Finite groups with a standard subgroup isomorphic to $U_4(2^n)$

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(Received Feb. 10, 1977)

Let $K$ be a subgroup of a finite group $G$. If $K$ has even order while $K \cap K^g$ has odd order for any $g \in G - N_G(K)$, then $K$ is said to be tightly embedded in $G$. Furthermore, a quasisimple subgroup $L$ of $G$ is said to be a standard subgroup if $K = C_o(L)$ is tightly embedded in $G$, $N_G(K) = N_G(L)$ and $[L, L^g] \neq 1$ for any $g \in G$. In this paper we prove the following theorem.

**THEOREM.** Let $G$ be a finite group and let $L$ be a standard subgroup of $G$. Assume that $L/Z(L) \simeq U_4(q)$, where $q = 2^n > 2$. Furthermore assume that $C_G(L)$ has a cyclic Sylow 2-subgroup. Let $X$ be the normal closure of $L$ in $G$. Then one of the following holds:

1. $O(G) < G$
2. $X \simeq L(q^2)$
3. $X \simeq U_4(q) \times U_4(q)$.

After the fundamental work of Aschbacher [2], much work has been done toward the classification of finite groups with a standard subgroup of known type (e.g. [5], [12], [16], [20], [21]). For details we refer the readers to [12]. As stated in the theorem above, the result of this paper classifies the groups with a standard subgroup isomorphic to $U_4(2^n)$ with $2^n > 2$ whose centralizer has a cyclic Sylow 2-subgroup.

In order to prove the theorem we may assume that $O(G) = 1$ (cf. Section 6). Then provided that $L \triangleleft G$, our aim is to obtain that $\langle L^g \rangle \simeq L \times L$ or $L(q^2)$. The method used in this paper is essentially a careful analysis of 2-local subgroups of $G$ which heavily depends on the structure of 2-local subgroups of $U_4(q)$. Let $A_1$ and $A_2$ be the largest normal 2-subgroups of the maximal parabolic subgroups of $L$. Then a major part of the proof is devoted to constructing the 2-subgroups $F_1$ and $F_2$ such that $N_G(A_i) < N(F_i)$ (cf. (3.5), (3.9)). Next, applying the previous classification theorem [16] to the groups $N_G(F_i)/F_i$, we construct the subgroups $L_i$ of $N_G(F_i)$, $i = 1, 2$ (cf. (3.12)). Here two cases occur:

1. $L_1/F_1 \simeq L(q^2) \times L(q^2)$, $L_2/F_2 \simeq L(q^2)$;
Case (1) is treated in Section 4 and Case (2) in Section 5. We note that $L_1$ is obtained as a normal closure of $N_{F}(A_{1})'$ in $N(F_{1})$ while $L_2$ is not obtained in the same way and some fusion argument is needed to prove $L_2L_{N}(F_{2})$.

The author would like to express his hearty thanks to Dr. Kensaku Gomi. The general line of the proof is shown by him in his paper [12] on which the author greatly depends. Also the author is grateful to Dr. Hiromichi Yamada for providing a lemma appearing as (1.6) in the present paper which contributes to the simplification of the argument on 2-groups in Section 3.

Our notation is standard and we refer the readers to [13] except possibly the following. If $X$ is a 2-group,

- $\mathcal{E}(X)$ = the set of the maximal elementary abelian subgroups of $X$,
- $J_{r}(X)$ = the subgroup generated by the elementary abelian subgroup of $X$ of maximal rank.

By $J(X)$ we denote the usual Thompson subgroup generated by the abelian subgroups of $X$ of maximal order. For any finite group $G$,

- $E(G)$ = the maximal normal semisimple subgroup of $G$,
- $G^{(i)}$ = the final term of the derived series of $G$,
- $G^{i}=\langle g^{i} | g \in G \rangle$ for an integer $i$,
- $I(X)$ = the set of the involutions of any subset $X$ of $G$.

If $X$ and $Y$ are subgroups of a group,

- $X*Y$ = a central product of $X$ and $Y$,
- $X \cdot Y$ = a semidirect product of $X$ by $Y$.

For subsets $X$ and $Y$ of a group,

- $X^{Y}=\{ y^{-1}xy | x \in X, y \in Y \}$.

If $X$ acts on a set $\Omega$, then $X^{\sigma}$ denotes the image of the permutation representation of $X$ on $\Omega$. By Goldschmidt groups we mean quasisimple groups called groups of type I and II in [10].

§ 1. Preliminaries

(1.1) Let $\langle t \rangle \times A$ be a normal 2-subgroup of a group $H$ and suppose $\langle t \rangle \times A$ is elementary abelian, $|H|C_{H}(t)|$ even and $C_{H}(t) \cap N_{H}(A)$ acts transitively on $A^{t}$ by conjugation. Let $\bar{H}=H/C_{H}(\langle t \rangle A)$. Furthermore suppose
that $C_H(t)$ has a cyclic normal subgroup $J$. Then $C_H(A)$ is elementary abelian of order $|A|$ and $t^Ht^{C(A)}=tA$.

**Proof.** We will consider the action of $H$ on $t^H$. Then $t^H=\{t\} \cup A^t$ or $tA$. In any case $H$ is doubly transitive on $t^H$ and contains a cyclic normal subgroup in its one-point-stabilizer. Hence by Theorem 3 of [1] either $H$ has a regular normal subgroup on $t^H$ or $\langle J^H \rangle \simeq L_n(r)$ for some prime $r$. Suppose that the former holds and let $M$ be the preimage of the regular normal subgroup and let $S$ be a Sylow 2-subgroup of $M$. Then $Z(M) \cap \langle t \rangle A = Z(S) \cap \langle t \rangle A$. Hence $Z(M) \cap \langle t \rangle A \neq 1$. $C_H(t) \cap N_H(A)$ acts on $Z(M) \cap \langle t \rangle A$ and this action yields that $Z(M) \cap \langle t \rangle A = A$ and $t^H=tA$. Suppose that the latter holds. Then, since $\langle J^H \rangle$ acts also on $(\langle t \rangle A)^t-t^H$, we get $\langle J^H \rangle \simeq L_n(7)$ and $\langle t \rangle A$ has order 16. If $t^H=tA$, then $\langle J^H \rangle$ is embedded in the parabolic subgroup of $GL_2(2)$ leaving the subspace $A$ invariant. Then by a property of Levi decomposition (cf. [6]) $\langle J^H \rangle$ has a fixed point on $tA$, which is a contradiction. If $t^H=\{t\} \cup A^t$, then a two-point-stabilizer of $tA^t$ in $\langle J^H \rangle$, which is a fours group, stabilizes a point on $\{t\} \cup A^t$. But a Sylow 2-subgroup of $L_n(7)$ acts regularly on $\{t\} \cup A^t$. This contradiction completes the proof.

(1.2) Let $q=2^n$. Suppose that $q^t-q^3+q^2-q+1=p^m$ for some prime $p$. Then $m=1$.

**Proof.** We prove by way of contradiction. First suppose that $m=2m_1+1$, $m_t \geq 1$. Then

$$q(q^3-q^2+q-1)=(p-1)(p^{m_1}+p^{m_1-1}+\ldots+1).$$
Since $q=2^n$, $q$ divides $p-1$. This yields that $m_t=1$. Put $p-1=qs$ for some integer $s$. Then

$$q^3-q^2+q-1=s((qs+1)^2+(qs+1)+1).$$
Hence

$$q(q^2-q-1)=s^3q^2+3s^2q+3s+1.$$ Then we have $q|3s+1$ and also $q\geq s^3+1$, which yield that $s=1$. So $q=2$ or 4. Then by direct calculation we can easily derive a contradiction in each case.

Next suppose that $m=2m_1$. Then

$$q(q^3-q^2+q-1)=(p^{m_1}+1)(p^{m_1}-1).$$
Hence $p^{m_1}-1$ or $p^{m_1}+1=\frac{1}{2}qs$ for some odd integer $s$. Then

$$q(q^2-q+1)=\frac{1}{2}qs^2+(1\pm s).$$
Now by direct calculation we may assume that \( q \geq 8 \). Hence \( s \pm 1 = \frac{1}{2}qr \) for some integer \( r \). Then

\[
(64-r^2)q^2 + (-64 \pm 8r)q + 48 \mp 16r = 0.
\]

In either case we have \( r < 8 \). Then checking for each \( r \), \( 0 \leq r \leq 8 \), we can easily obtain a contradiction.

(1.3) Let \( E \) be a normal 2-subgroup of a group \( H \). Suppose that \( E \) is elementary abelian of order \( q^2 \). Set \( \bar{H} = H/C_\mu(E) \) and suppose that \( \bar{H} \) has an abelian normal subgroup \( J \) which acts on \( E \) transitively. Let \( t \) be an involution in \( H - C_\mu(E) \) and let \( \bar{x} \) be an element of \( \bar{J} \). Then \( \bar{x}^t = \bar{x}^q \).

PROOF. We will define multiplication and addition \( + \) in \( E \) and show that \( (E, +, \cdot) \) is a field of \( q^2 \) elements. Name the identity element of \( E \) 0 and any one element of \( E \) 1. Define multiplication in \( E \) as follows. Let \( a \cdot 0 = 0 \) for all \( a \in E \). For \( b \in E \), \( b \neq 0 \), there exists a unique element \( u \) of \( J \) such that \( 1^u = b \), since \( J \) is regular on \( E \). Then let \( a \cdot b = a^u \) for \( a \in E \). Thus \( 0 \cdot b = 0 \). If \( 1^u = b \) and \( 1^v = c \) for \( u, v \in J \), then \( 1^{uv} = b^c \), and \( (a \cdot b) \cdot c = (a^u) \cdot c = a^{uv} = a^u \cdot 1^v = a \cdot b^v = a \cdot (b \cdot c) \). Clearly \( a \cdot 1 = a \), so 1 is a multiplicative identity of \( (E, +, \cdot) \). If \( 1^u = a \) and \( 1^{uv} = d \) for \( u \in J \), then \( a \cdot d = a^{uv} = 1^{uv-1} = 1 \). Also \( b \cdot c = 1^{uv} = 1^v = c \cdot b \). Hence \( E - 0 \) is an abelian group under the multiplication.

Next define addition as follows. \( a + b = ab \) for \( a, b \in E \). Then \( 0 + a = a + 0 = a \), since 0 is defined to be the identity of \( E \). Let \( a, b \) and \( c \) be in \( E \). If \( c = 0 \), then \( (a + b) \cdot c = 0 = a \cdot c + b \cdot c \). Now assume that \( c \neq 0 \) and as above we put \( 1^u = c \). Then we have

\[
(a + b) \cdot c = (a + b)^u = (ab)^u = a^u b^u = (a \cdot c)(b \cdot c) = a \cdot c + b \cdot c.
\]

Thus it is established that \( (E, +, \cdot) \) is a finite field of \( q^2 \) elements.

Suppose that \( h \in H \) such that \( 0^k = 0 \) and \( 1^k = 1 \). Let \( 1^u = a \) and let \( 1^v = b \). Then

\[
(a + b)^k = (ab)^k = a^k b^k = a^k + b^k,
\]

\[
(a \cdot b)^k = a^{2k} = 1^{k-16k} = 1^{k-16k} 1^{k-16k} = 1^{2k} = a^k \cdot b^k.
\]

Thus \( h \in \text{Aut} ((E, +, \cdot)) \). So \( a^k = a \cdot a \cdot \cdots \cdot a \) for any \( a \in E \) and for some integer \( i \). Put \( j = 2^i \). For any \( h \in H \) there exists \( u \in J \) such that \( 0^u = 0 \) and \( 1^u = 1 \). Then \( 1^{2u} = 1^{2j} \) for any \( x \in J \). On the other hand \( 1^{2u} = 1^{(a \cdot e) 12u} = 1^{k-12k} = 1^k \). Thus \( 1^{2u} = 1^{2j} \) and so \( \bar{x}^h = \bar{x}^q \). Hence in particular if \( t \in H - C_\mu(E) \) is an involution, then \( \bar{x}^t = \bar{x}^q \).
(1.4) Let $V$ be an elementary abelian group of order $2^{2m}$ for some integer $m$. Let $f$ be an element of order $2^i$ which acts on $V$. Suppose that $|C_V(f^{2i-1})|=2^m$. Then any element $x$ in $f^i V$ whose order is equal to $|f^i|$ is conjugate to $f^i$ in $V\langle f \rangle$.

PROOF. If $|x|=2$, then the assertion is easily checked. Suppose that $|x|>2$. Take $v \in V$ such that $x=f^iv$. Then by induction we may assume that there exists an element $u$ in $V$ such that $x^{2^m}=f^{2i}$. This implies that $[v,f^i]=[u,f^i]=[u,f^i,f^i]$. Hence $v[u,f^i] \in C_V(f^i)$. Since $C_V(f^i) \leq C_V(f^i^r)=[V,f^{2i-1}] \leq [V,f^i]$ by the assumption, we have $v[u,f^i]=[w,f^i]$ for some $w \in V$. Then $v=[uw,f^i]$ and consequently $x^{2^m}=f^i$.

The following is the well known (generalized) "Thompson transfer lemma" and the proof is omitted.

\[ (1.5) \quad \text{Let } S \text{ be a Sylow 2-Subgroup of a group } G \text{ and let } R \text{ be a subgroup of } S \text{ such that } S' \leq R. \text{ Suppose that } x \text{ is an involution of } S-R \text{ such that } x^\sigma \cap S \subseteq xR. \text{ Then } x \not\in O^i(G). \]

The following is due to H. Yamada [20] (2.4).

\[ (1.6) \quad \text{Let } X \text{ be a } p^\prime\text{-group acting on a } p\text{-group } A. \text{ Suppose that } C_A(X) \subseteq A. \text{ Then } A=C_A(X)*[A,X]. \]

§ 2. Properties of $U_4(q)$

In this section we fix the notation for various subgroups of $L\simeq U_4(q)$, $q=2^n > 2$, and obtain some basic properties about them and the automorphism group of $L$, which are required in the following sections.

We identify the elements of $L$ with $4 \times 4$ matrices $x$ with entries in $GF(q^2)$ satisfying the conditions

$$x^*rx=r \quad \text{and} \quad \det(x)=1,$$

where $r=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $x^*$ denotes the adjoint matrix of $x$ conjugated by the involutive automorphism of $GE(q^2)$. $L$ has order $q^9(q^2-1)(q^2+1)(q^2-1)$ (see for example [7, p. 8]). For $a \in GF(q^2)$ set $\bar{a}=a^*$ and denote by $GE(q^2)^{\times}$ the multiplicative subgroup of $GF(q^2)$. Denote by $P$ the subgroup of $L$ consisting of the matrices
where \( b \in GF(q) \) and \( d + \bar{d} + a\bar{e} + \bar{a}c = 0 \). Then \( P \) is a Sylow 2-subgroup of \( L \).

We define the following subgroups of \( P \):

\[
A_1 = \begin{bmatrix}
1 & 1 \\
1 & c \\
c & b \\
1 & d \\
\end{bmatrix}
\]

and

\[
A_2 = \begin{bmatrix}
1 & 1 \\
a & c \\
c & 1 \\
a & d \\
\end{bmatrix}.
\]

Then \( A_1 \) is elementary abelian. Moreover we have

\[
Z(P) = A'_1 = \begin{bmatrix}
1 & 1 \\
1 & 1 \\
1 & 1 \\
1 & 1 \\
\end{bmatrix}
\]

and

\[
Z_1(P) = A_1 \cap A_2 = \begin{bmatrix}
1 & 1 \\
c & 1 \\
c & \bar{c} \\
1 & \bar{d} \\
\end{bmatrix}.
\]

Denote by \( I \) the subgroup of \( L \) consisting of the matrices

\[
\begin{bmatrix}
a \\
b \\
\bar{b}^{-1} \\
\bar{a}^{-1}
\end{bmatrix},
\]

where \( ab\bar{a}^{-1}\bar{b}^{-1} = 1 \). Furthermore define

\[
I_1 = \begin{bmatrix}
a & a^{-1} \\
a & a^{-1} \\
a & a^{-1} \\
a & a^{-1}
\end{bmatrix} a \in GF(q)^x
\]

and

\[
I_2 = \begin{bmatrix}
a^2 & a^{q-1} & a^{q^2-1} & a \\
\bar{a}^{q-1} & \bar{a}^{q^2-1} & \bar{a}^{q^2-1} & \bar{a}
\end{bmatrix} a \in GF(q)^x.
\]
which are subgroups of $I$.

