**The Cauchy problem for operators with constant
coefficient hyperbolic principal part
and propagation of singularities**

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Introduction

Lax [16] and Mizohata [17] proved that for the non-characteristic Cauchy problem to be $\mathcal{E}$ -well posed it is necessary that the characteristic roots are real. Moreover Ivrii [11] proved that for the non-characteristic Cauchy problem for operators with analytic coefficients to be well-posed in Gevrey classes it is necessary that the characteristic roots are real. However, in general, the Cauchy problem for operators with hyperbolic principal part is not $\mathcal{E}$ -well posed. On the other hand Ivrii [10] proved that the Cauchy problem for such operators is well-posed in some Gevrey classes. So one must consider the Cauchy problem in Gevrey classes (or in the space of ultradistributions) if one imposes no conditions on the lower order terms. In this paper

we shall study the Cauchy problem for operators with constant coefficient hyperbolic principal part in Gevrey classes (or in the space of ultradistributions).

Let $P(x, D)$ be a partial differential operator of order m in n independent variables, where $x = (x_1, \dots, x_n)$ and $D = i^{-1}(\partial/\partial x_1, \dots, \partial/\partial x_n)$. We assume that

$$P(x, D) = P_m(D) + Q(x, D),$$

$$Q(x, D) = \sum_{j=0}^{m-1} Q_j(x, D), \quad Q_j(x, D) = \sum_{|\alpha|=j} a_\alpha(x) D^\alpha,$$

where α is a multi-index $(\alpha_1, \dots, \alpha_n)$. Let K be a regular compact set in $\mathbf{R}^n$ and let $1 < \kappa < \infty$ and $h > 0$. We denote by $\mathcal{E}^{(\kappa), h}(K)$ the space of all $f \in C^\infty(K)$ which satisfies

$$|D^\alpha f(x)| \leq Ch^{|\alpha|}(|\alpha|!), \quad x \in K, \quad |\alpha| = 0, 1, 2, \dots,$$

for some constant $C \geq 0$ and by $\mathcal{D}_K^{(\kappa), h}$ the space of all $f \in \mathcal{E}^{(\kappa), h}(K)$ with support in K . Then $\mathcal{E}^{(\kappa), h}(K)$ and $\mathcal{D}_K^{(\kappa), h}$ are Banach spaces under the norm defined by

$$\|f\|_{\mathcal{E}^{(\kappa), h}(K)} = \sup_{x \in K, \alpha} |D^\alpha f(x)| / (h^{|\alpha|}(|\alpha|!)).$$

Let Ω be an open set in $\mathbf{R}^n$. As locally convex spaces we define the following:

$$\begin{aligned} \mathcal{E}^{(\kappa)}(\Omega) &= \varprojlim_{K \subset \Omega} \mathcal{E}^{(\kappa)}(K), & \mathcal{E}^{(\kappa)}(K) &= \varprojlim_{h \rightarrow 0} \mathcal{E}^{(\kappa), h}(K), \\ \mathcal{E}^{(\kappa)}(\Omega) &= \varprojlim_{K \subset \Omega} \mathcal{E}^{(\kappa)}(K), & \mathcal{E}^{(\kappa)}(K) &= \varinjlim_{h \rightarrow \infty} \mathcal{E}^{(\kappa), h}(K), \\ \mathcal{D}^{(\kappa)}(\Omega) &= \varprojlim_{K \subset \Omega} \mathcal{D}_K^{(\kappa)}, & \mathcal{D}_K^{(\kappa)} &= \varprojlim_{h \rightarrow 0} \mathcal{D}_K^{(\kappa), h}, \\ \mathcal{D}^{(\kappa)}(\Omega) &= \varinjlim_{K \subset \Omega} \mathcal{D}_K^{(\kappa)}, & \mathcal{D}_K^{(\kappa)} &= \varinjlim_{h \rightarrow \infty} \mathcal{D}_K^{(\kappa), h}. \end{aligned}$$

We denote by $\mathcal{D}^{(\kappa)'}(\Omega)$ and $\mathcal{D}^{(\kappa)'}(\Omega)$ the strong dual spaces of $\mathcal{D}^{(\kappa)}(\Omega)$ and $\mathcal{D}^{(\kappa)}(\Omega)$, respectively (see [13]). We also write $\mathcal{E}^{(\kappa)}, \dots$, instead of $\mathcal{E}^{(\kappa)}(\mathbf{R}^n), \dots$, and put

$$\mathcal{E}^{(\infty)}(\Omega) = \mathcal{E}^{(\infty)}(\Omega) = \mathcal{E}(\Omega) (= C^\infty(\Omega)),$$

$$\mathcal{D}^{(\infty)'}(\Omega) = \mathcal{D}^{(\infty)'}(\Omega) = \mathcal{D}'(\Omega),$$

although

$$\mathcal{E}(\Omega) \ni \bigcup_{\kappa > 1} \mathcal{E}^{(\kappa)}(\Omega) = \bigcup_{\kappa > 1} \mathcal{E}^{(\kappa)}(\Omega).$$

Moreover, hereafter, we denote by $*$ either (κ) or $\{\kappa\}$, for example, $\mathcal{E}^{*\kappa} = \mathcal{E}^{(\kappa)}$ or $\mathcal{E}^{(\kappa)}$.

Let us consider the Cauchy problem for $P(x, D)$

$$(1) \quad \begin{cases} P(x, D)u(x) = f(x), \\ \text{supp } u(x) \subset \{x \in \mathbf{R}^n; x \cdot \vartheta \geq t\}, \end{cases}$$

where $\text{supp } f \subset \{x \in \mathbf{R}^n; x \cdot \vartheta \geq t\}$, $t \in \mathbf{R}$ and $\vartheta = (1, 0, \dots, 0) \in \mathbf{R}^n$. If $P(x, D) \equiv P(D)$ is an operator with constant coefficients, then a necessary and sufficient condition for the Cauchy problem (1) to be $\mathcal{E}^{*\kappa}$ -well posed was expressed as conditions on the zeros of $P_m(\xi)$ and $P(\xi)$ (by Gårding [4], [5] for $\kappa = \infty$ and Larsson [15] for $1 < \kappa < \infty$). Moreover Atiyah, Bott and Gårding [1] investigated the singular supports (wave front sets) and the lacunas of fundamental solutions for such operators hyperbolic in the sense of Gårding. On the other hand, for operators with variable lower-order coefficients Dunn [3] proved that the Cauchy problem (1) is $\mathcal{E}$ -well posed if

$$(2) \quad P_m(\xi) \text{ is a hyperbolic polynomial with respect to } \vartheta \text{ and } Q(x, \xi) \prec P_m(\xi) \text{ for every fixed } x \text{ in } \mathbf{R}^n$$

(see, also, [20]), where $q(\xi) \prec p(\xi)$ means that there is a positive number C such that

$$\tilde{q}(\xi) \equiv (\sum_a |q^{(a)}(\xi)|^2)^{1/2} \leq C \tilde{p}(\xi) \quad \text{for every } \xi \text{ in } \mathbf{R}^n.$$

Here we have used the notation $q^{(\alpha)}(\xi) = (\partial^\alpha / \partial \xi^\alpha) q(\xi)$. From Svensson [19] it follows that the condition (2) is equivalent to the condition

$$(2)' \quad P(x, \xi) \text{ is a hyperbolic polynomial with respect to } \vartheta \text{ in the sense of Gårding for each fixed } x \text{ in } \mathbf{R}^n.$$

Under the condition (2) the wave front sets of the solutions of the Cauchy problem (1) were studied in [20]. So, in this paper, we shall consider the following; (i) necessity of the condition (2) for the Cauchy problem (1) to be $\mathcal{E}$ -well posed, (ii) extension of Svensson's results in $\mathcal{E}^{*\kappa}$, (iii) well-posedness of the Cauchy problem (1) in $\mathcal{E}^{*\kappa}$, and (iv) singularities of the solutions of the Cauchy problem (1) in $\mathcal{E}^{*\kappa}$.

In general, the Cauchy problem (1) has no solution in $\mathcal{E}$ (or in $\mathcal{D}'$) even if $P_m(\xi)$ is hyperbolic with respect to ϑ and $f \in \mathcal{E}$ (or $\mathcal{D}'$). However, the solutions of (1) microlocally belong to $\mathcal{E}$ (or $\mathcal{D}'$) at almost every point in $T^*(\mathbf{R}^n) \setminus 0$. Let us consider the equation

$$\begin{cases} (D_1^2 + D_2)u(x) = \delta(x), \\ \text{supp } u(x) \subset \{x \in \mathbf{R}^2; x_1 \geq 0\}. \end{cases}$$

Then we have

$$u(x) = - \sum_{j=0}^{\infty} (-i)^j x_{1+}^{2j+1} / (2j+1)! \otimes \delta^{(j)}(x_2) \in \mathcal{D}^{(2)'}.$$

where $x_{1+} = x_1$ if $x_1 \geq 0$, $= 0$ if $x_1 < 0$. It can be proved that $u(x) = 0$ if $x_2 \neq 0$ and that $u(x)$ microlocally belongs to $\mathcal{D}'$ at (x, ξ) in $T^*(\mathbb{R}^2) \setminus 0$ if $(x, \xi) \neq ((x_1, 0), (0, \lambda))$ for all $\lambda > 0$ and $x \neq 0$. We shall show that this result can be extended to the general case.

The remainder of this paper is organized as follows. In chapter I we shall consider operators with constant coefficients, using the same arguments as in [1]. Larsson's result will be stated in § 1.1. The extension of Svensson's result will be given in § 1.2. In § 1.3 generalized wave front sets will be defined, which microlocally characterize singularities with respect to $\mathcal{E}^{**}$ and $\mathcal{D}^{**}$. The outer estimates of the generalized wave front sets of the solutions of (1) will be given in § 1.4. We shall give some remarks on the inner estimates of the generalized wave front sets of the fundamental solution of (1) in § 1.5. Chapter II will be devoted to operators with variable lower-order coefficients. In § 2.1 we shall construct solutions of (1) by successive iteration in the same way as in [20]. In § 2.2 necessary conditions for well-posedness in $\mathcal{E}^{**}$ will be given by the same arguments as in [11] when the coefficients of $P(x, D)$ are analytic. It will be shown in § 2.3 that the condition (2) is necessary and sufficient for the Cauchy problem (1) to be $\mathcal{E}$ -well posed and to have the finite propagation property, using the same arguments as in [12]. In § 2.4 the same results as in § 1.4 will be given for operators with variable lower-order coefficients. The class of hyperbolicity can be determined explicitly when $n \leq 3$. In § A.1 we shall give a procedure to determine the class of hyperbolicity when $n \leq 3$.

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Chapter I

Operators with constant coefficients

1.1. The Cauchy problem

In this chapter we assume that $P(D) \equiv P(x, D)$.

DEFINITION 1.1.1. Assume that $P_m(\xi)$ is a hyperbolic polynomial with respect to $\mathcal{D}$ and let Γ be a cone in $\mathbb{R}^n \setminus \{0\}$ and $\xi^0 \in \mathbb{R}^n \setminus \{0\}$. Then we define $\delta(P; \Gamma) \equiv \delta(\Gamma)$ and $\delta(P; \xi^0) \equiv \delta(\xi^0)$ by

$$\delta(\Gamma) = \inf \{ \delta; \delta \geq 0 \text{ and there exists } \gamma \text{ in } \mathbb{R} \text{ such that } P(\xi - is(1 + |\xi|)^{\delta} \mathcal{D}) \neq 0 \\ \text{for every } \xi \text{ in } \Gamma \text{ and } s > \gamma \},$$

$$\delta(\xi^0) = \inf \{ \delta(\Gamma); \Gamma \text{ is a conic neighborhood of } \xi^0 \}.$$

LEMMA 1.1.2. *Let Γ be a cone in $R^n \setminus \{0\}$ and $\xi^0 \in R^n \setminus \{0\}$. Then $\delta(\Gamma)$ and $\delta(\xi^0)$ have the following properties: (i) $\delta(\Gamma_1) \leq \delta(\Gamma_2)$ if $\Gamma_j, j=1, 2$, are cones in $R^n \setminus \{0\}$ and $\Gamma_1 \subset \Gamma_2$. (ii) $\delta(\xi^0) = \delta(\lambda \xi^0)$ for $\lambda > 0$. (iii) There is a conic neighborhood Γ_0 of ξ^0 such that $\delta(\xi^0) = \delta(\Gamma_0)$. (iv) There is a conic neighborhood Γ_0 of ξ^0 such that $\delta(\xi) \leq \delta(\xi^0)$ for every $\xi \in \Gamma_0$. (v) $\delta(\xi^0)$ is a rational number. (vi) If $\delta(\xi^0) > 0$, then for every conic neighborhood Γ of ξ^0 there are sequences $\{s_j\} \subset R$ and $\{\xi^j\} \subset \Gamma$ and a positive number C such that $|\xi^j| \rightarrow \infty, s_j = C |\xi^j|^{\delta(\xi^0)} (1 + o(1))$ as $j \rightarrow \infty$ and $P(\xi^j - is_j \partial) = 0$.*

PROOF. (i) and (ii) are obvious. Let us prove the assertion (iii). We may assume that $|\xi^0| = 1$. Consider the system

$$(1.1.1) \quad \begin{cases} P(\xi - is\partial) = 0, & s \in R, \\ |\xi|^2 = r^2, & r > 0, \\ |\xi - r\xi^0|^2 \leq \varepsilon^2 r^2, & 0 < \varepsilon \leq \varepsilon_0. \end{cases}$$

Seidenberg's lemma asserts that there are systems $T_1, \dots, T_N$ consisting of a finite number of real polynomial equations and inequalities in (s, r, ε) such that for fixed (s, r, ε) the existence of real solutions ξ of (1.1.1) is equivalent to the condition that (s, r, ε) satisfies at least one of the systems $T_1, \dots, T_N$ (see, e.g., [6]). Let $\{p_j\}$ be the collection of all polynomials in (s, r, ε) which are contained in the systems $T_1, \dots, T_N$. The polynomial $\prod_j p_j(s, r, \varepsilon)$ has a factorization

$$\prod_j p_j(s, r, \varepsilon) = p^1(s, r, \varepsilon)^{j_1} \cdots p^\nu(s, r, \varepsilon)^{j_\nu}.$$

Put $p(s, r, \varepsilon) = \prod_{j=1}^\nu p^j(s, r, \varepsilon)$ and let $\Delta(r, \varepsilon)$ be the discriminant of $p(s, r, \varepsilon)$ with respect to s . Since $\Delta(r, \varepsilon) \not\equiv 0$, there are positive numbers ε_1, r_1 and a positive integer l such that $\varepsilon_1 \leq \varepsilon_0$ and $\Delta(r, \varepsilon) \not\equiv 0$ for $r^{-1/l} \leq \varepsilon \leq \varepsilon_1, r \geq r_1$. Thus $p(s, r, \varepsilon) = 0$ has only simple roots for $(r, \varepsilon) \in D \equiv \{(r, \varepsilon) \in R^2; r \geq r_1 \text{ and } r^{-1/l} \leq \varepsilon \leq \varepsilon_1\}$. Let $\phi_j(r, \varepsilon), 1 \leq j \leq \omega$, be the roots of $p(s, r, \varepsilon) = 0$. Then the $\phi_j(r, \varepsilon)$ are analytic in D . Moreover $\phi_j(r, \varepsilon)$ is real for every (r, ε) in D if $\phi_j(r_2, \varepsilon_2)$ is real for some (r_2, ε_2) in D , since $p(s, r, \varepsilon)$ is a real polynomial. Thus we can assume that

$$\begin{aligned} \phi_1(r, \varepsilon) &< \phi_2(r, \varepsilon) < \cdots < \phi_\nu(r, \varepsilon), \\ \operatorname{Im} \phi_j(r, \varepsilon) &\not\equiv 0, \quad \nu + 1 \leq j \leq \omega, \quad (r, \varepsilon) \in D. \end{aligned}$$

If there is (ξ, s) satisfying (1.1.1) for some (r_2, ε_2) in D , then there is (ξ, s) satisfying (1.1.1) for each (r, ε) in D . Put

$$(1.1.2) \quad \hat{s} \equiv \hat{s}(r, \varepsilon) = \sup \{s; \xi \in R^n \text{ and } \xi \text{ satisfies (1.1.1) for } (s, r, \varepsilon)\}.$$

Then we have $\hat{s} = \phi_{j_0}(r, \varepsilon)$ in D for some $j_0, 0 \leq j_0 \leq \nu$, where $\phi_0(r, \varepsilon) = -\infty$. Here we have defined $\hat{s}(r, \varepsilon) = -\infty$ unless there is (ξ, s) satisfying (1.1.1) for fixed

(r, ε) . Write $p(s, r, \varepsilon) = \sum_{j,k} a_{jk}(\varepsilon) s^j r^k$. Then, modifying ε_1 , if necessary, we can assume that $a_{jk}(\varepsilon) \neq 0$ for $0 < \varepsilon \leq \varepsilon_1$ if $a_{jk}(\varepsilon) \neq 0$. For each ε with $0 < \varepsilon \leq \varepsilon_1$ there are real number $C_j(\varepsilon)$ and $\delta_j \in \mathbb{Q}$ such that

$$\phi_j(r, \varepsilon) = C_j(\varepsilon) r^{\delta_j} (1 + o(1)) \quad \text{as } r \rightarrow \infty, (r, \varepsilon) \in D, 1 \leq j \leq \nu.$$

Here the $C_j(\varepsilon)$ and δ_j are determined by the Newton polygon for $p(s, r, \varepsilon)$. Therefore the δ_j and $\text{sgn } C_j(\varepsilon)$ are independent of ε . Put $\delta_0 = 0$. Then we have

$$\delta(\Gamma) = \begin{cases} \max(\delta_{j_0}, 0), & \text{if } C_{j_0}(\varepsilon_1) > 0, \\ 0, & \text{otherwise,} \end{cases}$$

if Γ is a conic neighborhood of ξ^0 and $\Gamma \subset \{\xi \in \mathbb{R}^n; |\xi - |\xi| \xi^0| \leq \varepsilon_1 |\xi|\} \equiv \Gamma_0$. This proves (iii). The assertions (iv)-(vi) easily follow from the above. **Q.E.D.**

LEMMA 1.1.3. *The set $\{\delta(P; \xi); \xi \in \mathbb{R}^n \setminus \{0\}\}$ is a finite set.*

PROOF. In (1.1.1) we regard s, r, ε and ξ^0 ($|\xi^0| = 1$) as parameters and repeat the same arguments as in the proof of Lemma 1.1.2. Then we obtain the polynomial $p(s, r, \varepsilon, \xi^0)$ instead of $p(s, r, \varepsilon)$ in the proof of Lemma 1.1.2. For each fixed ξ^0 with $|\xi^0| = 1$ we define $\hat{s} \equiv \hat{s}(r, \varepsilon)$ by (1.1.2). Then we have $p(\hat{s}, r, \varepsilon, \xi^0) = 0$ or $\hat{s} = -\infty$. Regarding $p(s, r, \varepsilon, \xi^0)$ as a polynomial of (s, r) , the Newton polygon of $p(s, r, \varepsilon, \xi^0)$ depends on whether the coefficient of each monomial contained in $p(s, r, \varepsilon, \xi^0)$ vanishes or not. Thus we have at most finitely many Newton polygons as ξ^0 varies. From this Lemma 1.1.3 easily follows. **Q.E.D.**

DEFINITION 1.1.4. Let $1 < \kappa \leq \infty$. An operator $P(D)$ is called hyperbolic of class $*_\kappa$ with respect to $\mathcal{G}$ if there exist a closed cone K in $\mathbb{R}^n$ and $E(x) \equiv E(P, \mathcal{G}; x) \in \mathcal{D}^{*_{\kappa'}}$ such that $K \setminus \{0\} \subset \{x \in \mathbb{R}^n; x \cdot \mathcal{G} > 0\}$ and

$$P(D)E(x) = \delta(x), \text{ supp } E \subset K.$$

Then $E(P, \mathcal{G}; x)$ is called the fundamental solution.

REMARK. The fundamental solution $E(P, \mathcal{G}; x)$ is unique if it exists.

PROPOSITION 1.1.5 (Gårding [4], Larsson [15]). *Let $1 < \kappa \leq \infty$. $P(D)$ is hyperbolic of class $*_\kappa$ with respect to $\mathcal{G}$ if and only if $P_m(\xi)$ is hyperbolic with respect to $\mathcal{G}$ and $\kappa \leq 1/\delta(P)$ for $*_\kappa = (\kappa)$ and $\kappa < 1/\delta(P)$ for $*_\kappa = \{\kappa\}$, where*

$$\delta(P) = \max \{\delta(P; \xi); \xi \in \mathbb{R}^n \setminus \{0\}\}.$$

1.2. Hyperbolic polynomials

Let $H(\xi)$ be a hyperbolic polynomial with respect to $\mathcal{D}$, homogeneous of degree m and let $S(\xi)$ be a homogeneous polynomial of degree m' , $m' < m$. For $\xi^0 \in \mathbf{R}^n \setminus \{0\}$ we put

$$N_{\xi^0} = \{\hat{\gamma}(s); \hat{\gamma}(s) = s^{-1}(\xi^0 + \sum_{j=1}^{\infty} s^{j/l} \xi^j), \xi^j \in \mathbf{R}^n, l \in \mathbf{N} \text{ and } s\hat{\gamma}(s) \text{ is convergent in a neighborhood of } s=0\}.$$

DEFINITION 1.2.1. Let $\xi^0 \in \mathbf{R}^n \setminus \{0\}$. For each $\hat{\gamma}(s) \in N_{\xi^0}$ we define the rational number $n(\hat{\gamma}) \equiv n(H, S; \hat{\gamma})$ by

$$(1.2.1) \quad S(\hat{\gamma}(s) - i\mathcal{D})/H(\hat{\gamma}(s) - i\mathcal{D}) = c(\hat{\gamma})s^{-n(\hat{\gamma})}(1 + o(1)) \quad \text{as } s \rightarrow +0,$$

where $c(\hat{\gamma}) \neq 0$. Here we have defined $n(\hat{\gamma}) = -\infty$ if $S(\hat{\gamma}(s) - i\mathcal{D}) \equiv 0$. Moreover we define

$$n(\xi^0) \equiv n(H, S; \xi^0) = \sup \{n(\hat{\gamma}); \hat{\gamma} \in N_{\xi^0}\}.$$

DEFINITION 1.2.2. Let Γ be a cone in $\mathbf{R}^n \setminus \{0\}$. We define $n(\Gamma) \equiv n(H, S; \Gamma)$ by

$$n(\Gamma) = \inf \{k; |S(\xi - i\mathcal{D})/H(\xi - i\mathcal{D})| \leq C(1 + |\xi|)^k \text{ for every } \xi \in \Gamma\}^\dagger.$$

LEMMA 1.2.3. Let Γ be a cone in $\mathbf{R}^n \setminus \{0\}$ and $\xi^0 \in \mathbf{R}^n \setminus \{0\}$. Then $n(\Gamma)$ and $n(\xi^0)$ have the following properties: (i) $n(\Gamma_1) \leq n(\Gamma_2)$ if Γ_j , $j=1, 2$, are cones in $\mathbf{R}^n \setminus \{0\}$ and $\Gamma_1 \subset \Gamma_2$. (ii) $n(\xi^0) = n(\lambda \xi^0)$ for $\lambda \in \mathbf{R} \setminus \{0\}$. (iii) $n(\xi^0) \leq n(\Gamma)$ if Γ is a conic neighborhood of ξ^0 . (iv) There is a conic neighborhood Γ_0 of ξ^0 such that $n(\xi^0) = n(\Gamma_0)$. (v) $m' - m + \deg H_{\xi^0} - \deg S_{\xi^0} \leq n(\xi^0) \leq m' - m + \deg H_{\xi^0} \leq m'$. (vi) $n(\xi^0) = m' - m + \deg H_{\xi^0}$ if $S(\xi^0) \neq 0$. (vii) $n(\xi^0) = m' - m$ if $H(\xi^0) \neq 0$. (viii) We assume that $H(\xi)$ and $S(\xi)$ have no non-constant common divisors and that $H(\xi) = \prod_k h_k(\xi)^{j_k}$. Then $n(\xi^0) = m' - m + \deg H_{\xi^0}$ if $(\partial/\partial \xi_1)R(\xi^0) \neq 0$, where $R(\xi) = \prod_k h_k(\xi)$. Here H_{ξ^0} is the localization of H at ξ^0 which is defined by $H(s^{-1}\xi^0 + \xi) = s^r(H_{\xi^0}(\xi) + o(1))$ as $s \rightarrow 0$, $H_{\xi^0}(\xi) \neq 0$.

