Countability of the Douady Space of a Complex Space

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(Received February 20, 1978)

§1. Introduction

Let X be a complex space and $\mathcal{E}$ a coherent analytic sheaf on X . Then by Douady [1] there is a complex space $D_X(\mathcal{E})$ and a quotient analytic sheaf $\mathcal{R} = \mathcal{R}_X(\mathcal{E})$ of $p_i^*(\mathcal{E})$ on $D_X(\mathcal{E}) \times X$ ($p_i: D_X(\mathcal{E}) \times X \rightarrow X$ being the natural projection) which is flat and with proper support over $D_X(\mathcal{E})$, and with the following universal property; for any complex space T and any quotient analytic sheaf $\mathcal{F}$ of $p_T^*\mathcal{E}$ on $T \times X$, flat over T and with proper support over T , there is a unique morphism $h: T \rightarrow D_X(\mathcal{E})$ such that $(h \times \text{id}_X)^*\mathcal{R} \cong \mathcal{F}$, where $p_T: T \times X \rightarrow X$ is the natural projection. We call $D_X(\mathcal{E})$ the Douady space of quotient sheaves of $\mathcal{E}$ with compact supports on X . Then the purpose of this note is to prove the following:

THEOREM. *If X has a countable base of the open sets, then the same is true for $D_X(\mathcal{E})$.*

Our arrangement of this note is as follows. In §2 after summarizing some of the facts concerning the topology of $D_X(\mathcal{E})$ as described in [1], we give our plan of proof in (2.9). The next §3 is the main part of this note, where we show the separability of the sets $\mathcal{P}'(p, K)$ which will be defined in (3.5). From this we derive Theorem rather easily in §4.

§2. Topology of $D_X(\mathcal{E})$

(2.1) We begin with local situations. Let $C^n = C^n(z_1, \dots, z_n)$, $n > 0$, be the affine n -space. A compact subset $K \subseteq C^n$ is called a *polycylinder* if it is of the form $K = K_1 \times \dots \times K_n$, where $K_i \subseteq C(z_i)$ is a compact convex set with nonempty interior $\hat{K}_i$. When $n = 0$, we define $K = \{0\}$ to be a polycylinder in $C^0 = \{0\}$. If K is a polycylinder in C^n as above, then we denote by $B(K)$ the Banach space of all continuous functions on K which are holomorphic in the interior $\hat{K}$ of K , endowed with the sup norm. Further we use the following notations. We denote by $\mathcal{C}_{oh}(K)$ the set of (equivalence classes of) co-

*) Supported by the Sakkokai Foundation.

herent analytic sheaves defined in some neighbourhoods of K . For $\mathcal{E} \in \mathcal{C}_{oh}(K)$, $\mathcal{C}_{oh}(\mathcal{E}, K)$ is the set of quotient analytic sheaves $\mathcal{F}$ of $\mathcal{E}$ with fixed surjective homomorphisms $\beta: \mathcal{E} \rightarrow \mathcal{F}$. If $\mathcal{E} = \mathcal{O}^p$, the direct sum of p copies of the structure sheaf $\mathcal{O} = \mathcal{O}_{C^n}$ of C^n , then we write $\mathcal{C}_{oh}(p, K)$ instead of $\mathcal{C}_{oh}(\mathcal{O}^p, K)$. For any $\mathcal{F} \in \mathcal{C}_{oh}(K)$ we write $\mathcal{F}(K) = \Gamma(K, \mathcal{F})$. In particular for $\mathcal{F} = \mathcal{O}$ we have $\mathcal{O}(K) = \Gamma(K, \mathcal{O})$. We set $B(K, \mathcal{F}) = B(K) \otimes_{\mathcal{O}(K)} \mathcal{F}(K)$. Then $B(K, \mathcal{O}^p) = B(K)^p$.

(2.2) Let K be a polycylinder and $\mathcal{F} \in \mathcal{C}_{oh}(K)$. Then K is called *privileged* for $\mathcal{F}$ or $\mathcal{F}$ -*privileged*, if there is a free resolution

$$(1) \quad 0 \rightarrow \mathcal{O}^{p_s} \rightarrow \mathcal{O}^{p_{s-1}} \rightarrow \dots \rightarrow \mathcal{O}^{p_0} \rightarrow \mathcal{F} \rightarrow 0$$

of $\mathcal{F}$ on K such that the associated sequence

$$(2) \quad 0 \longrightarrow B(K)^{p_s} \xrightarrow{d_s} B(K)^{p_{s-1}} \longrightarrow \dots \xrightarrow{d_1} B(K)^{p_0}$$

is a direct exact sequence of Banach spaces [1, 7.1]. In general, noting that $\text{Coker}(d_i) \cong B(K, \mathcal{F})$ in (2) for any resolution (1) of $\mathcal{F}$, we put the quotient topology on $B(K, \mathcal{F})$ induced from $B(K)^{p_0}$, which is actually independent of the resolutions. Thus if K is $\mathcal{F}$ -privileged, then $B(K, \mathcal{F})$ has the natural structure of a Banach space. We shall recall some fundamental properties of privilegedness established in [1] in the following lemma.

LEMMA 1. *Let*

$$0 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{E} \rightarrow \mathcal{F} \rightarrow 0$$

be an exact sequence of coherent analytic sheaves on K . If K is privileged both for $\mathcal{F}_1$ (resp. $\mathcal{E}$) and $\mathcal{F}$, then it is privileged also for $\mathcal{E}$ (resp. $\mathcal{F}_1$). Moreover in this case the following sequence

$$0 \rightarrow B(K, \mathcal{F}_1) \rightarrow B(K, \mathcal{E}) \rightarrow B(K, \mathcal{F}) \rightarrow 0$$

is a direct exact sequence of Banach spaces.

Proof. See [1, 7.2, Prop. 3, 4].

(2.3) (cf. [1, 2.1]). Let B be a Banach space and $\hat{G}rB$ the set of all the closed subspaces of B . On $\hat{G}rB$ we have the natural metric d defined as follows; for $E, F \in \hat{G}rB$

$$d(E, F) = \sup(\theta(E, F), \theta(F, E)),$$

$$\theta(E, F) = \sup_{\|x\| \leq 1} \inf_{\substack{y \in F \\ \|y\| \leq 1}} \|x - y\|.$$

Let GrB be the subset of $\hat{Gr}B$ consisting of all the direct subspaces of B . Then GrB is open in $\hat{Gr}B$ and has the natural structure of a Banach manifold whose underlying topology is compatible with the above metric.

(2.4) Let K be a polycylinder in C^n and $\mathcal{E} \in \mathcal{C}oh(K)$. We assume that K is privileged for $\mathcal{E}$. Then we set

$$\mathcal{P}(\mathcal{E}, K) = \{\mathcal{F} \in \mathcal{C}oh(\mathcal{E}, K); K \text{ is } \mathcal{F}\text{-privileged}\}.$$

Now let $\mathcal{F} \in \mathcal{P}(\mathcal{E}, K)$, $\beta_1: \mathcal{E} \rightarrow \mathcal{F}$ the quotient map and $\mathcal{F}_1$ the kernel of β_1 . Then by Lemma 1 we have the unique element $b_{\mathcal{E}, K}(\mathcal{F})$ of $GrB(K, \mathcal{E})$ by setting

$$b_{\mathcal{E}, K}(\mathcal{F}) = B(K, \mathcal{F}_1)$$

the latter being regarded as a direct subspace of $B(K, \mathcal{E})$.

LEMMA 2. The map $b_{\mathcal{E}, K}: \mathcal{P}(\mathcal{E}, K) \rightarrow GrB(K, \mathcal{E})$ is injective.

Proof. First, by Theorem A on K we see readily that $B(K, \mathcal{F}_1)$ determines uniquely $\mathcal{F}_1|_{\hat{K}}$ for all $\mathcal{F} \in \mathcal{P}(K, \mathcal{E})$. So the problem is on the boundary. Let $\mathcal{F}, \mathcal{F}' \in \mathcal{P}(\mathcal{E}, K)$ with $B(K, \mathcal{F}_1) = B(K, \mathcal{F}'_1)$ as subspaces of $B(K, \mathcal{E})$. Take any point $a \in \hat{K}$ and let $m_t: C^n \rightarrow C^n$ be the homothety defined by $m_t(x) = (1-t)a + tx$, $t \in C^*$. Then by [1, 7.3 Cor.] there is a neighbourhood V of 1 in C^* such that if we put $K_t = m_t(K)$, then $\mathcal{E}, \mathcal{F}$ and $\mathcal{F}'$ are defined on K_t and K_t is privileged for all of them. Moreover we may assume that $\mathcal{B} = \bigcup_{t \in V} B(K_t, \mathcal{E})$ is a trivial vector bundle over V with typical fiber $B(K, \mathcal{E})$ (cf. [1, p. 65]). Fix an isomorphism $\lambda: \mathcal{B} \cong B(K, \mathcal{E}) \times V$ over V . Let $\lambda_t = \lambda|_{B(K_t, \mathcal{E})}: B(K_t, \mathcal{E}) \cong B(K, \mathcal{E})$ be the induced isomorphisms. Then we get the analytic map $\varphi: V \rightarrow GrB(K, \mathcal{E})$ by $\varphi(t) = \lambda_t(B(K_t, \mathcal{F}_1))$ where we consider $B(K_t, \mathcal{F}_1)$ naturally a closed subspace of $B(K_t, \mathcal{E})$ (cf. Lemma 1). Similarly we have the analytic map $\varphi': V \rightarrow GrB(K, \mathcal{E})$ starting from $\mathcal{F}'$. Then our assumption means that $\varphi(1) = \varphi'(1)$. Furthermore since $\mathcal{F}|_{\hat{K}} = \mathcal{F}'|_{\hat{K}}$ (as submodules of $\mathcal{E}|_{\hat{K}}$) by the remark at the beginning of the proof, for $t < 1$ ($t \in V$) we also have the equality $\varphi(t) = \varphi'(t)$. On the other hand, as follows from [1, p. 63, Prop. 7] the set $\{t \in V; \varphi(t) = \varphi'(t)\}$ defines a germ of analytic subset of V at 1. Thus $\varphi(t) = \varphi'(t)$ holds true in a neighbourhood of 1 in V . In particular for $t > 1$ sufficiently near to 1 we have $B(K_t, \mathcal{F}_1) = B(K_t, \mathcal{F}'_1)$. Then again by the same remark $\mathcal{F}_1|_{\hat{K}_t} = \mathcal{F}'_1|_{\hat{K}_t}$. Hence $\mathcal{F}_1|_K = \mathcal{F}'_1|_K$ since $K \subseteq \hat{K}_t$. Q.E.D.

