ for any $x \in L_2 - N_{L_2}(R)$ by the structure of L_2/F_2 . Therefore $|\bigcup_{x \in L_2} (F^\#)^x| = q^8 + q^8 - q^4 - 1$. By (4.1) $C_R(y) \leq F_2$ for any $y \in (U_0U_0^t)^\#$. Hence applying (4.4.1), we have $|I(F_2)| = (q^4 - 1)q^4 + q^6 - 1 = q^8 + q^6 - q^4 - 1$. Thus (1) holds. Then (2) follows from (4.3).

By (1) for any $x \in I(R)$ $m(C(x)) \geq 8n \geq 4n + 1 = m(C)$, so t does not fuse into R . Since $RT/R \simeq T/P$, $m(RT/R) = 1$ or 2 . If $m(RT/R) = 1$, then $I(RT - R) = I(tR)$ and we can apply (19.4) of [4] to R/F_1 and F_1 to obtain $t^R = I(tR)$. If $m(RT/R) = 2$, then $I(RT - R) = I(tR) \cup I(tuR) \cup I(uR)$. Set $X = N_{L_1}(R)T$ and set $\Omega = E^\#$. Then $(x^u)^\Omega = (x^q)^\Omega$ for any $x \in J_1$. Since X satisfies the assumption of (1.3), we have $(x^t)^\Omega = (x^q)^\Omega$. So $(x^{tu})^\Omega = x^\Omega$ for any $x \in J_1$, which implies $tu \in C(E)$. By (3.17) $F_1 = E \times Y \times Y^t \times E^{r_1}$, where $Y = Z(U) \cap Z(U^{r_1})$. Since $r_1u = ur_1$, $E, E^{r_1} \leq C(tu)$. Since tu interchanges Y and Y^t , $C_{Y^{r_1}}(tu) = \{zz^{tu} | z \in Y\}$. Consequently $|C_{F_1}(tu)| = q^6$. Then applying (19.4) of [4] to R/F_1 , we obtain that any element of $I(tuR)$ is R -conjugate to one of $I(tuF_1)$. Thus for any $x \in I(tuR)$, $|C_{F_1}(x)| = q^6$. Since $m(C) = 4n + 1$ by (3.1.1), t does not fuse into $I(tuR)$. Next we inspect $C_{F_1}(u)$. By (3.17.2) $E \times Y \times Y^t = F$. So $Z_\varepsilon(P) = Z(P)C_{Y^{r_1}}(t)$ and $|C_{Y^{r_1}}(t, u)| = q$. Hence $|C_Y(u)| = |C_{Y^t}(u)| = q$. Since $C_E(t) = C_E(u)$, we get $|C_{F_1}(u)| = q^4$ and $I(uR) = u^R$ as above. This gives (4).

Then in order to prove (3) it is enough to show that t is not conjugate to u . Suppose false. Then, since $B_1 = Jr(T)$, $C_{F_1}(u)\langle u \rangle$ is conjugate to B_1 . But $N_C(A_1)^{(\infty)} = A_1N_1$, while $C_{L_1}(u)/F_1 \simeq SL_2(q) \times SL_2(q)$, a contradiction. Thus (3) is proved.

(4.6) Let $b \in U_0U_0^t - U_0 - U_0^t$ and let $b_i = b^{x_i}$, where $\langle x \rangle = J_1 \cap K_1$. Set $\mathcal{C}(C_R(b_i)) = \{V_i, EU_0U_0^t\}$ and set $B = Z(U)U_0$. Then V_i , $i = 1, 2, \dots, q^2 - 1$, B and B^t are normal abelian subgroup of L_2 contained in F_2 . Moreover L_2/E acts on B/E in the same way as the natural action of $SL_2(q^2)$ on a 2-dimensional vector space over $GF(q^2)$.

PROOF. First we will show that $\{V_i - E, B - E, B^t - E | i = 1, \dots, q - 1\}$ makes a partition of the involutions of $F_2 - E$. By (4.4.1) $F_1 \cap V_i \cap V_j \leq$

$C_R(b_i, b_j) = C_R(b^i, b_i b_j) = E$ if $i \neq j$, since $b_i b_j \in F_1 U_0$. Then by (4.4.3) $V_i \cap V_j = E$. Similarly we have $B \cap V_i = B^i \cap V_i = E$ for any i by (4.4.4). Then counting argument gives the claim as in (4.5.1). In particular we may assume that $\langle z_2, z_2^{r_2} \rangle < V_1$. By (4.4.3) $C_R(z_2, z_2^{r_2}) = V_1$ and $C_R(z_2) = F_1 V_1$. Hence by (4.1.1) $V_1 \triangleleft R$. Therefore we have $V_1 \triangleleft \langle R, R^{r_2} \rangle = L_2$. Since J_1 normalizes L_2 by the proof of (3.14), this implies that all the V_i 's are normal in L_2 . Hence by (4.4.4) $U_0 \cup U_0^t \subseteq B^x \cup B^{tx}$ for any $x \in L_2$ and by (4.4.2) $B^x = B$ or B^t . By (3.13.3) L_2/E has no subgroup of index 2, which yields that $B^x = B$. Since $|(B/E) \cap (F/E)| = q^2$, the last assertion follows from (1K) of [12].

DEFINITION. $G_0 = \langle L^G \rangle$.

(4.8) (1) $F_1 = Jr(R)$.

(2) $R \in \text{Syl}_2(G_0)$.

(3) G_0 has just two conjugacy classes of involutions, which are represented by z_1 and z_2 . Furthermore $C_R(z_2)$ is a Sylow 2-subgroup of $C_{G_0}(z_2)$.

(4) B and B^t are weakly closed in R with respect to G_0 .

PROOF. (1) follows from (4.1.1) and (4.7). By (4.5.3) $N(RT) = RT(C \cap N(RT))$. Hence $RT \in \text{Syl}_2(G)$. Since $RT/R \simeq T/P$ is abelian, we have $t \notin O^2(G)$ by (1.5). By (3.13.3) $R < O^2(G)$. Suppose that $R \notin \text{Syl}_2(O^2(G))$. Then $\langle u \rangle R$ or $\langle tu \rangle R$ is contained in $O^2(G)$. Hence there exists an element x in R such that $x = u^g$ or $(tu)^g$ for some $g \in G$. By (4.5.1) and (4.5.4) we have

$$\begin{aligned} m(C(y)) &\geq 8n & \text{if } y \in I(R), \\ m(C_{RT}(y)) &\leq 6n + 1 & \text{if } y \in I(RT - R). \end{aligned}$$