PROOF. The assertions (i)-(iii) are obvious. Let us prove the assertion (iv). We may assume that $|\xi^0| = 1$ and that $S(\xi^0) \neq 0$. Consider the system

$$(1.2.2) \quad \begin{cases} \phi |H(\xi - i\mathcal{D})|^2 - |S(\xi - i\mathcal{D})|^2 = 0, & \phi \in \mathbf{R}, \\ |\xi|^2 = r^2, & r > 0, \\ |\xi - r\xi^0|^2 \leq \varepsilon^2 r^2, & 0 < \varepsilon \leq \varepsilon_0. \end{cases}$$

We repeat the same arguments as in the proof of Lemma 1.1.2. Put

$$\hat{\phi} \equiv \hat{\phi}(r, \varepsilon) = \max \{\phi; \xi \in \mathbf{R}^n \text{ and } \xi \text{ satisfies (1.2.2) for } (\phi, r, \varepsilon)\}$$

[†] Here and in the sequel C denotes a positive constant.

for each fixed (r, ε) and

$$D = \{(r, \varepsilon) \in \mathbf{R}^2; r \geq r_1 \text{ and } r^{-1/l} \leq \varepsilon \leq \varepsilon_1\}$$

for sufficiently small ε_1 and sufficiently large $r_1 > 0$ and $l \in \mathbf{N}$. Then there is a polynomial $p(\phi, r, \varepsilon)$ such that $\hat{\phi}(r, \varepsilon)$ is a simple root of $p(\phi, r, \varepsilon) = 0$ if $(r, \varepsilon) \in D$. We can write $\hat{\phi}(r, \varepsilon)$ in the form

$$(1.2.3) \quad \hat{\phi}(r, \varepsilon) = C(\varepsilon)r^\rho + r^\rho\psi(r, \varepsilon), \quad \psi(r, \varepsilon) = o(1) \quad \text{as } r \rightarrow \infty, \text{ for } (r, \varepsilon) \in D,$$

where $C(\varepsilon) \neq 0$ and ρ are determined by the Newton polygon of $p(\phi, r, \varepsilon)$ with respect to (ϕ, r) , since $\hat{\phi}(r, \varepsilon) > 0$. Modifying ε_1 , if necessary, ρ can be taken to be independent of ε in (1.2.3) and $C(\varepsilon)$ is represented as a Puiseux series of ε . (1.2.3) implies that

$$n(\Gamma_0) = \inf \{n(\Gamma); \Gamma \text{ is a conic neighborhood of } \xi^0\} = \rho/2,$$

where $\Gamma_0 = \{\xi \in \mathbf{R}^n; |\xi - \xi^0| \leq \varepsilon_1 |\xi^0|\}$. Let us first prove that there are a positive numbers ν and C such that

$$(1.2.4) \quad |\psi(r, \varepsilon)| \leq Cr^{-\nu} \quad \text{in } D.$$

Assume that $\psi(r, \varepsilon) \neq 0$. Put

$$q(\psi, r, \varepsilon) = r^{-L}p(C(\varepsilon)r^\rho + r^\rho\psi, r, \varepsilon),$$

where $p(C(\varepsilon)r^\rho + r^\rho\psi, r, \varepsilon) = O(r^\mu)$ as $r \rightarrow \infty$. Then $q(\psi, r, \varepsilon)$ is a polynomial of ψ and $r^{-1/L}$ and its coefficients are Puiseux series of ε , where $L \in \mathbf{N}$ is chosen suitably. We note that a Puiseux series $a(\varepsilon)$ of ε does not vanish for $0 < \varepsilon \leq \varepsilon_1$ if $a(\varepsilon) \neq 0$ and ε_1 is sufficiently small. Thus we can write $\psi(r, \varepsilon)$ in the form

$$(1.2.5) \quad \psi(r, \varepsilon) = C_1(\varepsilon)r^{-\nu'}(1 + o(1)) \quad \text{as } r \rightarrow \infty \text{ for } (r, \varepsilon) \in D,$$

where $C_1(\varepsilon) \neq 0$ and $\nu' > 0$ are determined by the Newton polygon of $q(\psi, r, \varepsilon)$ with respect to $(\psi, r^{-1/L})$. Modifying ε_1 , if necessary, ν' can be chosen to be independent of ε in (1.2.5). Put $\hat{\psi}(r, \varepsilon) = r^{\nu'}\psi(r, \varepsilon)$. Then we have

$$q_1(\hat{\psi}, \varepsilon) + r^{-\nu_1}q_2(\hat{\psi}, r, \varepsilon) = 0,$$

where $\nu_1 > 0$, $q_1(\psi, \varepsilon)$ is a polynomial of ψ , $q_2(\psi, r, \varepsilon)$ is a polynomial of ψ and $r^{-1/L}$ and the coefficients of $q_1(\psi, \varepsilon)$ and $q_2(\psi, r, \varepsilon)$ are Puiseux series of ε . There are positive numbers k, ε_1 and C and $k_1 \in \mathbf{R}$ such that

$$|q_1(\psi, \varepsilon)| \geq C\varepsilon^{-k_1} \quad \text{if } |\psi| \geq \varepsilon^{-k} \text{ and } 0 < \varepsilon \leq \varepsilon_1.$$

On the other hand we have

$$|q_2(\psi, r, \varepsilon)| \leq C\varepsilon^{-k_2}, \quad r \geq r_1, \quad 0 < \varepsilon \leq \varepsilon_1,$$

for some constants C and k_2 if $|\psi| \geq \varepsilon^{-k}$. Thus $|q_1(\psi, \varepsilon)| > r^{-\nu_1} |q_2(\psi, r, \varepsilon)|$ if $|\psi| \geq \varepsilon^{-k}$, $r \geq r_1$, $r^{-1/l} \leq \varepsilon \leq \varepsilon_1$ and $l \in \mathbb{N}$ is sufficiently large. This implies that $|\hat{\psi}(r, \varepsilon)| \leq r^{k/l}$ for $(r, \varepsilon) \in D$. Modifying l , if necessary, we have (1.2.4) with $\nu = \nu' - k/l > 0$. Since $C(\varepsilon) = A\varepsilon^a(1 + o(1))$, $\varepsilon \rightarrow +0$, for some $A \in \mathbb{R} \setminus \{0\}$ and $a \in \mathbb{Q}$, it follows from (1.2.4) that

$$(1.2.6) \quad \hat{\phi}(r, r^{-1/l}) = Ar^{\rho-a/l}(1 + o(1)) \quad \text{as } r \rightarrow \infty,$$

modifying l . Now assume that

$$n(\xi^0) < n(\Gamma_0) = \rho/2.$$

Then we can assume that $n(\xi^0) < (\rho - a/l)/2$, modifying l . Thus there are integers p and q such that $p > 0$, p and q are relatively prime and

$$(1.2.7) \quad 2n(\xi^0) < q/p < \rho - a/l.$$

Let l' be the least common multiple of p and l . Following the proof of Lemm 1.2 in [19], we consider the system

$$(1.2.8) \quad \begin{cases} sr^{q'} |H(\xi - i\vartheta)|^2 - |S(\xi - i\vartheta)|^2 = 0, \\ |\xi|^2 = r^{2l'}, & r \geq 1, \\ |\xi - r^{l'} \xi^0|^2 \leq r^{2l' - l''}, \end{cases}$$

where $q' = l'q/p$, $l'' = 2l'/l$. Then Seidenberg's lemma asserts for every j , $1 \leq j \leq n$, the existence of a condition H_j consisting of a finite number of systems of real polynomial equations and inequalities in $(\xi_1, \dots, \xi_j, s)$ such that for every $(\xi_1, \dots, \xi_j, s) \in \mathbb{R}^{j+1}$ the following conditions are equivalent:

(i) There exist real $\xi_{j+1}, \dots, \xi_n, r$ so that (ξ, s, r) satisfies the system (1.2.8).

(ii) At least one of the systems in H_j is satisfied by $(\xi_1, \dots, \xi_j, s)$.

Assume that for some j , $1 \leq j \leq n$, there are Puiseux series $\gamma_1(s), \dots, \gamma_{j-1}(s)$, convergent and real for all large real s , such that the system (1.2.8) is satisfied by some real $\xi = (\gamma_1(s), \dots, \gamma_{j-1}(s), \xi_j, \dots, \xi_n)$ and r for some arbitrarily large real s . It follows from (1.2.6) and (1.2.7) that there are real ξ and r satisfying (1.2.8) for some arbitrarily large real s . This means that the assumption is valid when $j=1$. If $(\gamma_1(s), \dots, \gamma_{j-1}(s), \xi_j, s)$ satisfies the condition H_j , then there is a Puiseux series $\gamma_j(s)$, convergent and real for all large real s , such that $(\gamma_1(s), \dots, \gamma_{j-1}(s), \gamma_j(s), s)$ satisfies the condition H_j (see [19]). Thus we obtain Puiseux series $\gamma_1(s), \dots, \gamma_n(s)$ such that $\xi = (\gamma_1(s), \dots, \gamma_n(s))$ and $r = |\xi|^{1/l'}$ satisfy the system (1.2.8) for all sufficiently large real s . Put $\eta(s) = (\gamma_1(s), \dots, \gamma_n(s))$. We can write $\eta(s)$ in the form

$$\eta(s) = Cs^{l'h/L}(\xi^0 + \sum_{j=1}^{\infty} s^{-j/L} \xi^j),$$

where $C > 0$ and $h, L \in \mathbb{N}$. In fact, we have $r(s) = C's^{h/L}(1+o(1))$ as $s \rightarrow \infty$, where $r(s) = |\gamma(s)|^{1/L'}$. Replacing $Cs^{h/L}$ by s^{-1} , we put

$$\hat{\gamma}(s) = s^{-1}(\xi^0 + \sum_{j=1}^{\infty} s^{j/L'} \tilde{\xi}^j),$$

where $L' = l'h$, $\tilde{\xi}^j = C^{j/L'} \xi^j$. Then we have

$$|S(\hat{\gamma}(s) - i\vartheta)/H(\hat{\gamma}(s) - i\vartheta)|^p = Cs^{-L/L' - q/p}(1+o(1)), \quad \text{as } s \rightarrow +0.$$

This implies that $2n(\xi^0) \geq q/p + L/L'$, which contradicts (1.2.7). This completes the proof of the assertion (iv). If $\hat{\gamma}(s) = s^{-1}\xi^0 + \xi^1$ and $S_{\xi^0}(\xi^1 - i\vartheta) \neq 0$, we have $n(\hat{\gamma}(s)) \geq m' - m + \deg H_{\xi^0} - \deg S_{\xi^0}$, which implies that $n(\xi^0) \geq m' - m + \deg H_{\xi^0} - \deg S_{\xi^0}$. On the other hand we have

$$|H(\hat{\gamma}(s) - i\vartheta)| \geq Cs^a, \quad 0 < s \leq s_0, \quad a = -m + \deg H_{\xi^0},$$

for $\hat{\gamma} \in N_{\xi^0}$ and some $s_0 > 0$. In fact, we have

$$|H(\hat{\gamma}(s) - i\vartheta)| \geq C\tilde{H}(\hat{\gamma}(s)) \geq C|H^{(\alpha)}(\hat{\gamma}(s))|$$

(see [7]). Moreover $H^{(\alpha)}(\xi^0) \neq 0$ if $\alpha = (\deg H_{\xi^0}, 0, \dots, 0)$ (see [1]). This proves the assertion (v). The assertion (vi) easily follows from (v). The assertion (vii) is obvious in virtue of (iv). Finally let us prove the assertion (viii). Assume that $H(\xi^0) = 0$. For every $\varepsilon > 0$ there exists $\xi_\varepsilon \in \mathbb{R}^n$ such that $H(\xi_\varepsilon) = 0$, $S(\xi_\varepsilon) \neq 0$ and $|\xi^0 - \xi_\varepsilon| < \varepsilon$, since $H(\xi)$ and $S(\xi)$ are relatively prime. Moreover the assumption of (viii) implies that $\deg H_{\xi^0} = \deg H_{\xi_\varepsilon}$ if ε is sufficiently small. Since

$$|S(s^{-1}\xi_\varepsilon - i\vartheta)/H(s^{-1}\xi_\varepsilon - i\vartheta)| \geq Cs^a, \quad 0 < s \leq s_0, \quad a = -m' + m - \deg H_{\xi_\varepsilon},$$

we have $n(\xi^0) \geq m' - m + \deg H_{\xi^0}$, which proves the assertion (viii). In fact, (viii) is an immediate consequence of (vii) if $H(\xi^0) \neq 0$. **Q.E.D.**

LEMMA 1.2.4. *Let $\xi^0 \in \mathbb{R}^n \setminus \{0\}$ and $\hat{\gamma}(s) \in N_{\xi^0}$ and put*

$$\hat{\gamma}_a^\kappa(s) = s^{\kappa/(1-\kappa)} \hat{\gamma}(s^{1/(1-\kappa)}) + a\vartheta \quad \text{for } 0 \leq \kappa < 1 \text{ and } a \in \mathbb{R}.$$

Then we have

$$(1.2.9) \quad (\hat{\gamma}_a^\kappa)_{a'}^{\kappa'}(s) = \hat{\gamma}_{a'+\kappa'-\kappa\kappa'}^{\kappa'+\kappa'-\kappa\kappa'}(s) + as^{\kappa'/(1-\kappa')}\vartheta, \quad 0 \leq \kappa, \kappa' < 1 \quad \text{and} \quad a, a' \in \mathbb{R}.$$

Moreover there exist a rational number κ_0 with $0 \leq \kappa_0 < 1$ and $a_0 \in \mathbb{R}$ such that

$$n(H, S; \hat{\gamma}_{a_0}^{\kappa_0}) \geq n(H, S; \hat{\gamma}_a^\kappa), \quad 0 \leq \kappa < 1, \quad a \in \mathbb{R},$$

where $n(H, S; \hat{\gamma}_a^\kappa)$ is defined by (1.2.1) even if $\hat{\gamma}_a^\kappa \notin N_{\xi^0}$.

PROOF. (1.2.9) easily follows from the definition of $\hat{\gamma}_a^\kappa(s)$. Let $\tau_j(s)$, $1 \leq$

$j \leq m$, and $\lambda_k(s)$, $1 \leq k \leq t$, be the roots of $H(s\hat{\gamma}(s) + \tau\vartheta) = 0$ and $S(s\hat{\gamma}(s) + \tau\vartheta) = 0$, respectively. Write $S(s\hat{\gamma}(s) + \tau\vartheta) = \sum_{j=0}^t q_j(s)\tau^{t-j}$. Then we have

$$\begin{aligned} H(\hat{\gamma}(s) + \tau\vartheta) &= s^{-m} H(\vartheta) \prod_{k=1}^m (s\tau - \tau_k(s)), \\ S(\hat{\gamma}(s) + \tau\vartheta) &= s^{-m'} q_0(s) \prod_{k=1}^t (s\tau - \lambda_k(s)). \end{aligned}$$

Put $\mu_k = \text{order } \tau_k(s)$, $\mu'_k = \text{order } \lambda_k(s)$ and $\alpha = \text{order } q_0(s)$, where $\mu = \text{order } \lambda(s)$ implies that $\lambda(s) = As^\mu(1 + o(1))$ as $s \rightarrow +0$ for some non-zero constant A if $\mu < \infty$ and that $\lambda(s) \equiv 0$ if $\mu = \infty$. Let $0 \leq \kappa < 1$. It is obvious that $\alpha = \text{order } q(s) \geq 0$. If $\alpha = \infty$, then $n(H, S; \hat{\gamma}_a^*) = -\infty$ for all $a \in \mathbb{R}$ and $0 \leq \kappa < 1$. So let us consider the case $\alpha < \infty$. We have

$$(1.2.10) \quad \begin{aligned} S(\hat{\gamma}(s) + s^{-\kappa}(\tau - i)\vartheta) / H(\hat{\gamma}(s) + s^{-\kappa}(\tau - i)\vartheta) &= s^\alpha (d_\kappa(\tau) + o(1)) \quad \text{as } s \rightarrow +0, \\ a = m - m' + \alpha - \sum_{k=1}^m \min(\mu_k, 1 - \kappa) + \sum_{k=1}^t \min(\mu'_k, 1 - \kappa), \end{aligned}$$

$$(1.2.11) \quad \begin{aligned} s^{-\kappa(m' - m)} S(s^\kappa \hat{\gamma}(s) + (\tau - i)\vartheta) / H(s^\kappa \hat{\gamma}(s) + (\tau - i)\vartheta) &= s^\alpha (d_{\tau, \kappa} + o(1)) \\ &\quad \text{as } s \rightarrow +0, \\ a = -\kappa(m' - m) - n(H, S; \hat{\gamma}_a^*)(1 - \kappa), \end{aligned}$$

where $d_\kappa(\tau)$ is a rational function of τ and $d_{\tau, \kappa}$ is a non-zero constant depending on τ and κ . Since the $\tau_k(s)$ are real for $s > 0$, it follows that $|d_\kappa(\tau)| < \infty$ for real τ . (1.2.10) and (1.2.11) imply that

$$(1.2.12) \quad \begin{aligned} n(H, S; \hat{\gamma}_a^*) &= m' - m + (1 - \kappa)^{-1} \{ -\alpha + \sum_{k=1}^m \min(\mu_k, 1 - \kappa) \\ &\quad - \sum_{k=1}^t \min(\mu'_k, 1 - \kappa) \} \equiv n(\kappa) \quad \text{if } d_\kappa(a) \neq 0, \end{aligned}$$

and that

$$n(H, S; \hat{\gamma}_a^*) < n(\kappa) \quad \text{if } d_\kappa(a) = 0.$$

Put $\beta = \min_{0 \leq j \leq t} \text{order } q_j(s)$. Then we have $\mu_k \geq 0$, $1 \leq k \leq m$, and $\sum_{\mu'_k \leq 0} \mu'_k = \beta - \alpha$ (see Lemma A.1.2). The set $\{\mu_k; 0 < \mu_k \leq 1\}$ is written as follows:

$$\{l_k; k = 1, \dots, N\} = \{\mu_k; 0 < \mu_k \leq 1\}, \quad 0 < l_1 < l_2 < \dots < l_N \leq 1.$$

Now assume that $\kappa_{k+1} \equiv 1 - l_{k+1} < \kappa < \kappa_k \equiv 1 - l_k$, $0 \leq k \leq N$, where $l_0 = 0$ and $l_{N+1} = 1$. Then we have

$$\begin{aligned} n(\kappa_{k+1}) - n(\kappa) &= (l_{k+1}^{-1} - (1 - \kappa)^{-1}) \{ -\beta + \sum_{\mu_j \leq l_k} \mu_j - \sum_{0 < \mu'_j \leq 1 - \kappa} \mu'_j \} \\ &\quad + \sum_{1 - \kappa < \mu'_j \leq l_{k+1}} (1 - \mu'_j / l_{k+1}), \quad 0 \leq k \leq N, \\ n(\kappa_k) - n(\kappa) &= (l_k^{-1} - (1 - \kappa)^{-1}) \{ -\beta + \sum_{\mu_j \leq l_k} \mu_j - \sum_{0 < \mu'_j \leq 1 - \kappa} \mu'_j \} \\ &\quad + \sum_{l_k < \mu'_j \leq 1 - \kappa} (\mu'_j / l_k - 1), \quad 1 \leq k \leq N. \end{aligned}$$

Since $\beta \geq 0$, we have $n(\kappa_1) \geq n(\kappa)$ for $\kappa_1 < \kappa < 1$. Moreover we have, for $\kappa_{k+1} < \kappa < \kappa_k$,

$$\begin{aligned} n(\kappa_k) &\geq n(\kappa), & \text{if } -\beta + \sum_{\mu_j \leq l_k} \mu_j - \sum_{0 < \mu'_j \leq 1 - \kappa} \mu'_j &\geq 0, \\ n(\kappa_{k+1}) &> n(\kappa), & \text{otherwise.} \end{aligned}$$

Therefore we have

$$(1.2.13) \quad \max_{1 \leq k \leq N+1} n(\kappa_k) = \sup_{0 \leq \kappa < 1} n(\kappa).$$

From (1.2.12) and (1.2.13) Lemma 1.2.4 easily follows. Q.E.D.

THEOREM 1.2.5. *Assume that $P_m(\xi)$ is hyperbolic with respect to $\mathcal{V}$ and let $\xi^0 \in \mathbb{R}^n \setminus \{0\}$. Then we have*

$$\begin{aligned} (1.2.14) \quad & \max(\delta(P; \xi^0), \delta(P; -\xi^0)) \\ &= \max_{0 \leq j \leq m-1} n_+(P_m, Q_j; \xi^0) / (m - j + n_+(P_m, Q_j; \xi^0)) \\ & (\leq \max((\deg P_{m, \xi^0} - 1) / \deg P_{m, \xi^0}, 0)), \end{aligned}$$

where $n_+(H, S; \xi^0) = \max(n(H, S; \xi^0), 0)$. In particular,

$$\delta(P) = \max_{|\xi|=1, 0 \leq j \leq m-1} n_+(P_m, Q_j; \xi) / (m - j + n_+(P_m, Q_j; \xi)).$$

REMARK. (i) Theorem 1.2.5 is an extension of Svensson's result ([19]). In fact, it follows from Proposition 1.1.5 that $P(D)$ is hyperbolic of class (κ) with respect to $\mathcal{V}$ if and only if $P_m(\xi)$ is hyperbolic with respect to $\mathcal{V}$ and

$$|Q_j(\xi - i\mathcal{V}) / P_m(\xi - i\mathcal{V})| \leq C(1 + |\xi|)^{(m-j)/(\kappa-1)}, \quad \xi \in \mathbb{R}^n, \quad 0 \leq j \leq m-1.$$

(ii) The left-hand side of (1.2.14) can not be replaced by $\delta(P; \xi^0)$. For example, if $P(\xi) = \xi_1^2 - \xi_2$ and $\xi^0 = (0, 1)$, then $\delta(P; \xi^0) = 0$ and $n(P_2, Q_1; \xi^0) = 1$.

PROOF. Let us first prove that

$$(1.2.15) \quad \delta(P; \xi^0) \leq \max_{0 \leq j \leq m-1} n_+(P_m, Q_j; \xi^0) / (m - j + n_+(P_m, Q_j; \xi^0)) \equiv \delta.$$

Let Γ_0 be a sufficiently small conic neighborhood of ξ^0 . Then it follows from Lemma 1.2.3 that

$$\begin{aligned} & |P_m(\xi - i\gamma(1 + |\xi|)^\delta \mathcal{V})| - |\sum_{j=0}^{m-1} Q_j(\xi - i\gamma(1 + |\xi|)^\delta \mathcal{V})| \\ & \geq |P_m(\xi - i\gamma(1 + |\xi|)^\delta \mathcal{V})| (1 - \sum_{j=0}^{m-1} C_j \gamma^{j-m} \\ & \quad \times (1 + |\xi|)^\delta (j - m - n_+(P_m, Q_j; \xi^0) + n_+(P_m, Q_j; \xi^0)) > 0 \end{aligned}$$

if $\xi \in \Gamma_0$ and γ is sufficiently large, since $\delta(j - m - n_+(P_m, Q_j; \xi^0) + n_+(P_m, Q_j; \xi^0)) \leq 0$, $0 \leq j \leq m-1$. This proves (1.2.15). We may assume that $\delta > 0$, that is, there is j with $0 \leq j \leq m-1$ such that $n(P_m, Q_j; \xi^0) > 0$. In fact, Theorem 1.2.5 easily follows from (1.2.15) if $\delta = 0$. Then for every ε , $0 < \varepsilon < \delta$, there is $\hat{\gamma}(s)$ in N_{ξ^0} such that

$$0 < \delta - \varepsilon < \max_{0 \leq j \leq m-1} n_+(P_m, Q_j; \hat{\gamma}) / (m - j + n_+(P_m, Q_j; \hat{\gamma})) \equiv \delta(\hat{\gamma}) \leq \delta < 1.$$

By Lemma 1.2.4 we can assume without loss of generality that

$$(1.2.16) \quad \begin{aligned} n(P_m, Q_j; \hat{\eta}) &\geq n(P_m, Q_j; \hat{\eta}_a^*) \\ \text{if } n_+(P_m, Q_j; \hat{\eta})/(m-j+n_+(P_m, Q_j; \hat{\eta})) &= \delta(\hat{\eta}), \quad 0 \leq \kappa < 1 \quad \text{and} \quad a \in \mathbf{R}. \end{aligned}$$

Moreover we can assume that

$$n(P_m, Q_j; \hat{\eta}) \geq n(P_m, Q_j; \hat{\eta}_a^0) \quad \text{for all } a \in \mathbf{R} \quad \text{and} \quad 0 \leq j \leq m-1.$$

Write

$$\begin{aligned} P_m(\hat{\eta}(s) + \tau \mathcal{G}) &= s^\rho (a_m \tau^{k_m} + b_m(\tau)) + o(s^\rho) \quad \text{as } s \rightarrow +0, \\ Q_j(\hat{\eta}(s) + \tau \mathcal{G}) &= s^{\rho - n(P_m, Q_j; \hat{\eta})} (a_j \tau^{k_j} + b_j(\tau)) + o(s^{\rho - n(P_m, Q_j; \hat{\eta})}) \quad \text{as } s \rightarrow +0, \end{aligned}$$

where $a_j \neq 0$ and the $b_j(\tau)$ are polynomials of degree less than k_j if $j = m$ or $n(P_m, Q_j; \hat{\eta}) > -\infty$. Put

$$\begin{aligned} p &\equiv p(\hat{\eta}) = \delta(\hat{\eta})/(1 - \delta(\hat{\eta})) \quad (> 0), \\ I(\tau; s, \gamma) &= \gamma^m s^{mp} P(\gamma^{-1} s^{-p} (\hat{\eta}(s) + \tau \mathcal{G})). \end{aligned}$$

Then we have

$$\begin{aligned} I(\tau; s, \gamma) &= s^\rho \{ a_m \tau^{k_m} + b_m(\tau) + \sum_{p(m-j) = n(P_m, Q_j; \hat{\eta})} \gamma^{m-j} (a_j \tau^{k_j} + b_j(\tau)) \} \\ &\quad + o(s^\rho) \quad \text{as } s \rightarrow +0. \end{aligned}$$

In fact, $p(m-j) - n(P_m, Q_j; \hat{\eta}) \geq 0$ for $0 \leq j \leq m-1$. Now let us prove that

$$(1.2.17) \quad \begin{aligned} k_j &\leq k_m - (1+p)(m-j) \\ \text{if } p(m-j) &= n(P_m, Q_j; \hat{\eta}) \quad \text{and} \quad 0 \leq j \leq m-1. \end{aligned}$$

For sufficiently small $\kappa > 0$ we have

$$(1.2.18) \quad P_m(\hat{\eta}(s) + s^{-\kappa} \tau \mathcal{G}) = s^{-m\kappa} P_m(s^\kappa \hat{\eta}(s) + \tau \mathcal{G}) = s^{\rho - k_m \kappa} (a_m \tau^{k_m} + o(1)) \quad \text{as } s \rightarrow +0,$$

$$(1.2.19) \quad \begin{aligned} Q_j(\hat{\eta}(s) + s^{-\kappa} \tau \mathcal{G}) &= s^{-j\kappa} Q_j(s^\kappa \hat{\eta}(s) + \tau \mathcal{G}) = s^{\rho - n(P_m, Q_j; \hat{\eta}) - k_j \kappa} (a_j \tau^{k_j} + o(1)) \\ &\quad \text{as } s \rightarrow +0, \end{aligned}$$

for each fixed $\tau \in \mathbf{R}$. Thus it follows from (1.2.18) and (1.2.19) that

$$\begin{aligned} m-j + n(P_m, Q_j; \hat{\eta}) &\leq k_m - k_j \\ \text{if } p(m-j) &= n(P_m, Q_j; \hat{\eta}) \quad \text{and} \quad 0 \leq j \leq m-1. \end{aligned}$$

This proves (1.2.17). Now put

$$I^0(\tau, \gamma) = a_m \tau^{k_m} + b_m(\tau) + \sum_{p(m-j) = n(P_m, Q_j; \hat{\eta})} \gamma^{m-j} (a_j \tau^{k_j} + b_j(\tau)).$$

Since $0 < (m-j)/(k_m - k_j) \leq 1/(1+p) < 1$ for $p(m-j) = n(P_m, Q_j; \hat{\eta})$, it follows

from the Newton polygon of $I^0(\tau; \gamma)$ that there is a root $\tau^0(\gamma)$ of $I^0(\tau; \gamma) = 0$ such that $\tau^0(\gamma) = A\gamma^\nu(1 + o(1))$ as $|\gamma| \rightarrow \infty$ for some rational number ν with $0 < \nu < 1$ and some non-zero constant A . Thus $I^0(\tau, \gamma_0)$ has a root τ_0 such that $\gamma_0 \operatorname{Im} \tau_0 < 0$, if γ_0 in $\mathbf{R}$ is suitably chosen. Then $I(\tau; s, \gamma_0) = 0$ has a root $\tau(s)$ such that $\tau(s) = \tau_0(1 + o(1))$ as $s \rightarrow +0$. Hence we have

$$P(\pm t^{-1}(\xi^0 + \sum_{j=1}^{\infty} (|\gamma_0|^{-1} t)^{j/((p+1)l)} \xi^j) \pm t^{-1}(|\gamma_0|^{-1} t)^{1/(p+1)} \tau(|\gamma_0|^{-1} t)^{1/(p+1)} \vartheta) = 0$$

if $\gamma_0 = \pm |\gamma_0|$.