Identifying $\mathcal{P}(\mathcal{E}, K)$ with its image by $b_{\mathcal{E}, K}$ by the above lemma, we put on $\mathcal{P}(\mathcal{E}, K)$ the induced topology from $GrB(K, \mathcal{E})$. When $\mathcal{E} = \mathcal{O}^p$ we write $\mathcal{P}(p, K)$ and $b_{p, K}$ instead of $\mathcal{P}(\mathcal{O}^p, K)$ and $b_{\mathcal{O}^p, K}$ respectively.

(2.5) Let K and E be as in (2.4). Suppose that there is a fixed surjective homomorphism $q: \mathcal{O}^p \rightarrow \mathcal{E}$. Then composed with q we have the natural inclusion $\mathcal{C}_{oh}(\mathcal{E}, K) \rightarrow \mathcal{C}_{oh}(p, K)$ which induces the inclusion $\alpha = \alpha_{\mathcal{E}, p, K}: \mathcal{P}(\mathcal{E}, K) \rightarrow \mathcal{P}(p, K)$. We shall show that α is homeomorphic onto a subspace of $\mathcal{P}(p, K)$ so that the topology of $\mathcal{P}(\mathcal{E}, K)$ defined above is equivalent to that induced from $\mathcal{P}(p, K)$ by α . In fact, we first observe the following commutative diagram of exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{F}_0 & \longrightarrow & \mathcal{O}^p & \longrightarrow & \mathcal{F} \longrightarrow 0 \\ & & \downarrow q_0 & & \downarrow q & & \parallel \\ 0 & \longrightarrow & \mathcal{F}_1 & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{F} \longrightarrow 0 \end{array}$$

where q_0 is induced by q and is surjective. This gives rise to the following direct exact sequence of Banach spaces (Lemma 1)

$$\begin{array}{ccccccc} 0 & \longrightarrow & B(K, \mathcal{F}_0) & \longrightarrow & B(K)^p & \longrightarrow & B(K, \mathcal{F}) \longrightarrow 0 \\ & & \downarrow B(q_0) & & \downarrow B(q) & & \parallel \\ 0 & \longrightarrow & B(K, \mathcal{F}_1) & \longrightarrow & B(K, \mathcal{E}) & \longrightarrow & B(K, \mathcal{F}) \longrightarrow 0. \end{array}$$

In general let $h: E \rightarrow F$ be a surjective and direct homomorphism of Banach spaces. Then it is easy to see that the map $\tilde{h}: GrF \rightarrow GrE$ defined by $\tilde{h}(F_1) = h^{-1}(F_1)$, $F_1 \in GrF$, is a homeomorphism of GrE onto the subset of GrE consisting of those direct subspaces of E which contain the kernel of h . Hence if we take $E = B(K)^p$, $F = B(K, \mathcal{E})$, and $h = B(q): B(K)^p \rightarrow B(K, \mathcal{E})$, we have $\tilde{h}b_{\mathcal{E}, K}(\mathcal{F}) = b_{p, K}\alpha(\mathcal{F}) = B(K, \mathcal{F}_0)$ by the above commutative diagram. Then from the definition of the topologies of $\mathcal{P}(\mathcal{E}, K)$ and $\mathcal{P}(p, K)$ our assertion follows.

(2.6) We turn to global situations. So let X be a complex space and $\mathcal{E}$ a coherent analytic sheaf on X . According to [1, 9.2] we define a *cuirasse* M on X by a data $M = ((\varphi_i), (K_i), (\varphi_{ij}), (K_{ija}), (V_i), L)$, $1 \leq i \leq s < \infty$, $1 \leq \alpha \leq t_{ij} < \infty$, where 1) φ_i (resp. φ_{ij}) are embeddings, $\varphi_i: X_i \rightarrow D_i$ (resp. $\varphi_{ij}: X_{ij} = X_i \cap X_j \rightarrow D_{ij}$), of open subsets X_i (resp. X_{ij}) of X into a domain D_i (resp. D_{ij}) of $\mathbb{C}^{n_i}$ (resp. $\mathbb{C}^{n_{ij}}$), 2) K_i (resp. K_{ija}) are polycylinders in D_i (resp. D_{ij}), 3) V_i are compact subsets of $\varphi_i^{-1}(\dot{K}_i)$ and 4) L is a closed subset of X , such that a) $V_i \cap V_j \subseteq \bigcup_{\alpha} \varphi_{ij}^{-1}(\dot{K}_{ija}) \subseteq \bigcup_{\alpha} \varphi_{ij}^{-1}(K_{ija}) \subseteq \varphi_i^{-1}(\dot{K}_i) \cap \varphi_j^{-1}(\dot{K}_j)$ and b) $L \cup \bigcup_i V_i = X$. Let $\mathcal{G}$ be any coherent analytic sheaf on X . Then M is called $\mathcal{G}$ -*semiprivileged* if K_i (resp. K_{ija}) are privileged for $\mathcal{G}_i$ (resp. $\mathcal{G}_{ij}$) for all i, j and α , where $\mathcal{G}_i = \varphi_{i*}(\mathcal{G}|_{X_i})$ (resp. $\mathcal{G}_{ij} = \varphi_{ij*}(\mathcal{G}|_{X_{ij}})$). M is called $\mathcal{G}$ -*privileged* if M is $\mathcal{G}$ -semiprivileged and $\text{Supp } \mathcal{G} \cap L = \emptyset$, where Supp denotes the support.

(2.7) Let X and $\mathcal{E}$ be as in (2.6). Let M be an $\mathcal{E}$ -semiprivileged cuirasse. Let $\mathcal{F}$ be a quotient analytic sheaf of $\mathcal{E}$ with compact support on X . Suppose

that M is $\mathcal{F}$ -privileged. Then $\mathcal{F}$ defines an element, $b_{s,M}(\mathcal{F})$, of $G_{s,M} = \prod_{i=1}^s \text{GrB}(K_i, \mathcal{E}_i)$ by $b_{s,M}(\mathcal{F}) = \prod_{i=1}^s b_{s_i, K_i}(\mathcal{F}_i)$ where $\mathcal{E}_i = \varphi_{i*}(\mathcal{E}|_{X_i})$, $\mathcal{F}_i = \varphi_{i*}(\mathcal{F}|_{X_i})$ and b_{s_i, K_i} are as in (2.4). Let $H_M = H_M(\mathcal{E})$ be the set of all quotient analytic sheaves of $\mathcal{E}$ with compact supports on X such that M is $\mathcal{F}$ -privileged. Then $b_{s,M}$ defines an injective map, $b_{s,M}: H_M \rightarrow G_{s,M}$. Identifying H_M with its image in $G_{s,M}$ we put the induced topology on H_M . Then by Douady's construction [1, § 9] we know that H_M with this topology is naturally identified with an open subset of $D_X(\mathcal{E})$ and that we have $D_X(\mathcal{E}) = \bigcup_M H_M$, where M run through all the $\mathcal{E}$ -semiprivileged cuirasses. Clearly $b_{s,M}(H_M)$ is contained in $\prod_i b_{s_i, K_i}(\mathcal{P}(\mathcal{E}_i, K_i))$ and we have the natural inclusions $H_M \subseteq \prod_{i=1}^s \mathcal{P}(\mathcal{E}_i, K_i)$ of topological spaces.

Now we fix for each i a surjective homomorphism $q_i: \mathcal{O}^{p_i} \rightarrow \mathcal{E}_i$ on K_i , where $\mathcal{O} = \mathcal{O}_{C^{n_i}}$. Then considering $\mathcal{F}_i$ a quotient sheaf of $\mathcal{O}^{p_i}$ via q_i as in (2.5) we get the injective map

$$b_M: H_M \rightarrow G_M = \prod_{i=1}^s \text{GrB}(K_i)^{p_i}, \quad b_M(\mathcal{F}) = \prod_{i=1}^s b_{p_i, K_i} \alpha_{\mathcal{E}_i, p_i, K_i}(\mathcal{F}_i).$$

As above the image of b_M is contained in $\prod_i b_{p_i, K_i}(\mathcal{P}(p_i, K_i))$ and we have the natural inclusion $H_M \subseteq \prod_i \mathcal{P}(p_i, K_i)$. Then by (2.5) we get that the topology of H_M defined above is equivalent to that induced from G_M by b_M . We define a metric $\tilde{d}_M$ on G_M by $\tilde{d}_M(x, y) = \sup d_i(\pi_i x, \pi_i y)$, $x, y \in G_M$, where d_i are the metrics on $\text{GrB}(K_i, \mathcal{O}^{p_i})$ defined in (2.3) and π_i are the projections to the i -th factors. Then $\tilde{d}_M$ induces a metric d_M on H_M via b_M , which is compatible with the given topology of H_M by the above remark. By definition if $\mathcal{F}, \mathcal{F}' \in H_M$, then we have

$$\tilde{d}_M(\mathcal{F}, \mathcal{F}') = \sup_i d_i(b_{p_i, K_i}(\mathcal{F}_i), b_{p_i, K_i}(\mathcal{F}'_i)).$$

(2.8) To show the countability of $D_X(\mathcal{E})$ we employ a reasoning of the following type.