Denote by $N_i$ the subgroup consisting of the matrices

$$\begin{pmatrix}
s & t \\
u & v
\end{pmatrix},$$

where $\begin{pmatrix}s & t \\
u & v\end{pmatrix} \in SL_2(q)$. Denote by $N_z$ the subgroup consisting of the matrices

$$\begin{pmatrix}
1 & s \\
u & v
\end{pmatrix},$$

where $\begin{pmatrix}s & t \\
u & v\end{pmatrix} \in SL_2(q)$. Finally define

$$r_1=\begin{pmatrix}1 & 1 \\1 & 1\end{pmatrix}, \quad r_2=\begin{pmatrix}1 & 1 \\1 & 1\end{pmatrix},$$

$$z_1=\begin{pmatrix}1 \\1 \\1\end{pmatrix}, \quad z_2=\begin{pmatrix}1 \\1 \\1\end{pmatrix}.$$

Let $A=\text{Aut} \langle L \rangle$. Then $A$ is a semidirect product of $L$ by a cyclic subgroup generated by a field automorphism $f$ of order $2n$ which acts on $L$ canonically. Let $u \in I(\langle f \rangle)$ and let $\bar{P} \in \text{Syl}_2(A)$ such that $\langle u, p \rangle \leq \bar{P}$.

Most of the results of this section are obtained by direct calculation and the proofs are omitted or given briefly.

1. $N_1(P)=PI$.
2. $Z(P)=C_p(I_i^*)$, $Z_2(P)=Z(P) \cdot C_p(I_i^*)$, $A_i=Z_2(P) \cdot C_p(I_{i-1})$, $P=A_1 \cdot C_p(I_1)$, $i=1, 2$.
3. $I$ acts transitively on $C_p(I_i^i)$ and on $C_p(I_i^{i-1})$, $i=1, 2$.
4. $N_1(A_i)=A_i \cdot (I_i \times N_i)$, $i=1, 2$.
5. $N_1(Z(P))=N_1(A_i)$ and $C_2(Z(P))=A_i \cdot (I_i^{-1} \times N_i)$. 
(2.2) (1) \( N_i \) acts on \( A_i \) irreducibly and \( I_i \times N_i \) has just two orbits \( z_{i,N_i}^1 \) and \( z_{i,N_i}^2 \) on \( A_i^2 \) whose lengths are \((q-1)(q^2+1)\) and \((q-1)q(q^2+1)\) respectively. In particular \( A_i N_i \) is perfect. Also it holds that \(|z_{i,N_i}^1|=(q-1)(q^2+1)\) and \(|z_{i,N_i}^2|=q(q^2+1)\).

(2) \( A_i N_i \) is perfect. Set \( N_i(Z(P))=N_i(Z(P))/Z(P) \). Let \( x_i \in I(A_i-Z(P)) \) and let \( x_i \in A_i-I(A_i) \). Then \(|\bar{x}_{i,N_i}^1|=(q+1)(q^2-1)\) and \(|\bar{x}_{i,N_i}^2|=(q-1)q(q^2-1)\).

(3) \( C_{L}(z_2)=C_{L}(Z(P)) \) and \( C_{N_i}(z_2)=C_{N_i}(u) \cong SL_2(q) \). Moreover \( C_{L}(z_2)'=(A_i \cap C(u))C_{N_i}(u) \) is perfect.

PROOF. By Lemma 4B of [8] \( A_i \) is an irreducible \( N_i \)-module. By calculation we have

\[
C_{N_i}(z_i) = \left\{ \begin{pmatrix} s & 0 & \bar{s} & 0 \\ u & v & \bar{u} & \bar{v} \end{pmatrix} \left| \begin{pmatrix} s & 0 \\ u & v \end{pmatrix} \in SL_2(q^2) \right. \right\}
\]

and

\[
C_{N_i}(z_2) = C_{N_i}(u) = \left\{ \begin{pmatrix} s & t \\ u & v \end{pmatrix} \left| \begin{pmatrix} s & t \\ u & v \end{pmatrix} \in SL_2(q) \right. \right\}.
\]

Also

\[
C_{I_i,N_i}(z_2) = \left\{ \begin{pmatrix} s & 0 & \bar{s} & 0 \\ u & v & \bar{u} & \bar{v} \end{pmatrix} \left| \begin{pmatrix} a & 1 \\ \bar{a} & a^{-1} \end{pmatrix} \in SL_2(q^2), a \in GF(q^2) \right. \right\}
\]

and

\[
C_{I_i,N_i}(z_i) = \left\{ \begin{pmatrix} s & t \\ u & v \end{pmatrix} \left| \begin{pmatrix} s & t \\ u & v \end{pmatrix} \in SL_2(q) \right. \right\}.
\]

Hence \(|z_{i,N_i}^1|=(q-1)(q^2+1)\) and \(|z_{i,N_i}^2|=(q-1)q(q^2+1)\). Since \((q-1)(q^2+1)+(q-1)q(q^2+1)=q^4-1\), (1) holds.

By calculation \( P'=Z(P) \). Since \( r_i \in N_i \) and since \( A_i=Z_i(Z(P))/Z_i \subset A_i \), we have \( A_i \leq (A_i N_i)' \). Which implies that \( A_i N_i \) is perfect. Similarly we have
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Hence \(|z^I_{N_2}| = (q+1)(q^2-1)\). Let \(x = \begin{pmatrix} 1 & 1 \\ c & 1 \\ d & 1 \end{pmatrix}, c \neq \bar{c}\). Then

\[
C_{I_2N_2}(\bar{x}) = \left\{ \begin{array}{ccc}
1 & a^s & a^{s-1} \\
s & t & u \\
u & v & 1
\end{array} \right\} = \left\{ \begin{array}{ccc}
1 & a^s & a^{s-1} \\
\bar{c} & a^{s-1} & a^{s^2} \\
u & v & 1
\end{array} \right\}.
\]

Hence in particular \(a^{q-3} + a^{-q+3} \in GF(q)\), so \(a^{q-3} + a^{-q+3} = a^{1-q^2} + a^{-1+q^2}\). This implies that \(a^{q-1}(a^q + a^{-q})/(a^{-q+1} + 1)^t = 0\). Hence \(a^{q+1} = 1\). Then, since \(t = (a^q + a^{-q})/(c + \bar{c})\), \(|x^I_{N_2}| = (q-1)q(q+1)(q^2-1)/(q+1) = (q-1)q(q-1)\). Now \((q+1)(q^2-1) + (q-1)q(q^2-1) = q^4 - 1 = |A^*_4|\). Thus (2) holds. (3) is obtained by direct calculation.

(2.3) (1) \(\delta(P \mod Z(P)) = \{A_1, A_2\}\).
(2) \(\text{Jr}(P) = A_1, \text{Jr}(P \mod A_1) = P\) and \(\text{Jr}(P \mod Z(P)) = A_2\).
(3) \(|C_{A_1}(x)| \leq q^2\) for \(x \in P - A_1\).
(4) \(|C_{A_2}(x)| \leq q^2\) for \(x \in P - A_2\).
(5) If \(B\) is a maximal abelian subgroup of \(P\) different from \(A_1\), then \(|B| \leq q^3\).

**Proof.** Set \(\bar{P} = P/Z(P)\). Then \(\bar{P} = \bar{A}_1\bar{A}_2\), where \(\bar{A}_1\) and \(\bar{A}_2\) are elementary abelian, and \(C_{\bar{P}}(\bar{x}) = \bar{A}_i\) for any \(\bar{x} \in \bar{A}_i - \bar{A}_2\). This implies (1). For (5) assume that \(B < A_2\). Set \((a, c, d) = \begin{pmatrix} 1 & 1 \\ c & 1 \\ d & 1 \end{pmatrix}\). Let \(x = (x_1, x_2, x_3) \in A_2 - Z(P)\). Then

\[
C_{A_2}(x) = \{(a, c, d)ax_1 + ax_2 + cx_3 + \bar{c}x_3\}.
\]

Hence \(|C_{A_2}(x)| = q^4\). Moreover we have \(C_{A_2}(x) = C_{A_4}(x')\) for any element \(x' = (kx_1, kx_2, x_3)\), where \(k \in GF(q)\). Suppose that \(y = (y_1, y_2, y_3) \in C_{A_4}(x) - Z(P)\) and that \(y \neq (kx_1, kx_2, x_3)\) for any \(k \in GF(q)\). Since \(|C_{A_4}(x)| = q^4\) and \(|C_{A_4}(kx_1, kx_2, x_3)| = q^2\), such \(y\) really exists.
Then $C_{s,t}(x,y) = \{(a, c, d) | a\bar{x} + \bar{a}x = c\bar{x} + \bar{c}x, \ a\bar{y} + \bar{a}y = c\bar{y} + \bar{c}y\}$. Hence $|C_{s,t}(x,y)| \leq q^r$. Suppose that $x \in P - (A_1 \cup A_2)$. Then it is easily obtained that $|C_s(x)| = q^2$. Thus (5) holds by (1). (3) and (4) follow from direct calculation. Now (3), (4) and (5) give that $\langle P\rangle = A_1$, since $q \geq 4$. By (1) we have $\langle P\rangle = A_1$. We get $\langle P\rangle = A_2$ easily.

The following is an immediate consequence of (2.2) and (2.3.2).

(2.4) $L$ has just two conjugacy classes of involutions which are represented by $z_1$ and $z_2$.

(2.5) (1) $C_s(I_1) = (I_1 \times N_1)\langle u \rangle$ if $q > 4$, and $C_s(I_1) = (I_1 \times N_1)\langle u \rangle$ if $q = 4$. $C_s(I_2) = I_2N_2$.

(2) $C_s(A_1) = C_s(Z(P)) = A_1$, $C_s(Z(P)) = A_1(I_2^{-1} \times N_2)\langle u \rangle$.

(3) $C_s(A_1N_1/A_1) = A_1I_1$, $C_s(A_2N_2/A_2) = A_2I_2\langle u \rangle$.

(4) $C_s(A_2I_2^{-1}/A_2) = C_s(A_2I_2^{-1}/A_2)$.

(5) $C_s(P/A_1) = PI_1$, $C_s(A_2/Z(P)) = A_2$.

Proof. (1) is obtained by direct calculation. Similarly we have $C_s(A_2) = C_s(Z(P)) = A_1$. Hence by the Frattini argument $C_s(Z(P)) = A_1(N_1(A_1) \cap C(Z(P))) = A_1(I_1N_1\langle f \rangle \cap C(Z(P))) = A_1$. By (2.1.5) $C_s(Z(P)) = A_1I_2^{-1}N_2(A_1(P) \cap C(Z(P))) = A_1I_2^{-1}N_2\langle u \rangle$. Thus (2) holds. By (2.1.4) $C_s(A_1N_1/A_1) = A_1(I_1N_1\langle f \rangle \cap C(I_1)) = A_1I_1$. Similarly $C_s(A_2I_2^{-1}/A_2) = A_2I_2\langle u \rangle$, and $C_s(A_2I_2^{-1}/A_2) = A_2(I_2N_2\langle f \rangle \cap C(I_2^{-1})) = A_2I_2N_2$, since any nontrivial element of $\langle f \rangle$ does not centralize $q + 1$ elements of $GF(q^2)$. This gives (4). (5) follows from (2.1.1) and (2.1.4).

The following is (19.8) (ii) of [4].

(2.6) Let $x \in I(A - L)$. Then $x$ is conjugate to $u$ or $uz$. Furthermore $C_s(u) \simeq PSp_4(q)$ and $C_s(uz)$ is isomorphic to the centralizer of a transvection in $PSp_4(q)$.

(2.7) Any complement of $A_1$ in $A_1N_1$ is conjugate to $N_1$ in $A_1N_1$.

Proof. Let $M$ be a complement. Set $i(a) = \begin{pmatrix} a & a^{-1} \\ \bar{a} & \bar{a} \end{pmatrix} \in I$ and set $I' = \{i(a) | a \in GF(q^2)\}$. Then $N_{A_1N_1}(P) = PI'$. So we may assume that $I' \leq M$ by Schur-Zassenhaus' theorem. Set $A_1N_1 = A_1N_1/A_1$. Since $I' \langle r_1 \rangle = N_{A_1N_1}(I') \geq N_{A_1N_1}(I^n)$, it follows that $N_{A_1N_1}(I') = N_{A_1}(I') \langle r_1 \rangle = C_s(I') \langle r_1 \rangle = I' \langle r_1 \rangle$. Therefore $N_{A_1}(I') = N_{A_1}(I') = I' \langle r_1 \rangle$. Then $M = \langle P \cap M, r_1 \rangle$ by the structure of $M \simeq SL_2(q^2)$. Hence in order to prove the assertion we will
show that $P \cap N_i = P \cap M$. Let $x = \begin{pmatrix} 1 & 1 \\ f & d \\ h & g \\ e & 1 \end{pmatrix} \in P \cap M$. Then, since $P = A_i \cdot (P \cap M)$ and $(P \cap M)'' = x''$ by the structure of $M$, we have $c \neq 0$. Hence $(P \cap M)'' = 1$, which yields $d = 0$ and so $g = f$. Since for any $a \in GF(q^2)^\times$ there exists an element $b$ in $GF(q^2)^\times$ such that $xx^{(a)} = x^{(b)}$, we have

\[
\begin{align*}
\frac{c + a'c}{c} &= b'a', \\
\frac{f + a'a^{-1}f}{f} &= b'b^{-1}f, \\
\frac{h + aah + a'cf + a^{-1}acf}{h} &= b'bh.
\end{align*}
\]

Then provided that $f \neq 0$, we have $a^2 = a$ from the first two equalities. This yields $f = 0$. Also we have $h = 0$, which implies that $x \in P \cap N_i$. Thus $(P \cap N_i)'' = (P \cap M)''$ and the proof is complete.

(2.8) Suppose that $x \in P - P$ and set $x = xP/P$. If $|x| = |x|$, then $m(C_p(x)) = 3m$, where $m = 2n/|x|$.

Proof. Applying (1.4) to $P/A_i$, we may assume that $x = f'a$ for some $a \in A_i$ and some integer $i$, where $f$ is the canonical field automorphism of $L$. Then $C_{A_i}(x) = C_{A_i}(f')$ and $m(C_{A_i}(x)) = 3m$. Moreover $m(C_{A_i}^{Z_2}(x)) = 2m$. Since $P/A_i \simeq A_i/Z_2(P)$ as $N_i(P)$-modules, we have $|C_{P/A_i}(x)| = |C_{A_i}^{Z_2}(f')(x)|$, so $m(C_{A_i}^{Z_2}(x)) = m$. Therefore $m(C_{A_i}(x)) \leq 3m$. Now (2.3.1) implies that any elementary abelian subgroup of $P$ is contained in either $A_i$ or $A_2$. Thus we have $m(C_{A_i}(x)) = 3m$.

§ 3. Fusion of the involution of $C_o(L)$

In this section we begin the proof of the theorem stated in the introduction under the following hypothesis.

HYPOTHESIS. Let $G$ be a finite group and let $L$ be a standard subgroup of $G$ such that $L/Z(L) \simeq U_i(q)$, where $q = 2^n \geq 4$. Suppose that $O(G) = 1$ and $L \triangleleft G$. Furthermore suppose that $C(L)$ has a cyclic Sylow 2-subgroup.

It is known (cf. [15] [18]) that the Schur multiplier of $U_i(q)$, $q \geq 4$, is trivial. So we may identify $L$ with the group of $4 \times 4$ matrices $x'$s defined in the beginning of Section 2. The symbols used in Section 2 for various objects of $U_i(q)$ and $Aut(U_i(q))$ will retain their meaning. Thus $P$ is a Sylow 2-subgroup of $L$.

Let $t$ be an involution of $C(L)$ and set $C = C(t)$. Let $Q$ be a Sylow 2-subgroup of $LC_o(L)$ containing $P$ and let $T$ be a Sylow 2-subgroup of $C$ containing $Q$. Finally, for $i = 1, 2$, we define $B_i = \langle t \rangle A_i$. 
(3.1) The following statements hold.
(1) $\text{Jr}(T)=B_i$.
(2) Set $C_r(B_i)=\tilde{B}_i$. Then $t^{x(\tilde{B}_i)}=t^q \cap \tilde{B}_i \neq \{t\}$.
(3) $\tilde{B}_i=B_i$ and $\langle t \rangle \in \text{Syl}_2(C(L))$.
(4) $I^q \cap L=\phi$.
(5) $Q=\text{Jr}(T \bmod B_i)$ and $B_2=\text{Jr}(Q \bmod Z(Q))$.
(6) $t^{x(\tilde{B}_i)}=tA_i$.