This means that

$$\max(\delta(\xi^0), \delta(-\xi^0)) \geq p/(p+1) = \delta(\gamma) > \delta - \varepsilon.$$

The proof is complete. **Q.E.D.**

From now on we assume that $P_m(\xi)$ is a hyperbolic polynomial with respect to ϑ and denote by $\Gamma(P_m, \vartheta)$ the connected component of the set $\{\xi \in \mathbf{R}^n; P_m(\xi) \neq 0\}$ which contains ϑ .

LEMMA 1.2.6. *Let M be a compact set in $\Gamma(P_m, \vartheta)$. Then there exists a positive number γ_M such that*

$$P(\xi - i\gamma(1 + |\xi|)^{\delta(P)}\eta) \neq 0 \quad \text{for } \gamma > \gamma_M \quad \text{and} \quad \eta \in M.$$

PROOF. From hyperbolicity of P_m and Lemma 4.1.1 in [7] it follows that

$$(1.2.20) \quad |P_m(\xi - i\gamma(1 + |\xi|)^{\delta(P)}\eta)| \geq C\gamma^m(1 + |\xi|)^{m\delta(P)}\tilde{P}_m(\gamma^{-1}(1 + |\xi|)^{-\delta(P)}\xi),$$

$\xi \in \mathbf{R}^n \quad \text{and} \quad \eta \in M.$

Moreover we have

$$(1.2.21) \quad |Q_j(\xi - i\gamma(1 + |\xi|)^{\delta(P)}\eta)| \leq C\gamma^j(1 + |\xi|)^{j\delta(P)}\tilde{Q}_j(\gamma^{-1}(1 + |\xi|)^{-\delta(P)}\xi),$$

$\xi \in \mathbf{R}^n \quad \text{and} \quad \eta \in M, \quad 0 \leq j \leq m-1.$

On the other hand we have

$$(1.2.22) \quad \tilde{Q}_j(\xi)/\tilde{P}_m(\xi) \leq C(1 + |\xi|)^{n_+(P_m, Q_j)}, \quad \xi \in \mathbf{R}^n, \quad 0 \leq j \leq m-1,$$

where $n_+(P_m, Q_j) = \max_{|\xi|=1} n_+(P_m, Q_j; \xi)$. In fact, it follows from Lemma 3.1.5 in [7] that

$$\tilde{Q}_j(\xi) \leq C'\tilde{Q}_j(\xi - i\vartheta) \leq C(1 + |\xi|)^{n_+(P_m, Q_j)}\tilde{P}_m(\xi), \quad \xi \in \mathbf{R}^n, \quad 0 \leq j \leq m-1.$$

By (1.2.20)–(1.2.22) we have

$$\begin{aligned} \tilde{P}_m(\gamma^{-1}(1 + |\xi|)^{-\delta(P)}\xi)^{-1} |P_m(\xi - i\gamma(1 + |\xi|)^{\delta(P)}\eta)| \\ - \sum_{j=0}^{m-1} |Q_j(\xi - i\gamma(1 + |\xi|)^{\delta(P)}\eta)| \geq C\gamma^m(1 + |\xi|)^{m\delta(P)}(1 - 1/\gamma), \end{aligned}$$

$\xi \in \mathbf{R}^n \quad \text{and} \quad \eta \in M, \quad \gamma \geq 1,$

which proves the lemma.

Q.E.D.

LEMMA 1.2.7. *Let $\xi^0 \in \mathbf{R}^n \setminus \{0\}$ and let M be a compact set in $\Gamma(P_{m\xi^0}, \mathcal{D})$ and $\max_{0 \leq j \leq m-1} n_+(P_m, Q_j; \xi^0)/(m-j+n_+(P_m, Q_j; \xi^0)) \leq \delta < 1$. Then there exist a conic neighborhood Γ of ξ^0 and positive numbers t_0, γ_0 and C such that*

$$P(\xi - i\gamma_0(1+|\xi|)^\delta \mathcal{D} - it|\xi|\eta) \neq 0 \quad \text{for } \xi \in \Gamma \text{ with} \\ |\xi| \geq C, \quad \eta \in M \text{ and } 0 \leq t \leq t_0.$$

PROOF. If Γ is a sufficiently small conic neighborhood of ξ^0 , there is $\gamma_0 > 0$ such that

$$(1.2.23) \quad P(\xi - i\gamma(1+|\xi|)^\delta \mathcal{D}) \neq 0 \quad \text{for } \xi \in \Gamma \text{ and } \gamma \geq \gamma_0 - 1.$$

Put

$$f(\nu, r, \zeta, t, \eta) = P(r\nu^{-1}\xi^0 + r\zeta - i\gamma_0(r\nu^{-1})^\delta \mathcal{D} - itr\nu^{-\delta}\eta)$$

when $\zeta \in \mathbf{R}^n, |\zeta| \leq \varepsilon, \eta \in M, 0 < \nu \leq \nu_0, r \geq r_0, |t| \leq t_0$ and $\operatorname{Re} t \geq 0$. Then we have

$$(1.2.24) \quad f(\nu, r, \zeta, t, \eta) = r^m \nu^a \{P_{m\xi^0}(\nu^\delta \zeta - i\gamma_0 r^{\delta-1} \mathcal{D} - it\eta) + o(1)\} \\ \text{as } r \rightarrow \infty \text{ and } \nu \rightarrow +0, \\ a = -m + (1-\delta) \deg P_{m\xi^0}.$$

In fact,

$$(1.2.25) \quad Q_j(r\nu^{-1}\xi^0 + r\zeta - i\gamma_0(r\nu^{-1})^\delta \mathcal{D} - itr\nu^{-\delta}\eta) \\ = r^j \nu^{-j\delta} Q_j(\nu^{-1+\delta}\xi^0 + \nu^\delta \zeta - i\gamma_0 r^{\delta-1} \mathcal{D} - it\eta) \\ = r^j \nu^a \{Q_{j\xi^0}(\nu^\delta \zeta - i\gamma_0 r^{\delta-1} \mathcal{D} - it\eta) + o(1)\} \quad \text{as } \nu \rightarrow +0, \quad 0 \leq j \leq m-1, \\ a = -j + (1-\delta) \deg Q_{j\xi^0}.$$

From the assertion (v) of Lemma 1.2.3 it follows that $m-j-(\deg P_{m\xi^0} - \deg Q_{j\xi^0})(1-\delta) \geq 0$. Thus (1.2.24) follows from (1.2.25). (1.2.23) implies that

$$f(\nu, r, \zeta, s, t, \eta) \neq 0 \quad \text{when } |t| \leq t_0, \operatorname{Re} t = 0, \zeta \in \mathbf{R}^n, \\ |\zeta| \leq \varepsilon, \eta \in M, 0 < \nu \leq \nu_0 \text{ and } r \geq r_0,$$

with a modification of ν_0 and t_0 , if necessary. Moreover we have

$$(1.2.26) \quad f(\nu, r, \zeta, s, t, \eta) \neq 0 \quad \text{when } |t| = t_0, \operatorname{Re} t \geq 0, \zeta \in \mathbf{R}^n, \\ |\zeta| \leq \varepsilon, \eta \in M, 0 < \nu \leq \nu_0 \text{ and } r \geq r_0,$$

modifying ν_0 and r_0 , if necessary. In fact, from (1.2.24) it follows that

$$f(\nu, r, \zeta, t, \eta) = r^m \nu^{-m+(1-\delta)a} \{t^a P_{m\xi^0}(\nu^\delta t^{-1} \zeta - i\gamma_0 r^{\delta-1} t^{-1} \eta - i\eta) + o(1)\}$$

as $r \rightarrow \infty$ and $\nu \rightarrow +0$,

$$a = \deg P_{m\xi^0}.$$

Since $P_{m\xi^0}(-i\eta) \neq 0$, $\eta \in M$, we have (1.2.26). Now assume that there are $\eta^1 \in M$, $\zeta^1 \in \mathbb{R}^n$ with $|\zeta^1| \leq \varepsilon$, $t_1 \in \mathbb{C}$ with $|t_1| < t_0$ and $\operatorname{Re} t_1 > 0$, $r_1 \geq r_0$ and ν_1 with $0 < \nu_1 \leq \nu_0$ such that $f(\nu_1, r_1, \zeta^1, t_1, \eta^1) = 0$. We may assume that M is convex and $\mathcal{D} \in M$. Denote by $t(\theta)$ a root of $f(\nu_1, r_1, \zeta^1, t, (1-\theta)\eta^1 + \theta\mathcal{D}) = 0$ such that $t(0) = t_1$ and $t(\theta)$ is continuous. Then we can define $t(\theta)$ for $\theta \in [0, 1]$ since $\operatorname{Re} t(\theta) > 0$ and $|t(\theta)| < t_0$. Thus we have $f(\nu_1, r_1, \zeta^1, t(1), \mathcal{D}) = 0$, which is a contradiction to (1.2.23). This proves the lemma. Q.E.D.

1.3. Generalized wave front sets

Let us first state the Paley-Wiener theorem for $\mathcal{D}^{*\kappa}$ and $\mathcal{E}^{*\kappa'}$ (see, e.g., [13], [14]).

PROPOSITION 1.3.1. *Let K be a compact convex set in $\mathbb{R}^n$ and $1 < \kappa < \infty$. (i) An entire function $\hat{\phi}(\zeta)$ on $\mathbb{C}^n$ is the Fourier-Laplace transform of $\phi \in \mathcal{D}_K^{(\kappa)}$ (resp. $\mathcal{D}_K^{[\kappa]}$) if and only if for every $L > 0$ there is a positive number C (resp. there are positive numbers L and C) such that*

$$|\hat{\phi}(\zeta)| \leq C \exp[-L|\zeta|^{1/\kappa} + h_K(\operatorname{Im} \zeta)], \quad \zeta \in \mathbb{C}^n,$$

where $h_K(\eta) = \sup_{x \in K} x \cdot \eta$. (ii) The following conditions are equivalent for an entire function $\hat{f}(\zeta)$ on $\mathbb{C}^n$: (a) $\hat{f}(\zeta)$ is the Fourier-Laplace transform of $f \in \mathcal{D}_K^{(\kappa)'}$ (resp. $\mathcal{D}_K^{[\kappa]'}$), where $\mathcal{D}_K^{*\kappa'} = \{f \in \mathcal{D}^{*\kappa'}; \operatorname{supp} f \subset K\}$. (b) There are constants L and C (resp. for every $L > 0$ there is a constant C) such that

$$|\hat{f}(\zeta)| \leq C \exp[L|\zeta|^{1/\kappa} + h_K(\operatorname{Im} \zeta)], \quad \zeta \in \mathbb{C}^n.$$

(c) There are constants L and C (resp. for every $L > 0$ there is a constant C) such that

$$|\hat{f}(\xi)| \leq C \exp[L|\xi|^{1/\kappa}], \quad \xi \in \mathbb{R}^n,$$

and for every $\varepsilon > 0$ there is a constant C_ε such that

$$|\hat{f}(\zeta)| \leq C_\varepsilon \exp[h_K(\operatorname{Im} \zeta) + \varepsilon|\zeta|], \quad \zeta \in \mathbb{C}^n.$$

Let $\kappa_1 > 1$ and $f \in \mathcal{D}^{(\kappa_1)' }(\Omega)$, where Ω is an open set in $\mathbb{R}^n$. Proposition 1.3.1 leads naturally to the following definition.

DEFINITION 1.3.2. Let $\kappa_1 \leq \kappa < \infty$. (i) $WF_{(\kappa)}(f)$ (resp. $WF_{[\kappa]}(f)$) is defined as the complement in $T^*(\Omega) \setminus 0$ of the collection of all (x^0, ξ^0) in $T^*(\Omega) \setminus 0$ such

that there are a neighborhood U of x^0 in Ω and a conic neighborhood Γ of ξ^0 such that for every $\phi \in \mathcal{D}^{(\kappa_1)}(U)$ and every $L > 0$ there is a positive number C (resp. for every $\phi \in \mathcal{D}^{(\kappa_1)}(U)$ there are positive numbers L and C) satisfying

$$|\mathcal{F}[\phi f](\xi)| \leq C \exp[-L|\xi|^{1/\kappa}], \quad \xi \in \Gamma,$$

where $\mathcal{F}[f](\xi) \equiv \hat{f}(\xi)$ denotes the Fourier transform of f (see [8]). (ii) $WF^{(\kappa)}(f)$ (resp. $WF^{(\kappa)}(f)$) is defined as the complement in $T^*(\Omega) \setminus 0$ of the collection of all (x^0, ξ^0) in $T^*(\Omega) \setminus 0$ such that there are a neighborhood U of x^0 in Ω and a conic neighborhood Γ of ξ^0 such that for every $\phi \in \mathcal{D}^{(\kappa_1)}(U)$ there are constants L and C (resp. for every $\phi \in \mathcal{D}^{(\kappa_1)}(U)$ and every $L > 0$ there is a constant C) satisfying

$$|\mathcal{F}[\phi f](\xi)| \leq C \exp(L|\xi|^{1/\kappa}), \quad \xi \in \Gamma.$$

(iii) Moreover we define $WF_{(\infty)}(f) = WF_{\{\infty\}}(f) = WF(f)$ (resp. $WF^{(\infty)}(f) = WF^{\{\infty\}}(f)$) as the complement in $T^*(\Omega) \setminus 0$ of the collection of all (x^0, ξ^0) in $T^*(\Omega) \setminus 0$ such that there are a neighborhood U of x^0 in Ω and a conic neighborhood Γ of ξ^0 such that for every $\phi \in \mathcal{D}^{(\kappa_1)}(U)$ and every non-negative integer N there is a positive number C (resp. for every $\phi \in \mathcal{D}^{(\kappa_1)}(U)$ there are an integer N and $C > 0$) satisfying

$$|\mathcal{F}[\phi f](\xi)| \leq C(1 + |\xi|)^{-N}, \quad \xi \in \Gamma$$

(see [9]).

The following lemmas give some properties of $WF_{*\kappa}(f)$ and $WF^{*\kappa}(f)$. The proofs of these lemmas can be given in the same way as in [9], [2].

LEMMA 1.3.3. *Let π be the projection $(x, \xi) \mapsto x$ from $T^*(\Omega) \setminus 0$ onto Ω . Then we have*

$$\begin{aligned} \pi(WF_{*\kappa}(f)) &= \text{sing supp}_{*\kappa} f \equiv \Omega \setminus \{x \in \Omega; \text{there is a neighborhood } U \\ &\text{of } x \text{ such that } f|_U \in \mathcal{E}^{*\kappa}(U)\}, \end{aligned}$$

$$\begin{aligned} \pi(WF^{*\kappa}(f)) &= \text{sing supp}^{*\kappa} f \equiv \Omega \setminus \{x \in \Omega; \text{there is a neighborhood } U \\ &\text{of } x \text{ such that } f|_U \in \mathcal{D}^{*\kappa}(U)\}. \end{aligned}$$

LEMMA 1.3.4. *Let $\phi \in \mathcal{E}^{(\kappa_1)}(\Omega)$. Then we have*

$$WF_{*\kappa}(\phi f) \subset WF_{*\kappa}(f), \quad WF^{*\kappa}(\phi f) \subset WF^{*\kappa}(f).$$

Let $\Omega_j, j=1, 2$, be open sets in $\mathbb{R}^{n_j}$.

PROPOSITION 1.3.5 ([18]). *Let $\Phi: \Omega_1 \rightarrow \Omega_2$ be a $\mathcal{E}^{*\kappa}$ -mapping. Then $\Phi^*u(x) \equiv u(\Phi(x))$ belongs to $\mathcal{E}^{*\kappa}(\Omega_1)$ if $u \in \mathcal{E}^{*\kappa}(\Omega_2)$.*

LEMMA 1.3.6. Let $(x^0, \xi^0) \in T^*(\Omega) \setminus 0$ and $\kappa_1 \leq \kappa < \infty$. (i) (x^0, ξ^0) does not belong to $WF_{(\kappa)}(f)$ (resp. $WF_{(\kappa)}(f)$) if and only if there are a conic neighborhood Γ of ξ^0 and $\phi \in \mathcal{D}^{(\kappa)}(\Omega)$ such that $\phi(x^0) \neq 0$ and for every $L > 0$ there is a positive number C (resp. there are positive numbers L and C) satisfying

$$|\mathcal{F}[\phi f](\xi)| \leq C \exp[-L|\xi|^{1/\kappa}], \quad \xi \in \Gamma.$$

(ii) (x^0, ξ^0) does not belong to $WF^{(\kappa)}(f)$ (resp. $WF^{(\kappa)}(f)$) if and only if there are a conic neighborhood Γ of ξ^0 and $\phi \in \mathcal{D}^{(\kappa)}(\Omega)$ such that $\phi(x^0) \neq 0$ and there are constants L and C (resp. for every $L > 0$ there is a constant C) satisfying

$$|\mathcal{F}[\phi f](\xi)| \leq C \exp[L|\xi|^{1/\kappa}], \quad \xi \in \Gamma.$$

DEFINITION 1.3.7. Let Γ be a closed cone in $T^*(\Omega) \setminus 0$ and $1 < \kappa$. Define

$$\mathcal{D}_\Gamma^{(\kappa)}(\Omega) = \{u \in \mathcal{D}^{(\kappa)}(\Omega); WF(u) \subset \Gamma\}.$$

We introduce the topology in $\mathcal{D}_\Gamma^{(\kappa)}(\Omega)$ by the semi norms of the weak topology in $\mathcal{D}^{(\kappa)}(\Omega)$ together with the semi norms $u \mapsto \sup_{\xi \in \gamma, |\xi| \geq 1} |\mathcal{F}[\phi u](\xi)| |\xi|^N$, where K is a compact set in Ω , γ is a closed cone, $K \times \gamma \cap \Gamma = \emptyset$, $\phi \in \mathcal{D}^{(\kappa)}$ with $\text{supp } \phi \subset K$ and $N = 0, 1, 2, \dots$.

REMARK. $\mathcal{E}^{(\kappa)}(\Omega)$ is dense in $\mathcal{D}_\Gamma^{(\kappa)}(\Omega)$.

LEMMA 1.3.8. Let $\Phi: \Omega_1 \rightarrow \Omega_2$ be a real analytic mapping. Define $N_\Phi = \{(\Phi(y), \xi) \in T^*(\Omega_2) \setminus 0; {}^t D\Phi_y \xi = 0\}$. Let Γ be a closed cone in $T^*(\Omega_2) \setminus 0$ such that $\Gamma \cap N_\Phi = \emptyset$. Then the pull-back $\Phi^*: C^\infty(\Omega_2) \rightarrow C^\infty(\Omega_1)$ has a unique continuous extension: $\mathcal{D}_\Gamma^{(\kappa)}(\Omega_2) \rightarrow \mathcal{D}^{(\kappa)}(\Omega_1)$. Let $f \in \mathcal{D}^{(\kappa)}(\Omega_2)$ and $\kappa_1 \leq \kappa$. If $WF_{*\kappa}(f) \cap N_\Phi = \emptyset$, we have

$$WF_{*\kappa}(\Phi^* f) \subset \Phi^* WF_{*\kappa}(f) \equiv \{(y, {}^t D\Phi_y \xi); (\Phi(y), \xi) \in WF_{*\kappa}(f)\}.$$

If $WF(f) \cap N_\Phi = \emptyset$, we have

$$WF_{*\kappa}(\Phi^* f) \subset \Phi^* WF_{*\kappa}(f).$$

REMARK. $WF_{*\kappa}(f)$ and $WF^{*\kappa}(f)$ are invariant under real analytic coordinate transformations.

LEMMA 1.3.9. Let $\Phi: \Omega_1 \rightarrow \Omega_2$ be a real analytic proper mapping. The push-forward $\Phi_* = {}^t(\Phi^*)^*: \mathcal{D}^{*\kappa_1}(\Omega_1) \rightarrow \mathcal{D}^{*\kappa_1}(\Omega_2)$ can be defined by

$$\langle \Phi_* f, \phi \rangle = \langle f, \Phi^* \phi \rangle, \quad \phi \in \mathcal{D}^{*\kappa_1}(\Omega_2),$$

for $f \in \mathcal{D}^{*\kappa_1}(\Omega_1)$. Then we have

$$WF_{*\kappa}(\Phi_* f) \subset \{(x, \xi) \in T^*(\Omega_2) \setminus 0; x = \Phi(y), (y, {}^t D\Phi_y \xi) \in WF_{*\kappa}(f) \\ \text{for some } y \in \Omega_1 \text{ or } x = \Phi(y), {}^t D\Phi_y \xi = 0 \text{ for some } y \in \Omega_1\},$$

$$WF_{*\kappa}^*(\Phi_*f) \subset \{(x, \xi) \in T^*(\Omega_2) \setminus 0; x = \Phi(y), (y, {}^tD\Phi_y\xi) \in WF_{*\kappa}^*(f) \\ \text{for some } y \in \Omega_1 \text{ or } x = \Phi(y), {}^tD\Phi_y\xi = 0 \text{ for some } y \in \Omega_1\}.$$

LEMMA 1.3.10. Let $u_j \in \mathcal{D}^{*\kappa_1}(\Omega_j)$, $j=1, 2$, and $1 < \kappa_1 \leq \kappa$. Then we have

$$\begin{aligned} WF_{*\kappa}^*(u_1 \otimes u_2) &\subset \bigcap_{\kappa_1 \leq s \leq \kappa} \{WF_{*\kappa_1}^{*s}(u_1) \times WF_{*\kappa_1}^*(u_2) \cup WF_{*\kappa_1}^*(u_1) \times WF_{*\kappa_1}^{*s}(u_2) \\ &\quad \cup WF_{*\kappa_1}^*(u_1) \times WF_{*\kappa_1}^*(u_2) \cap \text{supp}_0^{*s} u_1 \times WF_{*\kappa_1}^*(u_2) \\ &\quad \cup \text{supp}_0 u_1 \times WF_{*\kappa_1}^*(u_2) \cap WF_{*\kappa_1}^*(u_1) \times \text{supp}_0^{*s} u_2 \\ &\quad \cup WF_{*\kappa_1}^*(u_1) \times \text{supp}_0 u_2\}, \\ WF_{*\kappa}^*(u_1 \otimes u_2) &\subset \bigcap_{\kappa_1 \leq s \leq \kappa} \{WF_{*\kappa_1}^{*s}(u_1) \times WF_{*\kappa_1}^*(u_2) \cup WF_{*\kappa_1}^*(u_1) \times WF_{*\kappa_1}^{*s}(u_2) \\ &\quad \cup WF_{*\kappa_1}^*(u_1) \times WF_{*\kappa_1}^{*s}(u_2) \cup WF_{*\kappa_1}^*(u_1) \times WF_{*\kappa_1}^*(u_2) \\ &\quad \cup WF_{*\kappa_1}^*(u_1) \times \text{supp}_0 u_2 \cup WF_{*\kappa_1}^*(u_1) \times \text{supp}_0^{*s} u_2 \\ &\quad \cup WF_{*\kappa_1}^*(u_1) \times \text{supp}_0^{*s} u_2 \cup \text{supp}_0 u_1 \times WF_{*\kappa_1}^*(u_2) \\ &\quad \cup \text{supp}_0^{*s} u_1 \times WF_{*\kappa_1}^*(u_2) \cup \text{supp}_0^{*s} u_1 \times WF_{*\kappa_1}^*(u_2)\}, \end{aligned}$$

where $\text{supp}_0 u = \{(x, 0); x \in \text{supp } u\}$ and $\text{supp}_0^{*s} u = \{(x, 0); x \in \text{sing supp }^{*s} u\}$. Here we have identified $T^*(\Omega_1 \times \Omega_2)$ with $T^*(\Omega_1) \times T^*(\Omega_2)$.

Let π be the projection: $(x, y) \mapsto x$ from $\Omega_1 \times \Omega_2$ onto Ω_1 and Δ be a mapping: $(x, y) \mapsto (x, y, y)$ from $\Omega_1 \times \Omega_2$ into $\Omega_1 \times \Omega_2 \times \Omega_2$. Moreover let $K(x, y) \in \mathcal{D}^{*\kappa_1}(\Omega_1 \times \Omega_2)$ and $u \in \mathcal{E}^{*\kappa_1}(\Omega_2)$. We define formally

$$(1.3.1) \quad Ku = \pi_* \circ \Delta^*(K \otimes u).$$

If $u \in \mathcal{D}^{*\kappa_1}$, then we have

$$\langle Ku, \phi \rangle = \langle K(x, y), \phi(x)u(y) \rangle, \quad \phi \in \mathcal{D}^{*\kappa_1}.$$

Put

$$WF_{*\kappa_1\Omega_2}'(K) = \{(y, \eta) \in T^*(\Omega_2) \setminus 0; ((x, y), (0, -\eta)) \in WF_{*\kappa_1}(K) \text{ for some } x \in \Omega_1\}.$$

LEMMA 1.3.11. Assume that $WF_{*\kappa_1\Omega_2}'(K) \cap WF_{*\kappa_1}^*(u) = \emptyset$. Then Ku is well-defined by (1.3.1) and belongs to $\mathcal{D}^{*\kappa_1}(\Omega_1)$. Moreover we have

$$\begin{aligned} WF_{*\kappa}^*(Ku) &\subset \bigcap_{\kappa_1 \leq s \leq \kappa} \{WF_{*\kappa_1}^{*s'}(K) \circ WF_{*\kappa_1}^*(u) \cup WF_{*\kappa_1}'(K) \circ WF_{*\kappa_1}^{*s}(u) \\ &\quad \cup WF_{*\kappa_1}'(K) \circ WF_{*\kappa_1}^*(u) \cup WF_{*\kappa_1}'(K) \circ \text{supp}_0^{*s} u \\ &\quad \cup WF_{*\kappa_1}'(K) \circ \text{supp}_0 u\}, \\ WF_{*\kappa}^*(Ku) &\subset \bigcap_{\kappa_1 \leq s \leq \kappa} \{WF_{*\kappa_1}^{*s'}(K) \circ WF_{*\kappa_1}^*(u) \cup WF_{*\kappa_1}^{*s'}(K) \circ WF_{*\kappa_1}^*(u) \\ &\quad \cup WF_{*\kappa_1}'(K) \circ WF_{*\kappa_1}^{*s}(u) \cup WF_{*\kappa_1}'(K) \circ WF_{*\kappa_1}^*(u) \\ &\quad \cup WF_{*\kappa_1}^{*s'}(K) \circ \text{supp}_0 u \cup WF_{*\kappa_1}'(K) \circ \text{supp}_0^{*s} u \\ &\quad \cup WF_{*\kappa_1}'(K) \circ \text{supp}_0^{*s} u\}, \end{aligned}$$

where $W' = \{((x, \xi), (y, \eta)) \in T^*(\Omega_1) \times T^*(\Omega_2); ((x, y), (\xi, -\eta)) \in W\}$ for $W \subset T^*(\Omega_1 \times \Omega_2)$.