LEMMA 3. Let Y be a topological space and $\{Y_\alpha\}_{\alpha \in \mathbb{N}}$ a countable open covering of Y . Suppose that for each α there exist 1) a subspace, Z_α , of Y_α , not necessarily open, such that $\{Z_\alpha\}$ still is a covering of Y , 2) a metric d_α on Y_α , and 3) metric spaces $\tilde{Y}_\alpha, \tilde{Z}_\alpha$ with the following commutative diagram of isometric inclusions

$$\begin{array}{ccc} Y_\alpha & \subseteq & \tilde{Y}_\alpha \\ \cup & & \cup \\ Z_\alpha & \subseteq & \tilde{Z}_\alpha \end{array}$$

such that $\tilde{Z}_\alpha$ is separable, i.e., has a dense countable subset, where Z_α has the induced metric from Y_α . Then Y has a countable base of the open sets.

Proof. Fix for each α a countable dense subset N_α of $\tilde{Z}_\alpha$. Let $\tilde{d}_\alpha$ be the metric of $\tilde{Y}_\alpha$. For any point $\tilde{y} \in \tilde{Y}_\alpha$ let $\tilde{U}_\alpha(\tilde{y}, r) = \{\tilde{y}' \in \tilde{Y}_\alpha; \tilde{d}_\alpha(\tilde{y}, \tilde{y}') < r\}$, $r > 0$. Define a countable collection, $\mathcal{C}$, of open subsets of Y by $\mathcal{C} = \{U_\alpha(\tilde{y}, 1/n) = \tilde{U}_\alpha(y, 1/n) \cap Y_\alpha; \tilde{y} \in N_\alpha, \alpha \in \mathfrak{A}, n = 1, 2, \dots\}$. We show that $\mathcal{C}$ is a base of the open sets of Y . Let U be any open subset of Y and $y \in U$ any point. It is enough to show that there exists some $U_\alpha(\tilde{y}, 1/n)$ in $\mathcal{C}$ such that $y \in U_\alpha(\tilde{y}, 1/n) \subseteq U$. First by 1) $y \in Z_\alpha$ for some α . Since Y_α is open, we may assume that $U \subseteq Y_\alpha$. Take an open subset $\tilde{U}$ of $\tilde{Y}_\alpha$ with $\tilde{U} \cap Y_\alpha = U$. Take any sufficiently large n such that $\tilde{U}_\alpha(y, 2/n) \subseteq \tilde{U}$. Since N_α is dense in $\tilde{Z}_\alpha$, there exists a point $\tilde{y} \in N_\alpha \cap \tilde{U}_\alpha(y, 1/n)$. Then $U_\alpha(\tilde{y}, 1/n) \subseteq U_\alpha(y, 2/n) \subseteq U$. Since $U_\alpha(\tilde{y}, 1/n)$ is a member of $\mathcal{C}$, this completes the proof. Q.E.D.

(2.9) We shall use this lemma in proving Theorem in the following manner. We first define 1) a countable set, $\mathfrak{S}$, of $\mathcal{C}$ -semiprivileged cuirasses on X (§ 4) such that $D_X(\mathcal{C}) = \bigcup_{M \in \mathfrak{S}} H_M$, and 2) for each $M \in \mathfrak{S}$ a subset, H'_M , of H_M and subsets, $\mathcal{P}'(p_i, K_i)$, of $\mathcal{P}(p_i, K_i)$ such that $H'_M \subseteq \prod_{i=1}^s \mathcal{P}'(p_i, K_i)$ (cf. (3.5) and § 4). Moreover we show that $D_X(\mathcal{C}) = \bigcup_{M \in \mathfrak{S}} H'_M$ and that $\mathcal{P}'(p_i, K_i)$ are separable for all i . Then in the above lemma, setting $\mathfrak{A} = \mathfrak{S}$, $Y = D_X(\mathcal{C})$, $Y_\alpha = H'_M$, $Z_\alpha = H'_M$, $\tilde{Y}_\alpha = \prod_{i=1}^s \mathcal{P}(p_i, K_i)$ and $\tilde{Z}_\alpha = \prod_{i=1}^s \mathcal{P}'(p_i, K_i)$, and considering the natural metrics on these spaces, we conclude by the lemma that $D_X(\mathcal{C})$ has a countable base of the open sets.

§ 3. Definition and Separability of $\mathcal{P}'(p, K)$

After some preparations we define in (3.4) the notion of adaptedness of a polycylinder for a coherent analytic subsheaf of $\mathcal{O}^p$. Then using this notion we define the set $\mathcal{P}'(p, K)$, and state some of its properties in Proposition 1, the proof of which is the main purpose of this section.

(3.1) We go back to the local situation of (2.1). So let $K = K_1 \times \dots \times K_n$ be a polycylinder in $\mathbb{C}^n$. As in § 1 we write $\mathcal{O}$ for the structure sheaf $\mathcal{O}_{\mathbb{C}^n}$ of $\mathbb{C}^n$. Write $K^i = K_1 \times \dots \times K_i$, $1 \leq i \leq n$, and let $\pi_i: \mathbb{C}^n \rightarrow \mathbb{C}^i$ be the projection to the first i factors. An analytic subspace X of K (i.e. X is defined in some neighbourhood of K) is called *finite* over K^i if $|X| \cap (K^i \times \partial(K_{i+1} \times \dots \times K_n)) = \emptyset$, where $|X|$ and ∂ denote the underlying topological space and the boundary respectively. In this case there is a connected neighbourhood U of K^i in $\mathbb{C}^i$ such that the morphism $\pi_U: X \cap \pi^{-1}(U) \rightarrow U$ induced by π_i is a finite morphism. We call the rank of the coherent analytic sheaf $(\pi_U)_* \mathcal{O}$, on U the *degree* of X over K^i , which is independent of the choice of U . Let $f \in \mathcal{O}(K)$. Then f is called *finite of degree p* over K^{n-1} if the subspace of K defined by the ideal (f) is (nonempty and) finite of degree p over K^{n-1} . This implies of course that f does not vanish on $K^{n-1} \times \partial K_n$.

(3.2) By the same techniques as in the proof of Weierstrass preparation theorem and extended Weierstrass division theorem in [2, pp. 68, 80] we get the following:

LEMMA 4. Suppose that $f_0 \in \mathcal{O}(K)$ and f_0 is finite of degree p over K^{n-1} . Then: 1) f_0 can be written uniquely in the form $f_0 = uf$, where u is a nonvanishing function in $\mathcal{O}(K)$ and f is a monic polynomial in z_n of degree p with coefficients in $\mathcal{O}(K^{n-1})$. 2) Every $g \in \mathcal{O}(K)$ can be written uniquely in the form $g = hf + g'$, $h, g' \in \mathcal{O}(K)$, where g' is a polynomial in z_n of the form $g' = \sum_{i=1}^{p-1} a_i z_n^i$, $a_i \in \mathcal{O}(K^{n-1})$. Furthermore there is a constant $c > 0$, independent of g , such that $\|h\| \leq c \|g\|$, and $\|a_i\| \leq c \|g\|$, where $\|\cdot\|$ denotes the sup norm.

For later use we recall how to define h and g' in the above lemma (cf. [2, pp. 70, 81]);

$$\begin{cases} h(z_1, \dots, z_n) = 1/2\pi i \int_{\partial K_n} (g(z_1, \dots, z_{n-1}, \zeta)/f(z_1, \dots, z_{n-1}, \zeta)) d\zeta / (\zeta - z_n) \\ a_i(z_1, \dots, z_{n-1}) \\ \quad = 1/2\pi i \int_{\partial K_n} (f_j^*(z_1, \dots, z_{n-1}, \zeta)/f(z_1, \dots, z_{n-1}, \zeta)) g(z_1, \dots, z_{n-1}, \zeta) d\zeta \end{cases}$$

where $f_j^*(z_1, \dots, z_{n-1}, \zeta)$ are certain polynomials in ζ of degree $\leq p-1$ with coefficients being linear combinations over $\mathbb{C}$ of b_i , with b_i being defined by $f = z_n^p + \sum_{i=0}^{p-1} b_i(z_1, \dots, z_{n-1}) z_n^i$. Moreover if we put

$$r = 1/2\pi i \int_{\partial K_n} d\zeta, \quad c_j = \sup_{K^{n-1} \times \partial K_n} |f_j^*(z_1, \dots, z_{n-1}, \zeta)/f(z_1, \dots, z_{n-1}, \zeta)| \quad \text{and} \\ m = \inf_{\partial K_1 \times \dots \times \partial K_n} |f(z)|,$$

then we can take

$$c = \max \left(rc_1, \dots, rc_{n-1}, 1/m \left(1 + \sum_i r^{i+1} c_i \right) \right).$$

(3.3) We shall fix some notations which will be used in the rest of this section. For any $\mathcal{F} \in \mathcal{C}_{oh}(p, K)$ with the quotient map $\beta: \mathcal{O}^p \rightarrow \mathcal{F}$ we denote by $\mathcal{F}_0(\mathcal{F})$ the kernel of β . Let $\mathcal{O}_{n-1}[z_n]_p$ be the coherent analytic sheaf on $\mathbb{C}^{n-1}$ defined by the germs of polynomials in z_n of degree $< p$ with coefficients in $\mathcal{O}_{n-1} = \mathcal{O}_{\mathbb{C}^{n-1}}$. Then the map $\phi: \mathcal{O}_{n-1}[z_n]_p \rightarrow \mathcal{O}_{n-1}^p$ with $\phi(\sum_{i=0}^{p-1} a_i z_n^i) = (a_0, \dots, a_{p-1})$, gives a natural isomorphism of these $\mathcal{O}_{n-1}$ -modules. We consider $\mathcal{O}_{n-1}[z_n]_p$ also an $\mathcal{O}_{n-1}$ -submodule of $\mathcal{O}$ by the natural injection $\pi_{n-1}^*: \mathcal{O}_{n-1}[z_n]_p \rightarrow \mathcal{O}$. On the other hand, let $\mathcal{F}_0$ be a coherent analytic subsheaf of $\mathcal{O}^p$ on K . Consider the natural inclusions, $\mathcal{O}^i \subseteq \mathcal{O}^p$, of the first i factors. Then we put $\mathcal{F}_0^{(i)} = \mathcal{F}_0 \cap \mathcal{O}^i$, $i = 0, \dots, p$. This defines an increasing filtration $\{\mathcal{F}_0^{(i)}\}$ of $\mathcal{F}_0 = \mathcal{F}_0^{(p)}$.