Proof. Let $T_0=C_r(L)$. Then $Q=T_0 \times P$ and $\text{Jr}(Q)=B_i$ by (2.3.2). If $B$ is an abelian subgroup of $T$ of rank at least $4n+1$, then $m(BT_0/T_0) \geq 4n$, since $T_0$ is assumed to be cyclic by the hypothesis. So again by (2.3.2) $B \leq Q$. This proves (1). If $t^{x} \cap C=[t]$, then $t \in Z(G)$ by [9], since we are assuming $O(G)=1$. But then $C=G$ and $L=\langle G \rangle$, contrary to our hypothesis. So we may take $t^q \in C-\text{LC}(L)$. Then by (2.6) $t^{x(t)}=u, uz_i, tu$ or $tuz_i$ for some $h \in C$. Let $x=\frac{1}{a} \begin{pmatrix} 1 \\ c \\ d \end{pmatrix} \in L$. Then $t^{x(h)}=t^h \begin{pmatrix} 1 \\ c+\tilde{c} \\ c-\tilde{c} \end{pmatrix} \in L^{x(h)}$. Hence $t^{x(h)} \in L^{x(h)}C(L^{x(h)})-\{t^{x(h)}\}$. So we may assume that $t^q \in L \cap C(L)$. Then by (2.4) $t^{x} = z_k, z_{k+1}, t$ or $z_{k+1}t$ for some $k \in C$. Thus $t^q \cap B_i \neq \{t\}$. Let $s \in t^q \cap B_i \neq \{t\}$ and set $s^{y}=t$. Since $T \in \text{Syl}_2(C)$, we can choose $g' \in C$ such that $C_r(s)^{t^q} \leq T$. In particular $\tilde{B}_i^{t^q} \leq T$. Since $B_i=T_0 \times A_i \geq B_1, \tilde{B}_i$ is weakly closed in $T$. Hence $\tilde{B}_i^{t^q}=\tilde{B}_i$ and $g' \in N(\tilde{B}_i)$. Thus (2) holds. If $|T_0| \neq 2$, then $\langle t \rangle$ is a characteristic subgroup of $\tilde{B}_i$. But this contradicts (2). So (3) holds. Now (4) follows from (2.2.3), and (5) follows from (3) and (2.3.2).

By (2), (4) and (2.2.1) we have $t^{x(\tilde{B}_i)} \subseteq tA_i$, and $|t^{x(\tilde{B}_i)}|=q^q-q^q+q^p-q^q+q^p-q+1$ or $q^p$. Set $\Omega=t^{x(\tilde{B}_i)}$. If either of the former two cases holds, then $N(B_i)^{\Omega}$ is doubly transitive and the one-point-stabilizer $N_c(B_i)^{\Omega}$ has a cyclic normal subgroup $I_2$. Notice that $N_c(B_i)^{\Omega}$ is not solvable. Hence by theorem 3 of [1] together with (1.2) neither of these cases hold. Thus $t^{x(\tilde{B}_i)}=tA_i$.

(3.2) (1) $N(Z(Q))=O_4(N(Z(Q)))N_c(Z(Q))$ and $|O_4(N(Z(Q)))|=2q^6$.
(2) $N(Q)=O_d(N(Z(Q)))N_c(Q)$.
(3) $N(B_i)=N(Z(Q))$.

Proof. Set $\Omega=t^{x} \cap Z(Q)$. Then $\Omega=t \cap Z(P)$ by (3.1.4) and (3.1.6). (3.1.6) also implies that $T$ is not a Sylow 2-subgroup of $G$. Hence by (3.1.5) $t^{x(\cap Z(Q))} \neq \{t\}$. By (2.1.3) $N(Z(Q))$ is doubly transitive and the one-point-stabilizer
Let $M$ be the preimage of the regular normal subgroup and let $X$ be a Sylow 2-subgroup of $M$. Then $M \approx X^u$, so $[X^u, I_i^u] = X^u$. Now $C(Z(Q)) = B_2 \cdot (I_i^u \times N_i) \cdot O(C) \langle u \rangle$ by (2.5.2), where $u \neq 1$ or $u$ according as $Q = T$ or not. Since $C(I_i^u \times N_i) \geq I_i$, we have $[X, I_i] \leq C(B_2 I_i^u \times N_i O(C)/B_2)$ by three subgroup lemma. Let $X_i \in \text{Syl}_2([X, I_i]B_2)$. Since $u$ acts on $I_i^u$ nontrivially, it follows that $u \in [X, I_i]B_2$. Thus we have $M = X_i \cdot (I_i^u \times N_i) O(C) \langle u \rangle$ and (1) holds.

Since $Q$ is characteristic in $T$ by (3.1.5), $N(Q)^o$ is transitive as well as $N(Z(Q))^o$. So $|N(Q):N_o(Q)| = |N(Z(Q)):N_o(Z(Q))| = q$. Since $N(Q) \subseteq N(Z(Q))$, (2) follows. Since $Z(B_2) = Z(Q)$, we have (3) similarly.

(3.3) There exists a 2-subgroup $E$ of $C(P)$ of order $q^2$ such that $O_2(N(Z(Q))) = EB_2$, $E \cap B_2 = Z(P)$ and $E/Z(P) \cong Z(P)$ as $N_o(Z(Q))$-modules.

Proof. Set $M = O_2(N(Z(Q)))$. Then $Z(M) \cap Z(P) \neq 1$, since $Z(Q) \leq M$ and $t^u = Z(Q) - Z(P)$ by (3.2). Hence $Z(P) \leq Z(M)$ by (2.1.3). Now $M \cap C = B_2$ and $Z(B_2) = Z(Q)$, so $Z(M) \leq Z(Q)$. Thus $Z(P) = Z(M)$ and $t \in Z(M)$.

Then the map $[\cdot, t]$ from $M$ to $Z(P)$ is an $N_o(Z(Q))$-homomorphism. Clearly the kernel is $B_2$ and the image is $Z(P)$. Hence $M = B_2 \cap C$ and $C = Z(P)$ by (2.1.3) and (3.2.2) $Z_2(Q) \leq Z(M \mod Z(Q))$. So $EZ(Q) \leq EZ_2(Q)$. Since $Z(P) = Z(M) \cap Z(P)$, we have $[EZ(Q), Z_2(P)] = 1$ by (1.6). By definition $E$ is $r_2$-invariant. Hence $[EZ(Q), A_2] = 1$, since $A_2 = Z_2(P)/Z(P)^o$. Then $EZ(Q) = C_2(A_2) \cap N(Z(Q))$ and $E = [EZ(Q), I_2] = [EZ(Q), M_2] \leq N(Z(Q))$. Hence $E/Z(P) \cong Z(P)$ as $N_o(Z(Q))$-modules. Moreover by (1.6) we have $MQ = EZ(Q) * [M, I_2]$. Since $P = [P, I_2]$ by (2.1.2), $MQ = EZ(Q) * P$ and the proof is complete.

(3.4) Set $D_i = O_2(N(B_i))$. Then $N(B_i) = D_i N_o(B_i)$ and $|D_i| = 2q^2$. Furthermore $Z(D_i) = A_i$ and $D_i/B_i \cong A_i$ as $N_o(B_i)$-modules.

Proof. Set $t^{u_i} = 1$. Then by (2.2.1) and (3.1.6) $N(B_i)^o$ is primitive. Since $A_i = B_i - A_i$, we have $A_i \leq N(B_i)$. Hence $(C(A_i) \cap N(B_i))^o$ is a nontrivial normal subgroup of $N(B_i)^o$, since it contains $E^o$. Hence it is transitive.

Since $(C(A_i) \cap N(B_i))^o = B_2 O(C)$ by (2.5.2), $(C(A_i) \cap N(B_i))^o$ is a regular normal subgroup of $N(B_i)^o$. Thus $N(B_i) = (C(A_i) \cap N(B_i)) N_o(B_i)$ and $(C(A_i) \cap N(B_i))^o$
$\simeq A_1$ as $N_c(B_1)$-modules by the $[,]$-homomorphism. Hence $N_1$ acts on $(C(A_1) \cap N(B_1))^o$ irreducibly by (2.2.1). Since $N_1 \leq C(B_1, O(C)/B_1)$, we obtain that $C(A_1) \cap N(B_1)$ is 2-closed just as in (3.2). Therefore $D_1 = O_1(C(A_1) \cap N(B_1)),$ $t^{o_1} = tA_1$ and $|D_1| = 2^{q_n}$. Now it is clear that $D_1/B_1 \simeq (C(A_1) \cap N(B_1))^o$ as $N_c(B_1)$-modules, which implies the last assertion.

(3.5) Set $F_i = (D_i, I_i)$, $P_i = F_iP$, $Q_i = F_iQ$, $T_i = F_iT$ and $F = EC_{F_i}(I_i)$. Then the following statements hold.

(1) $D_i = F_i[t], F_i \subset N(B_1)$ and $F_i/A_1 \simeq A_i$ as $N_c(B_1)$-modules.
(2) $Z(P_i) = E, Z_2(P_i) = F$ and $Z_3(P_i) = P_i$.
(3) $F_1$ is elementary or homocyclic.
(4) $T_i \in \text{Syl}_1(N(B_1)), J(T_i) = F_i$ and $Jr(T_i \mod F_i) = Q_i$.
(5) $N_{F_i}(P) = EP, N_{F_i}(EP) = FP$ and $N_{F_i}(FP) = P_i$.
(6) $E = C_{F_i}(I_i), F = E \cdot C_{F_i}(I_i), F_i = F_i \cdot C_{F_i}(I_i), P_i = F_i \cdot C_{F_i}(I_i)$.

Proof. By (3.4) $D_i(A_i) = \langle t \rangle$. Hence in order to prove (1) it is enough to show that $t \in F_i$. Since $D_i$ is not elementary, $D_i^2 \neq 1$. So by (2.2.1) $A_i \leq D_i \leq B_i$. Suppose $D_i = B_i$. Then $t$ has a square root in $D_i$, since $t^{o_1} = tA_1$. But this yields that $D_i/A_1$ is elementary. So if we set $N(B_i) = N(B_1)/A_1$, then $D_i = \langle t \rangle \times [D_i, I_i]$ and (1) follows.

By (2.1.2) $A_i = C_{F_i}(I_i^{2-1}) \times C_{F_i}(I_i)$. Hence $F_i = C_{F_i}(I_i^{2-1})C_{F_i}(I_i)$ and $C_{F_i}(I_i^{2-1}) \cap C_{F_i}(I_i) = 1$. Since $[I_i^{2-1}, I_i] = 1$, by (1) of [12] we have $F_i = C_{F_i}(I_i^{2-1})C_{F_i}(I_i)$. Since $C_{F_i}(I_i^{2-1}) = C_{F_i}(I_i^{2-1}) \times C_{F_i}(I_i)$, $C_{F_i}(I_i^{2-1}) = C_{F_i}(I_i^{2-1}) \times C_{F_i}(I_i)$. Consequently we have $F_i = C_{F_i}(I_i^{2-1}) \times C_{F_i}(I_i) \times C_{F_i}(I_i)$ and in particular $E = C_{F_i}(I_i) \cap F_i$. Now by (2.1.5) and (2.2.3) $Z(P_i) \cap Z(P_i^{+}) = 1$ for $x \in N_c(A_1) \leq N(Z(P_i))$, so it follows that $[E, E^+] = 1$. By the structure of $U_i(q)$ it is easily seen that $A_i = \langle Z(P_i)^+ \mid x \in N_c(A_1) \leq N(Z(P_i)) \rangle$. Hence $F_i = \langle E \mid x \in N_c(A_1) \leq N(Z(P_i)) \rangle$ and we have $E \leq Z(F_i)$, which implies that $F_i$ is abelian. Then, since $F_i^2 = 1$ or $A_i$ by (1), $F_i$ is elementary or homocyclic respectively.

$N_{F_i}(P) = N_{F_i}(P)P$. By (1) $N_{F_i}(P) = \{x \in F_i \mid [x, P] \leq A_i\} = \{x \in F_i \mid [x, t] \in Z(P)\} = EA_i$. Hence $N_{F_i}(P) = EP$ and $C_{F_1}(P) = E$. Similarly $N_{F_i}(EP) = \{x \in F_i \mid [x, t] \in Z(P)\}$. Since $|A, EC_{F_i}(I_i)\langle A_i\rangle| = q^n, |EC_{F_i}(I_i)\langle A_i\rangle| = q^n$. Also $|EC_{F_i}(I_i), t| = |E, t[A, EC_{F_i}(I_i), t]| = Z(P)C_{F_i}(I_i) = Z(P)$ by (2.1.2), since $|E, t| = Z(P)$. Thus we have $N_{F_i}(EP) = FP$. Then $N_{F_i}(EP) = N_{F_i}(EP)P = F_iP = P_i$. Since $E = C_{F_i}(I_i)$ and $E = Z(EP)$, we have by (1.6) $E = Z(EP)$. Hence inductively we have $E = Z(P_i)$. By (2.3.2) $Jr(EP \mod E) = EA_i$. Hence $N_{F_i}(EA_i) \leq N_{F_i}(EA_i A_i) = N_{F_i}(EP)$, which gives $N_{F_i}(EA_i) = N_{F_i}(EP)$. Since $P_i = F_i \cap EA_i = Z(A_i)$, $EP/EA_i$ is abelian and so $F$ is normal in $P_i$. Then applying (1.6), we have $F/EP \leq Z(P_i/E)$. Since $N_{F_i}(EP) = F_iP, Z(P_i/E) \leq FP/E$. Suppose $F/EP \leq Z(P_i/E)$. Then $EZ(P_i/E) \leq EP/E \cap Z(P_i/E)$, which is absurd.
Hence $F = Z_2(\mathbb{P}_1)$. By (11) of [12], $P_1/F = C_{\mathbb{P}_1} Z_2(I_1) \times C_{\mathbb{P}_1} Z_2(I_2)$. Note that $C_{\mathbb{P}_1} Z_2(I_1) \simeq C_2 Z_2(I_1)$ and $C_{\mathbb{P}_1} Z_2(I_2) \leq F$. So this gives $Z_2(\mathbb{P}_1) = P_1$. Thus we get (2), (3) and (5). Also (6) is proved implicitly.

It remains to show (4). It is clear that $T_1 \in \text{Syl}_2(N(B_1))$ by (3.4). (2.3.3) gives that $|C_{\mathbb{P}_1}(x)| \leq q^4$ for $x \in P_1 - F_1$. So $|C_{\mathbb{P}_1}(x)| \leq q^6$ for $x \in P_1 - F_1$. Hence $J(\mathbb{P}_1) = F_1$. Let $B$ be an abelian subgroup of $T_1$ different from $F_1$. The above argument shows that $B \cap P_1 \leq F_1$, or else $B \cap P_1 \leq q^4$. If $B \leq Q_1$ and $B \leq Q_1$, then $B \cap Z(\mathbb{P}_1) = Z(\mathbb{P}_1)$. Hence $|(B \cap P_1) Z(\mathbb{P}_1)| = |B \cap P_1| |Z(\mathbb{P}_1)| |B \cap Z(\mathbb{P}_1)|$ and so $|B| \leq 2nq^6 < q^8$, since $q > 4$. If $B \leq Q_1$, then $|B \cap P_1 \leq F_1|$, then by (2.3.4) $|C_{\mathbb{P}_1}(x)| \leq q^6$ for $x \in T_1 - Q$ and this yields that $|B| \leq 2nq^6$. Thus we have $J(T_1) = F_1$. Then (2.3.2) gives $Jr(T_1 \mod F_1) = Q_1$, since $T_1/F_1 \cong T_1/A_1$.

(3.6) $F_1$ is elementary.

Proof. First we will show that $T_1 \in \text{Syl}_4(G)$. By (3.3) $E \leq \langle t \rangle * A_4$. Hence $J(E \cap B_1) = E \cap A_4$, since $q \geq 4$. Then $E = Z(J(E \cap B_1))$. Set $N(EB_1) = N(EB_1)/E$ and set $Q = \langle t \rangle A_4$. Since $Q/B_1 \simeq P_1/A_1$, $N_0(EQ) = FQ$ by (3.5.5). Since $J(EQ \mod E) = EB_1$ by (2.3.2), we have $F \leq N_0(EQ)$. Suppose that $x' = x$ for $x \in Q$ and $f \in F$. Then by (3.5.2) $\bar{t} = \bar{t}$, which implies $f^q = 1^q$. Since $F = E \times C_2(I_2)$ and each of these direct factors is $t$-invariant, we have $C_2(I_2) = N_2(\langle t \rangle E) = EZ(\mathbb{P}_1)$. Hence $F^q$ is semiregular of order $q^4$. By (2.2.2) and (3.2) $N_0(EB_1)$ has three orbits on $Q$ whose lengths are 1, $(q+1)(q^2-1)$ and $(q-1)q(q^2-1)$. Now the action of $F^q$ yields that any orbit of $N(EB_1)$ has length divisible by $q^4$. Hence the only possibility remains that $N(EB_1)^q$ is transitive. Then the index $|N(EB_1) : N(Z(Q))|$ is divisible by $q^4$, since $N(Z(Q))^q$ is contained in the stabilizer of $\bar{t}$. By (2.3.1) and (3.3) $ET \in \text{Syl}_2(N(Z(Q)))$. Thus the order of a Sylow 2-subgroup of $N(EB_1)$ is not less than $q^4/|T_1|$, while $|T_1| = q^4/|T_1|$. Therefore we have $T_1 \in \text{Syl}_4(G)$.