1.4. Fundamental solutions and singularities of solutions

From now on we assume that $P_m(\xi)$ is hyperbolic with respect to $\mathcal{V}$. The fundamental solution $E(x) \equiv E(P, \mathcal{V}; x)$ can be represented as

$$(1.4.1) \quad E(P, \mathcal{V}; x) = (2\pi)^{-n} \int e^{ix \cdot \zeta} P(\zeta)^{-1} d\zeta,$$

where $\zeta = \xi - i\gamma(1 + |\xi|)^{\delta(P)} \mathcal{V}$, $\xi \in \mathbf{R}^n$, $\gamma > \gamma_0$ and γ_0 is sufficiently large. (1.4.1) is interpreted as follows:

$$(1.4.2) \quad \langle E(x), \phi(x) \rangle = (2\pi)^{-n} \int_{\xi \in \mathbf{R}^n, \zeta = \xi - i\gamma(1 + |\xi|)^{\delta(P)} \mathcal{V}} \hat{\phi}(-\zeta) P(\zeta)^{-1} d\zeta, \\ \gamma > \gamma_0, \quad \phi \in \mathcal{D}^{(\kappa_0)},$$

where $\kappa_0 = 1/\delta(P)$. In fact, it follows from Proposition 1.3.1 that (1.4.2) is meaningful and that $E(x) \in \mathcal{D}^{(\kappa_0)'}.$ Moreover by Lemma 1.2.6 it is easy to see that $\text{ch} [\text{supp } E(x)] = \Gamma(P_m, \mathcal{V})^*$, where $\Gamma(P_m, \mathcal{V})^*$ is the dual cone of $\Gamma(P_m, \mathcal{V})$, that is,

$$\Gamma(P_m, \mathcal{V})^* = \{x \in \mathbf{R}^n; x \cdot \eta \geq 0 \text{ for all } \eta \in \Gamma(P_m, \mathcal{V})\},$$

and $\text{ch } [M]$ denotes the convex hull of M , which proves the sufficiency of Proposition 1.1.5. The necessity of proposition 1.1.5 can be proved in the same way as in [1] (see [15]).

Let $\kappa_1 > 1$ and $x^0 \in \mathbf{R}^n$ and let U and U_1 be neighborhoods of x^0 and $U_1 \subset U$. Then there exist a sequence $\{\chi_N\}_{N=1,2,\dots}$ in $\mathcal{D}^{(\kappa_1)}(U)$ and a positive number C such that for every $h > 0$ there exists a positive number C_h satisfying

$$(1.4.3) \quad |D^{\alpha+\beta} \chi_N(x)| \leq C_h (CN)^{|\alpha|} h^{|\beta|} (|\beta|!)^{\kappa_1}, \quad |\alpha| \leq N,$$

and $\chi_N(x) = 1$ on U_1 (see [8]).

DEFINITION 1.4.1. Let $f \in \mathcal{D}^{(\kappa_1)' }(\Omega)$. Then the analytic wave front set $WF_A(f)$ of f is defined as the complement in $T^*(\Omega) \setminus 0$ of the collection of all (x^0, ξ^0) in $T^*(\Omega) \setminus 0$ such that for some sequence $\{\chi_N\}$ of the above type (1.4.3), where U is a sufficiently small neighborhood of x^0 in Ω , there are a conic neighborhood Γ of ξ^0 and a positive number C satisfying

$$|\mathcal{F}[\chi_N f](\xi)| \leq C (CN)^N (1 + |\xi|)^{-N}, \quad \xi \in \Gamma, \quad N = 1, 2, \dots$$

THEOREM 1.4.2. Assume that $(x^0, \xi^0) \in T^*(\mathbf{R}^n) \setminus 0$ and $x^0 \in \Gamma(P_{m, \xi^0}, \mathcal{V})^*$. Then we have $(x^0, \xi^0) \in WF_A(E)$, that is,

$$WF_A(E) \subset \bigcup_{\xi \in \mathbf{R}^n \setminus \{0\}} \Gamma(P_{m\xi}, \mathcal{D})^* \times \{\xi\}.$$

PROOF. By the assumption there are $\eta \in \Gamma(P_{m\xi^0}, \mathcal{D})$ and a neighborhood U of x^0 such that $x \cdot \eta < 0$ for $x \in U$. Let M be a compact convex set in $\Gamma(P_{m\xi^0}, \mathcal{D})$ which contains η and $\mathcal{D}$ and let Γ, t_0 and γ_0 be as described in Lemma 1.2.7 with $\delta = \delta(P)$. We put

$$v_i(\xi) = \gamma_0(1 + |\xi|)^{\delta(P)} \mathcal{D} - t|\xi| \Phi(\xi) \eta,$$

where $\Phi(\xi) \in C^\infty(\mathbf{R}^n)$ is positively homogeneous of degree 0 in $|\xi| \geq 1$ and $\Phi(\xi) = 1$ on a neighborhood of $\Gamma_1 \cap \{|\xi| \geq 1\}$, $\text{supp } \Phi(\xi) \subset \Gamma$, $0 \leq \Phi(\xi) \leq 1$, and $\Gamma_1 (\subset \Gamma)$ is a conic neighborhood of ξ^0 . Let $\{\chi_N\} \subset \mathcal{D}^{(\epsilon_0)}(U)$ be a sequence which satisfies (1.4.3). It is easy to see that for every $B > 0$ there are a positive numbers C_B, A and c such that

$$(1.4.4) \quad \begin{aligned} |\chi_N(-\xi + iv_i(\xi) + \hat{\xi})| &\leq C_B (CN)^N (1 + |\xi - \hat{\xi}| + |v_i(\xi)|)^{-N} \\ &\times \exp[-B|\xi - \hat{\xi}|^{1/\epsilon_0} + A(1 + |\xi|)^{\delta(P)} - ct|\xi| \Phi(\xi)], \\ &\xi \quad \text{and} \quad \hat{\xi} \in \mathbf{R}^n, \quad N = 1, 2, \dots \end{aligned}$$

Seidenberg's lemma and Lemmas 1.2.6 and 1.2.7 imply that

$$(1.4.5) \quad |P(\xi - iv_i(\xi))|^{-1} \leq C(1 + |\xi|)^a, \quad \xi \in \mathbf{R}^n, \quad 0 \leq t \leq t_0$$

for some constants C and a . Let $\Gamma_2 (\subset \Gamma_1)$ be a conic neighborhood of ξ_0 . Then we have

$$(1.4.6) \quad |\xi - \hat{\xi}| \geq \epsilon(|\xi| + |\hat{\xi}|) \quad \text{for some } \epsilon > 0 \quad \text{if } \xi \in \Gamma_1 \quad \text{and} \quad \hat{\xi} \in \Gamma_2.$$

On the other hand we have

$$(1.4.7) \quad \exp[-ct_0|\xi| \Phi(\xi)] = \exp[-ct_0|\xi|],$$

$$(1.4.8) \quad |\xi - \hat{\xi}| + |v_{i_0}(\xi)| \geq \epsilon(|\xi| + |\hat{\xi}|), \quad \epsilon > 0,$$

if $\xi \in \Gamma_1$ and $|\hat{\xi}| \geq 1$. It follows from (1.4.4)–(1.4.8) that

$$\begin{aligned} &|\langle E(x), \exp[-ix \cdot \hat{\xi}] \chi_N(x) \rangle| \\ &= \left| (2\pi)^{-n} \int \chi_N(\hat{\xi} - \zeta) P(\zeta)^{-1} d\zeta \right| \leq C(CN)^N (1 + |\hat{\xi}|)^{-N}, \\ &\hat{\xi} \in \Gamma_2, \quad \zeta = \hat{\xi} - iv_{i_0}(\xi), \quad \xi \in \mathbf{R}^n, \quad N = 1, 2, \dots \end{aligned}$$

This proves the theorem.

Q.E.D.

THEOREM 1.4.3. *Let $(x^0, \xi^0) \in T^*(\mathbf{R}^n) \setminus 0$. Then we have $(x^0, \xi^0) \notin WF^{(1/\delta(\xi^0))}(E)$, where $(x^0, \xi^0) \notin WF^{(\infty)}(E)$ if $\delta(\xi^0) = 0$.*

PROOF. Let Γ and Γ_1 be conic neighborhoods of ξ^0 and $\Gamma_1 \subset \Gamma$. Moreover

let $\Phi(\xi)$ be as described in the proof of Theorem 1.4.2. Put

$$v_t(\xi) = \gamma_0(1 + |\xi|)^{\delta(P)} \mathcal{D} + \gamma_0 t \Phi(\xi) \{(1 + |\xi|)^{\delta(\xi^0)} - (1 + |\xi|)^{\delta(P)}\} \mathcal{D}, \quad 0 \leq t \leq 1.$$

Then it follows from Lemma 1.2.6 and the definition of $\delta(\xi^0)$ that

$$|P(\xi - iv_t(\xi))|^{-1} \leq C(1 + |\xi|)^a, \quad \xi \in \mathbf{R}^n, \quad 0 \leq t \leq 1,$$

for some constants C and a . Let $\phi \in \mathcal{D}^{(\kappa_0)}$ and let $\Gamma_2 (\subset \Gamma_1)$ be a conic neighborhood of ξ^0 . When $\hat{\xi} \in \Gamma_2$, we have

$$|\xi - \hat{\xi}| \geq \varepsilon(|\xi| + |\hat{\xi}|), \quad \varepsilon > 0, \quad \text{if } \xi \notin \Gamma_1 \quad \text{or} \quad |\xi| \geq 2|\hat{\xi}|$$

and $v_t(\xi) = \gamma_0(1 + |\xi|)^{\delta(\xi^0)} \mathcal{D}$ if $\xi \in \Gamma_1$ and $|\xi| \geq 1$. Therefore Proposition 1.3.1 implies that

$$|\langle E(x), \exp[-ix \cdot \hat{\xi}] \phi(x) \rangle| \leq C \exp[C|\hat{\xi}|^{\delta(\xi^0)}] \quad \text{if } \hat{\xi} \in \Gamma_2 \quad \text{and} \quad \delta(\xi^0) > 0.$$

If $\delta(\xi^0) = 0$, then we easily obtain

$$|\langle E(x), \exp[-ix \cdot \hat{\xi}] \phi(x) \rangle| \leq C(1 + |\xi|)^b, \quad \hat{\xi} \in \Gamma_2,$$

for some constants C and b . This completes the proof. **Q.E.D.**

THEOREM 1.4.4. *Let $1 < \kappa_1 \leq \kappa_0 \equiv 1/\delta(P)$, $\kappa_1 \leq \kappa \leq \infty$, $f \in \mathcal{D}^{(\kappa_1)'}$ (resp. $\mathcal{D}^{(\kappa_1)'} \wedge$) with $\text{supp } f \subset \{x \in \mathbf{R}^n; x \cdot \mathcal{D} \geq 0\}$ and $\xi \in \mathbf{R}^n \setminus \{0\}$. Then we have*

$$\begin{aligned} & WF_{(\kappa)}(E*f)|_{\xi} \text{ (resp. } WF_{[\kappa]}(E*f)|_{\xi}) \\ & \subset \begin{cases} \{x \in \mathbf{R}^n; x-y \in \Gamma(P_{m\xi}, \mathcal{D})^*, (y, \xi) \in WF_{(1/\delta(\xi))}(f) \text{ for some } y \in \mathbf{R}^n\} & \text{if } \kappa \geq 1/\delta(\xi), \\ \{x \in \mathbf{R}^n; x-y \in \Gamma(P_{m\xi}, \mathcal{D})^*, (y, \xi) \in WF_{(\kappa)}(f) \text{ (resp. } WF_{[\kappa]}(f)) \\ \text{for some } y \in \mathbf{R}^n\} & \text{if } \kappa < 1/\delta(\xi), \end{cases} \\ & WF^{(\kappa)}(E*f)|_{\xi} \text{ (resp. } WF^{[\kappa]}(E*f)|_{\xi}) \\ & \subset \begin{cases} \{x \in \mathbf{R}^n; x-y \in \Gamma(P_{m\xi}, \mathcal{D})^*, (y, \xi) \in WF_{(1/\delta(\xi))}(f) \text{ for some } y \in \mathbf{R}^n\} & \text{if } \kappa > 1/\delta(\xi) \text{ (resp. } \kappa \geq 1/\delta(\xi)), \\ \{x \in \mathbf{R}^n; x-y \in \Gamma(P_{m\xi}, \mathcal{D})^*, (y, \xi) \in WF^{(\kappa)}(f) \text{ (resp. } WF^{[\kappa]}(f)) \\ \text{for some } y \in \mathbf{R}^n\} & \text{if } \kappa \leq 1/\delta(\xi) \text{ (resp. } \kappa < 1/\delta(\xi)). \end{cases} \end{aligned}$$

Here $WF^{(\infty)}(E*f)|_{\xi} = WF^{\{\infty\}}(E*f)|_{\xi} \subset \{x \in \mathbf{R}^n; x-y \in \Gamma(P_{m\xi}, \mathcal{D})^*, (y, \xi) \in WF^{(\infty)}(f) \text{ for some } y \in \mathbf{R}^n\}$ if $\delta(\xi) = 0$.

PROOF. The theorem easily follows from Theorems 1.4.2 and 1.4.3 and Lemma 1.3.11. Then in order to estimate $WF_{[\kappa]}(E*f)|_{\xi}$ and $WF^{[\kappa]}(E*f)|_{\xi}$ for $\kappa \geq 1/\delta(\xi)$, we use the fact that

$$\begin{aligned} WF_{\{\kappa\}}(E*f)|_{\xi} &\subset WF_{\{\kappa\}}(E*f)|_{\xi}, \\ WF^{(\kappa)}(E*f)|_{\xi} &\subset WF^{(s)}(E*f)|_{\xi}, \quad s > \kappa. \end{aligned} \quad \text{Q.E.D.}$$

The finite set $\{\delta(\xi); \xi \in \mathbf{R}^n \setminus \{0\}\}$ can be written in the form

$$\{1/\delta(\xi); \xi \in \mathbf{R}^n \setminus \{0\}\} = \{s_j; 1 \leq j \leq l\}, \quad 1 < \kappa_0 = s_1 < s_2 < \dots < s_l = \infty.$$

Define

$$\begin{aligned} K_j &= \{((x, \xi), (y, \xi)); x - y \in \Gamma(P_{m_\xi}, \mathcal{G})^*, \xi \in \mathbf{R}^n \setminus \{0\} \text{ and } 1/\delta(\xi) = s_j, \\ &\quad 1 \leq j \leq l, \\ K &= \bigcup_{j=1}^l K_j = \{((x, \xi), (y, \xi)); x - y \in \Gamma(P_{m_\xi}, \mathcal{G})^* \text{ and } \xi \in \mathbf{R}^n \setminus \{0\}\}. \end{aligned}$$

COROLLARY 1.4.5. Denote by $\omega(\kappa)$ (resp. $\omega\{\kappa\}$)

$$\max \{j; \kappa > s_j \text{ (resp. } \kappa \geq s_j)\},$$

where $s_0 = 1$. Then we have

$$\begin{aligned} WF_{*\kappa}(E*f) &\subset \bigcup_{k=1}^{\omega^*\kappa} K_k \circ WF_{(s_k)}(f) \cup \bigcup_{k=\omega^*\kappa+1}^l K_k \circ WF_{*\kappa}(f), \\ WF^{*\kappa}(E*f) &\subset \bigcup_{k=1}^{\omega^*\kappa} K_k \circ WF_{(s_k)}(f) \cup \bigcup_{k=\omega^*\kappa+1}^l K_k \circ WF^{*\kappa}(f), \end{aligned}$$

where $\omega^*\kappa$ denotes either $\omega(\kappa)$ or $\omega\{\kappa\}$.

COROLLARY 1.4.6. We have the following: (i) If $WF_{*\kappa}(f)|_{\xi} = \emptyset$ for $\kappa < 1/\delta(\xi)$, $\xi \in \mathbf{R}^n \setminus \{0\}$ and $WF_{(1/\delta(\xi))}(f)|_{\xi} = \emptyset$ for $\kappa \geq 1/\delta(\xi)$, $\xi \in \mathbf{R}^n \setminus \{0\}$, then $WF_{*\kappa}(E*f) = \emptyset$, that is, $E*f \in \mathcal{E}^{*\kappa}$. (ii) If $WF^{(\kappa)}(f)|_{\xi} = \emptyset$ (resp. $WF^{(\kappa)}(f)|_{\xi} = \emptyset$) for $\kappa \leq 1/\delta(\xi)$ (resp. $\kappa < 1/\delta(\xi)$), $\xi \in \mathbf{R}^n \setminus \{0\}$ and $WF_{(1/\delta(\xi))}(f)|_{\xi} = \emptyset$ for $\kappa > 1/\delta(\xi)$ (resp. $\kappa \geq 1/\delta(\xi)$), $\xi \in \mathbf{R}^n \setminus \{0\}$, then $WF^{(\kappa)}(E*f) = \emptyset$ (resp. $WF^{(\kappa)}(E*f) = \emptyset$), that is, $E*f \in \mathcal{D}^{(\kappa)'} \text{ (resp. } \mathcal{D}^{(\kappa)'} \text{)}$. Here $WF^{(\infty)}(E*f) = \emptyset$ if $WF^{(\infty)}(f)|_{\xi} = \emptyset$ for $\delta(\xi) = 0$, $\xi \in \mathbf{R}^n \setminus \{0\}$ and $WF_{(1/\delta(\xi))}(f)|_{\xi} = \emptyset$ for $\delta(\xi) > 0$, $\xi \in \mathbf{R}^n \setminus \{0\}$.

1.5. Some remarks

In § 1.4 we have given the outer estimates of the generalized wave front sets of the solutions. However, we could not give non-trivial inner estimates of the generalized wave front sets, which correspond to the localization theorem in [1]. The following lemma is not really satisfactory.

LEMMA 1.5.1. Let $(x^0, \xi^0) \in T^*(\mathbf{R}^n) \setminus 0$ and $\eta(s) = s^{-1}(\xi^0 + \sum_{j=1}^{\infty} s^{j/l} \xi^j)$, $\xi^j \in \mathbf{C}^n$, $j = 1, 2, \dots, l \in \mathbf{N}$. Define a polynomial $P(\zeta; \eta(s)) \equiv 0$ in ζ by

$$P(\eta(s) + \zeta) = s^{-r(\eta)} \{P(\zeta; \eta(s)) + o(1)\}, \quad s \rightarrow +0.$$

Assume that the following conditions hold: (i) The principal part of $P(\zeta; \eta(s))$

is hyperbolic with respect to $\mathcal{D}$. (ii) $x^0 \in \text{supp } E(P(\cdot; \eta(s)), \mathcal{D}; x)$. (iii) $x^0 \cdot \text{Im } \eta(s) \leq -s^{-\nu}(c + o(1))$, $s \rightarrow +0$, $\nu \geq 0$, where c is a positive number if $\nu > 0$ and c is any constant if $\nu = 0$. Then we have $(x^0, \xi^0) \in WF^{(1/\nu)}(E(P, \mathcal{D}; x))$ if $\nu > 0$ and $(x^0, \xi^0) \in WF(E) = WF_{(\infty)}(E)$ if $\nu = 0$.

PROOF. Put $F(x, s) = \exp[-ix \cdot \eta(s)]E(P, \mathcal{D}; x)$. Then we have

$$(1.5.1) \quad \begin{aligned} \mathcal{F}[\phi E](\eta(s)) &= \mathcal{F}[\phi F](0) = s^{r(\eta)} \{ \langle E(P(\cdot; \eta(s)), \mathcal{D}; x), \phi(x) \rangle + o(1) \}, \\ s &\rightarrow +0, \quad \phi \in \mathcal{D}^{(\kappa_1)}, \quad 1 < \kappa_1 \leq 1/\delta(P). \end{aligned}$$

Now we assume that $\nu > 0$ and $(x^0, \xi^0) \notin WF^{(1/\nu)}(E)$. Then for sufficiently small neighborhood U of x^0 we have

$$|\mathcal{F}[\phi E](\eta(s))| \leq C \exp[-cs^{-\nu}/4], \quad 0 < s \leq s_0, \quad \phi \in \mathcal{D}^{(\kappa_1)}(U),$$

which is a contradiction to (1.5.1). When $\nu = 0$, we can similarly prove the lemma. **Q.E.D.**

EXAMPLE 1.5.2. Let $P(\xi) = \xi_1(\xi_1 - \xi_2) + \xi_3$ and $\xi^0 = (0, 0, 1)$. Then we have

$$(1.5.2) \quad \begin{aligned} WF^{(2)}(E(P, \mathcal{D}; x))|_{\xi^0} &= WF_A(E)|_{\xi^0} = \{x \in \mathbf{R}^3; x = \alpha(1, 0, 0) \\ &\quad + \beta(1, -1, 0) \text{ and } \alpha, \beta \geq 0\} \end{aligned}$$

and $WF^{(2)}(E(P, \mathcal{D}; x)) = \emptyset$. In fact we take $\eta(s) = s^{-1}\xi^0 - is^{-1/2}(a, b, 0)$, where $a > 0$, $b \in \mathbf{R}$ and $1 - a(a - b) = 0$. Then $\eta(s)$ satisfies the conditions of Lemma 1.5.1 with $x^0 = \alpha(1, 0, 0) + \beta(1, -1, 0)$ for $\alpha, \beta > 0$ and $\nu = 1/2$. Since $WF^{(\kappa)}(f)$ is closed in $T^*(\mathbf{R}^n) \setminus 0$, we have (1.5.2).

EXAMPLE 1.5.3. Let $P(\xi) = \xi_1^3 + \xi_2 \xi_3$ and $\xi^0 = (0, 0, 1)$. Put $\xi^j = (0, 1/j, 1)$, $j = 1, 2, \dots$, and $\eta^j(s) = s^{-1}\xi^j + s^{-2/3}\omega j^{-1/3}\mathcal{D}$, where $\omega = (1 - i\sqrt{3})/2$, $\omega^3 = 1$. If ξ^0 and $\eta(s)$ are replaced by ξ^j and $\eta^j(s)$, respectively, the conditions of Lemma 1.5.1 are satisfied when $x^0 = \alpha\mathcal{D}$ for $\alpha \geq 0$ and $\nu = 2/3$. Thus we have

$$WF^{[3/2]}(E)|_{\xi^j} = WF_A(E)|_{\xi^j} = \{x \in \mathbf{R}^3; x = \alpha\mathcal{D}, \alpha \geq 0\}.$$

The closedness of the wave front set implies that

$$WF^{[3/2]}(E)|_{\xi^0} = WF_A(E)|_{\xi^0} = \{x \in \mathbf{R}^3; x = \alpha\mathcal{D}, \alpha \geq 0\}.$$

LEMMA 1.5.4. Let $\xi^0 \in \mathbf{R}^n \setminus \{0\}$. If $\max(\delta(\xi^0), \delta(-\xi^0)) = 0$, then $P_{\xi^0}(\xi)$ is hyperbolic with respect to $\mathcal{D}$ and the principal part of $P_{\xi^0}(\xi)$ is $P_{m_{\xi^0}}(\xi)$. Therefore we have

$$(1.5.3) \quad \begin{aligned} \text{supp } E(P_{\xi^0}, \mathcal{D}; x) &\subset WF(E(P, \mathcal{D}; x))|_{\xi^0} \subset WF_A(E(P, \mathcal{D}; x))|_{\xi^0} \\ &\subset \Gamma(P_{m_{\xi^0}}, \mathcal{D})^* = \text{ch} [\text{supp } E(P_{\xi^0}, \mathcal{D}; x)]. \end{aligned}$$

PROOF. Write

$$\begin{aligned} P_m(s^{-1}\xi^0 + \zeta) &= s^{-p}(P_{m\xi^0}(\zeta) + o(1)), & s \rightarrow +0, \\ Q(s^{-1}\xi^0 + \zeta) &= s^{-q}(Q_{\xi^0}(\zeta) + o(1)), & s \rightarrow +0. \end{aligned}$$

Then we have $p \geq q$, since $n_+(P_m, Q_j; \xi^0) = 0$. If $p > q$, then $P_{\xi^0}(\zeta) = P_{m\xi^0}(\zeta)$. If $p = q$, then $\deg P_{m\xi^0} = m - p > \deg Q_{\xi^0}$. Thus the principal part of $P_{\xi^0}(\zeta)$ is $P_{m\xi^0}(\zeta)$. By the assumption there are a conic neighborhood Γ of ξ^0 and a positive number γ_0 such that $P(\xi - i\gamma\vartheta) \neq 0$, $\xi \in \Gamma$, $\gamma > \gamma_0$. If we assume that $P_{\xi^0}(\xi^1 - i\gamma_1\vartheta) = 0$ for some $\gamma_1 > \gamma_0$ and $\xi^1 \in \mathbf{R}$, Rouché's theorem leads us to a contradiction. This proves the lemma, since (1.5.3) is an immediate consequence of Lemma 1.5.1. **Q.E.D.**

Chapter II

Operators with variable lower-order coefficients

2.1. Construction of solutions

Let $\xi^0 \in \mathbf{R}^n \setminus \{0\}$. Define

$$(2.1.1) \quad n_j(\xi^0) = \max_{x \in \mathbf{R}^n} n_+(P_m, Q_j(x, \cdot); \xi^0),$$

$$(2.1.2) \quad \kappa(\xi^0) = \min_{0 \leq j \leq m-1} (m - j + n_j(\xi^0)) / n_j(\xi^0),$$

$$(2.1.3) \quad \kappa_0 = \min_{|\xi|=1} \kappa(\xi),$$

where $\kappa(\xi^0) = \infty$ if $n_j(\xi^0) = 0$. Here we have used the fact that the set

$$\{n(P_m, Q_j(x, \cdot); \xi); |\xi| = 1, x \in \mathbf{R}^n \text{ and } 0 \leq j \leq m-1\}$$

is finite. Indeed, applying the same argument as in the proof of Lemma 1.1.3 to the system

$$\begin{cases} \phi |P_m(\xi - i\vartheta)|^2 - |\sum_{|\alpha| \leq m-1} a_\alpha (\xi - i\vartheta)^\alpha| = 0, & a_\alpha \in \mathbf{C}, \\ |\xi|^2 = r^2, & r > 0, \\ |\xi - r\xi^0| \leq \varepsilon^2 r^2, & 0 < \varepsilon \leq \varepsilon_0, \quad |\xi^0| = 1, \quad \xi^0 \in \mathbf{R}^n, \end{cases}$$

we can prove this fact. Put

$$V(P) = V_0(P) \oplus V_1(P) \oplus \cdots \oplus V_{m-1}(P),$$

$$V_j(P) = \{p(\xi) \in C[\xi]; p(\xi) \text{ is homogeneous of degree } j$$

$$\text{and } n_+(P_m, p; \eta) \leq n_j(\eta) \text{ for all } \eta \in \mathbf{R}^n \setminus \{0\}\}.$$

Let $\{p^j\}$ be a basis of the finite dimensional vector space $V(P)$. Here we can choose p^j to be homogeneous. Let $\kappa_1 > 1$ and assume that the coefficients of $Q(x, D)$ belong to $\mathcal{E}^{*\kappa_1}$. Then the following lemma is obvious.