Let $\mathcal{F}_i(\mathcal{F}_0) = \mathcal{F}_0^{(i)} / \mathcal{F}_0^{(i-1)}$ be the associated graded modules, which are naturally regarded as ideal sheaves of $\mathcal{O}$ (identified with the i -th factor of $\mathcal{O}^p$).

Let $\mathfrak{M} = \{(n, p); n \geq 0, p > 0, n, p \text{ integers}\}$. We put the lexicographical order on the set $\mathfrak{M}$. Hence $(n, p) > (n', p')$ if and only if either $n > n'$ or $n = n'$ and $p > p'$. In the following we often use the induction indexed by $\mathfrak{M}$, considering three typical cases separately: *Case 1.* $n=0$ (beginning of the induction). *Case 2.* $n > 0$ and $p=1$. We reduce this case to the case $(n-1, q)$ for some $q > 0$. *Case 3.* $n > 0, p > 1$. We reduce this case to the case $(n, 1)$ or $(n, p-1)$. With these notations we shall now give the following:

(3.4) Definition 1. Let K be a polycylinder in C^n and $\mathcal{F}_0$ a coherent analytic subsheaf of $\mathcal{O}^p, p > 0$, on K . Then we define the *adaptedness* of K for $\mathcal{F}_0$ as follows. We proceed by induction on (n, p) as explained above. *Case 1.* $K = \{0\}$ is adapted for every $\mathcal{F}_0$. *Case 2.* Note that in this case $\mathcal{F}_0$ is a coherent sheaf of ideals of $\mathcal{O}$. If $\mathcal{F}_0 = \{0\}$ (resp. $\mathcal{O}$) then by definition K is adapted for $\mathcal{F}_0$. Suppose that $\mathcal{F}_0 \neq \{0\}, \neq \mathcal{O}$. Then K is said to be adapted if there exists an element $f \in \mathcal{F}_0(K) \cap \mathcal{O}(K^{n-1})[z_n]_{p+1}$ for some $p > 0$ such that 1) f is monic of degree p as a polynomial in z_n and is finite over K^{n-1} of degree p and 2) K^{n-1} is adapted for the $\mathcal{O}_{n-1}$ -module ${}_{n-1}\mathcal{F}_0 = \phi(\mathcal{F}_0 \cap \mathcal{O}_{n-1}[z_n]_p) \subseteq \mathcal{O}_{n-1}^p$. We write ${}_{n-1}\mathcal{F}_0$ as ${}_{n-1}\mathcal{F}_0 = {}_{n-1}\mathcal{F}_0(f)$, when we want to note the dependence of ${}_{n-1}\mathcal{F}_0$ on (the degree of) f . We call f with the above property a *reducing function* for $\mathcal{F}_0$ on K . *Case 3.* K is adapted for $\mathcal{F}_0$ if K is adapted for all $\mathcal{F}_i(\mathcal{F}_0)$ in the notation above. Note that in this case K is adapted for all the $\mathcal{F}_0^{(i)}$ as well, as follows immediately from the definition.

(3.5) For any subset $G = \{F_1, \dots, F_q\}$ of $\mathcal{O}(K)^p$ define $\mathcal{F}_0(G)$ to be the submodule of $\mathcal{O}^p$ generated by $F_1, \dots, F_q$. Note that each F_i can be written uniquely in the form $F_i = (f_{i1}, \dots, f_{ip})$ with $f_{ij} \in \mathcal{O}(K)$. Then we set

$$\mathcal{P}'(p, K) = \{\mathcal{F} \in \mathcal{C}_{oh}(p, K); K \text{ is adapted for } \mathcal{F}_0(\mathcal{F})\}$$

$$\mathcal{Q}(p, K) = \{\mathcal{F} \in \mathcal{C}_{oh}(p, K); \exists F_1, \dots, F_q \in \mathcal{O}(K)^p \text{ such that } \mathcal{F}_0(\mathcal{F}) = \mathcal{F}_0(\{F_1, \dots, F_q\}) \text{ and } f_{ij} \in \mathcal{Q}(\sqrt{-1})[z_1, \dots, z_n] \text{ for all } 1 \leq i \leq q, 1 \leq j \leq p\}$$

and

$$\mathcal{Q}'(p, K) = \mathcal{Q}(p, K) \cap \mathcal{P}'(p, K),$$

where $\mathcal{Q}(\sqrt{-1})[z_1, \dots, z_n]$ is the polynomial ring over $\mathcal{Q}(\sqrt{-1})$. Clearly $\mathcal{Q}(p, K)$, and hence $\mathcal{Q}'(p, K)$ also, is a countable set. Then the main purpose of this section is to prove the following:

PROPOSITION 1. 1) Let U be an open set in C^n and $\mathcal{F} \in \mathcal{C}_{oh}(p, U)$ which is defined in the same way as $\mathcal{C}_{oh}(p, K)$ (cf. (1.1)). Then for every point $x \in U$,

after a suitable linear change of coordinates with center x there is a polycylinder $K \subseteq U$ with $x \in K$ such that $\mathcal{F} \in \mathcal{P}'(p, K)$.

2) $\mathcal{P}'(p, K) \subseteq \mathcal{P}(p, K)$

3) $\mathcal{P}'(p, K)$ is dense in $\mathcal{P}(p, K)$, where $\mathcal{P}'(p, K)$ has the induced topology from $\mathcal{P}(p, K)$ by 2). In particular $\mathcal{P}'(p, K)$ is separable.

(3.6) We prove 1) in a little more precise form in the following lemma. We consider elements of $GL(n, C)$ as linear changes of coordinates of C^n .

LEMMA 5. Let U be an open subset of C^n with $0 \in U$, and $\mathcal{F}_i$, $1 \leq i \leq m$, be a finite number of coherent analytic subsheaves of $\mathcal{O}^p$ on U . Then there is an element $h \in GL(n, Q(\sqrt{-1}))$ with the property that after the linear change of coordinates by h , $(\exists U_n)(\forall K_n \subseteq U_n)(\exists U_{n-1})(\forall K_{n-1} \subseteq U_{n-1}) \cdots (\exists U_1)(\forall K_1 \subseteq U_1)$, such that the polycylinder $K = K_1 \times \cdots \times K_n$ is adapted for all the $\mathcal{F}_i$, where U_i are neighbourhoods of 0 in $C(z_i)$ and K_i are compact convex subsets of $C(z_i)$ with $0 \in K_i$.

PROOF. We prove the lemma by induction on (n, p) as in (3.3) except that we assume that $n \geq 1$. In particular we deal only with the cases 2 and 3. (The start of the induction is the case $(n, p) = (1, 1)$.) Case 2. Let $\pi_{n-1}: C^n \rightarrow C^{n-1}$ be the projection to the first $(n-1)$ factors. Set $\mathcal{O}'_0 = \mathcal{O}_{C^{n-1}, 0}$ and consider this as a subring of $\mathcal{O}_0$ via the injection $\pi_{n-1}^*: \mathcal{O}'_0 \rightarrow \mathcal{O}_0$. Since every K is adapted for $\mathcal{F}_i = \{0\}$ (resp. $\mathcal{O}$), disregarding such $\mathcal{F}_i$ if necessary we may assume that $\mathcal{F}_i \neq \{0\}$ (resp. $\mathcal{O}$) for any i . Take any nonzero element $f_i \in \mathcal{F}_{i,0} \subseteq \mathcal{O}_0$ for each i . Then there is a nonempty Zariski open subset V of $GL(n, C)$ such that after a linear change of coordinates by any element of V , $\mathcal{O}'_0 \cap (f_i) = \{0\}$ and $\mathcal{O}_0/(f_i)$ is finite over $\mathcal{O}'_0$ for each i (cf. [2, p. 13]). Thus for every $h_0 \in V$ after a linear change of coordinates by h_0 if we take $U_n \subseteq C(z_n)$ sufficiently small around 0, then the following is true; for every $K_n \subseteq U_n$ there is a neighbourhood $U_0 = U_0(K_n)$ of 0 in C^{n-1} with the property that each f_i is defined on $U_0 \times K_n$ and is finite of degree p_i , $p_i > 0$, over U_0 (cf. (3.1)). Now as in Lemma 4 1) We may assume that $f_i \in \mathcal{O}_{n-1}(U_0)[z_n]$. Hence for every polycylinder K_0 in U_0 we may take ${}_{n-1}\mathcal{F}_i = {}_{n-1}\mathcal{F}_i(f_i)$, which we consider an $\mathcal{O}_{U_0}$ -module (cf. (3.4)). If $n=1$, then U_0 is a point and clearly $K_n = K_1$ is adapted for $\mathcal{F}_i$. So suppose that $n > 1$. Then by induction there is an element $h_1 \in GL(n-1, Q(\sqrt{-1}))$ such that after a linear change of coordinates of C^{n-1} by h_1 , $(\exists U_{n-1})(\forall K_{n-1} \subseteq U_{n-1}) \cdots (\forall K_1 \subseteq U_1)$ such that $K_1 \times \cdots \times K_{n-1}$ is adapted for all the ${}_{n-1}\mathcal{F}_i(f_i)$. Regarding $GL(n-1, C) \subseteq GL(n, C)$ naturally, we put $h = h_1 h_0$. Then by the definition of adaptedness in case 2 of Definition 1, taking h_0 from $V \cap GL(n, Q(\sqrt{-1}))$ the lemma is true in this case. Case 3. By induction the lemma is true for the ideal sheaves $\mathcal{I}_j(\mathcal{F}_i)$, $0 \leq i \leq m$, $1 \leq j \leq p$, which in turn implies the truth of the lemma for $\mathcal{F}_i$ by the definition

of adaptedness in case 3 of Definition 1.