Suppose that $F$ is homocyclic. Then $F_1 = A_1$ and $D_1 = C_{T_1}(A_1)$ char $T_1$ by (2.5.4) and (3.5.4). If $ttxt = 1$ for some $x \in F_1 - A_1$, then $ttxt = 1$ for any $x \in F_1 - A_1$, since $F_1 = \langle x^{y_1(b_1)} \rangle$ by (2.2.1). Hence if $ttxt = 1$ for some $x \in F_1 - A_1$, then $ttxt = 1$ for any $x \in F_1 - A_1$. In this case we get $Jr(D_1) = B_1$. Then $B_1$ char $D_1$ char $T_1$, which is a contradiction, since $T_1 \in \text{Syl}_4(N(B_1))$ and $T_1 \in \text{Syl}_4(G)$. Therefore we have $ttxt = 1$ for any $x \in F_1 - A_1$. Now the first paragraph shows that $N(EB_1)$ is transitive on $tA_2$. So $t$ is $N(EB_1)$-conjugate to $txa$ for some $x \in E$ and $a \in A_1 - I(A_1)$. But $txatxa = ttxa^2 = a^2 \neq 1$, a contradiction. Thus by (3.5.3) $F_1$ is elementary.

(3.7) Set $O_2(N(Q)) = S$. Then $N(Q_1) = SN_0(Q)$ and $S/Q_1 \simeq P/A_1$, as $N_0(Q)$-modules. Furthermore set $R = \langle S \rangle$. Then $S = R\langle t \rangle$. 


PROOF. Set $Q=(Q_1/F_1)-(P_1/F_1)$. Since $T_i \in \text{Syl}_i(G)$, we can apply (1.1) to obtain that $N(Q)^g$ has a regular normal subgroup. By (3.6) $\delta(D_i)=\{B_i, F_i\}$. So $N(D_i)=N(B_i)=D_iN_i(A_i)$. Then, since $Q_1=D_1Q$, we have $N(D_1) \cap N(Q_1)=Q_1N_i(Q)$. Thus $C(Q_1/F_1)=Q_1C(Q_1/Q_1)=Q_1O(C)$ by (2.5.5). Therefore as in (3.2) we can obtain that the preimage of the regular normal subgroup is 2-closed and that $N(Q_1)=SN_i(Q)$. By the $[t, t]$-homomorphism we have $S/Q_1 \simeq P_i/F_i$. Hence $S/Q_1 \simeq P/A_i$ as $N_i(Q)$-modules. Since $N(D_1)=N(B_1)$, the last assertion is obtained as in (3.5.1).

(3.8) Set $D_2=C_s(EA_2/E) \cap N(EB_2)$. Then $D_2=O_s(N(EB_2))$, $N(EB_2)=D_2N_i(B_2)$ and $D_2/EB_2 \simeq A_i/Z(P)$ as $N_i(B_2)$-modules.

PROOF. By (3.7) $R \cap Q_1=P_i$, so $E, F \lhd R$. Hence $FC_s(I_1)/E=(EC_s(I_1)/E) \times (EC_s(I_1)/E)$ by (11) of [12]. Then $EA_2 \leq Z(FC_s(I_1) \mod E)$, since $FC_s(I_1) \cap P_1=FC_s(I_1)=FC_s(I_1) \cong EA_2$ and since $I$ acts on $C_s(I_1)/E$ transitively by (2.1.3). Hence by (3.5) and (3.7) $FC_s(I_1)=C_s(EA_2/E)$. Now $EB_2/E=(EZ_2(F)/E) \times (EC_s(I_1)/E)$. Moreover $EZ_2(P) \cap C(EA_2/E)=EB_2O(C)$ by (2.5.5). Therefore if we set $\Omega$ as in the proof of (3.6), then it follows that $(C_s(I_1/F))^g$ is a regular normal subgroup of $N(EB_2)$. Since $I_i \times N_i \lhd C(O(C))$ and it acts on $(C_s(I_1/F))^g$ irreducibly by (2.2.2), it is obtained that the preimage of $(C_s(I_1/F))^g$ is 2-closed as in the proof of (3.2.1). Hence $D_i=C_s(I_1/F)$ and $N(EB_2)=D_iN_i(B_2)$. Then by the $[t, t]$-homomorphism we have $D_2/EB_2 \simeq FA_2/E$ as $N_i(B_2)$-modules. Since $EA_2/E \simeq A_i/Z(P)$, the last assertion follows.

(3.9) (1) $Z(R)=E$, $Z_2(R)=F$, $Z_3(R)=R$, $F_1 \text{ char } R$.
(2) Set $F_i=D_2 \cap R$. Then $F_i \leq N(EB_2)$, $F_i/E$ is elementary and $F_i/EA_2 \simeq EA_2/E$ as $N_i(B_2)$-modules.
(3) $E=C_s(I_1^g)$, $F=E \cdot C_s(I_2^g)$, $F_i=F \cdot C_s(I_{i-1})$, $R=F_i \cdot C_s(I_1)$, $i=1, 2$.

PROOF. By (3.5.6) and (3.7) $E=C_s(I_1^g)$ and $F=E \cdot C_s(I_1^g)$. By (11) of [12] $R/F=(FC_s(I_1)/F) \times (FC_s(I_2)/F)$. Hence $F_i=F \cdot C_s(I_{i-1})$ and $R=F_i \cdot C_s(I_1)$, $i=1, 2$. Thus (3) holds.

By (1.6) $E \leq Z(R)$. Suppose that $E \not\leq Z(R)$. Then by (3) $Z(R) \not\leq P_i$ and $[Z(R), I_i] \not\leq P_i$. Hence $Z(R) \cap C(I_i) \neq 1$. Since $Z(R) \cap C(I_i)$ is $t$-invariant, we have $Z(R) \cap C(I_i) \cap C \neq 1$, which contradicts (2.1.2). Similarly we have $Z_s(R)=F_i$. By (3.7) and (3.8) $F_i=(D_i)^g$, so $F_i \leq N(EB_2)$, since $N(EB_2)=D_iN_i(B_2) \leq D_iI_i$. Then $(F_i/E)^g=(C_s(I_1/E)/E)^g \times (F/E)^g \leq (C_s(I_1/E)/E)^g < EA_2/E$. Since $I_i \times N_i$ acts on $EA_2/E$ irreducibly, we have $F_i \leq E$. Thus (2) holds and at the same time we have $Z_s(R)=R$. By (3.6) $C_s(F)^g \geq F_i$. If $C_s(F)/F_i$, then by (3)
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$C_R(F) \cap C_R(I) \neq 1$. Hence $C \cap C_R(F) \cap C_R(I) \neq 1$. This yields that $C_R(Z(P)) \cap C_R(I) \neq 1$, which contradicts (2.1.2) and (2.5.2). Thus $C_R(F) = F_1$ and (1) holds.

$$ (3.10) \quad \mathcal{C}(D_i) = \{F_1, B_1\} \quad \text{and} \quad \mathcal{C}(D_i \mod Z(D_i)) = \{F_2, EB_2\}. \quad \text{In particular} \quad N(D_i) = N(B_i) \quad \text{and} \quad N(D_i) = N(EB_2). \quad \text{PROOF.} \quad \text{By (3.6) \( \mathcal{C}(D_i) = \{F_1, B_1\} \). \ Z(D_i) \leq Z(C_{D_i}(t)) = Z(B_i) = Z(Q). \ Since \ t \in Z(D_i), \ Z(P) = Z(D_i) \ \text{by \ (2.1.3). \ Then} \ E \leq Z(D_i), \ since \ E \leq D_i \ \text{and since} \ I \ acts \ on \ E/|Z(P) \ irreducibly \ by \ (3.3). \ If \ E < Z(D_i), \ then \ EA_i = Z(D_i), \ since \ by \ (3.8) \ N_i(B_i) \ acts \ irreducibly \ on \ EA_i \mod E. \ This \ yields \ that \ A_i F/Z(P) \simeq A_i A_i F/A_i \simeq FP/A_i, \ which \ is \ not \ abelian \ by \ (3.5.5). \ This \ is \ a \ contradiction. \ Thus \ E = Z(D_i) \ \text{and by} \ (3.9.2) \ \text{we have} \ \mathcal{C}(D_i \mod Z(D_i)) = \{F_i, EB_i\}. \quad \text{PROOF.} \quad \text{By (3.9.3) \( E^i = C_R(I^1) \subseteq C_R(I^1) \subseteq C_R(I_1) \), \ since \ F_1 \ is \ \text{r-invariant}. \ Then \ (1) \ follows \ from \ (3.9.3) \ \text{again. \ We \ can \ obtain} \ (2) \ \text{similarly.} \quad \text{PROOF.} \quad \text{By \ (2.5.1) \ and \ (3.5.1) \ N_1 \ is \ a \ standard \ subgroup \ of \ \langle N(F) \cap C(I) \rangle \ \text{and} \ \langle t \rangle \ \text{is \ a \ Sylow \ 2-subgroup \ of} \ \langle N(F) \cap C(I) \rangle \ \text{and} \ \langle C(N) \rangle. \ By \ (3.9) \ t \ is \ not \ a \ central \ involution \ in \ \langle N(F) \cap C(I) \rangle \ \text{and} \ C_R(I) \ \text{is \ elementary} \ \text{We \ also \ have} \ [C_R(I), I \cap N] = C_R(I). \ \text{Hence \ we \ can \ appeal \ to \ the \ result \ of [16] \ to \ obtain} \ (1). \ \text{(2) \ is \ obtained \ similarly.} \quad \text{DEFINITION.} \quad L_1 = F_1 \cdot M_1 \ \text{and} \ L_2 = F_2 \cdot M_2. \quad \text{(3.13) \ (1) \ \( L_1 \simeq N(F_1). \) \ (2) \ \( E = Z(L_1). \) \ (3) \ \( L_1 \ \text{and} \ L_2 \mod E \ \text{are \ perfect.} \quad \text{PROOF.} \quad \text{Set} \ \bar{N}(F) = N(F)/F_1. \ \text{By \ (2.5.3) \ and \ (3.10) \ \bar{N}_1 \ \text{is \ a \ standard \ subgroup \ of} \ \bar{N}(F) \ \text{and} \ \langle t \rangle \ \text{is \ a \ Sylow \ 2-subgroup \ of} \ \bar{N}(F)^{(F)}(\bar{N}_1). \ \text{Then \ as \ in \ (3.12) \ \text{we \ have} \ \langle \bar{N}_1^{(F)} \rangle \not\simeq \langle \bar{N}_1 \rangle \ \text{and} \ \langle \bar{N}(F)^{(F)} \rangle \not\simeq \langle \bar{N}(F) \rangle. \ \text{Since} \ \bar{M}_1 \leq \langle \bar{N}_1 \rangle \ \text{and} \ L_1 \ \text{is \ the \ preimage \ of} \ \langle \bar{N}_1 \rangle, \ \text{so} \ L_1 \ \text{is \ a \ perfect \ subgroup \ of} \ \bar{N}(F) \ \text{and} \ \langle t \rangle \ \text{acts \ on} \ \bar{N}(F) \ \text{as} \ L_1 \ \text{is \ a \ perfect \ group} \ \text{of} \ \bar{N}(F). \ \text{Now} \ E \leq Z(F_2). \ \text{Suppose \ that} \ E \leq Z(F_2). \ \text{Then \ by \ (3.9.2) \ \( Z(F_2); A_i = F_i. \) \end{ref}
Since $Z(F) \cap A = Z(P)$ and $|F| = q^{10}$, it follows that $|Z(F)| = q^9$. By (3.9.3) $R = F_1 F_2$ and $F_1 \cap F_2 = F$. Therefore $Z(F) \cap F = E$. Then considering the order, we have $Z(F) F = F_2$, a contradiction. Hence $E = Z(F)$. By (2.1.5) and (3.3) $N_2 < C(E)$, since $N_2 = O^-(N_2)$. So $C_{N_2}(E)$ is a nontrivial $t$-invariant normal subgroup of $M_2$. Then the structure of $M_2$ yields that $C_{M_2}(E) = M_2$. Thus $E = Z(L)$. By (2.2.1) and (3.5.1) $F_1 N_1$ is perfect. Hence $L_1 = F_1 M_1$ is perfect. Similarly $L_2/E$ is perfect by (2.2.2) and (3.9.2).

(3.14) (1) $R \in Syl_2(L) \cap Syl_2(L_2)$.
(2) $N_{L_2}(R)$ and $N_{L_2}(R)$ normalize each other.
(3) $N_{L_2}(R) \cap N_{L_2}(R) = R$.

Proof. (1) is a direct consequence of (3.12). By (3.9.1) and (3.13.1) $N_{L_2}(R)$ normalizes $N_{L_2}(R)$. Set $J = N_{L_2}(R) \cap N_{L_2}(R)$. Then $N_{L_2}(R) = (R \cap M_1) \cdot J_1$ and $I \cap M_1 < J_1$. Since $J_1 \simeq Z_{q^{10}-1} \times Z_{q^{10}-1}$ or $Z_{q^{10}-1}$ and since $I = I_1 \times (I \cap N_1)$, we have $I_1 \times J_1 < C(I)$. By (3.9.3) $I = (RI_2)^*$. Hence $J_1 < N_2(R) \cap C(I)$. This proves (2).

By the choice of $J_1$, $r_1$ inverts the elements of $J_1$. Set $N_{L_2}(R) = (R \cap M_1) \cdot J_1$. Then $(J_1 \cap R J_1)^* = J_1 \cap R J_1$. By (3.13.2) $E \leq C(J_1 \cap R J_1)$. By (3.11.1) $F_1 = F \cdot E^*$. Therefore $J_1 \cap R J_2$ acts on $F_1/F$ trivially. Since $F_1/F \simeq R/F_2$ and since any non-identity element of $J_1$ acts on $R/F_2$ nontrivially, we have $J_1 \cap R J_2 = 1$. Since $R J_1 \cap R J_2 = R J_1 \cap R J_2$, (3) is obtained.

(3.15) $L_2/F_1 \simeq SL_2(q) \times SL_2(q^2)$.

Proof. Assume that the assertion does not hold. Let $J_1$ be as in (3.14). Then $I \cap J_1 = I \cap M_1 = I \cap N_1$. Suppose that $L_2/F_1 \simeq SL_2(q^2)$. Then $J_2 \simeq Z_{q^{10}-1}$. By (3.14) $J_1 J_2 R \simeq Z_{q^{10}-1} \times Z_{q^{10}-1}$. So we may take $J_2$ such that $J_1 J_2 = J_1 \times J_2$. Then $(J_1 J_2)^* = (J_1 J_2)^{q-1} = (I \cap N_1)^{q-1} J_1^* \simeq Z_{q^{10}-1} \times Z_{q^{10}-1}$. Since $q = 2^n$, $J_1 J_2 / C_{J_1 J_2}(L_1/F_1)$ acts on $L_2/F_2$ as an inner automorphism. Therefore $C(L_2/F_2) \cap (I \cap N_1)^{q-1} J_1^{q-1} \neq 1$. By (2.1.2) and (3.3) $(I \cap N_1)^{q-1} < C(E)$. So by (3.11.1) $(I \cap N_1)^{q-1} < C(F/F_2)$, since $I$ and $N_1$ are $r_1$-invariant. Also by (3.13.2) $J_1^* E \simeq C(E)$ and $J_1^* E \simeq C(F/F_2)$, hence $F/F \simeq R/F_2$. Consequently $F$ has a nontrivial $L_2$-invariant subspace $C_{J_1}(C(L_2/F_2)) \cap (I \cap N_1)^{q-1} J_1^{q-1}$. But this can not hold, since $J_1$ acts on $F_1$, irreducibly by Lemma 4B of [8]. Suppose that $L_2/F_1 \simeq SL_2(q) \times SL_2(q)$. Then $J_2 \simeq Z_{q^{10}-1} \times Z_{q^{10}-1}$ and by (3.14) $C_{J_1}(L_2/F_1) \neq 1$. Then we can derive a contradiction in the same way as above.

Definition. Set $L_1 = K_1 K_1$ and $U = R \cap K_1$, where $K_1 \leq L_1$ and $K_1/F_1 \simeq SL_2(q^2)$.

(3.16) (1) $|Z(U)| \geq q^4$. 

(2) If $|Z(U)|=q^4$, then $Z(U)<F$.

**Proof.** Suppose that $|Z(U)|<q^4$. Then by Lemma 4B of [8] $K_i \leq C(Z(U))$. By symmetry $K_i \leq C(Z(U'))$. Then $K_i K_i \leq C(Z(U) \cap Z(U'))$, which is a contradiction, since $Z(U) \cap Z(U')=E$ and $N_i \leq C(Z(P))$. Thus (1) holds.