So in any case we may take $C_{(RT)^g}(x) \leq RT$. Then $t^g \in RT$. Hence by (4.5.3) we may assume $g \in C$. This implies that u or tu fuses into P in C , which does not occur. Thus $R \in \text{Syl}_2(O^2(G))$. By (4.5.2) $\langle L_1, L_2 \rangle \leq G_0$. Therefore $R \in \text{Syl}_2(G_0)$ and also the first statement of (3) holds. Then by (4.2) and (4.4.1) we have $C_R(z_2) \in \text{Syl}_2(C_{G_0}(z_2))$. By (4.6) $B^* \subseteq z_1^{G_0}$ and $V_i^*, F^* \subseteq z_1^{G_0}$. So by (4.7), if $B^x < R$, then $B^x = B$ or B^t , or else $B^x < F_1$. Suppose that $B^x < F_1$. Then $B^x \cap F \subseteq Z(U) \cup Z(U^t)$ by (3.17.5). Since $|B^x \cap F| \geq q^4$, we may assume that $B^x \cap F = Z(U)$ without loss of generality. Now similarly $B^x \cap F^{r_1} \subseteq Z(U^{r_1}) \cup Z(U^{r_1 t})$. If $B^x \cap F^{r_1} = Z(U^{r_1})$, then $B^x = Z(U)Z(U^{r_1}) = F_1$ by (3.17.1), a contradiction. If $B^x \cap F^{r_1} = Z(U^{r_1 t})$, then $B^x = Z(U)Z(U^{r_1 t})$. By (4.1.4) this yields that $B^x = Z_2(UU^{r_1 t})$, also a contradiction. Thus $B^x = B$ or B^t . Then by a theorem of Burnside we have $x \in N_{G_0}(R)$ and by (4.6) $B^x = B$. Thus B is weakly closed in R with respect to G_0 . By symmetry B^t is weakly closed.

(4.9) $L_2 \triangleleft N(F_2)$.

PROOF. Set $\overline{N(F_2)} = N(F_2)/F_2$ and set $X = N(F_2) \cap G_0 \langle t \rangle$. By (4.8)

$Q \in \text{Syl}_2(C_X(t))$. Hence by (3.8) and (3.10) $\bar{N}_2$ is a standard subgroup of $\bar{X}$ and $\langle \bar{t} \rangle$ is a Sylow 2-subgroup of $C_X(\bar{N}_2)$. So, as in (3.12), we have $\langle \bar{N}_2^{\bar{X}} \rangle \simeq SL_2(q) \times SL_2(q)$ or $SL_2(q^2)$. Since $\bar{T}$ normalizes $\bar{N}_2$, $\langle \bar{N}_2^{N(F_2)} \rangle = \langle \bar{N}_2^{\bar{X}} \rangle$. Therefore, $\bar{L}_2 = \langle \bar{N}_2^{N(F_2)} \rangle \triangleleft \bar{N}(F_2)$.

Now we can appeal to an argument in [12] for the conclusion of this section. (4.10, 11, 12 and 13) below refer to $(4N, O, P, S)$ of [12].

$$(4.10) \quad N_{G_0}(E) = N_{G_0}(F_2)O(N_{G_0}(E)).$$

PROOF. Set $\overline{N_{G_0}(E)} = N_{G_0}(E)/O(N_{G_0}(E) \bmod E)$ and let X be the normal closure of F_2 in $N_{G_0}(E)$. Then by (4.1.1) and (4.8.2) we can appeal to (1H) of [12] to obtain that F_2 is strongly closed in R with respect to $N_{G_0}(E)$. Hence by the result of [10] we have $F_2 \in \text{Syl}_2(X)$, and either $\bar{X} = \bar{F}_2$ or $\bar{X} = O_2(\bar{X})\bar{X}_1 \cdots \bar{X}_k$, where $\bar{X}_i$ is a simple Goldschmidt group. Suppose the latter holds. Since $\bar{t}$ normalizes $\bar{X}$ and since $\mathcal{E}(\langle \bar{t} \rangle E) = \{E, Z(Q)\}$,

$$C(\bar{t}) \cap \bar{F}_2 \cap \bar{X}_1 \bar{X}_1^t = \bar{A}_2 \cap \bar{X}_1 \bar{X}_1^t \neq 1.$$

Let $\bar{x}_i \in (\bar{A}_2 \cap \bar{X}_i \bar{X}_i^t)^*$. If there exist $\bar{X}_1, \bar{X}_2$ and $\bar{X}_3$ such that $\bar{X}_1 \bar{X}_1^t \neq \bar{X}_2 \bar{X}_2^t \neq \bar{X}_3 \bar{X}_3^t \neq \bar{X}_1 \bar{X}_1^t$, then $\bar{x}_1, \bar{x}_1 \bar{x}_2$ and $\bar{x}_1 \bar{x}_2 \bar{x}_3$ are not conjugate mutually in $\bar{N}_{G_0}(E)$. But $\bar{N}_L(E)$ has only two orbits on $\bar{A}_2^*$ by (2.2.2). This contradiction yields that $E(\bar{X})$ has at most four components. Therefore $\overline{C_L(E)'}$ normalizes each $\bar{X}_i$, since $\overline{C_L(E)'}$ is perfect. In particular it normalizes $\bar{F}_2 \cap \bar{X}_i$ which is a Sylow 2-subgroup of $\bar{X}_i$. Consequently $\overline{C_L(E)'}$ centralizes each $\bar{X}_i$. But this yields that $\overline{C_L(E)'}$ centralizes $\bar{A}_2$, a contradiction. Therefore $\bar{F}_2 \triangleleft N_{G_0}(E)$. Then, since $E = Z(F_2)$, we have $F_2 O(N_{G_0}^*(E)) \triangleleft N_{G_0}(E)$ and the Frattini argument completes the proof.

DEFINITION. Let $C_i = C_{G_0}(z_i)$, $i = 1, 2$.

$$(4.11) \quad C_1 = N_{C_1}(E)O(C_1) \text{ and } C_1 \text{ is 2-constrained.}$$

PROOF. Suppose an element z of E is C_1 -conjugate to an element x of $R - E$. Then by (3.13.2), (4.1.1) and (4.5.1) we may take x in $F_1 - E$. Since F_1 is weakly closed in R , z and x are conjugate in $N_{C_1}(F_1)$. But by (3.17.4) we have $N_{C_1}(F_1) \leq N(R) \leq N(E)$, a contradiction. So E is strongly closed in R with respect to C_1 .

Let $X = \langle E^{C_1} \rangle$ and let $\bar{C}_1 = C_1/O(C_1)$. Then by the result of [10] we have either $\bar{X} = \bar{E}$ or $\bar{X} = O_2(\bar{X})\bar{X}_1 \cdots \bar{X}_k$, where $\bar{X}_i$ is a quasisimple Goldschmidt group. Furthermore we have $E = \mathcal{Q}_1(R \cap X)$. If $E < R \cap X$, then by (3.9.1) $E < F \cap X \leq \mathcal{Q}_1(R \cap X)$, a contradiction. Hence $E \in \text{Syl}_2(X)$. If $\bar{X} \neq \bar{E}$, then $\bar{L}_2$ normalizes each $\bar{X}_i$ by (3.13.2). Now by (3.17.3) $N_{L_1}(R)$ acts on E^* transitively. Hence $E \leq R'$ and so L_2 is perfect by (3.13.3). Therefore $\bar{L}_2$ centralizes

each $\bar{X}_i$. Then $\bar{E} \cap \bar{X}_i \leq \bar{L}_2 \cap \bar{X}_i \leq Z(\bar{X}_i)$, which does not hold. Hence $\bar{X} = \bar{E}$, and it follows that $C_1 = N_{C_1}(E)O(C_1)$. Since $C_R(F_2) < F_2 < R$ by (4.1.1), $N_{G_0}(F_2)$ is 2-constrained and so is $N_{G_0}(E)$ by (4.10). Thus C_1 is 2-constrained.

