A remark on the tensor product of irreducible representations of a compact Lie group

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§ 1. Introduction, and statement of the results

Let G be a connected compact Lie group with Lie algebra $\mathfrak{g}$. Then G is necessarily reductive. Assuming here that G is not abelian, we study some problems concerning the tensor product of irreducible representations of G , which are originally suggested by the first author's work [2]. Inspired by the Kirillov's orbital method for certain types of Lie groups, we consider orbits in $\mathfrak{g}$ under the adjoint action of G . (Note that $\mathfrak{g}$ can be identified with its dual space $\mathfrak{g}^*$ through the Killing form on $[\mathfrak{g}, \mathfrak{g}]$ and an appropriate inner product on the center of $\mathfrak{g}$, and the adjoint action on $\mathfrak{g}$ with the coadjoint one on $\mathfrak{g}^*$.) Let T be a maximal torus of G , and $\mathfrak{t}$ its Lie algebra. Then any orbit O in $\mathfrak{g}$ intersects with $\mathfrak{t}$. If $\lambda \in O \cap \mathfrak{t}$, we denote O by $O(\lambda)$. Put $G(\lambda) = \{g \in G; \text{Ad}(g)\lambda = \lambda\}$. Then $O(\lambda)$ is isomorphic to $G/G(\lambda)$, and the representation $T_{O(\lambda)}$ of G constructed naturally on $L^2(O(\lambda); d\nu_{O(\lambda)})$ is equivalent to the quasi-regular representation on $L^2(G/G(\lambda); d\bar{g})$, where $d\nu_{O(\lambda)}$ and $d\bar{g}$ ($\bar{g} = gG(\lambda)$) denote invariant measures on $O(\lambda)$ and on $G/G(\lambda)$ respectively. On the other hand, we identify $\mathfrak{t}$ with its dual space $\mathfrak{t}^*$ similarly as above, then the highest weight of an irreducible representation U of G is given in the form $\sqrt{-1}\mu$ with $\mu \in \mathfrak{t}$. Now our problem in its original form was to establish the following conjecture.

CONJECTURE. *Let U be an irreducible representation of G with highest weight $\sqrt{-1}\mu$, and denote by U^* its contragredient. Then the tensor product $U \otimes U^*$ is contained isomorphically in $T_{O(\lambda)}$ as a G -module, if and only if $G(\mu)$ contains a conjugate of $G(\lambda)$.*

Here we define the contragredient U^* of U as follows: realize U as a representation by matrices $U(g)$ ($g \in G$), then we put $U^*(g) = {}^tU(g)^{-1}$, where for a matrix A , tA denotes its transposed.

Note that for any $G(\lambda)$, there exists a dominant integral form $\sqrt{-1}\lambda'$ with

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$\lambda' \in \mathfrak{t} \cong \mathfrak{t}^*$ such that $G(\lambda)$ is conjugate to $G(\lambda')$. Further $G(\lambda)$ and $G(\mu)$ contain T , and are connected [3, Ch. VII, Cor. 2.8]. Let $N_G(T)$ be the normalizer of T in G , then $W = N_G(T)/T$ is called the Weyl group of G . For $w \in W$, we put $G(\lambda)^w = g_w G(\lambda) g_w^{-1}$ and $w\lambda = \text{Ad}(g_w)\lambda$, taking a representative $g_w \in N_G(T)$ of w . Then $G(\mu)$ contains a conjugate of $G(\lambda)$ if and only if $G(\mu) \supset G(\lambda)^w = G(w\lambda)$ for some $w \in W$. Thus the conjecture can be written as " $U \otimes U^* \subset T_{o(\lambda)}$ if and only if $G(\mu) \supset G(w\lambda)$ for some $w \in W$."

It is worth while remarking here that there exists a unique character χ_μ of $G(\mu)$ such that $\chi_\mu(\exp X) = \exp(\sqrt{-1}\langle \mu, X \rangle)$ for $X \in \mathfrak{g}(\mu)$, the Lie algebra of $G(\mu)$, where $\langle \mu, X \rangle$ denotes the Killing form, and that, for a highest weight vector v_0 for U ,

$$(1.1) \quad U(h)v_0 = \chi_\mu(h)v_0 \quad (h \in G(\mu)).$$

Conversely, let H be a connected subgroup of G such that $U(h)v_0 \in Cv_0$ for $h \in H$, then H is contained in $G(\mu)$.

For an irreducible representation V of G , denote by $[T_{o(\lambda)}: V]$ the multiplicity of V in $T_{o(\lambda)}$. Then, by the Frobenius reciprocity theorem, we have $[T_{o(\lambda)}: V] = [V|G(\lambda): 1_{G(\lambda)}]$, where $1_{G(\lambda)}$ denotes the trivial representation of $G(\lambda)$. Thus " $U \otimes U^* \subset T_{o(\lambda)}$ " is equivalent to

$$(1.2) \quad [U \otimes U^*: V] \leq [V|G(\lambda): 1_{G(\lambda)}] \text{ for any irreducible representation } V \text{ of } G.$$

In case λ is regular, we have $G(\lambda) = T$, and the conjecture says that

$$(1.3) \quad \text{for any irreducible representation } U \text{ of } G, U \otimes U^* \text{ is contained isomorphically in the representation } \text{Ind}_T^G 1_T \text{ induced from } T \text{ by } 1_T.$$

In this case, we prove the following theorem. Let w_0 be the unique element in W such that for any positive root α of $(\mathfrak{g}_c, \mathfrak{t}_c)$, $w_0\alpha$ is negative (with respect to a previously fixed order).