Since $F=Z_2(R)$ and $E=Z_2(R)<Z(U)$ by (3.9.1), it follows that $E<Z(U) \cap F$. By (3.9.3) $I_i /I_i$ acts on $F/E$ fixed-point-freely. Then, since $I_i =I_i \times I_i$, any $I_i$-orbit on $(F/E)^i$ is of length $q^2-1$. This yields that $(Z(U) \cap F)/E$ is of order at least $q^2$. Thus (2) holds.

For the following we refer the readers to (4I) of [12].

(3.17) Suppose $Z(U) \leq F$. Then the following statements hold.

1. $F_i = Z(U) \times Z(U')$.
2. $F = Z(U)Z(U')$.
3. $K_i$ acts on $Z(U)$ and on $Z(U')$ as $SL_2(q^2)$ acts naturally on a 2-dimensional vector space over $GF(q^2)$.
4. If $R \neq R' \in \text{Syl}_2(L_1)$, then $Z(R) \cap Z(R')=1$.
5. If $z \in E^i$, then $z^{(F_i)} \cap F = Z(U)^i \cup Z(U')^i$.

**Proof.** Suppose $Z(U) \cap Z(U') \neq E$. Then, since $K_i$ normalizes $Z(U)$ and $Z(U')$, we have $Z(U) \cap Z(U') \cap Z(U') \neq 1$. This implies that $1 \neq E \cap Z(U') \leq E \cap F^i$, which contradicts (3.11.1). Thus (3.16.1) gives (1). Since $Z(U) \cap Z(U') = E$, (2) follows. In particular $Z(U) \neq Z(U')$. Hence $K_i$ acts on $Z(U)$ nontrivially. So we can appeal to (1K) of [12] to obtain (3). By the structure of $SL_2(q^2) \times SL_2(q^2)$, $R'$ is $R$-conjugate to one of $R'$, $UU'$ and $U'U'$. Hence for the proof of (4) we may assume that $R'$ is one of those. Then in any case we have $E \cap Z(U') \neq 1$, $E \cap Z(U') \leq E \cap F^i=1$ by (3.11.1). Thus (4) holds. Since $F=Z_2(R)$, arguing as in (4), we see that it is enough for (5) to show that $Z(R') \cap F \subseteq Z(U) \cup Z(U')$ for $R'$ mentioned in (4), and this obviously holds.

§ 4. Case for $L_5(q^2)$

In this section we assume the following and we will obtain that $\langle L \rangle \simeq L_5(q^2)$ under the hypothesis made at the beginning of Section 3.

**Hypothesis.** $L_i /F_i \simeq SL_2(q^2)$.

(4.1) (1) $\langle R \mod E \rangle = \{F_i, F_i \}$.
(2) $Z(U)<F$.

**Proof.** If we show $I(R)=I(F_i) \cup I(F_i)$, then (3.6) and (3.9.2) will give (1). Suppose false and let $x \in I(R)-I(F_i)$. Then by our hypothesis in this
section $J_1$ acts on $(R/F_1)$ transitively. Since $F_1 \text{char } R$ (see (3.9.1)), we have $x^y \in I(R) - I(F_1) - I(F_2)$ for any $y \in J_1$. So we may take $x = fa \in F_1 \cup F_2$ for some $f \in F$, and $a \in A \cup \Z_2(P)$. Then by (3.9.2) $[f, t] \in Z_2(P) E$. Hence $f \in FA_2$. Set $f = a f'$ with $a \in A_1$, $f' \in F$ and set $R = R/E$. Then $\tilde{a} = (\bar{a} f'')^y = (\bar{a} a')^y$ by (3.9.1). But, since $F_1 P / E \cong P / Z_2(P)$, this yields that $a \in A_1 \cap A_2$ and so $x \in F_1$, a contradiction. Thus (1) holds. Then (2) follows from (1), since $F = F_1 \cap F_2 < U$.

(4.2) $C_{J_1 L_1}(z_2) = C_{L_1}(z_2)$ and $C_{L_1}(z_2)/F_1 \cong SL_2(q^2)$.

Proof. By (2.2.3) $SL_2(q) \cong C_{J_1}(z_2) \cong C_{L_1}(z_2)$. Since $C_{J_1}(z_2)$ is $t$-invariant, $C_{J_1}(z_2) \cong SL_2(q)$, $SL_2(q) \times SL_2(q)$ or $SL_2(q^2)$ by the structure of $M_1$. If $C_{J_1}(z_2) \cong SL_2(q)$, then $|z_2| = |M_1| : C_{J_1}(z_2) > |F_1|$. Hence this can not occur. So in any case $C_{J_1}(z_2)$ is maximal among the subgroups of $M_1$ which are $t$-invariant.

Now by (3.14) $J_1 L_1 / F_1 \cong SL_2(q^2) \times SL_2(q^2)$. Set $J_1J = J_1 J_2 R / R$ and set $J = C_{J_1 J_2}(U / F_1)$. Then $J \cong Z_2(q^2)$. Let $\langle x \rangle = \bar{J}$ and let $\langle y \rangle = \bar{J}_2$. Then $\langle xy \rangle \in J_1 \cup K_1$ for some integer $i$. So $\bar{x}$ acts on $U_1 / F_1$ as $y^i$ acts on it. Since $t$ acts on $L_1 / F_1$ as a field automorphism, we have $\bar{y} = \bar{y}^q$. Hence $\bar{x}$ centralizes $U_1 / F_1$ and acts on $U_1 / F_1$ as $y^i$ acts on it. Therefore $\langle xy \rangle$ centralizes $R / F_1$. Set $\Omega = E$ and we will show $\langle xy \rangle^\Omega \cong Z_2(q^2)$. By (3.13.2) $\langle xy \rangle^\Omega = \langle (xy)^{q^i} \rangle^\Omega$. Since $\langle t \rangle N_{L_1}(R)$ normalizes $E$ and since $N_{L_1}(R)$ is transitive, we can apply (1.3) to obtain $\langle (xy)^q \rangle^\Omega = \langle (xy)^i \rangle^\Omega$. Hence $\langle xy \rangle^\Omega = \langle (xy)^i \rangle^\Omega$. Then, since $(J_1 \cap K_1)^\Omega$ is faithful by (17.3), we have $\langle xy \rangle^\Omega \cong \langle \bar{y} \rangle$. So in order to see $\langle xy \rangle^\Omega \cong Z_2(q^2 - 1)$ it is enough to show that $\tilde{J} \cap \tilde{J}_2 = 1$. Since $J_2$ is cyclic, $\tilde{J} \cap \tilde{J}_2$ is $t$-invariant. So $\tilde{J} \cap \tilde{J}_2 \cong C(R / F_1)$. Hence if $R J$ denotes the preimage of $\tilde{J} \cap \tilde{J}_2$, then $J_2 \cong C(L_1 / F_1)$. By (3.13.2) $J_2 F_1 \cong C(E)$ and so by (17.3) and (4.1.2) we have $J_2 F_1 \cong C(F_1)$. In particular $J_2 \cong C(R / F_1)$, since $F_1 / F_1 \cong R / F_1$. This implies $J = 1$ by the structure of $L_1 / F_1$. Thus $\langle xy \rangle^\Omega = \langle \bar{J}_1 \rangle \langle \bar{J}_2 \rangle (R / F_1) \cong Z_2(q^2 - 1)$. In other words $C_{J_1 L_1}(L_1 / F_1)^\Omega \cong Z_2(q^2 - 1)$. By (2.2.1) $\langle z_{1i} \rangle = A_i$, and so $\langle z_{1i} \rangle = \langle A_{1i} \rangle = F_1$. Notice that $C_{J_1 L_1}(L_1 / F_1)$ acts on the set of the orbits of $L_1$ on $F_1$. Let $J' F_1$ be the subgroup of $C_{J_1 L_1}(L_1 / F_1)$ which leaves the orbit $z_{1i}$ invariant. Set $J'' = C_{J''}(z_{1i})$. Then $J'' F_1 = 1$, $\langle z_{1i} \rangle = F_1$. Hence the result of the previous paragraph yields $J'' = J'$. Since $\langle z_{1i} \rangle = F_1$, we get $J'' = J'$. Hence the result of the previous paragraph yields $J'' = J'$ is faithful on $z_{1i}$. So, if we set $d = z_{1i}^t$, then $(J' L_1)^d = J' \times K_1 \times K_1$. Since $|d| = |J' L_1| : C_{J' L_1}(z_1) / C_{J' L_1}(z_2) = |J' / C_{J' L_1}(z_2)|$. Then by the result obtained in the first paragraph we have $C_{J' L_1}(z_2) = J' C_{J' L_1}(z_2)$. Since $J'$ is normal and contained in a one-point-stabilizer of the transitive permutation group $(J' L_1)^d$, we have $J' = 1$ and $C_{J' L_1}(z_2) = C_{J' L_1}(z_2)$. Therefore $|z_{1i}^{q^2 - 1}| = (q^2 - 1)q^2(q^2 + 1)$ or $(q^2 - 1)q^2(q^2 + 1)$ according as $C_{J' L_1}(z_2) \cong SL_2(q) \times SL_2(q)$ or $SL_2(q^2)$. But $(q^2 - 1)q^2(q^2 + 1) > q^2$. Consequently $C_{J' L_1}(z_2) / F_1 \cong SL_2(q^2)$ and the proof is complete.
(4.3) $z_1$ and $z_2$ are the representatives of $N(F_1)$-conjugacy classes of the elements of $F_1$ and the fusion is controled by $J_zL_1$. Furthermore $|z_1^{N(F_1)}|=(q^2-1)(q^2+1)^2$ and $|z_2^{N(F_1)}|=(q^2-1)q^2(q^2+1)$.

**Proof.** By (3.17.4) and (4.1.2) $|\bigcup_{x \in U_0} (E_0^x)|=(q^2-1)(q^2+1)$. By (4.2) we have

$$|z_1^{L_1}|=(q^2-1)(q^2+1)^2$$

Hence $L_1$ has one orbit of length $(q^2-1)(q^2+1)^2$ and $q^2-1$ orbits of length $(q^2-1)q^2(q^2+1)$ on $F_1$, and the orbits of length $(q^2-1)q^2(q^2+1)$ are permuted transitively by $J_z$. Since $L_z \leq N(F_1)$, the assertion follows.

(4.4) Set $U \cap M_1 = U_0$. Then the following statements hold.

(1) $C_{F_1}(a) = Z(U)$ for $a \in U_0$ and $|C_{F_1}(b)| = q^4$ for $b \in U_0U'_0 - U_0 - U'_0$. In particular $C_{F_1}(a, b) = E$.

(2) $\Delta(C_{F_1}(a), U_0U'_0) = \{Z(U), EU_0U'_0\}$ and $\Delta(C_{F_1}(b)) = \{V, EU_0U'_0\}$, where $V = C_{F_1}(C_{F_1}(b)) \cap C_{F_1}(b)$.

(3) $C_{F_1}(x, b) = V$ for any $x \in C_{F_1}(b) - E$. Also $\Delta(C_{F_1}(c)) = \{V, EU_0U'_0\}$ for any $c \in V \cap U_0U'_0$.

(4) If $x \in Z(U) - E$, then $C_{F_1}(x) = U$. If $x \in F - Z(U) - Z(U')$, then $|C_{F_1}(x)| = q^{10}$.

**Proof.** Suppose $C_{F_1}(a) > Z(U)$. Then since $C_{F_1}(a) = C_{F_1}(a^2)$ for any $a \in K_1'$, we have $C_{F_1}(a) = F_1$ by (3.17.1, 3) and (4.1.2). This yields that $C_{F_1}(a) = R$, since $R = F_1(U_0 \times U'_0)$, which is against (3.9.1). Hence $C_{F_1}(a) = Z(U)$. By the structure of $M_1$ the elements of $U_0U'_0 - U_0 - U'_0$ are conjugate mutually in $N_{M_1}(R)$. So we may assume that $b \in C_{F_1}(z) < C_{M_1}(z)$. Now the proof of (4.2) shows that $C_{F_1}(L_1/F_1)$ acts on $F_1$ fixed-point-freely. Therefore if we denote $X = C_{F_1}(L_1/F_1)$, then $<z_1^y>E \leq C_{F_1}(b)$ and $|<z_1^y>E| \geq q^4$. By (4.1.1) $C_{F_1}(b) \leq F$. Hence, if $|C_{F_1}(b)| > q^4$, then by (3.17.5) $Z(U) \cap C_{F_1}(b) > E$. Let $c \in Z(U) \cap C_{F_1}(b) - E$ and let $b = b_ib_i'$ with $b_i \in K_1$ and $b_i' \in K'_1$. Since $bc = cb$ and $b,c = cb$, we have $b,c = cb$, which contradicts the first assertion. Thus (1) holds. Also it is obtained that $C_{F_1}(b) \cap Z(U) = E$ and (4) follows immediately.

By (1) $C_{F_1}(a) = (Z(U)U_0)(EU_0U_0)$, $Z(U)U_0 \cap EU_0U_0' = EU_0$ and $C_{F_1}(a, d) = EU_0U_0'$ for any $d \in EU_0U_0' - Z(U)U_0$. This implies that $\Delta(C_{F_1}(a)) = \{Z(U)U_0, EU_0U_0\}$. Now we may take $b = z_1^x$. Then $C_{F_1}(b) = <z_1^x>E$. Since $C_{F_1}(b) \cap Z(U) = E$, $C_{L_1}(x)/F_1 \simeq SL_2(q^2)$ for any $x \in C_{F_1}(b) - E$ by (3.17.5) and (4.2). Hence we have $C_2(x) = Syl_2(C_{L_1}(x))$ by the structure of $M_1$. Then $C_{F_1}(y) = <x^y>E = <z_1^y>E$ for any $y \in C(x) \cap U_0U_0'$ since $C(x) \cap (M_1 \cap K_1) \cap U_0U_0' = C(x) \cap (M_1 \cap K_1') \cap U_0U_0' = 1$ by the proof of (4.2). Hence $V = C_{F_1}(C_{F_1}(b)) \cap C_{F_1}(b)$ is abelian. So in order to prove (2) and (3) we only show that $\Delta(C_{F_1}(b)) = \{V, EU_0U_0'\}$. Let $x \in I(C_{F_1}(b))$ and let $x = x_1x_2x_3$, where $x_1 \in C_{F_1}(b)$, $x_2 \in U_0$ and
Furthermore we can choose \( x'_2 \) in \( U_2 \) such that \( x_2 x'_2 \in C(x_2) \). Then
\[ 1 = (x_2 x'_2 x_2 x'_2)^2 = (x_2 x'_2 x_2 x'_2)^2 = (x_2 x'_2 x_2 x'_2)^2, \]
which yields that \( x'^2 = 1 \) or \( x \in E \). This gives the claim.

(4.5) (1) \( I(F_i) = \bigcup_{x \in L_2} (F_i x)^2 \).

(2) If \( x \in I(R) \), then \( x \) is either conjugate to \( z_1 \) or to \( z_2 \) in \( L_1 \cup L_2 \).

(3) \( t^6 \cap RT = t^8 = I(tR) \).

(4) \(|C_\gamma(u)| = q^4 \) and \(|C_\gamma(ut)| = q^6 \), where the involution \( u \) defined in Section 2 is chosen to normalize each component of \( M_i \).

**Proof.** By (3.9.3) and (3.11.2) \( F \cap F^z = E \). Hence \( F \cap F^z = E \) for any \( x \in L_2 = N_{L_2}(R) \) by the structure of \( L_2/F_2 \). Therefore \( \bigcup_{x \in L_2} (F_i x)^2 = q^4 - q^4 - q^4 - 1 \).

By (4.1) \( C_\gamma(\gamma) \leq F_z \) for any \( y \in (U_i, \hat{U}_i)^0 \). Hence applying (4.4.1), we have
\[ |I(F_i)| = |I(\gamma)| = q^4 - q^4 - q^4 - 1. \]
Thus (1) holds. Then (2) follows from (4.3).

By (1) for any \( x \in I(R) \) \( m(C(x)) = 8n \leq 4n + 1 = m(C) \), so \( t \) does not fuse into \( R \). Since \( RT/R \simeq TP, m(RT/R) = 1 \) or 2. If \( m(RT/R) = 1 \), then \( I(RT - R) = I(tR) \) and we can apply (19.4) of [4] to \( R/F_i \), and \( F_i \) to obtain \( t^8 = I(tR) \).

If \( m(RT/R) = 2 \), then \( I(RT - R) = I(tR) \cup I(tuR) \cup I(uR) \). Set \( X = N_{L_1}(R)T \) and set \( \Omega = E^\gamma \). Then \( (x^\gamma)^2 = (x^\gamma)^2 \) for any \( x \in J_i \). Since \( X \) satisfies the assumption of (1.3), we have \( (x^\gamma)^2 = (x^\gamma)^2 \). So \( (x^\gamma)^2 = (x^\gamma)^2 \) for any \( x \in J_i \), which implies \( tu \in C(E) \). By (3.17) \( F_i = E^\times Y \times (E \times Y^\gamma) \), where \( Y = Z(U) \cap Z(U^\gamma) \).