(4.12) $C_2 = N_{C_2}(F_1)O(C_2)$ and C_2 is 2-constrained.

PROOF. By (4.8.3) $C_R(z_2) = F_1 V_1 \in \text{Syl}_2(C_2)$ and $\mathcal{C}(C_R(z_2)) = \{V_1, F_1\}$, where V_1 is defined in the proof of (4.6). Hence we can apply (1H) of [12] to obtain that F_1 is strongly closed in $C_R(z_2)$ with respect to C_2 . Let X be the normal closure of F_1 in C_2 and set $\bar{C}_2 = C_2/O_2(C_2)$. Then as in (4.10) we have $F_1 \in \text{Syl}_2(X)$ and either $\bar{X} = \bar{F}_1$ or $\bar{X} = O_2(\bar{X})\bar{X}_1 \cdots \bar{X}_k$, where $\bar{X}_i$ is a simple Goldschmidt group. Suppose that $\bar{X} \neq \bar{F}_1$. Then by (4.11) $C_1 \cap C_2$ is 2-constrained and so is $\bar{C}_1 \cap \bar{C}_2$. Hence $C_{\bar{X}}(\bar{z})$ is 2-constrained for any $z \in E^\#$. So, if we set $\bar{z} = \bar{x}\bar{y}_1 \cdots \bar{y}_k$, where $\bar{x} \in O_2(\bar{X})$ and $\bar{y}_i \in \bar{X}_i$, then it follows that $\bar{y}_i \neq \bar{1}$ for all i . Thus $|\bar{E}| \leq |\bar{F}_1 \cap \bar{X}_i|$. Since all the elements of $(\bar{F}_1 \cap \bar{X}_i)^\#$ are conjugate mutually in $N_{\bar{X}_i}(\bar{F}_1 \cap \bar{X}_i)$, F_1 contains at least $(q^2 - 1)^k$ elements conjugate to z . Then by (4.3) $(q^2 - 1)^k \leq (q^2 - 1)(q^2 + 1)^2$. Hence $k \leq 3$ and as in (4.10) we obtain that $\bar{C}_L(z_2)^\vee$ centralizes each $\bar{X}_i$ by (2.2.3). Then $\bar{z}_1 \in C_{\bar{X}}(E(\bar{X}))$. Since $\bar{C}_1 \cap \bar{C}_2$ is 2-constrained, this can not occur. Thus we have $\bar{X} = \bar{F}_1$ and the assertion follows easily.

(4.13) $G_0 \simeq L_4(q^2)$.

PROOF. By (4.8.3), (4.11) and (4.12) the centralizer of any involution in G_0 is 2-constrained. Furthermore $SCN_3(2)$ is not empty, as $F_1 \in SCN_3(2)$. Thus by a result of [14] we have $O(C_1) = O(C_2) = 1$, since we assumed $O(G) = 1$ at the beginning of Section 3. Hence $L_2 \triangleleft C_1$ by (4.9). Suppose that $B^g < C_1$ for some $g \in G_0$. Then $B^g < L_2$. Hence $B^{g^h} < R$ for some $h \in L_2$, and so by (4.8.4) $B^{g^h} = B$. Consequently we have $B^g = B$ by (4.6). Since $B^\# \subseteq z_1^{G_0}$ by (4.6), for any $a \in B^\# B$ is weakly closed in $C_{G_0}(a)$ with respect to G_0 . Since G_0 is simple (cf. Lemma 2.7 of [2]), it follows that $G_0 = \langle B^{G_0} \rangle$. Thus we can appeal to the result of Timmesfeld [19] to conclude that G_0 is isomorphic to $L_4(q^2)$.

§ 5. Case for $U_4(q) \times U_4(q)$

In this section we assume the following and we will obtain that $\langle L^G \rangle \simeq U_4(q) \times U_4(q)$ under the hypothesis made at the beginning of Section 3.

HYPOTHESIS. $L_2/F_2 \simeq SL_2(q) \times SL_2(q)$.

DEFINITION. Set $L_2 = K_2 K_2^t$, where $K_2 \triangleleft L_2$ and $K_2/F_2 \simeq SL_2(q)$.

- (5.1) (1) $F_1 = (Z(U) \cap Z(U^{r_1})) \times (Z(U^t) \cap Z(U^{r_1 t}))$.
 (2) K_1 acts on $Z(U) \cap Z(U^{r_1})$ trivially and on $Z(U^t) \cap Z(U^{r_1 t})$ irreducibly.
 (3) $L_1 = K'_1 \times K_1^t$.

PROOF. Suppose $|Z(U)| = q^4$. Then by (3.16.2) we can apply (3.17.3) to obtain that $J_1 \cap K_1$ acts on $(F_1/F)^*$ transitively. Since $R/F_2 \simeq F_1/F$, $C_{F_1/F}(J_2 \cap K_2) \neq 1$. Then by (3.14) $C_{F_1/F}(J_2 \cap K_2)$ is $J_1 \cap K_1$ -invariant. Hence this yields that $C_{F_1/F}(J_2 \cap K_2) = F_1/F$, so $C_{R/F_2}(J_2 \cap K_2) = R/F_2$. This can not occur by the structure of L_2/F_2 . Thus we have $|Z(U)| > q^4$ and $Z(U) \cap Z(U^{r_1}) \neq 1$. Since $\langle U, U^{r_1} \rangle = K_1$, $K_1 \leq C(Z(U) \cap Z(U^{r_1}))$. If $|Z(U) \cap Z(U^{r_1})| < q^4$, then by Lemma 4B of [8] $K_1^t \leq C(Z(U) \cap Z(U^{r_1}))$. But this contradicts (3.5.1). On the other hand we have $Z(U) \cap Z(U^{r_1}) \cap Z(U^t) \cap Z(U^{r_1 t}) = E \cap E^{r_1} = 1$ by (3.11.1). Thus (1) holds. Also (2) follows from (1). Then we have

$$K_1 = (Z(U) \cap Z(U^{r_1})) \times ((Z(U^t) \cap Z(U^{r_1 t})) \cdot (K_1 \cap M_1)).$$

Hence Gaschütz's theorem yields that $K'_1 = (Z(U^t) \cap Z(U^{r_1 t})) \cdot (K_1 \cap M_1)$ and so $L_1 = K'_1 \times K_1^t$.

(5.2) Set $W = R \cap K'_1$, $W_i = [W, I_i]$, $i = 1, 2$. Then the following statements hold.