LEMMA 2.1.1. *We can write*

$$Q(x, \xi) = \sum_j q_j(x) p^j(\xi), \quad q_j(x) \in \mathcal{E}^{*\kappa_1}.$$

Moreover the $q_j(x)$ are in $\mathcal{D}^{*\kappa_1}$ if the coefficients of $Q(x, D)$ are in $\mathcal{D}^{*\kappa_1}$.

We assume that

$$(A) \quad 1 < \kappa_1 < \kappa_0.$$

Let $\kappa_1 \leq \kappa \leq \kappa_0$ when $^{*}\kappa_1 = (\kappa_1)$ and $\kappa_1 \leq \kappa < \kappa_0$ when $^{*}\kappa_1 = \{\kappa_1\}$. Consider the Cauchy problem

$$(2.1.4) \quad \begin{cases} P(x, D)u(x) = f(x), \\ \text{supp } u(x) \subset \{x \in \mathbf{R}^n; x_1 \geq t\}, \end{cases}$$

where $f \in \mathcal{D}^{*\kappa'}$, $\text{supp } f \subset \{x \in \mathbf{R}^n; x_1 \geq t\}$ and t is a fixed real number. First we construct a solution of (2.1.4) by successive iteration when the $q_j(x)$ are in $\mathcal{D}^{*\kappa_1}$ and $f \in \mathcal{E}^{*\kappa'}$. Define u_l , $l=0, 1, 2, \dots$, by

$$\begin{aligned} P_m(D)u_0(x) &= f(x), \\ P_m(D)u_{l+1}(x) &= -Q(x, D)u_l(x), \\ \text{supp } u_l &\subset \text{supp } f + \Gamma(P_m, \mathcal{G})^*. \end{aligned}$$

It is obvious that $u_l \in \mathcal{D}^{*\kappa'}$.

LEMMA 2.1.2. *Assume that the condition (A) is satisfied, the $q_j(x)$ are in $\mathcal{D}^{(\kappa_1)}$ (resp. $\mathcal{D}^{[\kappa_1]}$) and that $f \in \mathcal{E}^{(\kappa)'}$ (resp. $\mathcal{E}^{[\kappa]'}$) with $\text{supp } f \subset \{x \in \mathbf{R}^n; x_1 \geq 0\}$. Then we have*

$$\begin{aligned} & |\hat{u}_l(\xi - i\gamma(1 + |\xi|)^{1/\kappa_0}\mathcal{G})| \\ & \leq C_0(C\gamma^{-\mu})^{l+1} \exp [L|\xi - i\gamma(1 + |\xi|)^{1/\kappa_0}\mathcal{G}|^{1/\kappa}], \quad \gamma \geq 1, \quad \xi \in \mathbf{R}^n, \end{aligned}$$

for some constants L and C_0 (resp. for every $L > 0$ and some constant C_0 depending on L), where C is a positive constant and

$$(2.1.5) \quad \begin{aligned} \mu &= \min_{0 \leq j \leq m-1} (m - j - n_j(P)(\kappa_1 - 1)) > 0, \\ n_j(P) &= \max_{|\xi|=1} n_j(\xi). \end{aligned}$$

Thus we can define $u(x) \in \mathcal{D}^{(\kappa)'}$ (resp. $\mathcal{D}^{[\kappa]'}$) by

$$(2.1.6) \quad \hat{u}(\xi - i\gamma(1 + |\xi|)^{1/\kappa_0}\mathcal{G}) = \sum_{l=0}^{\infty} \hat{u}_l(\xi - i\gamma(1 + |\xi|)^{1/\kappa_0}\mathcal{G}), \quad \gamma^\mu > C,$$

that is,

$$\begin{aligned} \langle u(x), \phi(x) \rangle &= (2\pi)^{-n} \int \hat{u}(\zeta) \hat{\phi}(-\zeta) d\zeta, \quad \zeta = \xi - i\gamma(1 + |\xi|)^{1/\kappa_0}\mathcal{G}, \\ \xi &\in \mathbf{R}^n, \quad \phi \in \mathcal{D}^{(\kappa)} \quad (\text{resp. } \mathcal{D}^{[\kappa]}). \end{aligned}$$

Then $u(x)$ is a solution of (2.1.4) and

$$\text{supp } u(x) \subset \text{supp } f(x) + \Gamma(P_m, \mathcal{G})^*.$$

REMARK. When $\kappa_0 = \infty$ we proved the lemma for $\kappa_1 \leq \infty$ (see [20]).

PROOF. From the above remark it suffices to prove the lemma in the case $\kappa_0 < \infty$. We have

$$\begin{aligned} & \hat{u}_i(\xi - i\gamma(1 + |\xi|)^{1/\kappa_0} \mathcal{G}) \\ &= (2\pi)^{-ln} \sum_{j_1, \dots, j_l} (-1)^l P_m(\xi - i\gamma(1 + |\xi|)^{1/\kappa_0} \mathcal{G})^{-1} \\ & \quad \times \int d\zeta^1 \hat{q}_{j_1}(\zeta^1) p^{j_1}(\xi - i\gamma(1 + |\xi|)^{1/\kappa_0} \mathcal{G} - \zeta^1) P_m(\xi - i\gamma(1 + |\xi|)^{1/\kappa_0} \mathcal{G} - \zeta^1)^{-1} \\ (2.1.7) \quad & \quad \times \left\{ \int d\zeta^2 \dots \right. \\ & \quad \times \left\{ \int d\zeta^l \hat{q}_{j_l}(\zeta^l) p^{j_l}(\xi - i\gamma(1 + |\xi|)^{1/\kappa_0} \mathcal{G} - \zeta^1 - \dots - \zeta^l) \right. \\ & \quad \times P_m(\xi - i\gamma(1 + |\xi|)^{1/\kappa_0} \mathcal{G} - \zeta^1 - \dots - \zeta^l)^{-1} \\ & \quad \left. \left. \times \hat{f}(\xi - i\gamma(1 + |\xi|)^{1/\kappa_0} \mathcal{G} - \zeta^1 - \dots - \zeta^l) \right\} \dots \right\}, \end{aligned}$$

where $\zeta^j = \eta^j + ia(1 + |\eta^j|)^{1/\kappa_1} \mathcal{G}$, $\eta^j \in \mathbf{R}^n$, $1 \leq j \leq l$, and $a > 0$ is to be determined later. Put

$$J_k^j = |p^j(\xi - i\gamma(1 + |\xi|)^{1/\kappa_0} \mathcal{G} - \zeta^1 - \dots - \zeta^k) P_m(\xi - i\gamma(1 + |\xi|)^{1/\kappa_0} \mathcal{G} - \zeta^1 - \dots - \zeta^k)^{-1}|.$$

Then we have

$$(2.1.8) \quad J_k^j \leq C\gamma^{-\mu}, \quad \gamma \geq 1.$$

In fact,

$$\begin{aligned} J_k^j &\leq C\{\gamma(1 + |\xi|)^{1/\kappa_0} + a \sum_{h=1}^k (1 + |\eta^h|)^{1/\kappa_1}\}^{\nu-m} \\ &\quad \times [1 + \{\gamma(1 + |\xi|)^{1/\kappa_0} + a \sum_{h=1}^k (1 + |\eta^h|)^{1/\kappa_1}\}^{-1} |\xi - \eta^1 - \dots - \eta^k|]^{n_\nu(P)}, \quad \gamma > 0, \end{aligned}$$

where $\deg p^j = \nu$. Thus

$$(2.1.9) \quad J_k^j \leq C\gamma^{\nu-m}, \quad \gamma \geq 1, \quad \text{if } |\eta^1| + \dots + |\eta^k| \leq |\xi|,$$

since $n_\nu(P) - (m - \nu + n_\nu(P))/\kappa_0 \leq 0$. Moreover we have

$$\begin{aligned} (2.1.10) \quad J_k^j &\leq C[\gamma^{\nu-m} + \gamma^{-\mu} \{a \sum_{h=1}^k (1 + |\eta^h|)^{1/\kappa_1}\}^{\nu-m+\mu+(\kappa_1-1)n_\nu(P)}] \\ &\leq C\gamma^{-\mu}, \quad \gamma \geq 1, \quad \text{if } |\eta^1| + \dots + |\eta^k| \geq |\xi|, \end{aligned}$$

since $(|\eta^1| + \dots + |\eta^k|)^{1/\kappa_1} \leq |\eta^1|^{1/\kappa_1} + \dots + |\eta^k|^{1/\kappa_1}$, $\nu - m + (\kappa_1 - 1)n_\nu(P) + \mu \leq 0$ and $m - \nu \geq \mu$. (2.1.9) and (2.1.10) give (2.1.8). Note that $\mu = n(P)(\kappa_0 - \kappa_1)$ if $n(P) \neq 0$,

$\mu=1$ if $n(P)=0$, where $n(P)=\max_{0\leq j\leq m-1} n_j(P)$. Thus $\mu>0$. Now Proposition 1.3.1 implies that

$$|\hat{q}_j(\zeta)|\leq C_1 \exp [A |\operatorname{Im} \zeta| - B_1 |\operatorname{Re} \zeta|^{1/\kappa_1}]$$

for every $B_1>0$ and some C_1 depending on B_1 (resp. for some $B_1>0$ and C_1), and

$$(2.1.11) \quad |\hat{f}(\xi - i\gamma\mathcal{G})|\leq C_2 \exp [B_2 |\xi - i\gamma\mathcal{G}|^{1/\kappa}], \quad \gamma>0,$$

for some B_2 and C_2 (resp. for every $B_2>0$ and some C_2 depending on B_2). Choose $a>0$ and $B_1>0$ (resp. $B_2>0$) so that $B_1\geq 2(aA + (1+a^{1/\kappa})B_2)$. Then it follows from (2.1.7) and (2.1.8) that

$$\begin{aligned} |\hat{u}_l(\xi - i\gamma(1+|\xi|)^{1/\kappa_0}\mathcal{G})| &\leq C |P_m(\xi - i\gamma(1+|\xi|)^{1/\kappa_0}\mathcal{G})|^{-1} \\ &\quad \times (C\gamma^{-\mu})^l \int d\gamma^1 \cdots d\gamma^l C_1^l C_2 \exp [aA \sum_{k=1}^l (1+|\gamma^k|)^{1/\kappa_1} \\ &\quad - B_1 \sum_{k=1}^l |\gamma^k|^{1/\kappa_1} + B_2 |\xi - i\gamma(1+|\xi|)^{1/\kappa_0}\mathcal{G} - \gamma^1 - \cdots - \gamma^l|^{1/\kappa} \\ &\quad + a^{1/\kappa} B_2 \sum_{k=1}^l (1+|\gamma^k|)^{1/(\kappa\kappa_1)}] \\ &\leq CC_2 |P_m(\xi - i\gamma(1+|\xi|)^{1/\kappa_0}\mathcal{G})|^{-1} \\ &\quad \times (CC_1\gamma^{-\mu})^l \exp [B_2 |\xi - i\gamma(1+|\xi|)^{1/\kappa_0}\mathcal{G}|^{1/\kappa}], \quad \gamma\geq 1. \end{aligned}$$

Here we have used the fact that $(a+b)^{1/\lambda}\leq a^{1/\lambda}+b^{1/\lambda}$ if $a, b\geq 0$ and $\lambda\geq 1$. This proves the first part of the lemma. It is obvious that $\hat{u}(\xi - i\gamma(1+|\xi|)^{1/\kappa_0}\mathcal{G})$ is well-defined for $\gamma>\gamma_0=(CC_1)^{1/\mu}$ and that

$$|\hat{u}(\xi - i\gamma(1+|\xi|)^{1/\kappa_0}\mathcal{G})|\leq C_\gamma \exp [L |\xi - i\gamma(1+|\xi|)^{1/\kappa_0}\mathcal{G}|^{1/\kappa}], \quad \gamma>\gamma_0,$$

for some L and C_γ (resp. for every $L>0$ and some C_γ depending on L and γ). If $\phi\in\mathcal{D}^{(\kappa)}$ (resp. $\mathcal{D}^{[\kappa]}$), then

$$|\hat{\phi}(-\xi + i\gamma(1+|\xi|)^{1/\kappa_0}\mathcal{G})|\leq C \exp [A\gamma(1+|\xi|)^{1/\kappa_0} - B|\xi|^{1/\kappa}]$$

for every $B>0$ and some C (resp. for some $B>0$ and C). This implies that $u\in\mathcal{D}^{(\kappa)'} for $\kappa_1\leq\kappa\leq\kappa_0$ (resp. $u\in\mathcal{D}^{[\kappa]}'$ for $\kappa_1\leq\kappa<\kappa_0$). The last assertion of the lemma is obvious. **Q.E.D.**$

LEMMA 2.1.3. *Let $P(x, D)$ be as described in Lemma 2.1.2. Assume that $f\in\mathcal{D}^{(\kappa)}$ (resp. $\mathcal{D}^{[\kappa]}$) with $\operatorname{supp} f\subset\{x\in\mathbf{R}^n; x_1\geq 0\}$. Then the solution $u(x)$ of (2.1.4) defined by (2.1.6) belongs to $\mathcal{E}^{(\kappa)}$ (resp. $\mathcal{E}^{[\kappa]}$).*

REMARK. The more general result will be given in Theorem 2.4.4.

PROOF. Instead of (2.1.11) we have

$$(2.1.12) \quad |\hat{f}(\xi - i\gamma\mathcal{G})|\leq C_2 \exp [-B_2 |\xi|^{1/\kappa}], \quad \gamma>0,$$

for every $B_2 > 0$ and some C_2 depending on B_2 (resp. for some $B_2 > 0$ and C_2). Choose $a > 0$ and $B_1 > 0$ (resp. $B_2 > 0$) in the proof of Lemma 2.1.2 so that $B_1 \geq 2(aA + B_2)$. Here we have used the fact that (2.1.12) with B'_2 holds if $B'_2 < B_2$. Repeating the same argument as in the proof of Lemma 2.1.2, we have

$$|\hat{u}_l(\xi - i\gamma(1 + |\xi|)^{1/\kappa_0} \mathcal{G})| \leq CC_2 |P_m(\xi - i\gamma(1 + |\xi|)^{1/\kappa_0} \mathcal{G})|^{-1} \\ \times (C_1 C \gamma^{-\mu})^l \exp[-B_2 |\xi|^{1/\kappa}], \quad \gamma \geq 1.$$

Thus

$$|\hat{u}(\xi - i\gamma(1 + |\xi|)^{1/\kappa_0} \mathcal{G})| \leq C_\gamma \exp[-B_2 |\xi|^{1/\kappa}], \quad \gamma > \gamma_0 = (C_1 C)^{1/\mu}.$$

If $\phi \in \mathcal{D}^{(\varepsilon)}$ (resp. $\mathcal{D}^{(\varepsilon)}$) and $\text{supp } \phi \subset \{x_1 \geq -\varepsilon\}$, $\varepsilon > 0$, then

$$|\hat{\phi}(\xi - \zeta)| \leq C \exp[\varepsilon \gamma(1 + |\eta|)^{1/\kappa_0} - B |\xi - \eta|^{1/\kappa}]$$

for every $B > 0$ and C depending on B (resp. for some $B > 0$ and C), where $\zeta = \eta - i\gamma(1 + |\eta|)^{1/\kappa_0} \mathcal{G}$ and $\xi, \eta \in \mathbb{R}^n$. Therefore we have

$$|\mathcal{F}[\phi u](\xi)| = |(2\pi)^{-n} \int \hat{\phi}(\xi - \zeta) \hat{u}(\zeta) d\zeta| \leq C_\gamma \exp[-B_3 |\xi|^{1/\kappa}], \\ \zeta = \eta - i\gamma(1 + |\eta|)^{1/\kappa_0} \mathcal{G}, \quad \eta \in \mathbb{R}^n, \quad \xi \in \mathbb{R}^n,$$

where $\varepsilon = 1/\gamma$, $\gamma > \gamma_0$, $B = B_2$ and $B_3 = B_2 - 1$ (resp. where $B_3 = \min(B/2, B_2/2)$). Here we have used the fact that

$$\exp[\varepsilon \gamma(1 + |\eta|)^{1/\kappa_0} - B_2 |\eta|^{1/\kappa}/2] \leq C(\kappa, \kappa_0, \varepsilon \gamma, B_2) \quad \text{if } \kappa_0 > \kappa. \quad \mathbf{Q.E.D.}$$

LEMMA 2.1.4. *Put*

$$\kappa(P, x; \xi^0) = \min_{0 \leq j \leq m-1} (m - j + n_+(P_m, Q_j(x, \cdot); \xi^0)) \\ \times 1/n_+(P_m, Q_j(x, \cdot); \xi^0), \quad x \in \mathbb{R}^n, \quad \xi^0 \in \mathbb{R}^n \setminus \{0\}.$$

Then we have

$$\kappa(P, x; \xi^0) = \kappa({}^t P, x; \xi^0),$$

where ${}^t P$ denotes the transposed operator of $P(x, D)$.

PROOF. Put $\nu_j = \deg p^j$. Then

$${}^t P(x, D)u(x) = (-1)^m P_m(D)u(x) + (-1)^{\nu_j} \sum_{j, \alpha} 1/\alpha! D^\alpha q_j(x) p^{j(\alpha)}(D)u(x).$$

From the definition of $n(P_m, p^j; \xi^0)$, hyperbolicity of $P_m(\xi)$ and Lemma 3.1.5 in [7] it follows that there exists a conic neighborhood Γ of ξ^0 such that

$$|p^{j(\alpha)}(\xi - i\mathcal{G})| \leq C \sup_{|\zeta| \leq 1} |p^j(\xi + \zeta - i\mathcal{G})| \\ \leq C(1 + |\xi|)^{n(P_m, p^j; \xi^0)} |P_m(\xi - i\mathcal{G})|, \quad \xi \in \Gamma.$$

This implies that $n(P_m, p^{j(\alpha)}; \xi^0) \leq n(P_m, p^j; \xi^0)$. Thus we have

$$\begin{aligned} (m - \nu_j + |\alpha| + n_+(P_m, p^{j(\alpha)}; \xi^0)) / n_+(P_m, p^{j(\alpha)}; \xi^0) \\ \geq (m - \nu_j + n_+(P_m, p^j; \xi^0)) / n_+(P_m, p^j; \xi^0), \end{aligned}$$

which proves $\kappa({}^tP, x; \xi^0) \geq \kappa(P, x; \xi^0)$. Since ${}^t({}^tP) = P$, the proof is complete.

Q.E.D.

By Lemmas 2.1.2–2.1.4 we can apply the same arguments as in Theorem 2.5 in [20] to obtain easily the following theorem.

THEOREM 2.1.5 (finite propagation property). *Assume that the condition (A)_x^{*} is satisfied. If $u \in \mathcal{D}^{*,*}$ with $\text{supp } u \subset \{x_1 \geq t\}$ satisfies the equation $P(x, D)u(x) = 0$ in a neighborhood of $x^0 - \Gamma(P_m, \mathcal{G})^*$, then $u = 0$ in a neighborhood of x^0 .*

THEOREM 2.1.6. *Under the condition (A) the Cauchy problem (2.1.4) has a unique solution u in $\mathcal{D}^{*,*}$.*

2.2. Necessary conditions for well-posedness

In this section we shall prove the following theorem by the same arguments as in [11].

THEOREM 2.2.1. *Assume that the coefficients of $Q(x, D)$ are real analytic. If the Cauchy problem (2.1.4) has a unique solution $u(x)$ in $\mathcal{E}^{*,*}$ for every $f \in \mathcal{E}^{*,*}$ with $\text{supp } f \subset \{x_1 \geq t\}$ and every $t \in \mathbf{R}$, then we have $\kappa \leq \kappa_0$, where κ_0 is defined by (2.1.3).*

REMARK. (i) Theorem 2.2.1 and the results in [11] imply that it is necessary and sufficient for the Cauchy problem (2.1.4) to be $\mathcal{E}^{(\kappa)}$ -well posed, $t \in \mathbf{R}$, that $\kappa \leq \kappa_0$, if the coefficients of $Q(x, D)$ are real analytic. (ii) When $n \leq 3$, we can prove that the condition that $\kappa < \kappa_0$ is necessary for the Cauchy problem to be $\mathcal{E}^{(\kappa)}$ -well posed (see the remark of Theorem A.1.8).

In this section we assume that the coefficients of $Q(x, D)$ are real analytic. Following Ivrii [11], we define the following

DEFINITION 2.2.2. The Cauchy problem

$$(2.2.1) \quad \begin{cases} P(x, D)u(x) = f(x), \\ \text{supp } u \subset \overline{\Omega}_T^+ \end{cases}$$

is said to be weakly $\mathcal{E}^{(\kappa)}$ -correct if for every $f \in \mathcal{D}^{(\kappa)}(\Omega_T^+)$ there is a solution u in $\mathcal{D}^{(\kappa)}(\Omega)$ satisfying (2.2.1) in Ω , where Ω is an open domain in $\mathbf{R}^n$, $\Omega_T^+ = \{x \in \Omega; x_1 \geq T\}$, $T \in \mathbf{R}$ and $1 < \kappa < \infty$.

Introduce the semi norm

$$|u; K, \kappa, \theta| = \sum_{\alpha} \theta^{|\alpha|} |D^{\alpha} u; L^2(K)| / (\alpha! (|\alpha|!)^{\kappa-1}),$$

where $\theta > 0$, $1 < \kappa < \infty$ and K is a compact set in $\mathbf{R}^n$. Define

$$\mathcal{D}^{[\kappa]}(K, \theta) = \{u \in \mathcal{D}^{[\kappa]}; \text{supp } u \subset K, |u; K, \kappa, \theta| < \infty\}.$$

Then $\mathcal{D}^{[\kappa]}(K, \theta)$ is a Banach space and we have

$$\mathcal{D}^{[\kappa]}(\Omega) = \lim_{\substack{\theta \rightarrow 0 \\ K \subset \Omega}} \mathcal{D}^{[\kappa]}(K, \theta), \quad \mathcal{D}^{[\kappa]'}(\Omega) = \lim_{\substack{\theta \rightarrow 0 \\ K \subset \Omega}} \mathcal{D}^{[\kappa]'}(K, \theta).$$

LEMMA 2.2.3 ([11]). Assume that the Cauchy problem (2.2.1) is weakly $\mathcal{E}^{[\kappa]}$ -correct. Let K and K' be compact sets in Ω such that K' is a neighborhood of K . Then there is a constant $t_0 > T$ such that the following inequality holds for all $t \in (T, t_0]$, $\theta > 0$, $\tau > 0$, with some constant $C = C_{\tau, \theta}$:

$$|u; C(K_t^-)|^2 \leq C |u; K_{t+\tau}^-, \kappa, \theta| |Pu; K_{t+\tau}^-, \kappa, \theta|$$

for all $u \in \mathcal{D}^{[\kappa]}(\Omega_T^+)$, where $K_t^- = \{x \in K; x_1 \leq t\}$.