Since \( r_u = u_r, E, E^\gamma \leq C(tu) \). Since \( tu \) interchanges \( Y \) and \( Y^\gamma \), \( C_{\gamma Y}(tu) = \{zz^t \mid z \in Y \} \). Consequently \(|C_{\gamma Y}(tu)| = q^4 \). Then applying (19.4) of [4] to \( R/F_i \), we obtain that any element of \( I(tuR) \) is \( R \)-conjugate to one of \( I(tuF_i) \). Thus for any \( x \in I(tuR), C_{\gamma Y}(x) = q^4 \). Since \( m(C) = 4n + 1 \) by (3.1.1), \( t \) does not fuse into \( I(tuR) \). Next we inspect \( C_{\gamma Y}(u) \). By (3.17) \( E^\times Y \times Y^\gamma \). Set \( Z(P) = Z(P)C_{\gamma Y}(t) \) and \(|C_{\gamma Y}(t, u)| = q \). Hence \(|C_{\gamma Y}(u)| = |C_{\gamma Y}(u)| = q \). Since \( C_\gamma(t) = C_\gamma(u) \), we get \(|C_{\gamma Y}(u)| = q^4 \) and \( I(uR) = u^R \) as above. This gives (4).

Then in order to prove (3) it is enough to show that \( t \) is not conjugate to \( u \). Suppose false. Then, since \( B_r = J_r(T), C_{\gamma Y}(u) \) is conjugate to \( B_r \).

By (3.11.2) \( L_2 / E \) acts on \( B \) in the same way as the natural action of \( SL_2(q^2) \) on a 2-dimensional vector space over \( GF(q^2) \).

**Proof.** First we will show that \( \{V_i - E, B - E, B^i - E \mid i = 1, \ldots, q - 1 \} \) makes a partition of the involutions of \( F_i - E \). By (4.4.1) \( F_i \cap V_i \cap V_i \leq \)}
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$C_g(b_i, b_j) = C_g(b_i, b_j) = E$ if $i \neq j$, since $b_i b_j \in F_i U_i$. Then by (4.4.3) $V_i \cap V_j = E$. Similarly we have $B \cap V_i = B' \cap V_i = E$ for any $i$ by (4.4.4). Then counting argument gives the claim as in (4.5.1). In particular we may assume that $(z_0, z_0^2) < V_i$. By (4.4.3), $C_{g_0}(z_0, z_0^2) = V_i$ and $C_{g_0}(z_0) = F_i V_i$. Hence by (4.1.1) $V_i < R$. Therefore we have $V_i < (R, R^i) = L_0$. Since $J_i$ normalizes $L_z$ by the proof of (3.14), this implies that all the $V_i$'s are normal in $L_z$. Hence by (4.4.4) $U_0 \cup U_0^i \subseteq B^i \cup B^i$, for any $x \in L_z$ and by (4.4.2), $B^i = B$ or $B'$. By (3.13.3) $L_0/E$ has no subgroup of index 2, which yields that $B^i = B$. Since $|(B/E) \cap (F/E)| = q_2^2$, the last assertion follows from (1K) of [12].

**Definition.** $G_0 = \langle L^0 \rangle$.

(4.8) (1) $F_i = J_i(R)$.
(2) $R \in \text{Syl}_1(G_0)$.
(3) $G_0$ has just two conjugacy classes of involutions, which are represented by $z_0$ and $z_0^2$. Furthermore $C_{g_0}(z_0)$ is a Sylow 2-subgroup of $C_{G_0}(z_0)$.
(4) $B$ and $B^i$ are weakly closed in $R$ with respect to $G_0$.

**Proof.** (1) follows from (4.1.1) and (4.7). By (4.5.3) $N(RT) = RT(C \cap N(RT))$. Hence $RT \in \text{Syl}_1(G)$. Since $RT/R \cong T/P$ is abelian, we have $t \in O^i(G)$ by (1.5). By (3.13.3) $R < O^i(G)$. Suppose that $R \in \text{Syl}_1(O^i(G))$. Then $\langle u \rangle R$ or $\langle tu \rangle R$ is contained in $O^i(G)$. Hence there exists an element $x$ in $R$ such that $x = u^g$ or $(tu)^g$ for some $g \in G$. By (4.5.1) and (4.5.4) we have

$$m(C(y)) \geq 8n \quad \text{if} \quad y \in I(R),$$

$$m(C_{RT}(y)) \leq 6n + 1 \quad \text{if} \quad y \in I(RT - R).$$

So in any case we may take $C_{(RT)}(x) \leq RT$. Then $t^i \in RT$. Hence by (4.5.3) we may assume $g \in C$. This implies that $u$ or $tu$ fuses into $P$ in $C$, which does not occur. Thus $R \in \text{Syl}_1(O^i(G))$. By (4.5.2) $\langle L_i, L_0 \rangle \leq G_i$. Therefore $R \in \text{Syl}_1(G_i)$ and also the first statement of (3) holds. Then by (4.2) and (4.4.1) we have $C_{g_0}(z_0) \in \text{Syl}_1(C_{G_0}(z_0))$. By (4.6) $B^i \subseteq z_0^2$ and $V_i^i \subseteq z_0^2$. So by (4.7), if $B^i < R$, then $B^i = B$ or $B'$, or else $B^i < F_i$. Suppose that $B^i < F_i$. Then $B^i \cap F_i \subseteq Z(U) \cup Z(U^i)$ by (3.17.5). Since $|B^i \cap F_i| = q^i$, we may assume that $B^i \cap F_i = Z(U)$ without loss of generality. Now similarly $B^i \cap F_i \subseteq Z(U^i) \cup Z(U^i)$. If $B^i \cap F_i = Z(U^i)$, then $B^i = Z(U)Z(U^i) = F_i$ by (3.17.1), a contradiction. If $B^i \cap F_i = Z(U^i)$, then $B^i = Z(U)Z(U^i)$. By (4.1.4) this yields that $B^i = Z(U(U^i))$, also a contradiction. Thus $B^i = B$ or $B'$. Then by a theorem of Burnside we have $x \in N_{g_0}(R)$ and by (4.6) $B^i = B$. Thus $B$ is weakly closed in $R$ with respect to $G_0$. By symmetry $B^i$ is weakly closed.

(4.9) $L_0 < \langle N(F_i) \rangle$.

**Proof.** Set $N(F_i) = N(F_i)/F_i$ and set $X = N(F_i) \cap G_0 \langle t \rangle$. By (4.8)
\[ Q \in \text{Syl}_4(C_\mathbb{F}(t)). \text{ Hence by (3.8) and (3.10) } \bar{N}_t \text{ is a standard subgroup of } \bar{X} \text{ and } \langle \bar{t} \rangle \text{ is a Sylow 2-subgroup of } C_\mathbb{F}(\bar{N}_t). \text{ So, as in (3.12), we have } \langle \bar{N}_t \rangle \cong \text{SL}_2(q) \times \text{SL}_2(q') \text{ or } \text{SL}_2(q'). \text{ Since } \bar{T} \text{ normalizes } \bar{N}_t, \langle \bar{N}_t^{\text{Syl}_2(q)} \rangle = \langle \bar{N}_t^{\text{Syl}_2(q')} \rangle. \text{ Therefore, } \bar{L}_q = \langle \bar{N}_t^{\text{Syl}_2(q')} \rangle \triangleleft \bar{N}(F_q). \]

Now we can appeal to an argument in [12] for the conclusion of this section. (4.10, 11, 12 and 13) below refer to (4N, O, P, S) of [12].

\[ (4.10) \quad N_{\alpha_0}(E) = N_{\alpha_0}(F_0)O(N_{\alpha_0}(E)). \]

**Proof.** Set \( \bar{N}_{\alpha_0}(E) = N_{\alpha_0}(E)/O(N_{\alpha_0}(E)) \mod E \) and let \( X \) be the normal closure of \( F_0 \) in \( N_{\alpha_0}(E) \). Then by (4.1.1) and (4.8.2) we can appeal to (1H) of [12] to obtain that \( F_0 \) is strongly closed in \( R \) with respect to \( N_{\alpha_0}(E) \). Hence by the result of [10] we have \( F_0 = \text{Syl}_2(X) \), and either \( X = F_0 \) or \( X = \text{O}_2(X)X_1 \cdots X_s \), where \( X_i \) is a simple Goldschmidt group. Suppose the latter holds. Since \( t \) normalizes \( X \) and since \( f(<t>E) = \{E, Z(\mathbb{Q})\} \),

\[ \text{Let } x_2 \in (A_2(X_1X_2)) \#. \text{ If there exist } X_1, X_2 \text{ and } X_3 \text{ such that } X_1X_1 = X_2X_2 = X_3X_3 \text{ and } X_iX_1 \text{ are not conjugate mutually in } N_{\alpha_0}(E). \text{ But } N_{\alpha_0}(E) \text{ has only two orbits on } \bar{A}_2 \text{ by (2.2.2). This contradiction yields } E(X) \text{ has at most four components. Therefore } C_i(E)' \text{ normalizes each } X_i, \text{ since } C_i(E)' \text{ is perfect. In particular it normalizes } \bar{F}_i \cap X_i \text{ which is a Sylow 2-subgroup of } X_i. \text{ Consequently } C_i(E)' \text{ centralizes each } X_i. \text{ But this yields that } C_i(E)' \text{ centralizes } \bar{A}_i, \text{ a contradiction. Therefore } \bar{F}_i < N_{\alpha_0}(E). \text{ Then, since } E = Z(F_0), \text{ we have } F_0O(N_{\alpha_0}(E)) < N_{\alpha_0}(E) \text{ and the Frattini argument completes the proof.} \]

**Definition.** Let \( C_i = C_{\alpha_0}(z_i), i = 1, 2. \)

\[ (4.11) \quad C_i = N_{\alpha_0}(E)O(C_i) \text{ and } C_i \text{ is 2-constrained.} \]

**Proof.** Suppose an element \( z \) of \( E \) is \( C_i \)-conjugate to an element \( x \) of \( R - E \). Then by (3.13.2), (4.1.1) and (4.5.1) we may take \( x \) in \( F_i - E \). Since \( F_i \) is weakly closed in \( R \), \( z \) and \( x \) are conjugate in \( N_{\alpha_0}(F_i) \). But by (3.17.4) we have \( N_{\alpha_0}(F_i) \leq N(R) \leq N(E) \), a contradiction. So \( E \) is strongly closed in \( R \) with respect to \( C_i \).

Let \( X = \langle E^{\alpha_i} \rangle \) and let \( \bar{C}_i = C_i/O(C_i) \). Then by the result of [10] we have either \( X = \bar{E} \) or \( X = \text{O}_2(X)X_1 \cdots X_s \), where \( X_i \) is a quasisimple Goldschmidt group. Furthermore we have \( E = \Omega_i(R \cap X) \). If \( E < R \cap X \), then by (3.9.1) \( E < F\cap X < \Omega_i(R \cap X) \), a contradiction. Hence \( E \in \text{Syl}_2(X) \). If \( X \neq \bar{E} \), then \( \bar{L}_i \) normalizes each \( X_i \) by (3.13.2). Now by (3.17.3) \( N_{\alpha_0}(R) \) acts on \( E^3 \) transitively. Hence \( E \leq R^3 \) and so \( L_i \) is perfect by (3.13.3). Therefore \( L_i \) centralizes
each $X_i$. Then $E \cap X_i \leq L_i \cap X_i \leq Z(X_i)$, which does not hold. Hence $X = \bar{E}$, and it follows that $C_1 = N_{C_i}(E)O(C_i)$. Since $C_1(F_2) < F_2 < R$ by (4.1.1), $N_{C_i}(F_2)$ is 2-constrained and so is $N_{C_i}(E)$ by (4.10). Thus $C_1$ is 2-constrained.

(4.12) $C_2 = N_{C_i}(F_2)O(C_i)$ and $C_2$ is 2-constrained.

**Proof.** By (4.8.3) $C_{n}(z_i) = F_i \in \text{Syl}_n(C_i)$ and $\sigma(C_n(z_i)) = \{V_i, F_i\}$, where $V_i$ is defined in the proof of (4.6). Hence we can apply (1H) of [12] to obtain that $F_i$ is strongly closed in $C_n(z_i)$ with respect to $C_i$. Let $X$ be the normal closure of $F_i$ in $C_2$ and set $C_i = C_i/O(C_i)$. Then as in (4.10) we have $F_i \in \text{Syl}_n(X)$ and either $\bar{X} = F_i$ or $\bar{X} = O_4(X)\bar{X}_1\cdots\bar{X}_s$, where $\bar{X}_i$ is a simple Goldschmidt group. Suppose that $\bar{X} \neq F_i$. Then by (4.11) $C_i \cap C_2$ is 2-constrained and so is $C_i \cap C_2$. Hence $C_2(x_i)$ is 2-constrained for any $x \in E$. So, if we set $z = x_1y_1\cdots y_s$, where $x \in O_4(X)$ and $y_i \in \bar{X}_i$, then it follows that $\overline{y}_i \neq \overline{1}$ for all $i$. Thus $|E| \leq |\bar{F}_i \cap \bar{X}_i|$. Since all the elements of $(\bar{F}_i \cap \bar{X}_i)$ are conjugate mutually in $N_{C_i}(|\bar{F}_i \cap \bar{X}_i|)$, $F_i$ contains at least $(q^2 - 1)^s$ elements conjugate to $z$. Then by (4.3) $(q^2 - 1)^s \leq (q^2 - 1)(q^2 + 1)^s$. Hence $k \leq 3$ and as in (4.10) we obtain that $C_3(x_i)$ centralizes each $\bar{X}_i$ by (2.2.3). Then $z \in C_3(E(X))$. Since $C_i \cap C_2$ is 2-constrained, this can not occur. Thus we have $X = F_i$ and the assertion follows easily.

(4.13) $G_i \simeq L_i(q^2).

**Proof.** By (4.8.3), (4.11) and (4.12) the centralizer of any involution in $G_i$ is 2-constrained. Furthermore $\text{SCN}_n(2)$ is not empty, as $F_i \in \text{SCN}_n(2)$. Thus by a result of [14] we have $O(G_i) = O(G_i) = 1$, since we assumed $O(G) = 1$ at the beginning of Section 3. Hence $L_i \leq C_i$ by (4.9). Suppose that $B^g < C_i$ for some $g \in G_i$. Then $B^g < L_i$. Hence $B^{gh} < R$ for some $h \in L_i$, and so by (4.8.4) $B^{gh} = B$. Consequently we have $B^g = B$ by (4.6). Since $B^g \leq z_i^{a_0}$ by (4.6), for any $a \in B^g$ is weakly closed in $C_0(a)$ with respect to $G_i$. Since $G_i$ is simple (cf. Lemma 2.7 of [2]), it follows that $G_i = \langle B^{a_0} \rangle$. Thus we can appeal to the result of Timmesfeld [19] to conclude that $G_i$ is isomorphic to $L_i(q^2)$.

§ 5. Case for $U_i(q) \times U_i(q)$

In this section we assume the following and we will obtain that $\langle L_i \rangle \simeq U_i(q) \times U_i(q)$ under the hypothesis made at the beginning of Section 3.

**Hypothesis.** $L_i/F_i \simeq SL_i(q) \times SL_i(q)$.

**Definition.** Set $L_i = K_iK_i'$, where $K_i \leq L_i$ and $K_i/F_i \simeq SL_i(q)$. 


(5.1) (1) $F_i = (Z(U) \cap Z(U')) \times (Z(U') \cap Z(U'^i))$.
(2) $K_i$ acts on $Z(U) \cap Z(U')$ trivially and on $Z(U') \cap Z(U'^i)$ irreducibly.
(3) $L_i = K'_i \times K''_i$.

Proof. Suppose $|Z(U)| = q^i$. Then by (3.16.2) we can apply (3.17.3) to obtain that $J_i \cap K_i$ acts on $(F_i/F)^i$ transitively. Since $R/F \simeq F_i/F$, $C_{R/F}(J_i \cap K_i) \neq 1$. Then by (3.14) $C_{R/F}(J_i \cap K_i) = J_i \cap K_i$-invariant. Hence this yields that $C_{F_i/F}(J_i \cap K_i) = F_i/F$, so $C_{R/F}(J_i \cap K_i) = R/F$. This can not occur by the structure of $L_i/F_i$. Thus we have $|Z(U)| > q^i$ and $Z(U) \cap Z(U'^i) \neq 1$. Since $\langle U, U'^i \rangle = K_i$, $K_i \leq C(Z(U) \cap Z(U'^i))$. If $|Z(U) \cap Z(U'^i)| < q^i$, then by Lemma 4B of [8] $K_i \leq C(Z(U) \cap Z(U'^i))$. But this contradicts (3.5.1). On the other hand we have $Z(U) \cap Z(U'^i) \cap Z(U') \cap Z(U'^i) = E \cap U = 1$ by (3.11.1). Thus (1) holds. Also (2) follows from (1). Then we have

$$K_i = (Z(U) \cap Z(U'^i)) \times ((Z(U') \cap Z(U'^i)) \cdot (K_i \cap M_i)).$$

Hence Gaschütz's theorem yields that $K'_i = (Z(U') \cap Z(U'^i)) \cdot (K_i \cap M_i)$ and so $L_i = K'_i \times K''_i$.