- (1) $E = Z(W) \times Z(W^t)$, $F = Z_2(W) \times Z_2(W^t)$ and $R = W \times W^t$.
 (2) $F_1 = W_1 \times W_1^t$ and $F_2 = W_2 \times W_2^t$.
 (3) $W_1 = O_2(K'_1) = Z(U^t) \cap Z(U^{r_1 t})$.

PROOF. (1) is clear by (5.1.3). By (3.9.3) $F_1 = [R, I_1] = [W \times W^t, I_1] = [W, I_1] \times [W^t, I_1] = W_1 \times W_1^t$. Also $F_2 = [R, I_2] = W_2 \times W_2^t$. Thus (2) holds. Then, since $|W_1| = |Z(U^t) \cap Z(U^{r_1 t})|$ and $W_1 \leq F_1 \cap K'_1$, we have (3).

$$(5.3) \quad L_2 = K_2'' \times K_2'''^t \text{ and } W \text{ is a Sylow 2-subgroup of } K_2'' \text{ or } K_2'''^t.$$

PROOF. By (3.14) $J_2/C_{J_2}(K'_1/W)$ acts on K'_1/W as an inner automorphism. Since $J_2 \simeq Z_{q-1} \times Z_{q-1}$, $|C_{J_2}(K'_1/W_1)| \geq q-1$. Set $J = C_{J_2}(K'_1/W_1)$. Suppose $J \not\subseteq K_2 \cup K_2^t$. Then $C_{R/F_2}(J) = 1$ and so $C_{F_1/F}(J) = 1$. Hence $C_{F_1}(J) \leq F$ and by (3.13.2) and (5.2) we have $1 \neq C_{W_1}(J) < W_1$. But this contradicts (5.1.2). Therefore $J \subseteq K_2 \cup K_2^t$. So we may assume that $J = J_2 \cap K_2^t$. Then $C_{W_1}(J) \neq 1$, which yields that $[W_1, J] = 1$. Since J stabilizes a normal series $W \triangleright W_1 \triangleright 1$, we have $[W, J] = 1$. Set $V = K_2 \cap R$. Then

$$V^t = F_2[R, J] = (W_2 \times W_2^t)([W, J] \times [W^t, J]) \leq W_2 \times W_2^t[W^t, J].$$

Comparing the orders of both sides, we have $V^t = W_2 \times W^t$. By symmetry $V = W \times W_2^t$. As a consequence $Z(V) = E$ and $Z_2(V) = Z_2(W) \times W_2^t$. Set $\bar{L}_2 = L_2/E$. By (3.9.2) $[\bar{I} \cap \bar{J}_2, \bar{F}_2] = \bar{F}_2$, so in particular $[\bar{J}_2, \bar{W}_2] = \bar{W}_2$. Since $J_2 =$

$(J_2 \cap K_2) \times J$ and $[\bar{J}, \bar{W}] = \bar{1}$, we have $[\overline{J_2 \cap K_2}, \bar{W}_2] = \bar{W}_2$. Notice that $|Z(\bar{V})| = q^6$ and that $K_2 = \langle V, V^{r_2} \rangle$. Hence $|Z(\bar{V}) \cap Z(\bar{V}^{r_2})| \geq q^4$ and $K_2 \leq C(Z(\bar{V}) \cap Z(\bar{V}^{r_2}))$. Therefore we have

$$\begin{aligned} Z(\bar{V}) \cap Z(\bar{V}^{r_2}) \cap \bar{W} &= Z(\bar{V}) \cap Z(\bar{V}^{r_2}) \cap \bar{W} \cap \bar{F}_2 \\ &= Z(\bar{V}) \cap Z(\bar{V}^{r_2}) \cap \bar{W}_2 = \bar{1}. \end{aligned}$$

Hence $\bar{V} = (Z(\bar{V}) \cap Z(\bar{V}^{r_2})) \times \bar{W}$ and Gaschütz's theorem yields that $\bar{K}_2' \cap \bar{V} = \bar{W}$. Then, since $EW = W \times Z(W^t)$, again by Gaschütz's theorem we have $K_2'' \cap V = W$ and so $L_2 = K_2'' \times K_2''^t$.

(5.4) Set $G_0 = \langle L_1, L_2 \rangle$ and set $G_1 = \langle K_1', K_2'' \rangle$, where K_2'' is chosen such that $W \in \text{Syl}_2(K_1') \cap \text{Syl}_2(K_2'')$. Then $G_0 = G_1 \times G_1^t$ and $G_1 \simeq U_4(q)$.

PROOF. We will show $[K_1'^t, K_2''] = 1$. We remark firstly that

$$\begin{aligned} F_1^{r_2} \cap F_1^{r_2 r_1 r_2} &= (F_1 \cap F_1^{r_2})^{r_1 r_2} = ((W_1 \times W_1^t) \cap (W_1^{r_2} \times W_1^{r_2 t}))^{r_1 r_2} \\ &= (W_1^{r_2} \cap W_1^{r_2 r_1 r_2}) \times (W_1^{r_2} \cap W_1^{r_2 r_1 r_2})^t \end{aligned}$$

and secondly by the structure of $A_2 N_2 = C_{L_2}(t)$ that

$$A_2 N_2 = \langle P, A_1^{r_2} \cap A_1^{r_2 r_1 r_2} \rangle.$$

Now by (5.3) the correspondence of x to xx^t for $x \in K_2''$ gives an isomorphism $K_2'' \simeq C_{L_2}(t)$. Then the first remark shows that $W_1^{r_2} \cap W_1^{r_2 r_1 r_2}$ corresponds to $A_1^{r_2} \cap A_1^{r_2 r_1 r_2}$. Clearly W corresponds to P . So by the second remark we have

$$K_2'' = \langle W, W_1^{r_2} \cap W_1^{r_2 r_1 r_2} \rangle.$$

For K_1' as above we have

$$\begin{aligned} F_1^{r_2 r_1} \cap F_2^{r_1} &= (W_1^{r_2 r_1} \cap W_2^{r_1}) \times (W_1^{r_2 r_1} \cap W_2^{r_1})^t, \\ K_1' \simeq C_{L_1}(t) &= A_1 N_1 = \langle P, A_1^{r_2 r_1} \cap A_2^{r_1} \rangle, \end{aligned}$$

and

$$K_1' = \langle W, W_1^{r_2 r_1} \cap W_2^{r_1} \rangle.$$

Then by (5.1.3) and (5.3)

$$[W^t, W_1^{r_2} \cap W_1^{r_2 r_1 r_2}] \leq [W^t, W_1^{r_2}] \leq [K_2''^t, K_2''] = 1$$

and

$$[W, (W_1^{r_2 r_1} \cap W_2^{r_1})^t] \leq [W, W_2^{r_1 t}] \leq [K_1', K_1'] = 1.$$

Of course $[W, W^t] = 1$. Moreover, since $(r_1 r_2)^4 = 1$, we have

$$\begin{aligned}
& [W_1^{r_2} \cap W_1^{r_2 r_1 r_2}, (W_1^{r_2 r_1} \cap W_2^{r_1})^t] \\
& \leq [W_1^{r_2 r_1 r_2}, W_1^{r_2 r_1 t}] = [W_1^{r_2 r_1 r_2 r_1}, W_1^{r_1 r_2 r_1 t}] = [W_1^{r_2}, W_1^t]^{r_1 r_2 r_1} \\
& \leq [K_2'', K_2'']^{r_1 r_2 r_1} = 1.
\end{aligned}$$

Therefore we have $[K_1', K_2''] = 1$ and consequently $G_0 = G_1 * G_1^t$. Then $C_{G_0}(t)$ is a homomorphic image of G_1 . Since G_1 is perfect, G_1 is a central extension of $U_4(q)$. Thus we have $G_0 = G_1 \times G_1^t$ and $G_1 \simeq U_4(q)$.