Let $\hat{\gamma}(s) = s^{-1}(\sum_{j=0}^l s^{j/l} \xi^j)$, where $\xi^j \in \mathbf{R}^n$, $\xi^0 \in \mathbf{R}^n \setminus \{0\}$ and $l \in \mathbf{N}$, and put

$$\sigma \equiv \sigma(\hat{\gamma}) = \max_{0 \leq j \leq m-1, x \in \mathbf{R}^n} n_+(P_m, Q_j(x, \cdot); \hat{\gamma}) / (m-j + n_+(P_m, Q_j(x, \cdot); \hat{\gamma})).$$

Now assume that $0 < \sigma (< 1)$. We can assume without loss of generality that

$$(2.2.2) \quad \sigma = \max_{0 \leq j \leq m-1} n_+(P_m, Q_j(0, \cdot); \hat{\gamma}) / (m-j + n_+(P_m, Q_j(0, \cdot); \hat{\gamma}))$$

and that

$$(2.2.3) \quad n_+(P_m, Q_j(0, \cdot); \hat{\gamma}) \geq n_+(P_m, Q_j(0, \cdot); \hat{\gamma}_a^\lambda)$$

if $\sigma = n_+(P_m, Q_j(0, \cdot); \hat{\gamma}) / (m-j + n_+(P_m, Q_j(0, \cdot); \hat{\gamma}))$, $0 \leq \lambda < 1$ and $a \in \mathbf{R}$. Moreover we may assume that

$$(2.2.4) \quad n_+(P_m, Q_j(x, \cdot); \hat{\gamma}(s) + \zeta) / (m-j + n_+(P_m, Q_j(x, \cdot); \hat{\gamma}(s) + \zeta)) \leq \sigma$$

for all $x \in \mathbf{R}^n$, $\zeta \in \mathbf{R}^n$ and $0 \leq j \leq m-1$. In fact, these easily follow from Lemma 1.2.4 and its proof. Put $\eta(s) = s^{-\sigma} \hat{\gamma}(s^{1-\sigma})$. Then we have

$$\text{order}_s P_m(\eta(s) - is^{-\sigma} \vartheta) \leq \text{order}_s Q_j(x, \eta(s) - is^{-\sigma} \zeta)$$

for all $x \in \mathbf{R}^n$, $\zeta \in \mathbf{R}^n$ and $0 \leq j \leq m-1$. Thus

$$\begin{aligned} P(x, D) (\exp [i\gamma^{-1}x \cdot \eta(s) + is^{-\sigma} \psi(x)] v(x)) \\ = \exp [i\gamma^{-1}x \cdot \eta(s) + is^{-\sigma} \psi(x)] \sum_{j=0}^m \sum_{\alpha} \gamma^{|\alpha| - j} s^{-j + |\alpha| + \nu_j a} / \alpha! \end{aligned}$$

$$\times (C_{j\alpha}(x) + o(1)) \sum_{\beta \leq \alpha} \alpha! / ((\alpha - \beta)! \beta!) \{ (s^{-\sigma} \psi_x)^{\alpha - \beta} + o(s^{-\sigma|\alpha - \beta|}) \} D^\beta v(x) \\ \text{as } s \rightarrow +0,$$

where

$$\begin{aligned} P_m^{(\alpha)}(s\gamma(s)) &= s^{\nu_{m\alpha}}(C_{m\alpha} + o(1)), & s \rightarrow +0, & C_{m\alpha} \not\equiv 0, \\ Q_j^{(\alpha)}(x, s\gamma(s)) &= s^{\nu_{j\alpha}}(C_{j\alpha}(x) + o(1)), & s \rightarrow +0, & C_{j\alpha}(x) \not\equiv 0, \\ & \text{if } Q_j^{(\alpha)}(x, s\gamma(s)) \equiv 0, & 0 \leq j \leq m-1. \end{aligned}$$

It follows from (2.2.2) and (2.2.4) that

$$j - |\alpha| - \nu_{j\alpha} + \sigma|\alpha| \leq \max_\beta (m - |\beta| - \nu_{m\beta} + \sigma|\beta|) \equiv \nu_1, \quad 0 \leq j \leq m-1,$$

and that there are j with $0 \leq j \leq m-1$ and a multi-index α such that $j - |\alpha| - \nu_{j\alpha} + \sigma|\alpha| = \nu_1$ and $C_{j\alpha}(0) \not\equiv 0$. Therefore we have

$$\begin{aligned} (2.2.5) \quad P(x, D) (\exp [i\gamma^{-1}x \cdot \gamma(s) + is^{-\sigma}\psi(x)] v(x)) \\ = s^{-\nu_1} \exp [i\gamma^{-1}x \cdot \gamma(s) + is^{-\sigma}\psi(x)] [\sum_\alpha 1/\alpha! T_\gamma^{(\alpha)}(x, \psi_x) s^{\sigma|\alpha|} D^\alpha v(x) \\ + \sum_{\alpha, j \geq 1} s^{\sigma|\alpha| + j\delta} a_{j\alpha}(x) D^\alpha v(x)], \end{aligned}$$

where

$$(2.2.6) \quad T_\gamma(x, \xi) = \sum_{j - |\alpha| - \nu_{j\alpha} + \sigma|\alpha| = \nu_1} \gamma^{|\alpha| - j} C_{j\alpha}(x) \xi^\alpha / \alpha!$$

and $\delta \in \mathbf{Q}$, $\delta > 0$. From the proof of Theorem 1.2.5 there are $x^0 \in \mathbf{R}^n$, a neighborhood U of x^0 , $\gamma^{0'} \in \mathbf{R}^{n-1}$, a neighborhood V of $\gamma^{0'}$ and $\gamma \in \mathbf{R}$ such that x^0 belongs to a sufficiently small neighborhood of 0 and the equation $T_\gamma(x, \tau, \gamma') = 0$ in τ has a root with negative imaginary part for $(x, \gamma') \in U \times V$ which has constant multiplicity q for $(x, \gamma') \in U \times V$. Then there is a complex analytic phase function $\psi(x)$ in U such that

$$\begin{aligned} T_\gamma^{(\alpha)}(x, \psi_x) &= 0, & |\alpha| < q, \\ (\partial/\partial \xi_1)^q T_\gamma(x^0, \psi_x(x^0)) &\not\equiv 0, & \text{Im } \psi(x^0) = 0, \\ \text{Im } \psi(x) &\geq \varepsilon(x_1^0 - x_1) + \varepsilon|x - x^0|^2 & \text{for } x_1 \leq x_1^0, \\ \text{Im } \psi(x) &\geq C(x_1^0 - x_1) + \varepsilon|x - x^0|^2 & \text{for } x_1 \geq x_1^0, \end{aligned}$$

where $\varepsilon > 0$. Thus we can write

$$\begin{aligned} P(x, D) (\exp [i\gamma^{-1}x \cdot \gamma(s) + is^{-\sigma}\psi(x)] v(x)) \\ = s^{-\nu_1} \exp [i\gamma^{-1}x \cdot \gamma(s) + is^{-\sigma}\psi(x)] P_{(s)}(x, D) v(x) \quad \text{in } U, \\ P_{(s)}(x, D) = s^{\sigma q} \{ Q_0(x, s, D) + \sum_{k=1}^{m-p} s^{\sigma k} Q_k(x, s, D) \}, \end{aligned}$$

where

$$\begin{aligned} Q_0(x, s, D) &= \sum_{|\alpha|=q} 1/\alpha! T_\gamma^{(\alpha)}(x, \psi_x) D^\alpha \\ &+ \sum_{j\delta + \sigma(|\alpha| - q) < \sigma, j \geq 1} s^{j\delta + \sigma(|\alpha| - q)} a_{j\alpha}(x) D^\alpha, \end{aligned}$$

$$Q_k(x, s, D) = \sum_{|\alpha|=q+k} \frac{1}{\alpha!} T_r^{(\alpha)}(x, \psi_x) D^\alpha + \sum_{|\alpha|=q+k, j \geq 1} s^{j\delta} a_{j\alpha}(x) D^\alpha \\ + \delta_{k1} \sum_{|\alpha| \leq q, \sigma \leq j\delta + \sigma(|\alpha|-q), j \geq 1} s^{j\delta + \sigma(|\alpha|-q-1)} a_{j\alpha}(x) D^\alpha, \quad 1 \leq k \leq m-q.$$

Here $\delta_{k1} = 1$ if $k=1$, $=0$ if $k \neq 1$. Choose a rational number σ' so that $\sigma - \delta/q \leq \sigma' < \sigma$ and $\sigma' > 0$. Then $Q_0(x, s, D)$ can be written in the form

$$Q_0(x, s, D) \equiv Q_0(x, \bar{\omega}; D) = \sum_{0 \leq k, k+|\alpha| \leq q} b_{k\alpha}(x, s) \bar{\omega}^k D^\alpha,$$

where $\bar{\omega} = s^{-\sigma'}$ and $|b_{k\alpha}(x, 0)| < \infty$, since $\delta + \sigma(|\alpha|-q) \geq \sigma'(|\alpha|-q)$. It is easy to see that the coefficient of D_1^q of $Q_0(x, \bar{\omega}; D)$ does not vanish in a neighborhood of x^0 if s is sufficiently small. Let us consider the equations

$$Q_0(x, \bar{\omega}; D) v_s^{(n)} = -g_s^{(n)}, \\ v_s^{(0)}|_{\Omega_t} = 1, \quad D_1^j v_s^{(0)}|_{\Omega_t} = 0, \quad 1 \leq j \leq q-1, \\ D_1^j v_s^{(n)}|_{\Omega_t} = 0, \quad 0 \leq j \leq q-1, \quad n=1, 2, \dots,$$

where $t = x_1^0$, $\Omega_t = \{x \in \Omega; x_1 = t\}$ and

$$g_s^{(n)} = \sum_{0 < k \leq \min(n, m-q)} s^{\sigma k} Q_k(x, s; D) v_s^{(n-k)}.$$

Define

$$\bar{\Phi}_s^{(n)}(\tau, \theta; \omega) = \sum_\alpha \tau^{\alpha_1} \theta^{|\alpha'|} / \alpha! \sum_{0 \leq k, k+|\beta| \leq q-1} |(D^{\alpha+\beta} v_s^{(n)})(x^0)| \omega^k,$$

where $\alpha' = (\alpha_2, \dots, \alpha_n)$ and $\omega \geq \bar{\omega} \gg 1$.

LEMMA 2.2.4 ([11]). *There exist positive numbers A, M, a_0 and b such that*

$$\bar{\Phi}_s^{(n)}(\tau, \theta; \omega) \ll A^{n+1} s^{\sigma n} \omega^{q-1} \sum_{k=0}^{2n} \omega^{n-k} k! (1 - \tau/a - \theta/b)^{-k-1} \exp[M\tau\omega], \\ n=0, 1, 2, \dots,$$

for sufficiently large ω , where $\sum_{j,k=0}^{\infty} a_{jk} \tau^j \theta^k \gg \sum_{j,k=0}^{\infty} b_{jk} \tau^j \theta^k$ means that $a_{jk} \geq |b_{jk}|$ for all j and k .

From Lemma 2.2.4 we have

$$\sum_{|\alpha| \leq m} |D^\alpha v_s^{(n)}; K'', \kappa, \theta| \leq B^{n+1} s^{\sigma n} \omega^m \sum_{k=0}^{2n} \omega^{n-k} k! \exp[M_1 \omega],$$

where K'' is a neighborhood of x^0 and B and M_1 are sufficiently large numbers depending on K'' . Thus, if $n \leq \omega/4$, then

$$\sum_{|\alpha| \leq m} |D^\alpha v_s^{(n)}; K'', \kappa, \theta| \leq 2B^{n+1} s^{\sigma n} \omega^{m+n} \exp[M_1 \omega].$$

Put $\omega = 4N$, $s^{-\sigma} = \omega B e^L$ and

$$V_s \equiv V_s^{(N)} = \sum_{n=0}^N v_s^{(n)},$$

where L will be determined later. Then we have

$$(2.2.7) \quad \begin{aligned} \sum_{|\alpha| \leq m} |D^\alpha V_s; K'', \kappa, \theta| &\leq CN^m \exp [4M_1 N], \\ |P_{(s)} V_s; K'', \kappa, \theta| &\leq C \exp [4M_1 N - LN]. \end{aligned}$$

Put

$$u = u_{(s)} = \exp [i\gamma^{-1}x \cdot \eta(s) + is^{-\sigma}\psi(x)] V_s \chi,$$

where $\chi \in \mathcal{D}^{(\kappa_1)}(\Omega_T^+)$, $1 < \kappa_1 < \kappa$, $\text{supp } \chi$ is contained in a small neighborhood of x^0 in K'' and $\chi=1$ in a neighborhood of x^0 . Then we have

$$(2.2.8) \quad \begin{aligned} |u_{(s)}; K'_{t+\tau}, \kappa, \theta| &\leq C |V_s; K'', \kappa, \theta| \exp [i\gamma^{-1}x \cdot \eta(s) \\ &\quad + is^{-\sigma}\psi(x)], K''_{t+\tau}, \kappa, 2\theta|. \end{aligned}$$

It follows from (2.2.7) and (2.2.8) and the lemmas in [11] that

$$|u_{(s)}; K'_{t+\tau}, \kappa, \theta| \leq CN^m \exp [4M_1 N + d(s^{-1}\theta)^{1/\epsilon} + c\tau s^{-\sigma}].$$

Therefore, if $\kappa \geq 1/\sigma$, $0 < \tau \leq \tau(L)$ and $0 < \theta \leq \theta(L)$, then

$$(2.2.9) \quad |u_{(s)}; K'_{t+\tau}, \kappa, \theta| \leq C \exp [5M_1 N].$$

On the other hand we have

$$(2.2.10) \quad |Pu_{(s)}; K'_{t+\tau}, \kappa, \theta| \leq C \exp [-6M_1 N]$$

if L is fixed sufficiently large, $0 < \tau \leq \tau(L)$, $0 < \theta \leq \theta(L)$ and $N \geq N_0$. Lemma 2.2.3 and (2.2.9) and (2.2.10) show that the Cauchy problem (2.2.1) is not weakly $\mathcal{E}^{(\kappa)}$ -correct if $\kappa \geq 1/\sigma$. Thus we have the following

THEOREM 2.2.5. *If the Cauchy problem (2.2.1) is weakly $\mathcal{E}^{(\kappa)}$ -correct, then $\kappa \leq \kappa_0$.*

From Theorem 2.2.5 Theorem 2.2.1 easily follows. We could not prove the existence of $\hat{\eta}(s)$ such that $\sigma(\hat{\eta}) = 1/\kappa_0$ unless $n \leq 3$.

2.3. Necessary and sufficient condition for $\mathcal{E}$ -well posedness

In this section we assume that the coefficients of $Q(x, D)$ are in $C^\infty (= \mathcal{E})$. Following Ivrii and Petkov [12], we shall show that the sufficient condition for Cauchy problem (2.1.4) to be $\mathcal{E}$ -well posed, which is given by Dunn [3], is also a necessary condition when the Cauchy problem (2.1.4) has the finite propagation property.

Consider the Cauchy problem

$$(2.3.1) \quad \begin{cases} P(x, D)u(x) = f(x) & \text{in } \Omega \cap \{x_1 < T_+\}, \\ \text{supp } u \subset \bar{G}_T^+, \end{cases}$$

where $T_- \leq T \leq T_+$, Ω is an open domain in $\mathbf{R}^n$ and $G = \Omega \cap \{T_- \leq x_1 \leq T_+\}$.

DEFINITION 2.3.1. The Cauchy problem (2.3.1) is said to be well-posed in G_T^+ if

(E) for every $f \in C_0^\infty$ with $\text{supp } f \subset \bar{\Omega}_T^+$ there exists a solution $u \in \mathcal{D}'(\Omega)$ satisfying (2.3.1),

and if

(U) $u=0$ in Ω_t^- for every t with $T < t \leq T_+$, provided that $u \in \mathcal{D}'(\Omega)$ satisfies (2.3.1) and that $Pu=0$ in Ω_t^- .

We say that the Cauchy problem (2.3.1) has the finite propagation property if there is a proper convex open cone Γ in $\mathbf{R}^n$ with its vertex at 0 such that $\bar{\Gamma} \subset \{x_1 > 0\} \cup \{0\}$ and the following statement holds for every $\hat{x} \in G_T^+$ with $\hat{x}_1 > T$ and $\{\hat{x} - \Gamma\} \cap G_T^+ \subseteq G_T^+$: If $u \in \mathcal{D}'(\Omega)$, $\text{supp } u \subset \bar{G}_T^+$ and $Pu=0$ in $\{\hat{x} - \Gamma\} \cap \Omega$, then $u=0$ in $\{\hat{x} - \Gamma\} \cap \Omega$.

LEMMA 2.3.2 ([12]). Assume that the Cauchy problem (2.3.1) is well-posed in G_T^+ and has the finite propagation property. Let K be a compact set in Ω and $\hat{x} \in K \cap G$ with $\hat{x}_1 > T$. Put $\Gamma(\hat{x}) = \hat{x} - \Gamma$. Let $(1, a_2, \dots, a_n) \in \Gamma$ and $\delta_1 > \delta_2 > \dots > 0$. We define a strictly hyperbolic operator M_p for each integer $p \geq 0$ by

$$\begin{aligned} M_p &= \prod_{j=1}^{p/2} ((D_1 + \sum_{k=2}^n a_k D_k)^2 - \delta_j^2 \sum_{k=2}^n D_k^2) & \text{if } p \text{ is even,} \\ M_0 &= 1, \quad M_p = (D_1 + \sum_{k=2}^n a_k D_k) M_{p-1} & \text{if } p \text{ is odd.} \end{aligned}$$

Then there exist non-negative integers p' and q' and a constant C such that

$$\|u\|_{C(\Gamma(\hat{x}))} \leq C \|PM_{p'} u\|_{C^{q'}(\Gamma(\hat{x}))}$$

for all $u \in C_0^\infty(K_T^+)$.

Making an asymptotic change of variables $y = s^{-\sigma'} x$, it follows from Lemma 2.3.2 that

$$(2.3.2) \quad \|u\|_{C(\Gamma(0))} \leq C s^{-\nu} \|P_s M_{p',s} u\|_{C^{q'}(\Gamma(0))}$$

for every $u \in C_0^\infty(W)$ and $0 < s \leq s_0(W)$ if $\hat{x}=0$, where W is a compact set in $\mathbf{R}^n$ and $P_s(y, D) = P(s^{\sigma'} y, s^{-\sigma'} D)$, $M_{p',s}(y, D) = M_{p'}(s^{\sigma'} y, s^{-\sigma'} D)$. We apply (2.3.2) to

$$(2.3.3) \quad u_s(y) = \sum_{j=0}^N s^{-j/\tau} v_j(y) E(y, s),$$

where $E(y, s) = \exp[i\{s^{-1}l^0(y) + \sum_{j=0}^{\mu-1} s^{-(\mu-j)/\tau} l^{j+1}(y)\}]$, $\mu, \tau \in \mathbf{N}$, $1 \leq \mu < \tau$, $0 \in W$ and $v_j(y) \in C^\infty(W)$ with $v_0(0) \neq 0$. Assume that

$$(2.3.4) \quad \text{Im}(s^{-1}l^0(y) + \sum_{j=0}^{\mu-1} s^{-(\mu-j)/\tau} l^{j+1}(y)) \geq -\varepsilon s^{-\delta} y_1$$

for $y \in W_0^- \cap \Gamma(0)$ and that

$$(2.3.5) \quad P_s M_{p',s} u_s = O(s^{N_1}) E(y, s),$$

where N_1 tends to ∞ as $N \rightarrow \infty$. Let $\phi \in C_0^\infty(W)$ such that $\phi=1$ in a neighborhood of 0. Then (2.3.2) is not valid for ϕu_s . Therefore it suffices to construct $u_s(y)$ satisfying (2.3.4) and (2.3.5) in order to prove that the Cauchy problem (2.3.1) is not well-posed in G_T^+ when the Cauchy problem has the finite propagation property. In this section we shall prove the following

THEOREM 2.3.3. *The Cauchy problem (2.1.4) is $\mathcal{E}$ -well posed for every $t \in \mathbf{R}$ and has the finite propagation property if and only if $\kappa_0 = \infty$, that is, $P(x, \xi)$ is hyperbolic with respect to $\mathcal{D}$ for each fixed x in $\mathbf{R}^n$.*

REMARK. The sufficiency of $\mathcal{E}$ -well posedness is given in [3]. If the coefficients of $Q(x, D)$ are in $\mathcal{E}^{(\kappa)}$ for some $\kappa < \kappa_0$, then the finite propagation property follows from Theorem 2.1.5.

Now assume that $\kappa_0 < \infty$. Then there is $\hat{\gamma}(s) = s^{-1} \sum_{j=0}^l s^{j/l} \xi^j$ with $\xi^j \in \mathbf{R}^n$, $\xi^0 \neq 0$ and $l \in \mathbf{N}$ such that

$$\sigma \equiv \max_{0 \leq j \leq m-1, x \in \mathbf{R}^n} n_+(P_m, Q_j(x, \cdot); \hat{\gamma}) / (m-j+n_+(P_m, Q_j(x, \cdot); \hat{\gamma})) > 0.$$

We can assume without loss of generality that (2.2.2)–(2.2.4) hold. Put $\eta(s) = s^{-\sigma} \hat{\gamma}(s^{1-\sigma})$. Then we have also (2.2.5). It follows from the proof of Theorem 1.2.5 that $T_r(0, \tau, \eta^0) = 0$ has a root τ_0 with $\text{Im } \tau_0 < 0$ for a fixed $\eta^0 \in \mathbf{R}^{n-1}$ and some $\gamma \in \mathbf{R}$. Let q be the multiplicity of the root τ_0 and put $\eta_0 = (\tau_0, \eta^0)$ and $\psi(x) = x \cdot \eta^0$. Moreover we may assume that $T_r^{(\alpha)}(0, \eta^0) = 0$ for $|\alpha| < q$, choosing η^0 and γ suitably. We make an asymptotic change of variables $y = s^{-\sigma'} x$, where $0 < \sigma' < \sigma$ and σ' is determined later. Then we have

$$\begin{aligned} P_s(y, D) (\exp [i\gamma^{-1} s^{\sigma'} y \cdot (\eta(s) + \gamma s^{-\sigma} \eta^0)] v_s(y)) &= s^{-\nu_1} \exp [i\gamma^{-1} s^{\sigma'} y \cdot (\eta(s) + \gamma s^{-\sigma} \eta^0)] \\ &\times \left\{ \sum_{|\alpha| \geq q} 1/\alpha! s^{(\sigma-\sigma')|\alpha|} T_r^{(\alpha)}(0, \eta^0) D^\alpha v_s(y) \right. \\ &\quad + \sum_{|\alpha| \leq q} \sum_{|\beta| \leq N} s^{(\sigma-\sigma')|\alpha| + \sigma'|\beta|} y^\beta / (\alpha! \beta!) T_{r(\beta)}^{(\alpha)}(0, \eta^0) D^\alpha v_s \\ &\quad \left. + \sum_{|\alpha|, j \geq 1, |\beta| \leq N} s^{(\sigma-\sigma')|\alpha| + j\delta + \sigma'|\beta|} y^\beta / \beta! a_{j\alpha(\beta)}(0) D^\alpha v_s + s^{\sigma'N} R_N(y, s, D) v_s \right\}, \end{aligned}$$

where $T_{r(\beta)}^{(\alpha)}(x, \xi) = (\partial^{|\alpha|+|\beta|} / \partial \xi^\alpha \partial x^\beta) T_r(x, \xi)$ and $R_N(y, s, D)$ is a differential operator of order m with smooth coefficients which together with their derivatives are uniformly bounded with respect to s in each domain $K \subset \mathbf{R}^n$, $0 < s \leq s_0(K)$. Here the $a_{j\alpha}(x)$ also depend on η^0 . Thus $a_{j\alpha}(x) \equiv a_{j\alpha}(x, \eta^0)$ and $a_{j\alpha(\beta)}(0) \equiv a_{j\alpha(\beta)}(0, \eta^0)$. Choose a rational number σ' so that $0 \leq q(\sigma - \sigma') < \sigma' < \sigma$ and $\delta > q(\sigma - \sigma')$, that is, $\max \{\sigma/(q+1), \sigma - \delta/q\} < \sigma' < \sigma$. Then we have

$$\begin{aligned} P_s(y, D) (\exp [i\gamma^{-1} s^{\sigma'} y \cdot (\eta(s) + \gamma s^{-\sigma} \eta^0)] v_s(y)) \\ = s^{-\nu_1 + q(\sigma - \sigma')} \exp [i\gamma^{-1} s^{\sigma'} y \cdot (\eta(s) + \gamma s^{-\sigma} \eta^0)] \\ \times \left\{ \sum_{|\alpha| \geq q} 1/\alpha! T_r^{(\alpha)}(0, \eta^0) D^\alpha v_s + \sum_{j=1}^{N_1-1} s^{j\delta'} c_j(y, D) v_s + s^{N_1\delta'} \hat{R}_{N_1}(y, s, D) v_s \right\}, \end{aligned}$$

where δ' is a positive rational number, $c_j(y, D)$ is a differential operator of order m with polynomial coefficients of y and $\hat{R}_{N_1}(y, s, D)$ has the same properties as $R_N(y, s, D)$. In the above argument we may replace P by $PM_{p'}$. Let $E(y, s) = \exp[i\gamma^{-1}s^{\sigma'}y \cdot (\gamma(s) + \gamma s^{-\sigma}\eta^0)]$. Then we construct an asymptotic solution u_s of the form (2.3.3) satisfying (2.3.5). Thus Theorem 2.3.3 easily follows, since $\text{Im} \{\gamma^{-1}s^{\sigma'}y \cdot (\gamma(s) + \gamma s^{-\sigma}\eta^0)\} = s^{\sigma'-\sigma}y_1 \text{Im } \tau_0$.

2.4. Wave front sets of solutions

In this section we assume that the condition (A) is fulfilled. Let $E(x, y)$ be the fundamental solution for $P(x, D)$, that is,

$$\begin{aligned} P(x, D_x)E(x, y) &= \delta(x - y), \\ \text{supp } E &\subset \{(x, y) \in \mathbf{R}^{2n}; x - y \in \Gamma(P_m, \mathcal{O})^*\}. \end{aligned}$$

Define $E_l(x, y)$ by

$$\begin{aligned} P_m(D)E_0(x, y) &= \delta(x - y), \\ P_m(D)E_{l+1}(x, y) &= -Q(x, D)E_l(x, y), \\ \text{supp } E_l(x, y) &\subset \{(x, y) \in \mathbf{R}^{2n}; x - y \in \Gamma(P_m, \mathcal{O})^*\} \quad l = 0, 1, 2, \dots \end{aligned}$$

Then, repeating the same arguments as in § 2.1, we have

$$E(x, y) = \sum_{l=0}^{\infty} E_l(x, y) \quad \text{in } \mathcal{D}^{(\kappa_0)'}(\mathbf{R}^{2n}).$$

THEOREM 2.4.1. *Under the condition (A) we have*

$$WF_{*\kappa_1}(E(x, y)) \subset \{((x, y), (\xi, -\xi)) \in T^*\mathbf{R}^{2n} \setminus 0; x - y \in \Gamma(P_{m\xi}, \mathcal{O})^*\}.$$

Let $x^0, y^0 \in \mathbf{R}^n$ and $\xi^0 \in \mathbf{R}^n \setminus \{0\}$. Let us first prove that $((x^0, y^0), (\xi^0, -\xi^0)) \notin WF_{*\kappa_1}(E(x, y))$ if $x^0 - y^0 \notin \Gamma(P_{m\xi^0}, \mathcal{O})^*$. Assume that $x^0 - y^0 \in \Gamma(P_{m\xi^0}, \mathcal{O})^*$. Then there are neighborhoods U_1 and U_2 of x^0 and y^0 , respectively, and $\eta^0 \in \Gamma(P_{m\xi^0}, \mathcal{O})$ such that

$$(2.4.1) \quad (x - y) \cdot \eta^0 < 0 \quad \text{for all } x \in U_1 \text{ and } y \in U_2.$$

We need the following lemma in order to prove the above statement.

LEMMA 2.4.2. *Put $\nu = \deg p^j$. Then there exist an open convex conic neighborhood Γ of ξ^0 and positive numbers t_0 and C such that*

$$(2.4.2) \quad |p^j(\xi - i(t|\xi|\eta^0 + \gamma(1 + |\xi|)^{1/\kappa_0}\eta^0)) / P_m(\xi - i(t|\xi|\eta^0 + \gamma(1 + |\xi|)^{1/\kappa_0}\eta^0))| \leq C\gamma^{\nu-m}$$

if $\xi \in \Gamma$, $0 \leq t \leq t_0$ and $\gamma \geq 1$.

REMARK. We can also prove Lemma 1.2.7 by the same method as used here.