(5.2) Set $W = R \cap K'_i$, $W_i = [W, I_i]$, $i = 1, 2$. Then the following statements hold.

(1) $E = Z(W) \times Z(W')$, $F = Z(W) \times Z(W')$ and $R = W \times W'$.
(2) $F_i = W_i \times W_i'$ and $F_i = W_i \times W_i'$.
(3) $W_i = O_2(K'_i) = Z(U') \cap Z(U'^i)$.

Proof. (1) is clear by (5.1.3). By (3.9.3) $F_i = [R, I_i] = [W \times W', I_i] = [W, I_i] \times [W', I_i] = W_i \times W_i'$. Also $F_i = [R, I_i] = W_i \times W_i'$. Thus (2) holds. Then, since $|W_i| = |Z(U') \cap Z(U'^i)|$ and $W_i \leq F_i \cap K'_i$, we have (3).

(5.3) $L_i = K''_i \times K''_i$ and $W$ is a Sylow 2-subgroup of $K''_i$ or $K''_i$.

Proof. By (3.14) $J_i/C_{J_i}(K_i/W)$ acts on $K_i/W$ as an inner automorphism. Since $J_i \simeq Z_{q-1} \times Z_{q-1}$, $|C_{J_i}(K_i/W_i)| \geq q - 1$. Set $J = C_{J_i}(K_i/W)$. Suppose $J \not\subseteq K_i \cup K'_i$. Then $C_{R/F}(J) = 1$ and so $C_{R/F}(J) = 1$. Hence $C_{J_i}(J) \leq F$ and by (3.13.2) and (5.2) we have $1 \neq C_{W_i}(J) \leq W_i$. But this contradicts (5.1.2). Therefore $J \subseteq K_i \cup K'_i$. So we may assume that $J = J_i \cap K_i$. Then $C_{W_i}(J) \neq 1$, which yields that $[W_i, J] = 1$. Since $J$ stabilizes a normal series $W_i \geq W_i \geq 1$, we have $[W, J] = 1$. Set $V = K_i \cap R$. Then

$$V' = F_i[R, J] = (W_i \times W_i')([W, J] \times [W', J]) \leq W_i \times W_i'[W', J].$$

Comparing the orders of both sides, we have $V' = W_i \times W_i'$. By symmetry $V = W \times W_2$. As a consequence $Z(V) = E$ and $Z_i(V) = Z_i(W) \times W_i$. Set $L_i = L_i/E$. By (3.9.2) $[I \cap J_i, F_i] = F_i$, so in particular $[J_i, W_i] = W_i$. Since $J_i =
(J_2 \cap K_2) \times J and [J, W] = 1, we have [J_2 \cap K_2, W] = W. Notice that \(|Z(V)| = q^6\) and that \(K_2 = \langle V, V^r \rangle\). Hence \(|Z(V) \cap Z(V^r)| \geq q^4\) and \(K_2 \subseteq C(Z(V) \cap Z(V^r))\). Therefore we have

\[Z(V) \cap Z(V^r) \cap W = Z(V) \cap Z(V^r) \cap W \cap F_2 = Z(V) \cap Z(V^r) \cap W_2 = 1.\]

Hence \(\bar{V} = (Z(V) \cap Z(V^r)) \times W\) and Gaschütz's theorem yields that \(\bar{K}_2 \cap \bar{V} = \bar{W}\). Then, since \(EW = W \times Z(W^r)\), again by Gaschütz's theorem we have \(K_2'' \cap V = W\) and so \(L_1 = K_2'' \times K_3''\).

(5.4) Set \(G_0 = \langle L_1, L_2 \rangle\) and set \(G_i = \langle K_i', K_i'' \rangle\), where \(K_i''\) is chosen such that \(W \in \text{Syl}_q(K_i') \cap \text{Syl}_q(K_i'')\). Then \(G_0 = G_i \times G_i'\) and \(G_i \simeq U_4(q)\).

PROOF. We will show \([K_i', K_i''] = 1\). We remark firstly that

\[F_i' \cap F_i'' = (F_i' \cap F_i'') \cap (W_i \times W_i^r) \cap (W_i' \times W_i'^r) = (W_i' \times W_i'^r) \cap (W_i' \times W_i'^r)^r = (W_i' \times W_i'^r) \cap (W_i' \times W_i'^r)^r = (W_i' \times W_i'^r)^r = 1.\]

and secondly by the structure of \(A_2 \times C_{t_1}\) that

\[A_2 \times C_{t_1} = \langle P, A_2 \cap A_2^{r_1} \rangle.\]

Now by (5.3) the correspondence of \(x\) to \(xx^r\) for \(x \in K_2''\) gives an isomorphism \(K_2'' \simeq C_{t_1}\). Then the first remark shows that \(W_i' \times W_i'^r \cap W_i'' \times W_i''^r\) corresponds to \(A_2 \cap A_2^{r_1} \cap A_2^{r_1 r_2}\). Clearly \(W\) corresponds to \(P\). So by the second remark we have

\[K_2'' = \langle W, W_i' \cap W_i'^r \rangle.\]

For \(K_2'\) as above we have

\[F_i' \cap F_i'' = (W_i' \cap W_i'^r) \times (W_i' \cap W_i'^r)^r = (W_i' \cap W_i'^r)^r,\]

\[K_2' \simeq C_{t_1} = A_2 \times A_2^{r_1} = \langle P, A_2^{r_1} \cap A_2^{r_1} \rangle,\]

and

\[K_2' = \langle W, W_i' \cap W_i' \rangle.\]

Then by (5.1.3) and (5.3)

\([W, W_i' \cap W_i'^r] \leq [W, W_i'] \leq [K_2'', K_2'] = 1\]

and

\([W, (W_i' \cap W_i'^r)] \leq [W, W_i'] \leq [K_2', K_2'] = 1.\]

Of course \([W, W_1] = 1\). Moreover, since \((r_1, r_2)^2 = 1\), we have
Therefore we have \([K'_1, K''_1] = 1\) and consequently \(G_0 = G_1 \times G_1^t\). Then \(C_{g_0}(t)\) is a homomorphic image of \(G_1\). Since \(G_1\) is perfect, \(G_1^t\) is a central extension of \(U_q(q)\). Thus we have \(G_0 = G_1 \times G_1^t\) and \(G_1 \cong U_q(q)\).

(5.5) Let \(S_0\) be a Sylow 2-subgroup of \(N(F_1)\) containing \(R\) and set \(R_0 = N_{S_0}(K'_1)\). Then

1. \(F_1 \in \text{Syl}_2(C(L_1/F_1))\)
2. \(S_0 \in \text{Syl}_2(G_0)\).

Proof. (1) is clear by the proof of (3.13.1), since \(t \in C(L_1/F_1)\). We will show that \(F_1 = J_{r(S_0)}(S_0)\) and then (2) follows.

Let \(x \in I(S_0 - R)\). Then by (1) \(x\) acts on \(L_1/F_1\) nontrivially. Hence \(m(S_0/R) \leq 2\). This implies \(I_r(S_0) = J_r(R_0)\). Suppose \(m(C_a(A_1A_2)/A_2) = 1\). Then \(C_{N(F_1)N_2/F_1}(F_1N_2/F_1)\) has a cyclic Sylow 2-subgroup and \(F_1N_2/F_1\) is a standard subgroup of \(N(F_1)/F_1\). Hence we have \(L_1 = F_1 < N_2 < N(F_1)/F_1\) as in (3.13). Since \(S_0\) normalizes \(G_0\). Then we have \(S_0 < \text{Aut}(G_0)\) and \(R_0 < N(G_0)\). Thus (2) holds by (2.3.2, 4 and 5). Consequently we may assume that \(m(C_a(A_1A_2)/A_2) = 2\), so \(I(T - P) = I(tP) \cup I(uP) \cup I(tuP)\), where \(u\) is defined in Section 2. We also may assume without loss of generality that \(u\) acts on both components of \(L_1/F_1\) as a field automorphism. Then \(I(uP) \subseteq I(R_0)\) and \(I(tP) \cup I(tuP) \subseteq I(S_0 - R_0)\). Since \(F_2 = J_r(R \text{ mod } E)\) and since \(u\) normalizes \(I_2\), it follows that \(u \in N(R_0)\). Hence \(u \in N(G_0)\). Then as above we have \(J_r(\langle u \rangle R) = F_1\).

Therefore, if \(I(R_0 - R) = I(uR)\), then by the result of the previous paragraph there is nothing more to say. So we may assume that \(m(R_0/R) = 2\) and then we can choose an element \(x \in R_0 - \langle u \rangle R\) such that \(x \in F_1\), \(x\) acts on one of the components of \(L_1/F_1\) trivially and on the other as a field automorphism. Suppose that \(x\) acts on \(K_1/F_1\) trivially. Let \(z \in Z(W)\) and let \(\Omega = zK_1\). Then by (2.2.1) and (5.4) \(|\Omega| = (q - 1)(q^5 + 1)\). Since \(\langle x \rangle K_1^t \simeq Z_{\times SL_2(q^5)}\), \(x\) is semiregular. Hence \(x^2 = 1\), since \(|\Omega| = \text{odd}\). So we have \(x \in C(W), \langle \Omega \rangle = W_1\). Then by (2.7) \(xy \in C(K_1)\) for some \(y \in W_1\). Set \(s = (xy)^t\). Then

\[K_1^t \langle s \rangle \cap K_1^t \langle s' \rangle \leq C(L_1)\]

By (1) \(|C(L_1)| = \text{odd}\). Hence we have

\[K_1^t \langle s \rangle \cap K_1^t \langle s' \rangle = 1\]
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So

\[ L_1\langle s, s'\rangle = K_1'\langle s\rangle \times K_1'\langle s'\rangle, \]
\[ \varpi_1(R_0 \mod R) = R\langle s, s'\rangle = W\langle s\rangle \times W\langle s'\rangle \]

and

\[ K_1'\langle s\rangle \simeq C(t) \cap L_1\langle s, s'\rangle = A_1N_1(u). \]

Here we can choose \( s \) to be an involution. Moreover we may assume that

\[ u = ss'. \]

Then for any involution \( x \) of \( sR \) we have \( m(C_{K_1'}(x)) = 3n \) by (2.3.4, 5) and so \( m(C_{K_1'}(x)) \leq 7n + 1 \). Therefore \( J_1(R_0) = J_1(R) = F_1 \). Thus the proof is complete.

(5.6) (1) The fusion of \( F_1 \) in \( G \) is controlled by \( \langle t \rangle J_1L_1 \).

(2) \( L_1 \triangleleft N(F_1) \).

**Proof.** Set \( \varpi = F_1' \). Then by (2.2.1) and (5.4) \( L_1^2 \) has two orbits of length \((q-1)(q^2+1)\) and \(2(q-1)\) orbits of length \(q(q^2+1)\) in \( (W_1 \cup W_i)^t \), and one orbit of length \((q-1)^2(q+1)^2\), \(2(q-1)\) orbits of length \((q-1)q(q^2+1)^2\) and \((q-1)^2\) orbits of length \(q^2(q^2+1)^2\) in \( F_1 - (W_1 \cup W_i) \). Also we have that \( \langle t \rangle J_1L_1 \) permutes transitively the orbits of \( L_1^2 \) of the same length, that is, \( \langle t \rangle J_1L_1 \) has five orbits which are of length \(2(q-1)(q^2+1), 2(q-1)q(q^2+1)\) in \( (W_1 \cup W_i)^t \) and \((q-1)^2(q^2+1)^2, 2(q-1)^2q(q^2+1), (q-1)^2q^2(q^2+1)^2\) in \( F_1 - (W_1 \cup W_i) \). Since \( F_1 \) is weakly closed in \( S_n \), the fusion of the elements of \( F_1 \) is controlled by \( N(F_1) \), which permutes the orbits of \( L_1^2 \) of the same length. Hence the fusion is already controlled by \( \langle t \rangle J_1L_1 \).

Then \( W_1 \cup W_i \) is weakly closed in \( F_1 \) with respect to \( G \), since \( F_1^2 \cap W_1^2 = I(W_1 \cup W_i) \) and \( \langle I(W_1) \rangle = W_1 \). Therefore, if we set \( X = N(F_1) \cap N(W_1) \), then \( |N(F_1) : X| = 2 \) and \( R_0 \in \text{Syl}_2(X) \). Let \( Y = \varpi(R_0 \mod R) \). In order to prove \( L_1 \triangleleft N(F_1) \) we may assume that \( m(C_{s}(A_1N_1/A_1)|A_1) = 2 \) as in the proof of (5.5).

Hence we have

\[ Y = \langle u \rangle R \text{ if } m(R_0|R) = 1, \text{ or} \]
\[ Y = \langle s, s' \rangle R = W\langle s\rangle \times W\langle s'\rangle \text{ if } m(R_0|R) = 2, \]

where \( s \) is chosen as in (5.5). In any case \([R|F_1, s] = [R|F_1, u] = 1 \) by (2.5.3). So we have

\[ Y = J_1(R_0 \mod F_1) = \varpi(R_0 \mod F_1) \]

and also \( Y/F_1 \) is abelian. Therefore the fusion of \( I(R_0|F_1) \) in \( X \) is controlled by \( N_1(Y) \). Since \( Y > F_1 = J_1(S_0) \), we have
which implies that $N_x(R)$ controls the fusion of $I(R_o/F_i)$ in $X$. Consequently any involution of $R_o - R$ does not fuse into $R$ in $X$. Suppose that $u^x \leq uR$. Since $R_o/R$ is abelian, we have $u \in O^i(X)$ by (1.5). Suppose $R \in \text{Syl}_2(O^i(X))$. Then $R \langle s \rangle$ or $R \langle s' \rangle < O^i(X)$, since $R < L_z \leq O^i(X)$. This yields that $u \in R \langle s, s' \rangle \leq O^i(X)$, since $O^i(X)$ is $t$-invariant, a contradiction. Hence $R \in \text{Syl}_1(O^i(X))$. Then $F_iN_i/F_i$ is a standard subgroup of $O^i(X) \langle t \rangle / F_i$ and $\langle t \rangle F_i/F_i$ is a Sylow 2-subgroup of $C(F_iN_i/F_i) \cap O^i(X) \langle t \rangle / F_i$. Hence as in (4.9) we have $L_z \langle N(F_i) \rangle$.

(5.7) (1) $N(F_i) \leq N(G_o)$, $i = 1, 2$.

(2) $N(G_o)$ controls the $G$-fusion of involutions of $G_o$.

(3) If $t \in N(G_o)$, then $g \in N(G_o)$.

(4) $|C(G_o)| = \text{odd}$.

Proof. (4) holds obviously by (5.5.1). Since $L_z \langle N(F_i) \rangle$ for $i = 1$ and 2, the Frattini argument yields that $N(F_i) = L_z(N(R) \cap N(F_i))$. By (5.4) $F_i = J_{R}(R \text{mod} E)$. Hence $F_i \text{char} R$, $i = 1, 2$. Therefore $N(F_i) = L_z(N(F_{i-1}) \cap N(F_i))$, which normalizes $G_o$. Thus (1) holds. By (2.4) and (5.4) any involution of $G_o$ fuses into $F_i$. Hence (2) follows from (1) and (5.6.1).

By (1) $S_0 < \text{Aut}(G_o)$. If an involution $x$ of $G_o$ does not change $G_i$ and $G_i$, then by (2.3.4) $m(C(x)) \geq 6n + 1$. Since $m(C) = 4n + 1$, $t$ interchanges $G_i^t$ and $G_i^t$, whenever $t \in N(G_o)$. Hence $C(t) \cap G_i^t = \{xx^t \mid x \in G_i^t\}$ and $L \simeq C(t) \cap C_i \leq C^{(\infty)} = L$. Therefore we have $C(t) \cap G_i^t = L$. Let $R' \langle t \rangle$ be a Sylow 2-subgroup of $G_o \langle t \rangle$ containing $Q$. Then $N_{R' \langle t \rangle}(Q) > Q$. Hence by Lemma 2.5.3 of [2] we have $\langle N_{R' \langle t \rangle}(Q), L \rangle = G_i^t$. On the other hand by (3.2.2) and (3.3) $N(Q) = EN_c(t) \leq N(R)$.