Q.E.D.

(3.7) As for 2) we first need the following lemma.

LEMMA 6. *Let K be a polycylinder and $\mathcal{I}$ a coherent sheaf of ideals of $\mathcal{O}$ such that the analytic subspace of K defined by $\mathcal{I}$ is finite over $K^i = K_1 \times \cdots \times K_i$ for some $i \geq 1$ in the sense of (3.1). If K^i is privileged for $\pi_*(\mathcal{O}/\mathcal{I})$, then K is privileged for $\mathcal{O}/\mathcal{I}$, where $\pi: K \rightarrow K^i$ is the natural projection.*

PROOF. We shall use the following criterion of privilegedness due to Pourcin [4, Théoreme 3.2] and Siu [5]. First for every integer $p \geq 0$ define $K_{(p)} \subseteq K$ by $K_{(p)} = \{(z_i) \in K; \#\{i | z_i \in \partial K_i\} \geq p\}$. On the other hand, for any $\mathcal{F} \in \mathcal{O}h(K)$ let $S_p(\mathcal{F}) = \{x \in U; \text{depth}_x \mathcal{F} \leq p\}$. Now the criterion mentioned above is: K is privileged for $\mathcal{F}$ if for every $n-1 \geq p \geq 0$, $S_p(\mathcal{F}) \cap K_{(p+1)} = \emptyset$. In our case where $\mathcal{F} = \mathcal{O}/\mathcal{I}$ is finite over K^i , we have for every p that $S_p(\mathcal{F}) \cap K_{(p+1)} = S_p(\mathcal{F}) \cap \pi^{-1}(K_{(p+1)}^i)$. Then by the above criterion and the privilegedness of K^i for $\pi_*(\mathcal{O}/\mathcal{I})$ it suffices to show that $S_p(\mathcal{F}) \subseteq \pi^{-1}(S_p(\pi_*\mathcal{F}))$. Indeed, this follows immediately from the well-known relation that $\text{depth}_y \pi_*\mathcal{F} = \inf_x \text{depth}_x \mathcal{F}$, $x \in \pi^{-1}(y) \cap \text{supp } \mathcal{F}$, where $\text{supp } \mathcal{F}$ is the support of $\mathcal{F}$. Q.E.D.

(3.8) PROOF OF 2). Let $\mathcal{F}_0$ be a coherent analytic subsheaf of $\mathcal{O}^p$ defined on a polycylinder K and $\mathcal{F} = \mathcal{O}^p/\mathcal{F}_0$. Then what we have to show is the following: *If K is adapted for $\mathcal{F}_0$, then it is privileged for $\mathcal{F}$.* We proceed by induction on (n, p) as in (3.4). *Case 1.* K is privileged for every $\mathcal{F}$ so that there is nothing to prove. *Case 2.* If $\mathcal{F}_0 = \{0\}$ (resp. $\mathcal{O}$), then the claim is obvious because K is privileged for $\mathcal{F} = \mathcal{O}^p$ (resp. $\{0\}$). Let $\mathcal{I} = \mathcal{F}_0 \neq \{0\}$, $\neq \mathcal{O}$ and take a reducing function f for $\mathcal{I}$ as in Definition 1. Let $\mathcal{J} = (f)$. Then we have the following exact sequence of $\mathcal{O}$ -modules on K

$$0 \rightarrow \mathcal{I}/\mathcal{J} \rightarrow \mathcal{O}/\mathcal{J} \rightarrow \mathcal{O}/\mathcal{I} \rightarrow 0.$$

Let $\pi = \pi_{n-1}: K \rightarrow K^{n-1}$ be the natural projection. Taking the direct image of this sequence by π we obtain the following exact sequence on K^{n-1}

$$0 \rightarrow \pi_*(\mathcal{I}/\mathcal{J}) \rightarrow \pi_*(\mathcal{O}/\mathcal{J}) \rightarrow \pi_*(\mathcal{O}/\mathcal{I}) \rightarrow 0.$$

Let p be the degree of f in z_n . Then we have the natural isomorphism $\pi_*(\mathcal{O}/\mathcal{J}) \cong \mathcal{O}_{n-1}^p$ which induces the isomorphism $\pi_*(\mathcal{I}/\mathcal{J}) \cong {}_{n-1}\mathcal{F}_0(f)$ (cf. Def. 1). Since K^{n-1} is adapted for ${}_{n-1}\mathcal{F}_0(f)$, by induction we see that K^{n-1} is privileged for $\pi_*(\mathcal{O}/\mathcal{J})$. Since $\mathcal{O}/\mathcal{I}$ is finite over K^{n-1} , this in turn implies the privilegedness of K for $\mathcal{O}/\mathcal{I}$ by the previous lemma. *Case 3.* We have the natural exact sequence

$$0 \rightarrow \mathcal{F}_0^{(p-1)} \rightarrow \mathcal{F}_0 \rightarrow \mathcal{F}_p(\mathcal{F}_0) \rightarrow 0.$$

Since $\mathcal{F}_0^{(p-1)}$ (resp. $\mathcal{I}_p(\mathcal{F}_0)$) is naturally the subsheaf of $\mathcal{O}^{p-1}$ (resp. $\mathcal{O}$) and K is adapted for it (cf. Def. 1 Case 3), by induction K is privileged for the quotient $\mathcal{O}^{p-1}/\mathcal{F}_0^{(p-1)}$ (resp. $\mathcal{O}/\mathcal{I}_p(\mathcal{F}_0)$). Then from the exact sequence

$$0 \rightarrow \mathcal{O}^{p-1}/\mathcal{F}_0^{(p-1)} \rightarrow \mathcal{O}^p/\mathcal{F}_0 \rightarrow \mathcal{O}/\mathcal{I}_p(\mathcal{F}_0) \rightarrow 0$$

and from Lemma 1 it follows that K is privileged for $\mathcal{O}^p/\mathcal{F}_0 = \mathcal{F}$. Q.E.D.

(3.9) For the proof of 3) we need some preparations.

Let $\mathcal{F}_0$ be a coherent analytic subsheaf of $\mathcal{O}^p$ on a polycylinder K . Let $G = \{F_1, \dots, F_q\}$ be a set of generators of $\mathcal{F}_0(K)$ over $\mathcal{O}(K)$. We consider the following condition on $\mathcal{F}_0$:

(*) *There is a constant $c > 0$ such that every $F \in \mathcal{F}_0(K)$ can be written in the form $F = \sum_{i=1}^q g_i F_i$ for some $g_i \in \mathcal{O}(K)$ with $\|g_i\| \leq c \|F\|$.*

Let $B(K, \mathcal{F}_0)^*$ be the closure of $\mathcal{F}_0(K)$ in $B(K)$. Then we denote by $(*)'$ the condition obtained from $(*)$ by replacing $\mathcal{F}_0(K)$ and $\mathcal{O}(K)$ by $B(K, \mathcal{F}_0)^*$ and $B(K)$ respectively. Clearly both conditions are independent of the choice of G and depends only on $\mathcal{F}_0$. We need an easy half of the following:

LEMMA 7. *(*) and $(*)'$ are equivalent.*

PROOF. Suppose that $(*)$ is true. Let $F \in B(K, \mathcal{F}_0)^*$ be any element. Take a sequence $\{F_\nu\}$, $F_\nu \in \mathcal{F}_0(K)$, converging to F in $B(K)^p$ and write $F_\nu = \sum g_{\nu i} F_i$ with $g_i \in \mathcal{O}(K)$ and $\|g_{\nu i}\| \leq c \|F_\nu\|$. Then since $\|g_{\nu i}\|$ are bounded, taking a subsequence if necessary, we may assume that $g_{\nu i}$ converge to elements $g_i \in B(K)$. Then we get that $F_0 = \sum_{i=1}^q g_i F_i$ with $\|g_i\| \leq c \|F\|$. To prove the converse we shall interpret $(*)'$ another way. Let $\alpha_0: \mathcal{O}(K)^q \rightarrow \mathcal{O}(K)^p$ be defined by G i.e. $\alpha_0(g_1, \dots, g_q) = \sum_{i=1}^q g_i F_i$, and $\alpha: B(K)^q \rightarrow B(K)^p$ be induced by α_0 . Then $(*)'$ is equivalent to saying that i) $\text{Im } \alpha = B(K, \mathcal{F}_0)^*$, or equivalently, $\text{Im } \alpha$ is closed in $B(K)^p$ since $\mathcal{F}_0(K)$ is dense in $\text{Im } \alpha$, and ii) the conorm $q(\alpha)$ of $\alpha := \inf_x \{\|\alpha(x)\| / \inf_{x' \in \alpha^{-1}\alpha(x)} \|x'\|\} > 0$. Now we show that the natural morphism $\rho: B(K, \mathcal{F}_0) \rightarrow B(K, \mathcal{F}_0)^*$ which is surjective by i) is injective. First note that ii) implies that the topologies induced on $\mathcal{F}_0(K)$ from $\mathcal{O}(K)^q$ as a quotient and from $\mathcal{O}(K)^p$ as a subspace are equivalent. There are natural inclusions $i: \mathcal{F}_0(K) \rightarrow B(K, \mathcal{F}_0)$ and $j: \mathcal{F}_0(K) \rightarrow B(K, \mathcal{F}_0)^*$ such that $i \cdot \rho = j$. Then the above remark means that $\rho|_{i(\mathcal{F}_0(K))}$ is homeomorphic. Since $i(\mathcal{F}_0(K))$ is dense in $B(K, \mathcal{F}_0)$ it follows then easily that ρ is in fact injective. Let $\mathcal{F}_1 = \text{Ker } \alpha$. Fixing any set G_1 of generators of $\mathcal{F}_1(K)$ over $\mathcal{O}(K)$ we have homomorphisms $\beta_0: \mathcal{O}(K)^r \rightarrow \mathcal{O}(K)^q$ and $\beta: B(K)^r \rightarrow B(K)^q$ as above, where $r = \#G_1$. Then the injectivity of ρ is equivalent to the exactness of the natural sequence $B(K)^r \rightarrow B(K)^q \rightarrow B(K)^p$. Now suppose that $(*)'$ is true. Let $F \in \mathcal{F}_0(K)$. We can write $F = \sum_{i=1}^q g_i F_i$ for some $g_i \in B(K)$ with $\|g_i\| \leq c \|F\|$.

Hence it is enough to show that the set $\{(g_1, \dots, g_q) \in \mathcal{O}(K)^q; \sum_{i=1}^q g_i F_i = F\}$ is dense in the set $\{(g_1, \dots, g_q) \in B(K)^q; \sum_{i=1}^q g_i F_i = F\}$. To show this, by translation we may assume that $F=0$. Namely it suffices to show that $\text{Ker } \alpha_0$ is dense in $\text{Ker } \alpha$, which in turn follows from the equalities $\text{Ker } \alpha_0 = \text{Im } \beta_0$ and $\text{Ker } \alpha = \text{Im } \beta$ and the obvious fact that $\text{Im } \beta_0$ is dense in $\text{Im } \beta$. Q.E.D.

REMARK. The conditions i) and ii) in the above proof are equivalent by the open mapping theorem, so by i), $(*)'$ is also equivalent to the separation of $B(K, \mathcal{F})$. Then the above proof shows that for any $\mathcal{F} \in \mathcal{Coh}(K)$ if $B(K, \mathcal{F})$ is separated, then for any free resolution (1) of $\mathcal{F}$ on K the associated sequence (2) is an exact sequence of Banach spaces (cf. 2.2). Note that the latter condition was called K being weakly $\mathcal{F}$ -privileged in [4] and [5]. Actually [4, Th. 3.2] even shows that the separation of $B(K, \mathcal{F})$ is equivalent to the $\mathcal{F}$ -privilegedness of K .

(3.10) Now consider $B(K, \mathcal{F}_0)^*$ an element of $\widehat{Gr}B(K)^p$ (cf. 2.3).

LEMMA 8. Let $\mathcal{F}_0$ and $G = \{F_1, \dots, F_q\}$ be as above. Suppose that $(*)$ is true for $\mathcal{F}_0$, and take and fix $c > 0$ satisfying $(*)$. Then for every $\varepsilon > 0$ if we put $\eta = \varepsilon/2qc$, for any subset $G' = \{F'_1, \dots, F'_q\}$ of $\mathcal{O}(K)^p$ with $\|F'_i - F_i\| < \eta$, $1 \leq i \leq q$, we have that

$$\theta(B(K, \mathcal{F}_0)^*, B(K, \mathcal{F}'_0)^*) < \varepsilon$$

where θ is as in (2.3) and $\mathcal{F}'_0 = \mathcal{F}_0(G')$.

PROOF. For any $F \in B(K, \mathcal{F}_0)^*$ write $F = \sum_{i=1}^q g_i F_i$ with $\|g_i\| \leq c \|F\|$ and $g_i \in B(K)$ as in $(*)'$ (cf. the proof of the previous lemma). Then we set $F' = \sum_{i=1}^q g_i F'_i$ and $F'' = 1/(1+qc\eta)F'$, regarded as elements of $B(K, \mathcal{F}'_0)^*$. Then we have that $\|F - F'\| = \|\sum_{i=1}^q g_i(F_i - F'_i)\| \leq \sum_{i=1}^q \|g_i\| \|F_i - F'_i\| \leq qc\eta \|F\|$ and hence $\|F''\| \leq \|F\|$. Moreover if $\|F\| \leq 1$, then $\|F - F''\| = \|F - F' + qc\eta F''\| \leq 2qc\eta = \varepsilon$. These imply that for every $F \in B(K, \mathcal{F}_0)^*$ with $\|F\| \leq 1$, we have that $\inf \|F - F_1\| \leq \|F - F''\| < \varepsilon$, where F_1 ranges over the elements of $B(K, \mathcal{F}'_0)^*$ with $\|F_1\| \leq 1$. Since F was arbitrary, the lemma follows.

In the above lemma if $(*)$ is also true for $\mathcal{F}'$ for some $c' < \varepsilon/2q\eta$ then by symmetry we could even get that $d(B(K, \mathcal{F}_0)^*, B(K, \mathcal{F}'_0)^*) < \varepsilon$. In fact we want to show less precisely that we can find c' satisfying $(*)$, arbitrary close to c as η tends to 0. Our next purpose is to show that this is true if we take a 'suitable' set of generators $\{F_1, \dots, F_q\}$ of $\mathcal{F}(K)$ and its 'suitable' approximation $\{F'_1, \dots, F'_q\}$. We shall first make precise the meaning of 'suitable' in the following two definitions, and then prove the desired result in Lemma 11.

(3.11) DEFINITION 2. Let $\mathcal{F}_0$ be a coherent analytic subsheaf of $\mathcal{O}^p$ on a polycylinder K and G a set of generators of $\mathcal{F}_0(K)$ over $\mathcal{O}(K)$. Then G is said to be *admissible* (Notation: $G \in \text{Adm}(\mathcal{F}_0, K)$) if the following conditions are satisfied: (As in Definition 1 we proceed by induction on (n, p) and distinguish three cases.) *Case 1.* No condition, i.e., every G is admissible. *Case 2.* If $\mathcal{F}_0 = \{0\}$ (resp. $\mathcal{O}$), then $G = \{0\}$ (resp. $\{1\}$). Suppose that $\mathcal{F}_0 \neq \{0\}$, $\neq \mathcal{O}$. Then there is a reducing function $f \in \mathcal{F}_0(K)$ and an element $G_0 \in \text{Adm}({}_{n-1}\mathcal{F}_0(f), K^{n-1})$ such that $G = \{f\} \cup \phi^{-1}(G_0)$. *Case 3.* Let $\alpha_i: \mathcal{F}_0^{(i)} \rightarrow \mathcal{F}_i(\mathcal{F}_0)$ be the natural projections. Then there are $G_i \in \text{Adm}(\mathcal{F}_i(\mathcal{F}_0), K)$ and subsets $\tilde{G}_i$ of $\mathcal{F}_0^{(i)}(K)$ which are mapped bijectively to G_i by α_i , such that $G = \bigcup_{i=1}^p \tilde{G}_i$.

(3.12) Let $\mathcal{F}_0$ be as above and $G = \{F_1, \dots, F_q\}$ be an (ordered) set of generators of $\mathcal{F}_0(K)$ over $\mathcal{O}(K)$. For any $\eta > 0$ an η -approximation of G is by definition a system $G' = \{F'_1, \dots, F'_q\}$ of elements of $\mathcal{O}(K)^p$ such that $\|F_i - F'_i\| < \eta$ for all i .

DEFINITION 3. Let $\mathcal{F}_0$ be a coherent analytic subsheaf of $\mathcal{O}^p$ on a polycylinder K and $G \in \text{Adm}(\mathcal{F}_0, K)$. For any $\eta > 0$ an *admissible* η -approximation of G is an η -approximation G' of G prescribed by the following rule (Notation: $G' \in a.\eta.A(G)$). We use induction on (n, p) and distinguish three cases: *Case 1.* G' has no further property. *Case 2.* If $\mathcal{F}_0 = \{0\}$ (resp. $\mathcal{O}$), then $G' = \{0\}$ (resp. $\{1\}$). Suppose that $\mathcal{F}_0 \neq \{0\}$, $\neq \mathcal{O}$ and write $G = \{f\} \cup \phi^{-1}(G_0)$ as in Definition 2, with $f = z_n^p + \sum_{i=0}^{p-1} a_i z_n^i$, $a_i \in \mathcal{O}(K^{n-1})$. Then $G' = \{f'\} \cup \phi^{-1}(G'_0)$, where $f' = z_n^p + \sum_{i=0}^{p-1} a'_i z_n^i$, $a'_i \in \mathcal{O}(K^{n-1})$, with $\|a_i - a'_i\| < \eta$ on K^{n-1} , and $G'_0 \in a.\eta.A(G_0)$. *Case 3.* Let $G = \bigcup_{i=1}^p G_i$ and G_i be as in Definition 2. Then $G' = \bigcup_{i=1}^p \tilde{G}'_i$, where $\tilde{G}'_i$ are η -approximations of G_i such that $\tilde{\alpha}_i(\tilde{G}'_i) \in a.\eta.A(G_i)$, $\tilde{\alpha}_i: \mathcal{O}(K)^i \rightarrow \mathcal{O}(K)$ (the last factor) being the natural projections.