PROOF. Denote by J the left-hand side of (2.4.2). Then we have

$$(2.4.3) \quad J = \{\gamma(1+|\xi|)^{1/\kappa_0}\}^{\nu-m} |p^j(\xi - i(t|\xi|\eta^0 + \mathcal{G})) / P_m(\xi - i(t|\xi|\eta^0 + \mathcal{G}))|,$$

where $\hat{\xi} = \{\gamma(1+|\xi|)^{1/\kappa_0}\}^{-1}\xi$. From Lemma 3.4 in [20] it follows that

$$\tilde{p}^j(\hat{\xi} - i(t|\hat{\xi}|\eta^0 + \mathcal{G})) \leq C(1+|\hat{\xi}|)^{n_\nu(P)} |P_m(\hat{\xi} - i(t|\hat{\xi}|\eta^0 + \mathcal{G}))|, \quad \xi \in \Gamma,$$

if $t|\hat{\xi}| \leq \rho$, where a positive number ρ is fixed and Γ is a suitable conic neighborhood of ξ^0 . Moreover we have

$$\begin{aligned} \tilde{p}^j(t^{-1}|\hat{\xi}|^{-1}\hat{\xi} - i(\eta^0 + t^{-1}|\hat{\xi}|^{-1}\mathcal{G})) &\leq C(1+t^{-1}|\hat{\xi}|^{-1}|\hat{\xi}|)^{n_\nu(P)} \\ &\times |P_m(t^{-1}|\hat{\xi}|^{-1}\hat{\xi} - i(\eta^0 + t^{-1}|\hat{\xi}|^{-1}\mathcal{G}))|, \quad \xi \in \Gamma, \end{aligned}$$

if $t|\hat{\xi}| \geq \rho$. Thus we have

$$(2.4.4) \quad |p^j(\xi - i(t|\xi|\eta^0 + \mathcal{G}))| \leq C(1+|\xi|)^{n_\nu(P)} |P_m(\xi - i(t|\xi|\eta^0 + \mathcal{G}))|, \quad \xi \in \Gamma.$$

Since $n_\nu(P) - (n_\nu(P) + m - \nu)/\kappa_0 \leq 0$, (2.4.2) follows from (2.4.3) and (2.4.4).

Q.E.D.

PROOF OF THEOREM 2.4.1. We may assume that the coefficients of $Q(x, D)$ are in $\mathcal{D}^{*\kappa_1}$. Let $\phi_j \in \mathcal{D}^{(\kappa_1)}(U_j)$, $j=1, 2$. We have

$$\begin{aligned} \mathcal{F}_{(x,y)}[\phi_1(x)\phi_2(y)E_l(x,y)](\xi, \eta) &= \sum_{j_1, \dots, j_l} (-1)^l (2\pi)^{-(1+l)n} \\ &\times \int d\zeta^0 \hat{\phi}_1(1/2\xi - \zeta^0) P_m(1/2\xi + \zeta^0)^{-1} \\ &\times \left\{ \int d\zeta^1 \hat{q}_{j_1}(\zeta^1) p^{j_1}(1/2\xi + \zeta^0 - \zeta^1) P_m(1/2\xi + \zeta^0 - \zeta^1)^{-1} \right. \\ (2.4.5) \quad &\times \left\{ \int d\zeta^2 \dots \right. \\ &\times \left\{ \int d\zeta^l \hat{q}_{j_l}(\zeta^l) p^{j_l}(1/2\xi + \zeta^0 - \zeta^1 - \dots - \zeta^l) \right. \\ &\times P_m(1/2\xi + \zeta^0 - \zeta^1 - \dots - \zeta^l)^{-1} \hat{\phi}_2(1/2\xi + \eta + \zeta^0 - \zeta^1 - \dots - \zeta^l) \} \dots \left. \right\}, \\ & \quad l=0, 1, 2, \dots, \end{aligned}$$

where $\zeta^0 = -1/2\xi + w^0 - i\gamma(1+|w^0|)^{1/\kappa_0}\mathcal{G}$, $\zeta^j = w^j + ia(1+|w^j|)^{1/\kappa_1}\mathcal{G}$, $1 \leq j \leq l$, $w^j \in \mathbf{R}^n$, $0 \leq j \leq l$, and $a > 0$. Write

$$\begin{aligned} \mathcal{F}_{(x,y)}[\phi_1\phi_2E_l](\xi, \eta) &= \sum_{j_1, \dots, j_l} (-1)^l (2\pi)^{-(1+l)n} \\ &\times \left\{ \int_{|w^1| + \dots + |w^l| < \epsilon|\xi|} + \int_{|w^1| + \dots + |w^l| > \epsilon|\xi|} \right\} d\zeta^0 \dots d\zeta^l \\ &\equiv I_1 + I_2. \end{aligned}$$

We first estimate $|I_1|$. Let I^1 and I^2 be convex conic neighborhoods of ξ^0 such

that $I^1 \subset I^2 \subset I$ and $1/2\xi + \zeta \in I$ for $\xi \in I^2$ and $\zeta \in R^n$ with $|\zeta| < \varepsilon|\xi|$, where $\varepsilon > 0$ is chosen suitably. Denote by V_t , $0 \leq t \leq t_0$, the chain $\{\zeta_t^0 \in C^n; \zeta_t^0 = -1/2\xi + w^0 - i(t| -1/2\xi + w^0| \Phi(-1/2\xi + w^0)\gamma^0 + \gamma(1+|w^0|)^{1/\kappa_0})\mathcal{G}, w^0 \in R^n\}$, where $\Phi(\zeta) \in C^\infty(R^n)$ is positively homogeneous of degree 0 in $|\zeta| \geq 1$ and $\Phi(\zeta) = 1$ on a neighborhood of $I^1 \cap \{|\zeta| \geq 1\}$, $\text{supp } \Phi(\zeta) \subset I^2 \cap \{|\zeta| \geq 1/2\}$, $0 \leq \Phi(\zeta) \leq 1$. It follows from Proposition 1.3.1. and (2.4.1) that

$$(2.4.6) \quad \begin{aligned} & |\dot{\phi}_1(1/2\xi - \zeta_t^0)\dot{\phi}_2(\gamma + 1/2\xi + \zeta_t^0 - \zeta^1 - \dots - \zeta^l)| \\ & \leq C_1 \exp [A(\gamma(1+|w^0|)^{1/\kappa_0} + a \sum_{j=1}^l (1+|w^j|)^{1/\kappa_1}) \\ & \quad - ct| -1/2\xi + w^0| \Phi(-1/2\xi + w^0) - B_1|\xi - w^0|^{1/\kappa_1} \\ & \quad - B_1|\gamma + w^0 - w^1 - \dots - w^l|^{1/\kappa_1}] \end{aligned}$$

for every $B_1 > 0$ and some C_1 depending on B_1 , where

$$\zeta_t^0 = -1/2\xi + w^0 - i(t| -1/2\xi + w^0| \Phi(-1/2\xi + w^0)\gamma^0 + \gamma(1+|w^0|)^{1/\kappa_0})\mathcal{G}$$

and A and c are positive constants. Moreover we have

$$(2.4.7) \quad |\hat{q}_j(\zeta)| \leq C_2 \exp [A|\text{Im } \zeta| - B_2|\text{Re } \zeta|^{1/\kappa_1}],$$

where (2.4.7) holds for every $B_2 > 0$ and some C_2 depending on B_2 if $*_{\kappa_1} = (\kappa_1)$, and for some positive constants B_2 and C_2 if $*_{\kappa_1} = \{\kappa_1\}$. Let $\xi \in I^1$. Then we have

$$(2.4.8) \quad |p^{jk}(1/2\xi + \zeta_t^0 - \zeta^1 - \dots - \zeta^k)/P_m(1/2\xi + \zeta_t^0 - \zeta^1 - \dots - \zeta^k)| \leq C\gamma^{-\mu}$$

if $|w^1| + \dots + |w^k| < \varepsilon|\xi|$, $\gamma \geq 1$ and $0 \leq t \leq t_0$, where μ is defined by (2.1.5). In fact, (2.4.8) is proved in the proof of Lemma 2.1.2 if $\Phi(-1/2\xi + w^0) = 0$. On the other hand we have $-1/2\xi + w^0$, $w^0 \in I^2$ and $|w^0| \geq |\xi|/2$ if $\Phi(-1/2\xi + w^0) \neq 0$ and I^2 is sufficiently small. Thus we have

$$\gamma(1+|w^0|)^{1/\kappa_0} \geq \gamma/2(1+|w^0 - w^1 - \dots - w^k|)^{1/\kappa_0}$$

if $\Phi(-1/2\xi + w^0) \neq 0$, $|w^1| + \dots + |w^k| < \varepsilon|\xi|$ and ε is sufficiently small. Then Lemma 2.4.2 gives (2.4.8). From Stokes' formula and (2.4.6)–(2.4.8) it follows that

$$(2.4.9) \quad \begin{aligned} & |I_1| \leq \left| \int d\zeta^0 \int_{|w^1| + \dots + |w^l| < \varepsilon|\xi|} dw^1 \dots dw^l \dots \right| \\ & \leq CC_1(CC_2\gamma^{-\mu})^{l+1} \int dw^0 \dots dw^l \\ & \quad \times \exp [A(\gamma(1+|w^0|)^{1/\kappa_0} + 2a \sum_{j=1}^l (1+|w^j|)^{1/\kappa_1}) \\ & \quad - \delta B_1|\xi|^{1/\kappa_1} - \delta B_1|w^0|^{1/\kappa_1} - B_1|\gamma + w^0 - w^1 - \dots - w^l|^{1/\kappa_1} \\ & \quad - B_2(|w^1|^{1/\kappa_1} + \dots + |w^l|^{1/\kappa_1})], \quad \zeta^0 \in V_{t_0}, \quad \xi \in I^1, \quad \gamma \geq 1. \end{aligned}$$

Here we have used the inequality

$$\begin{aligned} & \exp [-ct_0 | -1/2\xi + w^0 | \Phi(-1/2\xi + w^0) - B_1 |\xi - w^0|^{1/\kappa_1}] \\ & \leq C \exp [-\delta B_1 |\xi|^{1/\kappa_1} - \delta B_1 |w^0|^{1/\kappa_1}], \quad \xi \in \Gamma^1, \end{aligned}$$

for some $\delta > 0$. Choose $a > 0$ so that $2aA \leq B_2/2$. Then by (2.4.9) we obtain

$$(2.4.10) \quad |I_1| \leq C_r C_1 (CC_2 \gamma^{-\mu})^{l+1} \exp [-\delta B_1 |\xi|^{1/\kappa_1}], \quad \xi \in \Gamma^1, \quad \gamma \geq 1,$$

since $\exp [A\gamma(1+|w^0|)^{1/\kappa_0} - \delta B_1 |w^0|^{1/\kappa_1}/2] \leq C_r$. $|I_2|$ can be estimated without a change of the integral chain. We have

$$\begin{aligned} |I_2| & \leq CC_1 (CC_2 \gamma^{-\mu})^{l+1} \int_{|w^1| + \dots + |w^l| > \varepsilon |\xi|} dw^0 \dots dw^l \\ & \quad \times \exp [A(\gamma(1+|w^0|)^{1/\kappa_0} + 2a \sum_{j=1}^l (1+|w^j|)^{1/\kappa_1}) \\ & \quad - B_1 |\xi - w^0|^{1/\kappa_1} - B_1 |\gamma + w^0 - w^1 - \dots - w^l|^{1/\kappa_1} \\ & \quad - B_2 (|w^1|^{1/\kappa_1} + \dots + |w^l|^{1/\kappa_1})]. \end{aligned}$$

Choose $a > 0$ so that $2aA \leq B_2/4$. Then we obtain

$$(2.4.11) \quad |I_2| \leq C_r C_1 (CC_2 \gamma^{-\mu})^{l+1} \exp [-\varepsilon B_2 |\xi|^{1/\kappa_1}/4], \quad \gamma \geq 1.$$

Here we have used the inequality

$$\exp [A\gamma(1+|w^0|)^{1/\kappa_0} - B_1 |\xi - w^0|^{1/\kappa_1}/2 - \varepsilon B_2 |\xi|^{1/\kappa_1}/4] \leq C_r.$$

(2.4.10) and (2.4.11) mean that $((x^0, y^0), (\xi^0, \eta^1)) \in WF_{*\kappa_1}(E(x, y))$ if $x^0 - y^0 \in \Gamma(P_{m\xi^0}, \mathcal{V})^*$. Finally let us prove that

$$(2.4.12) \quad ((x^0, y^0), (\xi^0, \eta^1)) \in WF_{*\kappa_1}(E(x, y)) \quad \text{if } \xi^0 \neq -\eta^1.$$

Let $\tilde{\Gamma}$ be a conic neighborhood of (ξ^0, η^1) in $\mathbf{R}^{2n} \setminus \{0\}$. Then we may assume that $|\xi + \eta| \geq 2\varepsilon |(\xi, \eta)|$, $\varepsilon > 0$ if $(\xi, \eta) \in \tilde{\Gamma}$. Write

$$\begin{aligned} (2.4.13) \quad \mathcal{F}_{(x,y)}[\phi_1(x)\phi_2(y)E_l(x,y)](\xi,\eta) &= \sum_{j_1, \dots, j_l} (-1)^l (2\pi)^{-(1+l)n} \\ & \quad \times \int d\zeta^0 \hat{\phi}_1(\xi - \zeta^0) P_m(\zeta^0)^{-1} \\ & \quad \times \left\{ \int d\zeta^1 \hat{q}_{j_1}(\zeta^1) P^{j_1}(\zeta^0 - \zeta^1) P_m(\zeta^0 - \zeta^1)^{-1} \right. \\ & \quad \times \left\{ \int d\zeta^2 \dots \right. \\ & \quad \times \left\{ \int d\zeta^l \hat{q}_{j_l}(\zeta^l) P^{j_l}(\zeta^0 - \zeta^1 - \dots - \zeta^l) P_m(\zeta^0 - \zeta^1 - \dots - \zeta^l)^{-1} \right. \\ & \quad \times \left. \hat{\phi}_2(\eta + \zeta^0 - \zeta^1 - \dots - \zeta^l) \right\} \dots \left. \right\}, \end{aligned}$$

where $\zeta^0 = w^0 - i\gamma(1 + |w^0|)^{1/\kappa_0}\mathcal{G}$, $\zeta^j = w^j + ia(1 + |w^j|)^{1/\kappa_1}\mathcal{G}$, $1 \leq j \leq l$, and $w^j \in \mathbb{R}^n$, $0 \leq j \leq l$. Thus (2.1.8) and Proposition 1.3.1 give

$$\begin{aligned} |\mathcal{F}[\phi_1\phi_2E_t](\xi, \eta)| &\leq CC_1(CC_2\gamma^{-\mu})^{l+1} \int dw^0 \cdots dw^l \\ &\quad \times \exp [A(\gamma(1 + |w^0|)^{1/\kappa_0} + 2a \sum_{j=1}^l (1 + |w^j|)^{1/\kappa_1}) \\ &\quad - B_1|\xi - w^0|^{1/\kappa_1} - B_1|\eta + w^0 - w^1 - \cdots - w^l|^{1/\kappa_1} \\ &\quad - B_2(|w^1|^{1/\kappa_1} + \cdots + |w^l|^{1/\kappa_1})], \quad \gamma \geq 1. \end{aligned}$$

Choose positive numbers a and B_1 so that $B_1 = B_2/2$ and $2aA = B_2/4$. Since

$$\begin{aligned} &|\xi - w^0|^{1/\kappa_1} + |\eta + w^0 - w^1 - \cdots - w^l|^{1/\kappa_1} + (|w^1|^{1/\kappa_1} + \cdots + |w^l|^{1/\kappa_1}) \\ &\geq 2\varepsilon' |(\xi, \eta)|^{1/\kappa_1}, \quad \varepsilon' > 0, \quad \text{for } (\xi, \eta) \in \tilde{I}, \\ &\exp [A\gamma(1 + |w^0|)^{1/\kappa_0} - B_1|\xi - w^0|^{1/\kappa_1}/2 - \varepsilon'|\xi|^{1/\kappa_1}/2] \leq C_\gamma, \end{aligned}$$

we have

$$|\mathcal{F}[\phi_1\phi_2E_t](\xi, \eta)| \leq C_\gamma C_1(CC_2\gamma^{-\mu})^{l+1} \exp [-\varepsilon' B_2 |(\xi, \eta)|^{1/\kappa_1}/4]$$

if $(\xi, \eta) \in \tilde{I}$ and $\gamma \geq 1$. This proves (2.4.12). The proof of the theorem is complete. **Q.E.D.**

THEOREM 2.4.3. *Under the condition (A) we have $((x^0, y^0), (\xi^0, -\xi^0)) \in WF^{(\kappa(\xi^0))}(E(x, y))$ for $x^0, y^0 \in \mathbb{R}^n$ and $\xi^0 \in \mathbb{R}^n \setminus \{0\}$, where $\kappa(\xi^0)$ is defined by (2.1.2).*

PROOF. Let U_1 and U_2 be neighborhoods of x^0 and y^0 , respectively, and let $\phi_j \in \mathcal{D}^{(\kappa)}(U_j)$, $j = 1, 2$, $\kappa_1 < \kappa < \kappa_0$. We may assume that the coefficients of $Q(x, D)$ are in $\mathcal{D}^{*\kappa_1}$. Therefore they are in $\mathcal{D}^{(\kappa)}$. Then we have (2.4.13) with $\zeta^0 = w^0 - i\gamma(1 + |w^0|)^{1/\kappa_0}\mathcal{G}$, $\zeta^j = w^j + i(1 + |w^j|)^{1/\kappa_1}\mathcal{G}$, $1 \leq j \leq l$, and $w^j \in \mathbb{R}^n$, $0 \leq j \leq l$. Let Γ^j , $j = 0, 1, 2$, be conic neighborhoods of ξ^0 such that $\Gamma^0 \subset \Gamma^1 \subset \Gamma^2$ and $\xi + \zeta \in \Gamma^2$ if $\xi \in \Gamma^1$ and $|\zeta| \leq \varepsilon|\xi|$, where $\varepsilon > 0$ is sufficiently small. Choose $\Phi(\zeta) \in C^\infty(\mathbb{R}^n)$ so that $\Phi(\zeta)$ is positively homogeneous of degree 0 in $|\zeta| \geq 1$, $\text{supp } \Phi(\zeta) \subset \Gamma^1 \cap \{|\zeta| \geq 1/2\}$, $\Phi(\zeta) = 1$ on a neighborhood of $\Gamma^0 \cap \{|\zeta| \geq 1\}$ and $0 \leq \Phi(\zeta) \leq 1$. Put

$$\zeta_t^0 = w^0 - i\gamma\{(1 + |w^0|)^{1/\kappa_0} + t\Phi(w^0)((1 + |w^0|)^{1/\kappa(\xi^0)} - (1 + |w^0|)^{1/\kappa_0})\}\mathcal{G}, \quad 0 \leq t \leq 1.$$

Let us first prove the inequality

$$(2.4.14) \quad J \equiv |p^j(\zeta_t^0 - \zeta^1 - \cdots - \zeta^k)/P_m(\zeta_t^0 - \zeta^1 - \cdots - \zeta^k)| \leq C\gamma^{-\mu}, \quad \gamma \geq 1,$$

where μ is defined by (2.1.5) with κ_1 replaced by κ . (2.4.14) follows from (2.1.8) if $w^0 \notin \Gamma^1$. Let $w^0 \in \Gamma^1$. Then

$$(2.4.15) \quad J \leq C\{\gamma(1 + |w^0|)^{1/\kappa(\xi^0)}\}^{\nu-m}(1 + \{\gamma(1 + |w^0|)^{1/\kappa(\xi^0)}\}^{-n_\nu(\xi^0)}|w^0|^{n_\nu(\xi^0)})$$

if $|w^1| + \cdots + |w^k| \leq \varepsilon|w^0|$ and Γ^2 and ε are sufficiently small, where $\nu = \deg p^j$ and $n_\nu(\xi^0)$ is defined by (2.1.1). Thus we have

$$J \leq C \gamma^{\nu-m} \quad \text{if } \gamma \geq 1 \quad \text{and} \quad |w^1| + \cdots + |w^k| \leq \varepsilon |w^0|.$$

Moreover we have

$$\begin{aligned} J &\leq C \{ \gamma (1 + |w^0|)^{1/\kappa(\xi^0)} + \sum_{h=1}^k (1 + |w^h|)^{1/\kappa} \}^{\nu-m} \\ &\quad \times (1 + \{ \gamma (1 + |w^0|)^{1/\kappa(\xi^0)} + \sum_{h=1}^k (1 + |w^h|)^{1/\kappa} \}^{-n_\nu(P)} \{ \sum_{j=1}^k (1 + |w^j|) \}^{n_\nu(P)}) \\ &\leq C \gamma^{-\mu} \quad \text{if } \gamma \geq 1 \quad \text{and} \quad |w^1| + \cdots + |w^k| \geq \varepsilon |w^0|. \end{aligned}$$

Let $\tilde{I}$ be a conic neighborhood of $(\xi^0, -\xi^0)$ in $\mathbf{R}^{2n} \setminus \{0\}$ such that $\{\xi; (\xi, \eta) \in \tilde{I} \text{ for some } \eta\} \subset \Gamma^0$. From Proposition 1.3.1 and (2.4.14) it follows that

$$\begin{aligned} |\mathcal{F}[\phi_1 \phi_2 E_t](\xi, \eta)| &\leq \left| \int_{V_1} d\xi^0 dw^1 \cdots dw^l \cdots \right| \leq C (CC_B \gamma^{-\mu})^{l+1} \int dw^0 \cdots dw^l \\ (2.4.16) \quad &\times \exp [A \{ \gamma (1 + |w^0|)^{1/\kappa_0} + \gamma \Phi(w^0) ((1 + |w^0|)^{1/\kappa(\xi^0)} - (1 + |w^0|)^{1/\kappa_1}) \\ &\quad + \sum_{j=1}^l (1 + |w^j|)^{1/\kappa} - B |\xi - w^0|^{1/\kappa} - B \sum_{j=1}^l |w^j|^{1/\kappa} \\ &\quad - B |\eta + w^0 - w^1 - \cdots - w^l|^{1/\kappa}], \quad \gamma \geq 1, \end{aligned}$$

where $V_t = \{\xi_t^0; w^0 \in \mathbf{R}^n\}$, $0 \leq t \leq 1$. It is easy to see that

$$(2.4.17) \quad (1 + |w^0|)^{1/\kappa(\xi^0)} \leq (1 + |\xi|)^{1/\kappa(\xi^0)} + (1 + |\xi - w^0|)^{1/\kappa(\xi^0)},$$

$$(2.4.18) \quad \exp [\gamma (1 + |\xi - w^0|)^{1/\kappa(\xi^0)} - |\xi - w^0|^{1/\kappa}] \leq C_\gamma.$$

If $\Phi(w^0) \asymp 1$, then we have

$$(2.4.19) \quad 1 + |\xi - w^0|^{1/\kappa} \geq \delta (|\xi|^{1/\kappa} + |w^0|^{1/\kappa}), \quad (\xi, \eta) \in \tilde{I},$$

for some $\delta > 0$. Thus

$$\begin{aligned} (2.4.20) \quad &\gamma \{ (1 + |w^0|)^{1/\kappa_0} + \Phi(w^0) ((1 + |w^0|)^{1/\kappa(\xi^0)} - (1 + |w^0|)^{1/\kappa_1}) \\ &\quad - |\xi - w^0|^{1/\kappa} \} \leq \gamma (1 + |w^0|)^{1/\kappa_0} - \delta |\xi|^{1/\kappa} - \delta |w^0|^{1/\kappa} \end{aligned}$$

if $\Phi(w^0) \asymp 1$ and $(\xi, \eta) \in \tilde{I}$. It follows from (2.4.16) with $B = A + 1$ and (2.4.17)–(2.4.20) that

$$|\mathcal{F}[\phi_1 \phi_2 E_t](\xi, \eta)| \leq C_\gamma (C \gamma^{-\mu})^{l+1} \exp [A \gamma |\xi|^{1/\kappa(\xi^0)}], \quad \gamma \geq 1,$$

which proves the theorem. **Q.E.D.**

Theorems 2.4.2 and 2.4.3 together with Lemma 1.3.11 give the following

THEOREM 2.4.4. *Assume that the condition (A) is satisfied and $f \in \mathcal{D}^{(\kappa_1)'} (resp. \mathcal{D}^{(\kappa_1)})$ with $\text{supp } f \subset \{x \in \mathbf{R}^n; x_i \geq t\}$, $t \in \mathbf{R}$. Let u be the solution of (2.1.4) and $\kappa_1 \leq \kappa \leq \infty$. Then Theorem 1.4.4 is valid if $E * f$ and $\delta(\xi)$ are replaced by u and $1/\kappa(\xi)$, respectively.*

The finite set $\{\kappa(\xi); \xi \in \mathbf{R}^n \setminus \{0\}\}$ can be written in the form

$$\{\kappa(\xi); \xi \in \mathbb{R}^n \setminus \{0\}\} = \{s_j; 1 \leq j \leq l\}, \quad 1 < \kappa_0 = s_1 < s_2 < \dots < s_l = \infty.$$

Define

$$\begin{aligned} \tilde{K}_j &= \{((x, \xi), (y, \xi)); x - y \in \Gamma(P_{m\xi}, \mathcal{D})^*, \xi \in \mathbb{R}^n \setminus \{0\} \text{ and } \kappa(\xi) = s_j\}, \quad 1 \leq j \leq l, \\ K &= \bigcup_{j=1}^l \tilde{K}_j. \end{aligned}$$

COROLLARY 2.4.5. *Corollary 1.4.5 is valid if $E*f$ and the K_j are replaced by the solution u and the K_j , respectively.*

COROLLARY 2.4.6. *Corollary 1.4.6 is valid if $E*f$ and $\delta(\xi)$ are replaced by the solution u and $1/\kappa(\xi)$, respectively.*

Appendix

§ A.1. Procedure to determine the class of hyperbolicity in the case $n \leq 3$

Let $H(\xi)$ be a hyperbolic polynomial with respect to $\mathcal{D}$, homogeneous of degree m and let $S(\xi)$ be a homogeneous polynomial of degree m' , $m' < m$.

LEMMA A.1.1. *Let $n=2$ and $\xi^0 \in \mathbb{R}^2 \setminus \{0\}$. Then we have*

$$n(H, S; \xi^0) = \begin{cases} m' - m + \deg H_{\xi^0} - \deg S_{\xi^0} & \text{if } \deg H_{\xi^0} \geq \deg S_{\xi^0}, \\ m' - m, & \text{otherwise.} \end{cases}$$

This lemma is obvious and we shall calculate $n(H, S; \xi^0)$ when $n=3$.