(5.5) *Let S_0 be a Sylow 2-subgroup of $N(F_1)$ containing R and set $R_0 = N_{S_0}(K_1')$. Then*

- (1) $F_1 \in \text{Syl}_2(C(L_1/F_1))$ and
- (2) $S_0 \in \text{Syl}_2(G)$.

PROOF. (1) is clear by the proof of (3.13.1), since $t \notin C(L_1/F_1)$. We will show that $F_1 = Jr(S_0)$ and then (2) follows.

Let $x \in I(S_0 - R)$. Then by (1) x acts on L_1/F_1 nontrivially. Hence $m(S_0/R) \leq 2$. If x interchanges K_1' and $K_1'^t$, then $m(C_R(x)) = 4n$ and $m(C_{S_0}(x)) \leq 4n + 2$. This implies $Jr(S_0) = Jr(R_0)$. Suppose $m(C_C(A_2 N_2/A_2)/A_2) = 1$. Then $C_{N(F_2)/F_2}(F_2 N_2/F_2)$ has a cyclic Sylow 2-subgroup and $F_2 N_2/F_2$ is a standard subgroup of $N(F_2)/F_2$. Hence we have $L_2 = F_2 \langle N_2^{N(F_2)} \rangle \triangleleft N(F_2)$ as in (3.13). Since $R = L_1 \cap S_0 \triangleleft S_0$ and $F_2 = Jr(R \bmod E)$ by (5.4), S_0 normalizes L_2 . Hence S_0 normalizes G_0 . Then we have $S_0 < \text{Aut}(G_0)$ and $R_0 < N(G_0) \cap N(G_1)$. Thus (2) holds by (2.3.2, 4 and 5). Consequently we may assume that $m(C_C(A_2 N_2/A_2)/A_2) = 2$, so $I(T - P) = I(tP) \cup I(uP) \cup I(tuP)$, where u is defined in Section 2. We also may assume without loss of generality that u acts on both components of L_1/F_1 as a field automorphism. Then $I(uP) \subseteq I(R_0)$ and $I(tP) \cup I(tuP) \subseteq I(S_0 - R_0)$. Since $F_2 = Jr(R \bmod E)$ and since u normalizes I_2 , it follows that $u \in N(L_2)$. Hence $u \in N(G_0)$. Then as above we have $Jr(\langle u \rangle R) = F_1$.

Therefore, if $I(R_0 - R) = I(uR)$, then by the result of the previous paragraph there is nothing more to say. So we may assume that $m(R_0/R) = 2$ and then we can choose an element x of $R_0 - \langle u \rangle R$ such that $x^2 \in F_1$, x acts on one of the components of L_1/F_1 trivially and on the other as a field automorphism. Suppose that x acts on K_1/F_1 trivially. Let $z \in Z(W)$ and let $\Omega = z^{K_1}$. Then by (2.2.1) and (5.4) $|\Omega| = (q-1)(q^2+1)$. Since $(\langle x \rangle K_1)^q \simeq Z_2 \times SL_2(q^2)$, x^q is semiregular. Hence $x^q = 1^q$, since $|\Omega| = \text{odd}$. So we have $x \in C(W_1)$, since $\langle \Omega \rangle = W_1$. Then by (2.7) $xy \in C(K_1')$ for some $y \in W_1$. Set $s = (xy)^t$. Then

$$K_1' \langle s \rangle \cap K_1'^t \langle s^t \rangle \leq C(L_1).$$

By (1) $|C(L_1)| = \text{odd}$. Hence we have

$$K_1' \langle s \rangle \cap K_1'^t \langle s^t \rangle = 1.$$

So

$$\begin{aligned} L_1\langle s, s' \rangle &= K'_1\langle s \rangle \times K'_1\langle s' \rangle, \\ \Omega_1(R_0 \bmod R) &= R\langle s, s' \rangle = W\langle s \rangle \times W'\langle s' \rangle \end{aligned}$$

and

$$K'_1\langle s \rangle \simeq C(t) \cap L_1\langle s, s' \rangle = A_1 N_1\langle u \rangle.$$

Here we can choose s to be an involution. Moreover we may assume that

$$u = ss'.$$

Then for any involution x of sR we have $m(C_{K'_1}(x)) = 3n$ by (2.3.4, 5) and so $m(C_{R_0}(x)) \leq 7n + 1$. Therefore $Jr(R_0) = Jr(R) = F_1$. Thus the proof is complete.

- (5.6) (1) *The fusion of F_1 in G is controlled by $\langle t \rangle J_2 L_1$.*
 (2) $L_2 \triangleleft N(F_2)$.

PROOF. Set $\Omega = F_1^\#$. Then by (2.2.1) and (5.4) L_1^Ω has two orbits of length $(q-1)(q^2+1)$ and $2(q-1)$ orbits of length $q(q^2+1)$ in $(W_1 \cup W_1^t)^\#$, and one orbit of length $(q-1)^2(q^2+1)^2$, $2(q-1)$ orbits of length $(q-1)q(q^2+1)^2$ and $(q-1)^2$ orbits of length $q^2(q^2+1)^2$ in $F_1 - (W_1 \cup W_1^t)$. Also we have that $(\langle t \rangle J_2 L_1)^\Omega$ permutes transitively the orbits of L_1^Ω of the same length, that is, $(\langle t \rangle J_2 L_1)^\Omega$ has five orbits which are of length $2(q-1)(q^2+1)$, $2(q-1)q(q^2+1)$ in $(W_1 \cup W_1^t)^\#$ and $(q-1)^2(q^2+1)^2$, $2(q-1)^2q(q^2+1)^2$, $(q-1)^2q^2(q^2+1)^2$ in $F_1 - (W_1 \cup W_1^t)$. Since F_1 is weakly closed in S_0 , the fusion of the elements of F_1 is controlled by $N(F_1)$, which permutes the orbits of L_1^Ω of the same length. Hence the fusion is already controlled by $\langle t \rangle J_2 L_1$.