LEMMA 9. Let $\mathcal{F}_0$ be a coherent analytic subsheaf of $\mathcal{O}^p$ on a polycylinder K . Suppose that K is adapted for $\mathcal{F}_0$. Then $\text{Adm}(\mathcal{F}_0, K) \neq \emptyset$. Moreover if $G \in \text{Adm}(\mathcal{F}_0, K)$ with $G = \{F_1, \dots, F_q\}$, then for any $\eta > 0 \exists G' = \{F'_1, \dots, F'_q\} \in a.\eta.A(G)$ such that $f'_{ij} \in \mathcal{Q}(\sqrt{-1})[z_1, \dots, z_n]$ where $F'_i = (f'_{i1}, \dots, f'_{ip})$.

PROOF. In Definition 2 if we replace 'generators' by 'elements', we obtain the definition of admissible sets of elements of $\mathcal{F}_0(K)$. From the definition of adaptedness it follows immediately that we can always find an admissible set of elements of $\mathcal{F}_0(K)$. Then by virtue of Lemma 4 we can prove easily that such a set is in fact a set of generators of $\mathcal{F}_0(K)$ over $\mathcal{O}(K)$ by induction on (n, p) (cf. [2, p. 82 Theorem]). Finally the last assertion follows by induction from Definition 3 and the polynomial approximation theorem on a polycylinder.

LEMMA 10. Let $\mathcal{F}_0$ be a coherent analytic subsheaf of $\mathcal{O}^p$ on a polycylinder K with $\mathcal{F}_0 \neq \{0\}$. Suppose that K is adapted for $\mathcal{F}_0$ and $G \in \text{Adm}(\mathcal{F}_0, K)$ with $G = \{F_1, \dots, F_q\}$. Then there is an $\eta > 0$ such that for every $G' \in a.\eta.A(G)$, K is adapted also for $\mathcal{F}'_0 = \mathcal{F}'_0(G')$.

PROOF. By induction on (n, p) . *Case 1.* Trivial. *Case 2.* For $\mathcal{F}_0 = \{0\}$ or $\mathcal{O}$, the assertion is trivial, so we assume otherwise. Then we may take $f = F_1$ as a reducing function for $\mathcal{F}$ with $f = z_n^p + \sum_{i=0}^{p-1} a_i z_n^i$, $a_i \in \mathcal{O}(K^{n-1})$. Then by the continuous dependence of the roots of a polynomial on its coefficients there is an $\eta_1 > 0$ such that any $f' \in \mathcal{O}(K)$ with $f' = z_n^p + \sum_{i=0}^{p-1} a'_i z_n^i$, $a'_i \in \mathcal{O}(K^{n-1})$, $\|a_i - a'_i\| < \eta_1$ on K^{n-1} , is also finite over K^{n-1} of degree p . Next we consider ${}_{n-1}\mathcal{F}_0 = {}_{n-1}\mathcal{F}_0(f)$, for which $H = \{H_2, \dots, H_q\}$, $H_i = \phi(F_i)$, is an admissible set of generators on K^{n-1} . Then by induction there is an $\eta_2 > 0$ such that for any $H' = \{H'_2, \dots, H'_q\} \in a.\eta_2.A(H)$, K^{n-1} is adapted for the submodule $\mathcal{F}'_0 = \mathcal{F}'_0(H')$. Then if we set $\eta = \min(\eta_1, \eta_2)$, by the definitions of adaptedness and admissibility it is clear that for every $G' \in a.\eta.A(G)$, K is adapted for $\mathcal{F}'_0(G')$. *Case 3.* Let $G = \bigcup_{i=1}^p \tilde{G}_i$ and G_i be as in Definition 2. By induction there is an $\eta > 0$ such that every $G'_i \in a.\eta.A(G_i)$, K is adapted for $\mathcal{F}'_i = \mathcal{F}'_i(G'_i)$ for all i . Then from the definitions of adaptedness and admissibility it follows immediately that K is adapted for $\mathcal{F}'_0(G')$ for every $G' \in a.\eta.A(G)$. Q.E.D.

LEMMA 11. Let $\mathcal{F}_0$ be a coherent analytic subsheaf of $\mathcal{O}^p$ on a polycylinder K . Suppose that K is adapted for $\mathcal{F}_0$. Let $G = \{F_1, \dots, F_q\} \in \text{Adm}(\mathcal{F}_0, K)$. Then there is a constant $c > 0$ satisfying (*) for $\mathcal{F}_0$ and G (3.9), and with the following property; (**) for every $\varepsilon > 0$ we can find an $\eta > 0$ such that for every $G' \in a.\eta.A(G)$ there is a $c' > 0$ satisfying (*) for G' and $\mathcal{F}'_0(G')$ such that $c' < c + \varepsilon$.

PROOF. We prove the lemma by induction on (n, p) . *Case 1.* Trivial. *Case 2.* We assume that $\mathcal{F}_0 \neq \{0\}$, $\neq \mathcal{O}$ since the other cases are trivial. We may assume that $f = F_1$ is a reducing function for $\mathcal{F}_0$. Then by Lemma 4 2) for every $F \in \mathcal{F}_0(K)$ we can write $F = g_1 f + H$ for some $g_1 \in \mathcal{O}(K)$ and $H = \sum_{i=0}^{p-1} a_i z_n^i$, $a_i \in \mathcal{O}(K^{n-1})$, with $\|g_1\| \leq c_1 \|F\|$ and $\|a_i\| \leq c_1 \|F\|$ for some $c_1 > 0$ (independent of F). Further from the explicit formula for c_1 in (3.2) we infer readily the following. Let $f = z_n^p + \sum_{i=0}^{p-1} b_i z_n^i$, $b_i \in \mathcal{O}(K^{n-1})$. Then for any $\varepsilon_1 > 0$ there is an $\eta_1 > 0$ such that for any $f' = z_n^p + \sum_{i=0}^{p-1} b'_i z_n^i$, $b'_i \in \mathcal{O}(K^{n-1})$, with $\|f - f'\| < \eta_1$ and $\|b_i - b'_i\| < \eta_1$ we can write $F = g' f' + H'$ for some $g' \in \mathcal{O}(K)$ and $H' = \sum_{i=0}^{p-1} a'_i z_n^i$, $a'_i \in \mathcal{O}(K^{n-1})$ with $\|g'\| \leq c'_1 \|F\|$, $\|a'_i\| \leq c'_1 \|F\|$ for some c'_1 with $0 < c'_1 < c_1 + \varepsilon_1$. On the other hand, since $\phi(H) = (a_0, \dots, a_{p-1})$ and $\{\phi(F_2), \dots, \phi(F_p)\} \in \text{Adm}({}_{n-1}F_0, K^{n-1})$, by induction there are a constant $c_2 > 0$, not depending on $H = H(F)$, and elements $g_2, \dots, g_q \in \mathcal{O}(K^{n-1})$ with the property that $\phi(H) = \sum_{i=2}^q g_i \phi(F_i)$ with $\|g_i\| \leq c_2 \|\phi(H)\| = c_2 \max \|a_i\| \leq c_1 c_2 \|F\|$ and that for any $\varepsilon_2 > 0$,

$\exists \eta_2 > 0$ such that for any ${}_{n-1}G' = \{\phi(F'_2), \dots, \phi(F'_q)\} \in a.\eta_2.A(G)$ we can find $0 < c'_2 < c_2 + \varepsilon_2$ such that $\phi(H') = \sum_{i=2}^q g'_i \phi(F'_i)$ for some $g'_i \in \mathcal{O}(K^{n-1})$ with $\|g'_i\| \leq c'_2 \|\phi(H')\| \leq c'_1 c'_2 \|F'\|$. Hence if we put $c = \max(c_1, c_1 c_2)$ we have $F' = \sum_{i=1}^q g_i F'_i$ with $\|g_i\| \leq c \|F'\|$. Moreover if in the above we take ε_1 and ε_2 in such a way that $\varepsilon_1 < \varepsilon$, $\varepsilon_1 c_2 + \varepsilon_2 c_1 + \varepsilon_1 \varepsilon_2 < \varepsilon$ for the given ε and put $\eta = \min(\eta_1, \eta_2)$ and $c' = \min(c'_1, c'_2)$, then from the definition of admissible η -approximation it follows that for any $G' \in a.\eta.A(G)$, $(**)$ is true. *Case 3.* Let $G = \bigcup_{j=1}^q \tilde{G}_j$ and G_j be as in Definition 2. Write $F = (f_1, \dots, f_p)$. Let $G_j = (h_1^j, \dots, h_{k_j}^j)$ and $\tilde{G}_j = (H_1^j, \dots, H_{k_j}^j)$ with $\alpha_j(H_i^j) = h_i^j$. Since $f_p \in \mathcal{I}_p(\mathcal{F}_0)(K)$, by induction $\exists c_1 > 0$ such that we can write $f_p = \sum_{i=1}^{k_p} g_i h_i^p$, for some $g_i \in \mathcal{O}(K)$ with $\|g_i\| \leq c_1 \|f\|$, and that $(**)$ is true for the triple $\{\mathcal{I}_p(\mathcal{F}_0), G_p, c_1\}$. Then $F' = F - \sum_{i=1}^{k_p} g_i H_i^p \in \mathcal{O}(K)^{p-1}$. Hence by induction $\exists c_2 > 0$ and $g_i^j \in \mathcal{O}(K)$, $1 \leq j \leq p-1$, $1 \leq i \leq k_j$ such that $F' = \sum g_i^j H_i^j$ with $\|g_i^j\| \leq c_2 \|F'\|$ and that $(**)$ is true for the triple $\{\mathcal{F}_0^{(p-1)}, \bigcup_{j=1}^{p-1} \tilde{G}_j, c_2\}$. From these the lemma follows easily. Q.E.D.