LEMMA A.1.2. *Let $Q(\tau, s)$ be a polynomial. Write*

$$\begin{aligned} Q(\tau, s) &= \tau^{\alpha_1} s^{\alpha_2} \hat{Q}(\tau, s), \quad \hat{Q}(\tau, 0) \equiv 0, \quad \hat{Q}(0, s) \equiv 0 \\ \hat{Q}(\tau, s) &= \hat{q}_0(s) \tau^{\alpha_3} + \hat{q}_1(s) \tau^{\alpha_3-1} + \dots + \hat{q}_{\alpha_3}(s), \end{aligned}$$

where $Q(0, s) = \hat{q}_{\alpha_3}(s)$. Moreover write

$$\begin{aligned} \hat{Q}(0, s) &= s^{\alpha_4} q(s), & q(0) &\equiv 0, \\ \hat{q}_0(s) &= s^{\alpha_5} q_0(s), & q_0(0) &\equiv 0, \\ \hat{Q}(\tau, 0) &= \tau^{\alpha_6} q_1(\tau), & q_1(0) &\equiv 0. \end{aligned}$$

Let $\lambda_j(s)$, $1 \leq j \leq t$, be the roots of $Q(\tau, s) = 0$. Then we have $t = \alpha_1 + \alpha_3$. Put $\nu_j = \text{order } \lambda_j(s)$. We have

$$\begin{aligned} \sum_{\nu_j < \infty} \nu_j &= \alpha_4 - \alpha_5, & \sum_{0 < \nu_j < \infty} \nu_j &= \alpha_4, \\ \sum_{0 < \nu_j < \infty} 1 &= \alpha_6, & \sum_{\nu_j = \infty} 1 &= \alpha_1, & \sum_{\nu_j \leq 0} 1 &= \alpha_3 - \alpha_6. \end{aligned}$$

Moreover we have

$$Q(\tau, s) = \tau^{\alpha_1} s^{\alpha_2} q_0(s) \prod_{0 < \nu_j < \infty} (\tau - \lambda_j(s)) \prod_{\nu_j \leq 0} (s^{-\nu_j} \tau - s^{-\nu_j} \lambda_j(s)).$$

PROOF. Lemma A.1.2 easily follows from the Newton polygon of $Q(\tau, s)$.

Q.E.D.

DEFINITION A.1.3. Let $\tau_j(s)$, $j=1, 2$, be Puiseux series. Then we define

$$d(\tau_1, \tau_2) = \text{order}(\tau_1(s) - \tau_2(s)),$$

that is, $\tau_1(s) - \tau_2(s) = As^{d(\tau_1, \tau_2)}(1 + o(1))$ as $s \rightarrow +0$, $A \neq 0$.

The following lemma is obvious.

LEMMA A.1.4. Let $\tau_j(s)$, $j=1, 2, 3$, be Puiseux series. Then we have the following: (i) $d(\tau_1, \tau_2) = d(\tau_2, \tau_1)$. (ii) If $d(\tau_1, \tau_2) > d(\tau_1, \tau_3)$, $d(\tau_1, \tau_3) = d(\tau_2, \tau_3)$. (iii) If $d(\tau_1, \tau_2) = d(\tau_1, \tau_3)$, $d(\tau_2, \tau_3) \geq d(\tau_1, \tau_2)$. (iv) $d(\tau_1 + \tau_3, \tau_2 + \tau_3) = d(\tau_1, \tau_2)$. (v) $d(\tau_1 + as^\nu, \tau_2) = \nu$ if $a \neq 0$ and $\nu < d(\tau_1, \tau_2)$. (vi) $d(\tau_1 + as^\nu, \tau_2) = \min(\nu, d(\tau_1, \tau_2))$ for some a .

Let $\xi^0 = (0, 0, 1)$ and $\mathcal{G} = (1, 0, 0)$. Write $H(\tau, s, 1)$ and $S(\tau, s, 1)$ in the form

$$\begin{aligned} H(\tau, s, 1) &= H_1(\tau, s)R(\tau, s)A(\tau, s), & A(0, 0) &\neq 0, \\ S(\tau, s, 1) &= S_1(\tau, s)R(\tau, s)B(\tau, s), & B(0, 0) &\neq 0, \end{aligned}$$

where $H_1(\tau, s)$ and $S_1(\tau, s)$ are relatively prime. Let $\tau_j(s)$, $1 \leq j \leq r_1$, and $\lambda_j(s)$, $1 \leq j \leq r_2$, be the distinct roots of $H_1(\tau, s) = 0$ and $S_1(\tau, s) = 0$, respectively, such that $\text{order } \tau_j(s) > 0$ and $\text{order } \lambda_j(s) > 0$. Moreover let δ_j , $1 \leq j \leq r_1$, and γ_j , $1 \leq j \leq r_2$, be the multiplicities of the roots $\tau_j(s)$ and $\lambda_j(s)$, respectively. Here “the distinct roots” means that they are not identically equal. Hyperbolicity of $H(\xi)$ implies that the $\tau_j(s)$ are analytic and real-valued. Since H_1 and S_1 are relatively prime, we have

$$d(\tau_j, \tau_k) < \infty \quad \text{for } j \neq k, \quad d(\lambda_j, \lambda_k) < \infty \quad \text{for } j \neq k, \quad d(\tau_j, \lambda_k) < \infty.$$

Put $\eta(s) = \xi^0 \pm s\xi^1 + \sum_{j=1}^{\infty} a_j s^{\nu_j} \mathcal{G}$, where $\xi^1 = (0, 1, 0)$, $a_j \in \mathbf{R} \setminus \{0\}$ and $\nu_j \in \mathbf{Q}$, $0 < \nu_1 < \nu_2 < \dots$. Consider the order of $I_\gamma(\eta, l) \equiv S(\eta(s) - i\gamma s^l \mathcal{G}) / H(\eta(s) - i\gamma s^l \mathcal{G})$ with respect to s , where $l > 0$, $\gamma \in \mathbf{R} \setminus \{0\}$. It follows from Lemma A.1.2 that

$$\begin{aligned} \text{order } I_\gamma(\eta, l) &= \alpha + \text{order} \prod_{j=1}^{r_2} (\tau(s) - i\gamma s^l - \lambda_j(\pm s))^{\gamma_j} \\ &\quad - \text{order} \prod_{j=1}^{r_1} (\tau(s) - i\gamma s^l - \tau_j(\pm s))^{\delta_j}, \end{aligned}$$

where $\tau(s) = \sum_{j=1}^{\infty} a_j s^{\nu_j}$ and $S_1(\tau, s) = s^a \hat{S}_1(\tau, s)$, $\hat{S}_1(\tau, 0) \neq 0$. It is easily seen that

$$d(\tau(s) - i\gamma s^l, \tau_j(\pm s)) \leq l \quad \text{for } \gamma \in \mathbf{R} \setminus \{0\},$$

and that

$$(A.1.1) \quad d(\tau(s) - i\gamma s^l, \lambda_j(\pm s)) \leq l \quad \text{for some } \gamma \in \mathbf{R} \setminus \{0\}.$$

Choose $\gamma \in \mathbf{R} \setminus \{0\}$ so that (A.1.1) is fulfilled and put $\hat{\gamma}(s; l) = s^{-1}\gamma(s^{1/l})$. Then we have

$$\begin{aligned} \text{order } S(\hat{\gamma}(s; l) - i\gamma\partial)/H(\hat{\gamma}(s; l) - i\gamma\partial) &= -n(\gamma, l), \\ n(\gamma, l) &= m' - m + \rho(\gamma, l)/l, \\ \rho(\gamma, l) &= -\alpha + \sum_{j=1}^{r_1} \delta_j d(\tau(s) - i\gamma s^l, \tau_j(\pm s)) - \sum_{j=1}^{r_2} \gamma_j d(\tau(s) - i\gamma s^l, \lambda_j(\pm s)). \end{aligned}$$

Lemma A.1.5.

$$\begin{aligned} n(H, S, \xi^0) &= \sup \{n(\gamma, l); \gamma(s) = \xi^0 \pm s\xi^1 + \sum_{j=1}^{\infty} a_j s^{\nu_j} \partial, \\ & \quad l > 0, a_j \in \mathbf{R} \setminus \{0\}, \nu_j \in \mathbf{Q} \text{ and } 0 < \nu_1 < \nu_2 < \dots\}. \end{aligned}$$

PROOF. Let $\hat{\gamma}(s) = s^{-1}(\xi^0 + \sum_{j=1}^{\infty} s^{j/L} \eta^j) \in N_{\xi^0}$, $\eta^j \in \mathbf{R}^3$, $L \in \mathbf{N}$. Write $\hat{\gamma}(s) = a_0(s)(\xi^0 + a_1(s)\xi^1 + a_2(s)\partial)$, where $a_0(s) = s^{-1}(1 + o(1))$ as $s \rightarrow +0$ and $\text{order } a_j(s) > 0$, $j = 1, 2$. Assume that $a_1(s) \not\equiv 0$. Then put $t = |a_1(s)|$. If $a_1(s) = cs^{\mu}(1 + o(1))$, $c \not\equiv 0$, as $s \rightarrow +0$, we have $s = c't^{1/\mu}(1 + o(1)) \equiv s(t)$, $c' > 0$, as $t \rightarrow +0$. Thus

$$\hat{\gamma}(s(t)) = a_0(s(t))(\xi^0 \pm t\xi^1 + \sum_{j=1}^{\infty} b_j t^{\nu_j} \partial) \equiv a_0(s(t))\gamma(t), \quad \text{if } t = \pm a_1(s),$$

where $b_j \in \mathbf{R}$ and $\nu_j \in \mathbf{Q}$, $0 < \nu_1 < \nu_2 < \dots$. Since

$$\begin{aligned} |S(\hat{\gamma}(s) - i\partial)/H(\hat{\gamma}(s) - i\partial)| \\ &= |a_0(s(t))^{m'-m} S(\gamma(t) - ia_0(s(t))^{-1}\partial)/H(\gamma(t) - ia_0(s(t))^{-1}\partial)| \\ &\leq C |a_0(s(t))|^{m'-m} t^{-\rho(\gamma, 1/\mu)} (1 + o(1)) \quad \text{as } t \rightarrow +0, \end{aligned}$$

we obtain

$$(A.1.2) \quad n(H, S; \hat{\gamma}) \leq n(\gamma, 1/\mu).$$

When $a_1(s) \equiv 0$, we can apply the above argument to $\hat{\gamma}(s)$ replaced by $\hat{\gamma}(s) + a\xi^1$. In fact, we have

$$|S(\hat{\gamma}(s) - i\partial)/H(\hat{\gamma}(s) - i\partial)| \leq C |S(\hat{\gamma}(s) + a\xi^1 - i\partial)/H(\hat{\gamma}(s) + a\xi^1 - i\partial)|$$

for some $a \in \mathbf{R} \setminus \{0\}$, which shows that $n(H, S; \hat{\gamma}) \leq n(H, S; \hat{\gamma} + a\xi^1)$. On the other hand it is obvious that

$$(A.1.3) \quad n(H, S; \xi^0) \geq n(\gamma, l).$$

(A.1.2) and (A.1.3) imply the lemma.

Q.E.D.

LEMMA A.1.6. Let $\gamma(s) = \xi^0 \pm s\xi^1 + \tau(s)\partial$, $\text{order } \tau(s) > 0$. Assume that $d(\tau(s), \tau_j(\pm s)) < l$, $1 \leq j \leq r_1$. Then we have $n(\gamma, l) \leq \max(m' - m, n(\gamma, l_0))$, where

$$(A.1.4) \quad l_0 = \max_{1 \leq j \leq r_1} d(\tau(s), \tau_j(\pm s)).$$

PROOF. From Lemma A.1.4 it follows that $\rho(\eta, l) \leq \rho(\eta, l_0)$. In fact, we have

$$d(\tau(s) - i\gamma s^l, \tau_j(\pm s)) = d(\tau(s) - i\gamma s^{l_0}, \tau_j(\pm s)) < l, \quad \gamma \in R,$$

and

$$\begin{aligned} d(\tau(s) - i\gamma s^l, \lambda_j(\pm s)) &= \min(l, d(\tau(s), \lambda_j(\pm s))), \\ d(\tau(s) - i\gamma s^{l_0}, \lambda_j(\pm s)) &= \min(l_0, d(\tau(s), \lambda_j(\pm s))) \end{aligned}$$

for some $\gamma \in R \setminus \{0\}$. If $\rho(\eta, l_0) \leq 0$, then $n(\eta, l) \leq m' - m$. On the other hand we have $n(\eta, l) \leq n(\eta, l_0)$ if $\rho(\eta, l_0) \geq 0$. **Q.E.D.**

If $H_1(0, 0) \neq 0$, that is, $H_1(\tau, s) \equiv 1$, then $n(H, S; \xi^0) = m' - m$. Therefore we consider only the case $H_1(0, 0) = 0$. Then we have $n(H, S; \xi^0) > m' - m$. For it is easy to see that $n(\eta, l) \rightarrow m' - m + \delta_1$ as $l \rightarrow \infty$ if $\eta_j(s) = \xi^0 + s\xi^1 + \tau_1(s)\vartheta$. By Lemma A.1.6 we have

$$\sup_{l > 0} n(\eta, l) = \sup_{0 < l \leq l_0} n(\eta, l),$$

where $\eta_j(s) = \xi^0 \pm s\xi^1 + \tau(s)\vartheta$ and l_0 is defined by (A.1.4), if $\sup_{l > 0} n(\eta, l) > m' - m$. If $l \leq l_0$, then $\tau(s)$ can be written in the form

$$\tau(s) = \sum_{k < l, k \in N} a_{j_0 k} (\pm s)^k + s^{k_0} \sum_{k=1}^{\infty} a'_k s^{\nu'_k},$$

where $1 \leq j_0 \leq r_1$, $\tau_{j_0}(s) = \sum_{k=1}^{\infty} a_{j_0 k} s^k$, $k_0 = \max \{j \in Z; k < l\} (\geq 0)$, $0 < \nu'_1 < \nu'_2 < \dots$, $k_0 + \nu'_1 \geq l$. Therefore we have

$$\begin{aligned} d(\tau(s) - i\gamma s^l, \tau_j(\pm s)) &= d(\tau_{j_0}(s) - i\gamma s^l, \tau_j(s)), & \gamma \in R \setminus \{0\}, \\ d(\tau(s) - i\gamma s^l, \lambda_j(\pm s)) &= d(\tau_{j_0}(s) - i\gamma s^l, \lambda_j(s)) & \text{for some } \gamma \in R \setminus \{0\}. \end{aligned}$$

If we put $\eta_{j_0}(s) = \xi^0 + s\xi^1 + \tau_{j_0}(s)\vartheta$, we have $n(\eta, l) = n(\eta_{j_0}(s), l)$. Thus it suffices to choose $\eta_j(s) \equiv \xi^0 + s\xi^1 + \tau_j(s)\vartheta$, $1 \leq j \leq r_1$, as $\eta(s)$ in order to calculate $n(H, S; \xi^0)$. Define

$$\begin{aligned} d_{ij} &= d(\tau_i, \tau_j) & \text{for } 1 \leq i, j \leq r_1, \\ d_{ij+\tau_1} &= d(\tau_i, \lambda_j) & \text{for } 1 \leq i \leq r_1 \text{ and } 1 \leq j \leq r_2, \\ \delta_{j+\tau_1} &= -\gamma_j & \text{for } 1 \leq j \leq r_2. \end{aligned}$$

Moreover put

$$n_j(l) = \rho(\eta_j(s), l)/l = (\sum_{k=1}^{r_1+r_2} \delta_k \min(d_{jk}, l) - \alpha)/l, \quad 1 \leq j \leq r_1.$$

The set $\{d_{ij}; 1 \leq j \leq r_1 + r_2\}$ can be written as follows:

$$\begin{aligned} \{d_{ij}; 1 \leq j \leq r_1 + r_2\} &= \{l_{ij}; 1 \leq j \leq q_i\}, & 1 \leq i \leq r_1, \\ 0 &< l_{i1} < l_{i2} < \dots < l_{iq_i} = \infty. \end{aligned}$$

Put

$$\beta_{ij} = \sum_{d_{ik}=l_{ij}} \delta_k, \quad \beta_{i0}=0, \quad I_{ij} = \sum_{k=1}^j \beta_{ik} l_{ik}, \quad I_{i0}=0, \quad 1 \leq i \leq r_1, \quad 1 \leq j \leq q_i.$$

Then we have

$$n_i(l) = (I_{ii(l)} - \alpha)/l + \sum_{k=i(l)+1}^{q_i} \beta_{ik},$$

where $i(l) = \max \{j \in \mathbf{Z}; l_{ij} \leq l\}$, $l_{i0}=0$.

LEMMA A.1.7.

$$\sup_{i>0} n_i(l) = \max \{n_i(l); l = d_{ij}, 1 \leq j \leq r_1\}, \quad 1 \leq j \leq r_1,$$

where $n_i(\infty) = \delta_i$.

PROOF. Assume that $d_{ij_0} \equiv l_{ij_1} < l < d_{ij_2} \equiv l_{ij_3}$, $0 \leq j_0 \leq r_1 - 1$, $1 \leq j_2 \leq r_1$, and that (d_{ij_0}, d_{ij_2}) does not contain d_{ij} , $1 \leq j \leq r_1$, where $d_{i0}=0$, $d_{ii}=\infty$. Then we have

$$\begin{aligned} n_i(l_{ij_1}) - n_i(l) &= (1/l_{ij_1} - 1/l)(I_{ii(l)} - \alpha) + \sum_{k=j_1+1}^{i(l)} (\beta_{ik} - \beta_{ik} l_{ik}/l_{ij_1}) \\ (A.1.5) \quad & \text{for } j_0 \geq 1, \\ n_i(l_{ij_3}) - n_i(l) &= (1/l_{ij_3} - 1/l)(I_{ii(l)} - \alpha) + \sum_{k=i(l)+1}^{j_3-1} (\beta_{ik} l_{ik}/l_{ij_3} - \beta_{ik}). \end{aligned}$$

It is obvious that

$$\sum_{k=j_1+1}^{i(l)} (\beta_{ik} - \beta_{ik} l_{ik}/l_{ij_1}), \quad \sum_{k=i(l)+1}^{j_3-1} (\beta_{ik} l_{ik}/l_{ij_3} - \beta_{ik}) \geq 0.$$

If $l_{ij_3} = \min \{d_{ij}; 1 \leq j \leq r_1\}$, that is, $j_0=0$, then $n_i(l_{ij_3}) \geq n_i(l)$. In fact, $I_{ii(l)} - \alpha \leq 0$, since $\beta_{ik} \leq 0$ if $k \leq i(l) < j_3$. Moreover we have for $j_0 \geq 1$

$$\begin{aligned} n_i(l_{ij_1}) &\geq n_i(l) && \text{if } I_{ii(l)} - \alpha \geq 0, \\ n_i(l_{ij_3}) &\geq n_i(l), && \text{otherwise.} \end{aligned}$$

Thus the proof of the lemma is complete.

Q.E.D.

THEOREM A.1.8. We have

$$(A.1.6) \quad n(H, S; \xi^0) = m' - m + \max \{n_i(l); l = d_{ij}, 1 \leq i, j \leq r_1\}.$$

Moreover if $n(H, S; \xi^0) = m' - m + n_{i_0}(d_{i_0 j_0})$, $1 \leq i_0, j_0 \leq r_1$, then $I_{i_0 j_1} - \alpha \geq 0$ and $I_{i_0 j_1-1} - \alpha \leq 0$, where $d_{i_0 j_0} = l_{i_0 j_1}$.

REMARK. If $n(H, S; \xi^0) = m' - m + n_i(\infty) = m' - m + \delta_i$, there exists $\xi^1 \in \mathbf{R}^n$ such that $n(H, S; \hat{\eta}) = m' - m + \delta_i$, where $\hat{\eta}(s) = s^{-1} \xi^1$.

PROOF. (A.1.6) easily follows from Lemma A.1.7. If we put $l = l_{i_0 j_1}$ in (A.1.5), it follows that $I_{i_0 j_1} - \alpha \geq 0$. The same argument as in the proof of Lemma A.1.7 gives

$$n_{i_0}(l_{i_0j}) - n_{i_0}(l_{i_0j_1-1}) = (I_{i_0j_1-1} - \alpha)(1/l_{i_0j_1} - 1/l_{i_0j_1-1}),$$

which implies that $I_{i_0j_1-1} - \alpha \leq 0$.

Q.E.D.

By Theorem A.1.8 we obtain the following procedure to determine $n(H, S; \xi^0)$ when $n=3$:

PROCEDURE. 1. Make a change of variables so that $\xi^0 = (0, 0, 1)$ and $\vartheta = (1, 0, 0)$.

2. Write

$$\begin{aligned} H(\tau, s, 1) &= H_1(\tau, s)R(\tau, s)A(\tau, s), & A(0, 0) &\neq 0, \\ S(\tau, s, 1) &= S_1(\tau, s)R(\tau, s)B(\tau, s), & B(0, 0) &\neq 0, \\ S_1(\tau, s) &= s^a \hat{S}_1(\tau, s), & \hat{S}_1(\tau, 0) &\neq 0, \end{aligned}$$

where $H_1(\tau, s)$ and $S_1(\tau, s)$ are relatively prime.

3. Let $\tau_j(s)$, $1 \leq j \leq r_1$, and $\lambda_j(s)$, $1 \leq j \leq r_2$, be the distinct roots of $H_1(\tau, s) = 0$ and $S_1(\tau, s) = 0$, respectively, such that $\text{order } \tau_j(s) > 0$ and $\text{order } \lambda_j(s) > 0$. Moreover let δ_j , $1 \leq j \leq r_1$, and $-\delta_{j+r_1}$, $1 \leq j \leq r_2$, be the multiplicities of the roots $\tau_j(s)$ and $\lambda_j(s)$, respectively.

4. Put $d_{ij} = d(\tau_i, \tau_j)$ for $1 \leq i, j \leq r_1$ and $d_{ij+r_1} = d(\tau_i, \lambda_j)$ for $1 \leq i \leq r_1$, $1 \leq j \leq r_2$, where $d(\tau_i, \tau_j) = \text{order } (\tau_i(s) - \tau_j(s))$.

5. Write $\{d_{ij}; 1 \leq j \leq r_1 + r_2\} = \{l_{ij}; 1 \leq j \leq q_i\}$, $0 < l_{i1} < l_{i2} < \dots < l_{iq_i} = \infty$.

6. Put $\beta_{ij} = \sum_{d_{ik}=l_{ij}} \delta_k$, $I_{ij} = \sum_{k=1}^j \beta_{ik} l_{ik}$.

7. Let $i(k)$, $1 \leq k \leq p_i$, be integers such that $I_{ii(k)} - \alpha \geq 0$, $I_{ii(k)-1} - \alpha \leq 0$ and $l_{ii(k)} = d_{ij}$ for some j with $1 \leq j \leq r_1$.

8. Then we have

$$n(H, S; \xi^0) = m' - m + \max \{ (I_{ii(k)} - \alpha) / l_{ii(k)} + \sum_{j=i(k)+1}^{q_i} \beta_{ij}; 1 \leq i \leq r_1, 1 \leq k \leq p_i \},$$

where $(I_{ii(k)} - \alpha) / l_{ii(k)} + \sum_{j=i(k)+1}^{q_i} \beta_{ij} = \delta_i$ if $l_{ii(k)} = \infty$.

§ A.2. Examples

EXAMPLE A.2.1. Let $P(\xi) = (\xi_1^2 - \xi_2^2 + 2\xi_1\xi_3)(\xi_1^2 - 2\xi_2^2 + 2\xi_1\xi_3) + \xi_2\xi_3^2$ and $\xi^0 = (0, 0, 1)$. Then we obtain

$$\begin{aligned} \alpha &= 1, \quad \tau_1(s) = s^2/2 + \dots, \quad \tau_2(s) = s^2 + \dots, \\ (l_{ij}) &= \begin{pmatrix} 2 & \infty \\ 2 & \infty \end{pmatrix}, \quad (\beta_{ij}) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad (I_{ij}) = \begin{pmatrix} 2 & \infty \\ 2 & \infty \end{pmatrix}, \quad (I_{ij} - \alpha) = \begin{pmatrix} 1 & \infty \\ 1 & \infty \end{pmatrix}. \end{aligned}$$

Thus $n(P, Q; \xi^0) = -1 + 1/2 + 1 = 1/2$ and $\kappa(\xi^0) = (1 + 1/2)/(1/2) = 3$.

EXAMPLE A.2.2. Let $P(\xi) = (\xi_1^2 - \xi_2^2 + 2\xi_1\xi_3)(\xi_1^2 - 2\xi_2^2 + 2\xi_1\xi_3)(\xi_1^2 - 3\xi_2^2 + 2\xi_1\xi_3) + \xi_2^3\xi_3^2$. If ξ^0 is not parallel to $(0, 0, 1)$ and $(2, 0, -1)$, then $(\partial/\partial\xi_i)P_e(\xi^0) \neq 0$. Thus

it follows from Lemma 1.2.3 that $n(P_6, Q_5; \xi^0) = 0$ if ξ^0 is not parallel to $(0, 0, 1)$ and $(2, 0, -1)$. Let $\xi^0 = (0, 0, 1)$. Then we obtain

$$\alpha = 3, \quad \tau_1(s) = s^2/2 + \dots, \quad \tau_2(s) = s^2 + \dots, \quad \tau_3(s) = 3s^2/2 + \dots,$$

$$(l_{ij}) = \begin{pmatrix} 2 & \infty \\ 2 & \infty \\ 2 & \infty \end{pmatrix}, \quad (\beta_{ij}) = \begin{pmatrix} 2 & 1 \\ 2 & 1 \\ 2 & 1 \end{pmatrix}, \quad (I_{ij}) = \begin{pmatrix} 4 & \infty \\ 4 & \infty \\ 4 & \infty \end{pmatrix}, \quad (I_{ij} - \alpha) = \begin{pmatrix} \textcircled{1} & \infty \\ \textcircled{1} & \infty \\ \textcircled{1} & \infty \end{pmatrix}.$$

Thus $n(P_6, Q_5; \xi^0) = -1 + 1/2 + 1 = 1/2$. When $\xi^0 = (2, 0, -1)$, we have also $n(P_6, Q_5; \xi^0) = 1/2$. Therefore $\kappa_0 = 1/\delta(P) = 3$. We note that P_ξ is hyperbolic with respect to $\mathcal{V}$ for $\xi \in \mathbb{R}^3 \setminus \{0\}$, although P is not hyperbolic with respect to $\mathcal{V}$.

EXAMPLE A.2.3. Let $P(\xi) = (\xi_1^2 - \xi_2^2 + 2\xi_1\xi_3)(\xi_1^2 - 2\xi_2^2 + 4\xi_1\xi_3) + 2\xi_1\xi_3^2$ and $\xi^0 = (0, 0, 1)$. Then we obtain

$$\alpha = 0, \quad \tau_1(s) = s^2/2 - s^4/8 + \dots, \quad \tau_2(s) = s^2/2 - s^4/16 + \dots,$$

$$\lambda_1(s) \equiv 0, \quad (l_{ij}) = \begin{pmatrix} 2 & 4 & \infty \\ 2 & 4 & \infty \end{pmatrix}, \quad (\beta_{ij}) = \begin{pmatrix} -1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}, \quad (I_{ij}) = \begin{pmatrix} -2 & 2 & \infty \\ -2 & 2 & \infty \end{pmatrix},$$

$$(I_{ij} - \alpha) = \begin{pmatrix} -2 & \textcircled{2} & \infty \\ -2 & \textcircled{2} & \infty \end{pmatrix}.$$

Thus $n(P_4, Q_3; \xi^0) = -1 + 2/4 + 1 = 1/2$ and $\kappa(\xi^0) = 3$.

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