Then $W_2 \cup W_2^t$ is weakly closed in F_2 with respect to G , since $F_2^\# \cap W_1^\Omega = I(W_2 \cup W_2^t)$ and $\langle I(W_2) \rangle = W_2$. Therefore, if we set $X = N(F_2) \cap N(W_2)$, then $|N(F_2) : X| = 2$ and $R_0 \in \text{Syl}_2(X)$. Let $Y = \Omega_1(R_0 \bmod R)$. In order to prove $L_2 \triangleleft N(F_2)$ we may assume that $m(C_C(A_2 N_2 / A_2) / A_2) = 2$ as in the proof of (5.5). Hence we have

$$\begin{aligned} Y &= \langle u \rangle R \text{ if } m(R_0/R) = 1, \text{ or} \\ Y &= \langle s, s' \rangle R = W\langle s \rangle \times W'\langle s' \rangle \text{ if } m(R_0/R) = 2, \end{aligned}$$

where s is chosen as in (5.5). In any case $[R/F_2, s] = [R/F_2, u] = 1$ by (2.5.3). So we have

$$Y = Jr(R_0 \bmod F_2) = \Omega_1(R_0 \bmod F_2)$$

and also Y/F_2 is abelian. Therefore the fusion of $I(R_0/F_2)$ in X is controlled by $N_X(Y)$. Since $Y > F_1 = Jr(S_0)$, we have

$$N_x(Y) \leq N_x(F_1) = N_x(R),$$

which implies that $N_x(R)$ controls the fusion of $I(R_0/F_2)$ in X . Consequently any involution of $R_0 - R$ does not fuse into R in X . Suppose that $u^x \in sR$ for some $x \in X$. Then we may take $x \in N_x(R)$. But $\langle s \rangle R \neq \langle u \rangle R$. This gives a contradiction. Hence $u^x \subseteq uR$. Since R_0/R is abelian, we have $u \notin O^2(X)$ by (1.5). Suppose $R \notin \text{Syl}_2(O^2(X))$. Then $R\langle s \rangle$ or $R\langle s^t \rangle < O^2(X)$, since $R < L_2 \leq O^2(X)$. This yields that $u \in R\langle s, s^t \rangle \leq O^2(X)$, since $O^2(X)$ is t -invariant, a contradiction. Hence $R \in \text{Syl}_2(O^2(X))$. Then $F_2 N_2 / F_2$ is a standard subgroup of $O^2(X)\langle t \rangle / F_2$ and $\langle t \rangle F_2 / F_2$ is a Sylow 2-subgroup of $C(F_2 N_2 / F_2) \cap O^2(X)\langle t \rangle / F_2$. Hence as in (4.9) we have $L_2 \triangleleft N(F_2)$.

- (5.7) (1) $N(F_i) \leq N(G_0)$, $i=1, 2$.
 (2) $N(G_0)$ controls the G -fusion of involutions of G_0 .
 (3) If $t \in N(G_0)^g$, then $g \in N(G_0)$.
 (4) $|C(G_0)| = \text{odd}$.

Proof. (4) holds obviously by (5.5.1). Since $L_i \triangleleft N(F_i)$ for $i=1$ and 2 , the Frattini argument yields that $N(F_i) = L_i(N(R) \cap N(F_i))$. By (5.4) $F_2 = Jr(R \bmod E)$. Hence $F_i \text{ char } R$, $i=1, 2$. Therefore $N(F_i) = L_i(N(F_{3-i}) \cap N(F_i))$, which normalizes G_0 . Thus (1) holds. By (2.4) and (5.4) any involution of G_0 fuses into F_1 . Hence (2) follows from (1) and (5.6.1).

By (1) $S_0 < \text{Aut}(G_0)$. If an involution x of S_0 does not interchange G_1 and G_1^t , then by (2.3.4) $m(C(x)) \geq 6n+1$. Since $m(C) = 4n+1$, t interchanges G_1^g and $G_1^{t^g}$, whenever $t \in N(G_0)^g$. Hence $C(t) \cap G_0^g = \{xx^t \mid x \in G_1^g\}$ and $L \simeq C(t) \cap C_0^g \leq C^{(\infty)} = L$. Therefore we have $C(t) \cap G_0^g = L$. Let $R'\langle t \rangle$ be a Sylow 2-subgroup of $G_0\langle t \rangle$ containing Q . Then $N_{R'\langle t \rangle}(Q) > Q$. Hence by Lemma 2.5.3 of [2] we have $\langle N_{R'}(Q), L \rangle = G_0^g$. On the other hand by (3.2.2) and (3.3) $N(Q) = EN_c(Q) \leq N(R)$. Hence $N(Q) \leq N(Q_0)$ and consequently $G_0^g \leq N(G_0)$. Then $G_0^g \leq N(G_0)^{(\infty)} = G_0$ and clearly the equality holds.

(5.8) *There exists an abelian subgroup R_1 of $N(G_0) \cap N(G_1)$ such that $R \cdot R_1 \in \text{Syl}_2(N(G_0) \cap N(G_1))$, t normalizes R_1 and $C_{R_1}(t)$ acts on G_1 and G_1^t as a group of field automorphisms.*

PROOF. By (5.7.1) $S_0 < N(G_0)$ and $R_0 \in \text{Syl}_2(N(G_0) \cap N(G_1))$. Set $N(G_0) \cap C(G_1) = \tilde{G}_1$ and set $\overline{N(G_0) \cap N(G_1)} = N(G_0) \cap N(G_1) / \tilde{G}_1^t$. Then $\overline{N(G_0) \cap N(G_1)}$ is embedded in $\text{Aut}(U_4(q))$. Hence there exists an element $\bar{x}$ which is a generator of the group of field automorphisms in $\overline{N(G_0) \cap N(G_1)}$ and which normalizes $\bar{R}$. Let x be a preimage of $\bar{x}$ and set $X = \tilde{G}_1^t \langle x \rangle \cap \tilde{G}_1 \langle x^t \rangle$. Notice that $G_1 \leq \tilde{G}_1$ and $G_1 \tilde{G}_1^t \langle x \rangle = N(G_0) \cap N(G_1)$. Hence we have

$$G_0 X = G_1^t (G_1 \tilde{G}_1^t \langle x \rangle \cap \tilde{G}_1 \langle x^t \rangle) = G_1 \tilde{G}_1^t \langle x \rangle \cap G_1^t \tilde{G}_1 \langle x^t \rangle = N(G_0) \cap N(G_1)$$

and

$$G_0 \cap X = G_0 \cap \tilde{G}_1^t \langle x \rangle \cap \tilde{G}_1 \langle x^t \rangle = G_1^t \cap \tilde{G}_1 \langle x^t \rangle = 1.$$

Therefore $N(G_0) \cap N(G_1) = G_0 \cdot X$ and by the choice of x $X \cap C$ acts on G_1 and G_1^t as a group of field automorphisms. Since

$$R\tilde{G}_1^t \cap R\tilde{G}_1 = R(W\tilde{G}_1^t \cap \tilde{G}_1) = R(\tilde{G}_1^t \cap \tilde{G}_1) = R \times C(G_0),$$

X normalizes R . Thus if we choose a t -invariant Sylow 2-subgroup R_1 of X , then we have $R \cdot R_1 \in \text{Syl}_2(N(G_0) \cap N(G_1))$. By (5.7.4) R_0/R is abelian. Therefore R_1 is abelian.