PROOF OF PROPOSITION 1.3). Let $\mathcal{F} \in \mathcal{P}'(p, K)$ be any element. Let $\varepsilon > 0$. Take any $G \in \text{Adm}(\mathcal{F}_0, K)$ with $G = \{F_1, \dots, F_q\}$, where $\mathcal{F}_0 = \mathcal{F}_0(\mathcal{F})$. This is possible by Lemma 9. Take a constant $c > 0$ satisfying the condition of the above lemma for G and $\mathcal{F}_0$. Now take $\eta_1 > 0$ so small that the conclusions of Lemmas 10 and 11 are true for $\mathcal{F}_0, G$ and ε . Let $\eta = \min(\eta_1, 1/2q(c + \varepsilon))$. Then by Lemma 9 we can find a $G' \in a.\eta.A(G)$ with $G' = (F'_1, \dots, F'_q)$, $F'_i = (f'_{i1}, \dots, f'_{ip})$ such that $f'_{ij} \in \mathcal{Q}(\sqrt{-1})[z_1, \dots, z_n]$. Let $\mathcal{F}'_0 = \mathcal{F}'_0(G')$ and $\mathcal{F}' = \mathcal{O}^p/\mathcal{F}'_0$. Then $\mathcal{F}'_0 \in \mathcal{D}'(p, K)$ by Lemma 10 and 2) of Proposition 1. Further by the above lemma we can find $c' > 0$ satisfying $(*)$ for G' and $\mathcal{F}'_0$ and with $c' < c + \varepsilon$. Finally, since $\eta < \min(1/2qc, 1/2qc')$, from Lemma 8 it follows that $d(B(K, \mathcal{F}_0), B(K, \mathcal{F}'_0)) < \varepsilon$, where d is the metric on $\mathcal{P}(p, K)$ defined in (2.3). By the definition of the topology on $\mathcal{P}'(p, K)$ this completes the proof of 3).

§4. Proof of Theorem

(4.1) For every n -tuples $(r_1, \dots, r_n) \in \mathbf{R}^n$, $r_i > 0$, and $(a_1, \dots, a_n) \in \mathbf{C}^n$, let $\Delta(a_i; r_i) = \{(z_i) \in \mathbf{C}^n; |z_i - a_i| \leq r_i\}$. Then we define a countable set, $\mathfrak{P}_n$, of subsets of $\mathbf{C}^n$ by

$$\mathfrak{P}_n = \{K \subseteq \mathbf{C}^n; \exists h \in GL(n, \mathcal{Q}(\sqrt{-1})) \text{ such that after a linear change of coordinates by } h, K = \Delta(a_i; r_i) \text{ for some } a_i \in \mathcal{Q}(\sqrt{-1}) \text{ and } r_i \in \mathcal{Q}\}.$$

Let $\mathcal{F}$ be a coherent analytic sheaf on an open subset U of $\mathbf{C}^n$. Then we call $K \in \mathfrak{P}_n$, $K \subseteq U$, *adapted* for $\mathcal{F}$ if it is adapted with respect to some coordinates for which $K = \Delta(a_i; r_i)$ as above. Then from Lemma 5 we infer readily that for every point $x \in U$ we can find an element $K \in \mathfrak{P}_n$, $x \in K \subseteq K \subseteq U$, which is adapted for $\mathcal{F}$.

(4.2) Let X and $\mathcal{E}$ be as in (2.6) and suppose that X has a countable base of the open sets. We shall define a countable set, $\mathfrak{S}$, of $\mathcal{E}$ -semiprivileged cuirasses on X . First we note that in the definition of cuirasses in (2.6) we may clearly allow K_i (resp. K_{ija}) to be subsets of C^{n_i} (resp. $C^{n_{ija}}$) which become polycylinders after some linear changes of coordinates. So we assume this in the following. We fix once and for all locally finite countable open coverings $\{W'_\lambda\}$ and $\{W_\lambda\}$, $\lambda \in A$, of X such that for each λ , $\overline{W}_\lambda \subseteq W'_\lambda$ and we have an embedding $j_\lambda: \overline{W}'_\lambda \rightarrow \Delta_\lambda$, where $\overline{}$ denotes the closure and $\Delta_\lambda = \{(z_i) \in C^{n_\lambda}; |z_i| \leq 1\}$ for some $n_\lambda > 0$. Next we fix once and for all a surjective homomorphism, $q_\lambda: \mathcal{O}_{\Delta_\lambda}^{p_\lambda} \rightarrow \mathcal{E}_{\Delta_\lambda}$, on Δ_λ , where $\mathcal{E}_\lambda = j_{\lambda*}(\mathcal{E}|_{W'_\lambda})$. Then we define a set, $\mathfrak{S}_0$, of cuirasses as follows. Let $M = ((\varphi_i), (K_i), (\varphi_{ij}), (K_{ija}), (V_i), L)_{1 \leq i \leq s, 1 \leq a \leq t_{ij}}$ be a cuirasse as in (2.6). Then by definition $M \in \mathfrak{S}_0$ if and only if for each i (resp. (i, j)) there is a $\lambda = \lambda(i)$ such that 1) $\overline{X}_i \subseteq W_\lambda$, and $\varphi_i = j_\lambda|_{X_i}$ (resp. $\varphi_{ij} = j_\lambda|_{X_{ij}}$) so that in particular we can take D_i (resp. D_{ij}) subdomains of Δ_λ , where X_i, X_{ij}, D_i and D_{ij} are as in (2.6), 2) K_i (resp. K_{ija}) $\in \mathfrak{P}_{n_\lambda}$, 3) there is an X'_i (resp. V'_i) $\in \mathfrak{P}_{n_\lambda}$ with $X'_i \subseteq \Delta_i$ (resp. $V'_i \subseteq \Delta_i$) such that $\varphi_i^{-1}(X'_i) = X_i$ (resp. $\varphi_i^{-1}(V'_i) = V_i$), and finally 4) $L = X - \bigcup_i \overset{\circ}{V}_i$. Then we see readily that the set $\mathfrak{S}_0$ is countable. Finally we put $\mathfrak{S} = \{M \in \mathfrak{S}_0; M \text{ is } \mathcal{E} \text{ semiprivileged}\}$.

(4.3) Recalling the inclusions $H_M \subseteq \prod_{i=1}^s \mathcal{P}(p_i, K_i)$ and $\prod_{i=1}^s \mathcal{P}'(p_i, K_i) \subseteq \prod_{i=1}^s \mathcal{P}(p_i, K_i)$ (cf. (2.6) and (3.5)), we put $H'_M = H_M \cap \prod_{i=1}^s \mathcal{P}'(p_i, K_i)$ for each cuirasse M as above. We shall show that $D_X(\mathcal{E}) = \bigcup_{M \in \mathfrak{S}} H'_M$. Since $\mathcal{P}'(p_i, K_i)$ are separable by Proposition 1 this would complete the proof of our theorem by (2.9). Now let $\mathcal{F}$ be any quotient analytic sheaf of $\mathcal{E}$ with compact support on X . Let $\mathcal{F}_\lambda = j_{\lambda*}(\mathcal{F}|_{W'_\lambda})$, $\beta_\lambda: \mathcal{E}_\lambda \rightarrow \mathcal{F}_\lambda$ the induced quotient homomorphisms and $\mathcal{F}_{\lambda_0}$ the kernels of $\beta_\lambda q_\lambda$. Take any point $x \in \text{supp } \mathcal{F}$, the support of $\mathcal{F}$. Suppose that $x \in W_\lambda$ for some $\lambda \in A$. Then as remarked above we can find by Lemma 5 an element $K \in \mathfrak{P}_{n_\lambda}, j_\lambda(x) \in \overset{\circ}{K} \subseteq K \subseteq \Delta_\lambda$, such that K is adapted for $\mathcal{F}_{\lambda_0}$ and hence privileged for $\mathcal{F}_\lambda$ by 2) of Proposition 1. In particular $\mathcal{F}_\lambda \in \mathcal{P}'(p_\lambda, K)$. Starting from this local fact together with the compactness of $\text{supp } \mathcal{F}$, we can construct easily a cuirasse $M \in \mathfrak{S}$ such that $\mathcal{F} \in H'_M$. This implies the desired equality $D_X(\mathcal{E}) = \bigcup_{M \in \mathfrak{S}} H'_M$ and proves Theorem as was mentioned above.

(4.4) REMARK. 1) Our proof gives a little more than the theorem in the introduction: *Let X be a complex space which has a base of the open sets with cardinal number m and $\mathcal{E}$ a coherent analytic sheaf on X . Then $D_X(\mathcal{E})$ also has a base of the open sets with the same cardinal number m .* 2) Let X and S be complex spaces and $f: X \rightarrow S$ a morphism of complex spaces. Let $\mathcal{E}$ be a coherent analytic sheaf on X . Then by Pourcin [3] we have the 'relative Douady space' $D_{X/S}(\mathcal{E})$ of quotient analytic sheaves of $\mathcal{E}$ with proper

supports over S , which is a complex space over S and with the universal property analogous to $D_X(\mathcal{E})$ (cf. [3]). Since $D_{X/S}(\mathcal{E})$ can be considered essentially as a subspace of $D_X(\mathcal{E})$ (cf. [3, § 5]) our theorem and the above remark are true also for $D_{X/S}(\mathcal{E})$.

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