(5.8) There exists an abelian subgroup $R_1$ of $N(G_o) \cap N(G_i)$ such that $R \cdot R_1 \in \text{Syl}_2(N(G_o) \cap N(G_i))$, $t$ normalizes $R_1$, and $C_{R_1}(t)$ acts on $G_i$ and $G_i$ as a group of field automorphisms.

Proof. By (5.7.1) $S_0 \leq N(G_o)$ and $R_0 \in \text{Syl}_2(N(G_o) \cap N(G_i))$. Set $N(G_o) \cap C(G_i) = \tilde{G}_i$ and set $N(G_o) \cap N(G_i) = N(G_o) \cap N(G_i)/\tilde{G}_i$. Then $N(G_o) \cap N(G_i)$ is embedded in $\text{Aut}(U_i(q))$. Hence there exists an element $\tilde{x}$ which is a generator of the group of field automorphisms in $N(G_o) \cap N(G_i)$ and which normalizes $\tilde{R}$. Let $x$ be a preimage of $\tilde{x}$ and set $X = \tilde{G}_i \langle x \rangle \cap \tilde{G}_i \langle x' \rangle$. Notice that $G_i \leq \tilde{G}_i$ and $G_i \tilde{G}_i \langle x \rangle = N(G_o) \cap N(G_i)$.

Hence we have

$$G_o X = G_i \langle G_i \tilde{G}_i \langle x \rangle \cap \tilde{G}_i \langle x' \rangle \rangle = G_i \tilde{G}_i \langle x \rangle \cap G_i \tilde{G}_i \langle x' \rangle = N(G_o) \cap N(G_i)$$
and
\[ G_i \cap X = G_i \cap \tilde{G}_i \langle x \rangle \cap \tilde{G}_i \langle x' \rangle = G_i \cap \tilde{G}_i \langle x' \rangle = 1. \]

Therefore \( N(G_i) \cap N(G_i) = G_i \cdot X \) and by the choice of \( x \) \( X \cap C \) acts on \( G_i \) and \( G_i \) as a group of field automorphisms. Since

\[ R\tilde{G}_i \cap R\tilde{G}_i = R(W\tilde{G}_i \cap \tilde{G}_i) = R(\tilde{G}_i \cap \tilde{G}_i) = R \times C(G_i), \]

\( X \) normalizes \( R \). Thus if we choose a \( t \)-invariant Sylow \( 2 \)-subgroup \( R_i \) of \( X \), then we have \( R \cdot R_i \in \text{Syl}_2(N(G_0) \cap N(G_1)) \). By (5.7.4) \( R_i \cap R \) is abelian. Therefore \( R_i \) is abelian.

(5.9) Let \( \psi \) be the transfer homomorphism of \( G \) into \( S_\alpha \cap R \) relative to the subgroup \( S_\alpha \). Then

1. \( t, tu \in \text{Ker} \ \psi \) and
2. \( u^g \cap S_\alpha \subseteq uR \).

**Proof.** By the structure of \( G_\alpha \),

\[ m(C_{N(G_\alpha)}(x)) \geq 6n + 1 \quad \text{for } x \in I(R). \]

Hence \( t^g \cap R_\alpha = \phi \). Suppose \((tu)^g \in R_\alpha \) for some \( g \in G \). Since

\[ m(C_{N(G_\alpha)}(tu)) = 4n + 1, \]

we can choose \( C_{S_\alpha}((tu)^g) \) to be a Sylow \( 2 \)-subgroup of \( C((tu)^g) \). Furthermore we may assume that \( C_{S_\alpha}((tu)^g) \leq C_{S_\alpha}((tu)^g) \). Then \( t^g, t \in S_\alpha \). So by (5.7.3) \( g \in N(G_\alpha) \) and \((tu)^g \in R_\alpha \), which is a contradiction. Thus (1) holds.

By (5.8) \( \Omega_1(R_\alpha) = \langle s \rangle R \) or \( \langle s, s' \rangle R \) according as \( m(R_\alpha / R) = 1 \) or 2. By (2.8) we have

\[ m(C_{S_\alpha}(u)) \leq 6n + 2 \quad \text{and} \]

\[ m(C_{N(G_\alpha)}(x)) \geq 7n + 1 \quad \text{for } x \in I(\langle s \rangle R) \cup I(\langle s' \rangle R). \]

Therefore if \( u^g \in \langle s \rangle R \cup \langle s' \rangle R \) for some \( g \in G \), then we may take \( C_{S_\alpha}(u^g) \leq C_{S_\alpha}(u^g), t^g, t \in N(G_\alpha) \) and we have \( g \in N(G_\alpha) \) as above. This does not occur. Thus \( u^g \cap R_\alpha \subseteq uR \).

Now in order to prove (2) we will show \( u^g \cap (S_\alpha - R_\alpha) = \phi \). Take involutions \( w_i \) and \( w_i \) of \( W \) such that \( w_i w_i = z_i \) and \( w_i w_i = z_i \). Then by (5.7.2) the conjugacy classes of involutions of \( G_i \) in \( G \) are represented by \( w_i, w_i, z_i, z_i \) and \( w_i w_i \). By (2.8) \( m(C_{W}(x)) = m(C_{W}(x)) = 3n \) for any involution \( x \) of \( uR \). Hence we have

\[ m(C_{S_\alpha}(x)) \geq 6n + 1, \]

while
where $B$ is an elementary abelian subgroup of $C_{s_{o}}(x)$ such that

$$B' \subseteq z_{o}^{i} \cup z_{o}^{j} \cup (w_{i}w_{j})^{o}.$$ 

On the other hand $m(C_{s_{o}}(y)) = 4n$ for any involution $y$ of $S_{o} - R_{o}$, since $R = W \times W' = W \times W'$. Hence we have

$$m(C_{s_{o}}(y)) \leq 4n + 2, \text{ and } m(C(y) \cap W_{1}W_{i}) = 4n.$$ 

Note that

$$(C(y) \cap W_{1}W_{i})^{t} \subseteq z_{o}^{i} \cup z_{o}^{j} \cup (w_{i}w_{j})^{o}.$$ 

These facts imply that any involution of $S_{o} - R_{o}$ does not fuse into $uR$ if $n > 2$. Hence by the result of the second paragraph we have $u^{o} \cap S_{o} \subseteq uR$, if $q > 4$.

Suppose that $q = 4$ and $u$ fuses into $S_{o} - R_{o}$. Choose a conjugate $u'$ of $u$ such that $C_{s_{o}}(u') \in Syl_{2}(C(u'))$ and choose $u''$ such that $u'' \in u^{o} \cap (S_{o} - R_{o})$. We may assume that $u'' = u'$ and $C_{s_{o}}(u'')^{g} \subseteq C_{s_{o}}(u')$ for some $g \in G$. Set $B'' = Jr(C_{s_{o}}(u''))$ and set $B' = B''^{o}$. The result of the second paragraph gives that $u'' \in uR$. Then by (2.6) $C_{w}(u'') \subseteq C_{w}(u')$ is isomorphic to a Sylow 2-subgroup of $Sp_{4}(q)$. So, if we set $W_{i} = C_{w}(u')$ for $i = 1, 2$ and take $t' \in t' \cap C_{s_{o}}(u')$, then

$$\sigma(C_{s_{o}}(u')) = \{W_{3}W_{i}^{r}, W_{4}W_{i}^{r}, W_{5}W_{i}^{r}, W_{6}W_{i}^{r}\}$$

and

$$m(W_{i}) = m(W_{o}) = 3q.$$ 

Suppose that $B' \leq R_{o}$. Then $m(C_{s_{o}}(u', b)) = 4n$ whenever $b \in B' - R_{o}$. But $B' \subseteq B'' \leq C_{s_{o}}(u', B'')$ which gives that $m(C_{s_{o}}(u', B'')) = 4n + 1$. This contradicts the choice of $u'$. Hence $B' \leq R_{o}$. Since $O_{1}(R_{o}) = \langle s \rangle W \times \langle s' \rangle W'$, there exists a unique element $s'$ in $sW$ such that $s'u' \in s'W'$. Then for some $x, y \in C_{s}(u')$ we have

$$B' = (B' \cap R)\langle s'x \rangle \langle s'u'y \rangle.$$ 

Now by focal subgroup theorem, $m(R_{o} \cap O^{i}(G)/R) = 2$. Then it is clear that $m(C_{s_{o}}(u'')) \cap O^{i}(G)/C_{s_{o}}(u'')) = 1$. Therefore

$$|C_{s_{o}}(u'') \cap O^{i}(G)/C_{s_{o}}(u'', B') \cap O^{i}(G)| \geq 2q^{2}.$$ 

Next we calculate $|C_{s_{o}}(u') \cap N(B')/C_{s_{o}}(u', B')|$. Suppose that $B' < W_{i}W_{j}^{r}$ for any $i, j \in \{3, 4\}$. Then $C_{s}(u', B' \cap R) = W_{i}W_{j}^{r}$ in which $x$ and $y$ lie. Hence we have $C_{s}(u', B') = W_{i}W_{j}^{r}$ and so

$$|N_{s}(R \cap B') \cap C_{s}(u')/C_{s}(u', B')| = q.$$
Hence

$$|N_{S_0}(B') \cap C_{S_0}(u') \cap O^i(G)/C_{S_0}(u', B') \cap O^i(G)| \leq 4q \leq 2q^2.$$  

Again this contradicts the choice of $u'$. Thus $u$ does not fuse into $S_0 - R_0$, even if $q = 4$.

(5.10) \( R \in \text{Syl}_i(O^i(G)). \)

**Proof.** We may assume that $R_0 = R \cdot R_1$ by (5.8). We will show that any element $x$ of $R_1 \langle t \rangle - R$ does not fuse into $R_0$, and that any element $y$ of $R_1$ such that $O_i(\langle y \rangle) < C$ does not fuse into $S_0 - R_0$. These facts imply that $R_0 \in \text{Syl}_i(Ker \psi)$.

Suppose that $|x| > 2$. Then $x = xl$ for some $l \in R_1$ and $x^2 = x, x^4 \in C$. Hence $I(\langle x \rangle)$ is conjugate to $u$. Therefore for any $g \in G$ such that $x^g \in S_0$ we have $|x^g R/R| = |x|$ by (5.9.2). Hence if $x^g \in R_0$, then by (2.8)

$$C_R(x^g) = C_W(x^g) \times C_W(x)$$
and

$$m(C_W(x^g)) = m(C_W(x)) = 3m, \quad \text{where } m = 2n/|x|.$$  

Next if $x^h \in S_0 - R_0$ for some $h$, then

$$C_R(x^h) = \{ww^g | w \in C_W(x^{2g})\}.$$  

Now for a subgroup $X$ of $G$ we define the following:

$$m_1(X) = \text{Max} \{m(B) | B \leq X, B^g \leq z^g \cup z^g \cup (w, w)^g \};$$
and

$$m_2(X) = \text{Max} \{m(B') | B' \leq X, B'^g \leq w^g \cup w^g \}.$$  

Then the above result gives that

$$m_1(C_R(x^g)) = 3m, \quad m_1(C_R(x)) = 3m,$$

$$m_1(C_R(x^h)) = 6m, \quad m_2(C_R(x^h)) = 0.$$  

Therefore we have

$$m_1(C_{S_0}(x^g)) \leq 3m + 2, \quad m_1(C_{S_0}(x)) \geq 3m,$$

$$m_1(C_{S_0}(x^h)) \geq 6m, \quad m_1(C_{S_0}(x^h)) \leq 2.$$  

Consequently for any $x^g \in R_0$ and any $x^h \in S_0 - R_0$,

$$m_1(C_{S_0}(x^g)) < m_1(C_{S_0}(x^h))$$
and

$$m_2(C_{S_0}(x^g)) > m_2(C_{S_0}(x^h)),$$

which is absurd, since $C_{S_0}(x') \in \text{Syl}_i(C(x'))$ for some conjugate $x'$ of $x$ in $S_0$. Thus $x$ does not fuse into $R_0$. Similarly we can obtain that $y$ does not fuse.
into $S_5 - R_n$. Therefore $x \in \text{Ker } \psi$ for any $x$ of order at least 4. Let $R' = R_1 \langle t \rangle \cap \text{Ker } \psi$. Suppose $R' \neq R$. Then any element of $R' - R$ is an involution. This yields that $R' \cap R$ is cyclic and so $R'$ is a dihedral group by the structure of $\text{Aut } G_5$. Since $t \notin R'$, $R_1 \langle t \rangle$ is also dihedral and so $R'$ is cyclic. Then $\Omega_i(\langle y \rangle) < C$ for any $y$ of $R_i$. Since such elements do not fuse into $S_5 - R_n$, we have $R_i < \text{Ker } \psi$, a contradiction. Therefore $R' = R_i$ and $R_0 = S_5 \cap \text{Ker } \psi$. Then, since $R_i/R$ is abelian, $u \in O^i(G)$ by (5.9.2), and we can obtain the assertion as in the proof of (5.9.2).

For the argument in (5.11, 12) below we refer the readers to 4.23, 24 of [21].

(5.11) (1) $W_i$ and $W^i_i$ are weakly closed in $R$ with respect to $O^i(G)$.

(2) $W \in \text{Syl}_4(O^i(C_0(W_i)))$.

**Proof.** By (5.7.2), if $W_i \subset R$, then $W \subset W \cup W'$. If $W_i \subset W$, then $W = W_i$ by (2.3.2). If $W_i \subset W'$, then $W = W_i$. So, if $W_i = W$, then $N(R) \cap O^i(G)$ permutes $W_i$ and $W_i$ by a theorem of Burnside. But this can not occur, since $R \in \text{Syl}_4(O^i(G))$. Thus (1) holds. It is clear that $W \times W \in \text{Syl}_4(C(W_i) \cap O^i(G))$. Hence by Gaschütz's theorem we have $W \in \text{Syl}_4(O^i(C_0(W_i)))$.

(5.12) $G_5 \leq G$.

**Proof.** First we will show that $W$ and $W'$ are strongly closed in $R$ with respect to $O^i(G)$. Let $x \in W$ and suppose that $x^i \in R - W$ for some $g \in O^i(G)$. If $x$ is an involution, then by (5.6.1) we may take $x \in W_i$ and $x^i \in W^i_i$. Since $F_i = \text{Jr } (R)$, $x$ and $x^i$ are conjugate in $N(F_i) \cap O^i(G)$. But this contradicts (5.11.1).

Next we assume that $|x| = 2^i, i \geq 2$. Set $x^i = x_1 x_2$, where $x_1 \in W$ and $x_2 \in W'$. If $|x_1| \leq |x_2|$, then $x^{2i-1}$ which is an involution in $W$ is conjugate to $(x^i)^{2i-1}$ which is an involution in $R - W$. This contradicts the first paragraph. Hence $|x| > |x_2|$, and replacing $x$ by an appropriate power, we may assume $|x_2| = 2$. Furthermore we may assume that $x_1 \in W$ and $x_2 \in W_1$ by conjugation in $G_5$. Then $x, x^i \in C(W_i)$. Hence by (5.11.2) we have $x \in O^i(C(W_i))$ and $x^i \in O^i(C(W_i))$. Then (5.11.1) gives a contradiction. Thus $W$ is strongly closed in $R$ with respect to $O^i(G)$. By symmetry $W'$ is also strongly closed.

Now we are in a position to quote a result of Goldschmidt [11] in order to obtain the final result of this section. Set $X = \langle W_{0i}^i \rangle$. Then by [11] we have $[X, X] = 1$, since we are assuming $O(G) = 1$. Then $\{xx^i|x \in X\}$ is a homomorphic image of $X$ contained in $C$, and the kernel is contained in $X \cap X^i \leq Z(X)$. Therefore $X^{(n)} \simeq U_i(g)$. Since $\langle W_{0i}^i \rangle = G_i$, we have $G_5 = XX'$. Thus $G_5 \leq G$. 


§ 6. Proof of the theorem

We have assumed in Section 4 and 5 that $O(G)=1$. Now we will exclude this condition. Let $X$ denote the normal closure of $L$ in $G$ and let $\bar{G}=G/O(G)$. Then $\bar{L}$ is a standard subgroup of $\bar{G}$ whose centralizer has a cyclic Sylow 2-subgroup. So we have either $\bar{L} \leq \bar{G}$ or $X \cong L_4(q^2)$ or $U_4(q) \times U_4(q)$ by the results of Section 4 and 5. If one of the latter two cases holds, then in each case we can easily find a $t$-invariant 2-subgroup $A$ of $X$ such that $1 \neq [A, t] \leq L$. Then by (1J) of [12] we have $[A, t, O(G)]=1$. Hence in particular $C_{r}(O(X)) \leq O(X)$. Thus $O(X)=Z(X)$ and the theorem is established.

References


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