(5.9) *Let ψ be the transfer homomorphism of G into S_0/R_0 relative to the subgroup S_0 . Then*

- (1) $t, tu \notin \text{Ker } \psi$ and
- (2) $u^g \cap S_0 \subseteq uR$.

PROOF. By the structure of G_0 ,

$$m(C_{N(G_0)}(x)) \geq 6n+1 \quad \text{for } x \in I(R).$$

Hence $t^g \cap R_0 = \phi$. Suppose $(tu)^g \in R_0$ for some $g \in G$. Since

$$m(C_{N(G_0)}(tu)) = 4n+1,$$

we can choose $C_{S_0}((tu)^g)$ to be a Sylow 2-subgroup of $C((tu)^g)$. Furthermore we may assume that $C_{S_0}(tu)^g \leq C_{S_0}((tu)^g)$. Then $t^g, t \in S_0$. So by (5.7.3) $g \in N(G_0)$ and $(tu)^g \notin R_0$, which is a contradiction. Thus (1) holds.

By (5.8) $\Omega_1(R_0) = \langle u \rangle R$ or $\langle s, s^t \rangle R$ according as $m(R_0/R) = 1$ or 2 . By (2.8) we have

$$\begin{aligned} m(C_{S_0}(u)) &\leq 6n+2 \quad \text{and} \\ m(C_{N(G_0)}(x)) &\geq 7n+1 \quad \text{for } x \in I(\langle s \rangle R) \cup I(\langle s^t \rangle R). \end{aligned}$$

Therefore if $u^g \in \langle s \rangle R \cup \langle s^t \rangle R$ for some $g \in G$, then we may take $C_{S_0}(u)^g \leq C_{S_0}(u^g)$, $t^g, t \in N(G_0)$ and we have $g \in N(G_0)$ as above. This does not occur. Thus $u^g \cap R_0 \subseteq uR$.

Now in order to prove (2) we will show $u^g \cap (S_0 - R_0) = \phi$. Take involutions w_1 and w_2 of W such that $w_1 w_1^t = z_1$ and $w_2 w_2^t = z_2$. Then by (5.7.2) the conjugacy classes of involutions of G_0 in G are represented by w_1, w_2, z_1, z_2 and $w_1 w_2^t$. By (2.8) $m(C_w(x)) = m(C_{w^t}(x)) = 3n$ for any involution x of uR . Hence we have

$$m(C_{S_0}(x)) \geq 6n+1,$$

while

$$m(B) \leq 3n + 2,$$

where B is an elementary abelian subgroup of $C_{S_0}(x)$ such that

$$B^* \subseteq z_1^G \cup z_2^G \cup (w_1 w_2^t)^G.$$

On the other hand $m(C_R(x)) = 4n$ for any involution y of $S_0 - R_0$, since $R = W \times W^t = W \times W^y$. Hence we have

$$m(C_{S_0}(y)) \leq 4n + 2, \quad \text{and} \quad m(C(y) \cap W_1 W_1^y) = 4n.$$

Note that

$$(C(y) \cap W_1 W_1^t)^* \subseteq z_1^G \cup z_2^G \cup (w_1 w_2^t)^G.$$

These facts imply that any involution of $S_0 - R_0$ does not fuse into uR if $n > 2$. Hence by the result of the second paragraph we have $u^q \cap S_0 \subseteq uR$, if $q > 4$.

Suppose that $q = 4$ and u fuses into $S_0 - R_0$. Choose a conjugate u' of u such that $C_{S_0}(u') \in \text{Syl}_2(C(u'))$ and choose u'' such that $u'' \in u^q \cap (S_0 - R_0)$. We may assume that $u''^g = u'$ and $C_{S_0}(u'')^g \leq C_{S_0}(u')$ for some $g \in G$. Set $B'' = J_r(C_R(u''))$ and set $B' = B''^g$. The result of the second paragraph gives that $u' \in uR$. Then by (2.6) $C_W(u') \simeq C_{W^t}(u')$ is isomorphic to a Sylow 2-subgroup of $Sp_4(q)$. So, if we set $W_{i+2} = C_{W_i}(u')$ for $i = 1, 2$ and take $t' \in t^q \cap C_{S_0}(u')$, then

$$\mathcal{C}(C_R(u')) = \{W_3 W_3^{t'}, W_3 W_4^{t'}, W_4 W_3^{t'}, W_4 W_4^{t'}\}$$

and

$$m(W_3) = m(W_4) = 3n.$$

Suppose that $B' \leq R_0$. Then $m(C_{S_0}(u', b)) = 4n$ whenever $b \in B' - R_0$. But $B'' \langle u'' \rangle \leq C_{S_0}(u'', B'')$ which gives that $m(C_{S_0}(u'', B'')) = 4n + 1$. This contradicts the choice of u' . Hence $B' \not\leq R_0$. Since $\Omega_1(R_0) = \langle s \rangle W \times \langle s^t \rangle W^t$, there exists a unique element s' in sW such that $s'u' \in s^t W^t$. Then for some $x, y \in C_R(u')$ we have

$$B' = (B' \cap R) \langle s'x \rangle \langle s'u'y \rangle.$$

Now by focal subgroup theorem, $m(R_0 \cap O^2(G)/R) = 2$. Then it is clear that $m(C_{R_0}(u'') \cap O^2(G)/C_R(u'')) = 1$. Therefore

$$|C_{S_0}(u'') \cap O^2(G)/C_{S_0}(u'', B'') \cap O^2(G)| \geq 2q^2.$$

Next we calculate $|C_{S_0}(u') \cap N(B')/C_{S_0}(u', B')|$. Suppose that $B' < W_{i'} W_{j'}^{t'}$ for any $i', j' \in \{3, 4\}$. Then $C_R(u', B' \cap R) = W_{i'} W_{j'}^{t'}$ in which x and y lie. Hence we have $C_R(u', B') = W_{i'} W_{j'}^{t'}$ and so

$$|N_R(R \cap B') \cap C_R(u')/C_R(u', B')| = q.$$

Hence

$$|N_{S_0}(B') \cap C_{S_0}(u') \cap O^2(G) / C_{S_0}(u', B') \cap O^2(G)| \leq 4q \leq 2q^2.$$

Again this contradicts the choice of u' . Thus u does not fuse into $S_0 - R_0$, even if $q = 4$.

$$(5.10) \quad R \in \text{Syl}_2(O^2(G)).$$

PROOF. We may assume that $R_0 = R \cdot R_1$ by (5.8). We will show that any element x of $R_1 \langle t \rangle - R$ does not fuse into R_0 , and that any element y of R_1 such that $\Omega_1(\langle y \rangle) < C$ does not fuse into $S_0 - R_0$. These facts imply that $R_0 \in \text{Syl}_2(\text{Ker } \psi)$.

Suppose that $|x| > 2$. Then $x = x_1 t$ for some $x_1 \in R_1$ and $x^2 = x_1 x_1^t \in C$. Hence $I(\langle x \rangle)$ is conjugate to u . Therefore for any $g \in G$ such that $x^g \in S_0$ we have $|x^g R / R| = |x|$ by (5.9.2). Hence if $x^g \in R_0$, then by (2.8)

$$\begin{aligned} C_R(x^g) &= C_W(x^g) \times C_{W^t}(x^g) \quad \text{and} \\ m(C_W(x^g)) &= m(C_{W^t}(x^g)) = 3m, \quad \text{where } m = 2n/|x|. \end{aligned}$$

Next if $x^h \in S_0 - R_0$ for some h , then

$$C_R(x^h) = \{w w^{x^h} \mid w \in C_W(x^{2h})\}.$$

Now for a subgroup X of G we define the following:

$$\begin{aligned} m_1(X) &= \text{Max } \{m(B) \mid B \leq X, B^\# \subseteq z_1^G \cup z_2^G \cup (w_1 w_2^t)^G\}; \\ m_2(X) &= \text{Max } \{m(B') \mid B' \leq X, B'^\# \subseteq w_1^G \cup w_2^G\}. \end{aligned}$$

Then the above result gives that

$$\begin{aligned} m_1(C_R(x^g)) &= 3m, \quad m_2(C_R(x^g)) = 3m, \\ m_1(C_R(x^h)) &= 6m, \quad m_2(C_R(x^h)) = 0. \end{aligned}$$

Therefore we have

$$\begin{aligned} m_1(C_{S_0}(x^g)) &\leq 3m + 2, \quad m_2(C_{S_0}(x^g)) \geq 3m, \\ m_1(C_{S_0}(x^h)) &\geq 6m, \quad m_2(C_{S_0}(x^h)) \leq 2. \end{aligned}$$

Consequently for any $x^g \in R_0$ and any $x^h \in S_0 - R_0$,

$$\begin{aligned} m_1(C_{S_0}(x^g)) &< m_1(C_{S_0}(x^h)) \\ m_2(C_{S_0}(x^g)) &> m_2(C_{S_0}(x^h)), \end{aligned}$$

which is absurd, since $C_{S_0}(x') \in \text{Syl}_2(C(x'))$ for some conjugate x' of x in S_0 . Thus x does not fuse into R_0 . Similarly we can obtain that y does not fuse

into $S_0 - R_0$. Therefore $x \notin \text{Ker } \psi$ for any x of tR_1 of order at least 4. Let $R'_1 = R_1 \langle t \rangle \cap \text{Ker } \psi$. Suppose $R'_1 \neq R_1$. Then any element of $R'_1 - R$ is an involution. This yields that $R'_1 \cap R_1$ is cyclic and so R'_1 is a dihedral group by the structure of $\text{Aut}(G_0)$. Since $t \notin R'_1$, $R'_1 \langle t \rangle$ is also dihedral and so R_1 is cyclic. Then $\Omega_1(\langle y \rangle) < C$ for any y of R_1 . Since such elements do not fuse into $S_0 - R_0$, we have $R_1 < \text{Ker } \psi$, a contradiction. Therefore $R'_1 = R_1$ and $R_0 = S_0 \cap \text{Ker } \psi$. Then, since R_0/R is abelian, $u \notin O^2(G)$ by (5.9.2), and we can obtain the assertion as in the proof of (5.9.2).

For the argument in (5.11, 12) below we refer the readers to 4.23, 24 of [21].

- (5.11) (1) W_1 and W_1^t are weakly closed in R with respect to $O^2(G)$.
 (2) $W \in \text{Syl}_2(O^2(C_G(W_1)))$.

PROOF. By (5.7.2), if $W_1^t < R$, then $W_1^t \subseteq W \cup W^t$. If $W_1^t < W$, then $W_1^t = W_1$ by (2.3.2). If $W_1^t < W^t$, then $W_1^t = W_1^t$. So, if $W_1^t = W_1^t$, then $N(R) \cap O^2(G)$ permutes W_1 and W_1^t by a theorem of Burnside. But this can not occur, since $R \in \text{Syl}_2(O^2(G))$. Thus (1) holds. It is clear that $W \times W_1^t \in \text{Syl}_2(C(W_1) \cap O^2(G))$. Hence by Gaschütz's theorem we have $W \in \text{Syl}_2(O^2(C_G(W_1)))$.

$$(5.12) \quad G_0 \trianglelefteq G.$$

PROOF. First we will show that W and W^t are strongly closed in R with respect to $O^2(G)$. Let $x \in W$ and suppose that $x^g \in R - W$ for some $g \in O^2(G)$. If x is an involution, then by (5.6.1) we may take $x \in W_1$ and $x^g \in W_1^t$. Since $F_1 = J r(R)$, x and x^g are conjugate in $N(F_1) \cap O^2(G)$. But this contradicts (5.11.1).

Next we assume that $|x| = 2^i$, $i \geq 2$. Set $x^g = x_1 x_2$, where $x_1 \in W$ and $x_2 \in W^t$. If $|x_1| \leq |x_2|$, then $x^{2^{i-1}}$ which is an involution in W is conjugate to $(x^g)^{2^{i-1}}$ which is an involution in $R - W$. This contradicts the first paragraph. Hence $|x_1| > |x_2|$, and replacing x by an appropriate power, we may assume $|x_2| = 2$. Furthermore we may assume that $x_1 \in W$ and $x_2 \in W_1^t$ by conjugation in G_0 . Then $x, x^g \in C(W_1^t)$. Hence by (5.11.2) we have $x \in O^2(C(W_1^t))$ and $x^g \notin O^2(C(W_1^t))$. Then (5.11.1) gives a contradiction. Thus W is strongly closed in R with respect to $O^2(G)$. By symmetry W^t is also strongly closed.

Now we are in a position to quote a result of Goldschmidt [11] in order to obtain the final result of this section. Set $X = \langle W^{O^2(G)} \rangle$. Then by [11] we have $[X, X'] = 1$, since we are assuming $O(G) = 1$. Then $\{xx^t | x \in X\}$ is a homomorphic image of X contained in C , and the kernel is contained in $X \cap X^t \leq Z(X)$. Therefore $X^{(\infty)} \simeq U_4(q)$. Since $\langle W^{G_0} \rangle = G_1$, we have $G_0 = XX^t$. Thus $G_0 \trianglelefteq G$.

§ 6. Proof of the theorem

We have assumed in Section 4 and 5 that $O(G)=1$. Now we will exclude this condition. Let X denote the normal closure of L in G and let $\bar{G}=G/O(G)$. Then $\bar{L}$ is a standard subgroup of $\bar{G}$ whose centralizer has a cyclic Sylow 2-subgroup. So we have either $\bar{L} \trianglelefteq \bar{G}$ or $\bar{X} \simeq L_4(q^2)$ or $U_4(q) \times U_4(q)$ by the results of Section 4 and 5. If one of the latter two cases holds, then in each case we can easily find a t -invariant 2-subgroup A of X such that $1 \neq [A, t] \leq L$. Then by (1J) of [12] we have $[A, t, O(G)]=1$. Hence in particular $C_X(O(X)) \not\leq O(X)$. Thus $O(X)=Z(X)$ and the theorem is established.

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