THEOREM 1. *Let U_i ($i = 1, 2$) be irreducible representations of G with highest weights $\sqrt{-1}\mu_i$. Then the tensor product $U_1 \otimes U_2$ is contained isomorphically in $\text{Ind}_T^G \chi_\mu$ with $\mu = \mu_1 + w_0\mu_2$:*

$$U_1 \otimes U_2 \subset \text{Ind}_T^G \chi_{\mu_1}(\chi_{\mu_2})^{w_0}.$$

From this theorem, we get easily

COROLLARY. *The above conjecture (1.3) is true.*

In fact, let the highest weight of U be $\sqrt{-1}\mu$, then that of U^* is $-\sqrt{-1}w_0\mu$. Put $U_1 = U$, $U_2 = U^*$ in Theorem 1, then $\chi_{\mu_1}(\chi_{\mu_2})^{w_0}$ is equal to 1_T on T . Hence $U \otimes U^* \subset \text{Ind}_T^G 1_T$.

In the general case, we have not been able to get a complete answer for the conjecture. We get its "if" part from the next theorem as explained below.

THEOREM 2. *Let U_i ($i=1, 2$) be irreducible representations of G with highest weights $\sqrt{-1}\mu_i$. Then, the tensor product $U_1 \otimes U_2$ is contained in $\text{Ind}_H^G \chi_{\mu_1}(\chi_{\mu_2})^{w_0}$ with $H = G(\mu_1) \cap G(\mu_2)^{w_0}$ and $(\chi_{\mu_2})^{w_0}(h) = \chi_{\mu_2}(g_{w_0}^{-1} h g_{w_0})$, for $h \in G(\mu_2)^{w_0}$:*

$$U_1 \otimes U_2 \subset \text{Ind}_H^G \chi_{\mu_1}(\chi_{\mu_2})^{w_0}, \quad H = G(\mu_1) \cap G(\mu_2)^{w_0}.$$

As a corollary of this theorem, we get

COROLLARY. *Let U be an irreducible representation of G with highest weight $\sqrt{-1}\mu$. Then*

$$U \otimes U^* \subset \text{Ind}_{G(\mu)}^G 1_{G(\mu)}.$$

In fact, put $U_1 = U$, $U_2 = U^*$ in Theorem 2. Then $\mu_1 = \mu$, $\mu_2 = -w_0\mu$ as before. Therefore $G(\mu_2)^{w_0} = G(w_0\mu_2) = G(-\mu_1) = G(\mu_1)$. Whence $H = G(\mu)$. Moreover $\chi_{\mu_1}(\chi_{\mu_2})^{w_0} = 1_H$ because $\mu_1 + w_0\mu_2 = 0$. Thus we get $U \otimes U^* \subset \text{Ind}_H^G 1_H$ with $H = G(\mu)$.

From this corollary, we get the "if" part of the conjecture.

PROPOSITION. *Let U be an irreducible representation of G with highest weight $\sqrt{-1}\mu$. If $G(\mu)$ contains a conjugate of $G(\lambda)$, then the tensor product $U \otimes U^*$ is contained isomorphically in $\text{Ind}_{G(\lambda)}^G 1_{G(\lambda)}$.*

In fact, let H' be a conjugate of $G(\lambda)$ contained in $H = G(\mu)$. Then $\text{Ind}_{H'}^G 1_{H'}$ is equivalent to $\text{Ind}_{G(\lambda)}^G 1_{G(\lambda)}$, and contains isomorphically $\text{Ind}_H^G 1_H$.

We must remark that we take here the following definition of the induced representation.

DEFINITION. Let H be a closed subgroup of G , and χ a character of H . Then $\text{Ind}_H^G \chi$ is realized by right translations on the space of functions $\{f \in L^2(G); f(hg) = \chi(h)f(g) \ (h \in H, g \in G)\}$.

We note that Theorem 2 can also be stated in the following form. For this interpretation, we are indebted to Prof. N. Tatsuuma.

THEOREM 3. *Let the notations be as in Theorem 2. Then the tensor product $U_1 \otimes U_2$ is always cyclic and has a cyclic vector v satisfying*

$$(U_1 \otimes U_2)(h)v = (\chi_{\mu_1}(\chi_{\mu_2})^{w_0})(h)v \quad (h \in H = G(\mu_1) \cap G(\mu_2)^{w_0}).$$

In particular, $U_1 \otimes U_1^$ has a cyclic vector invariant under $G(\mu_1)$.*

In fact, similarly as (1.2), the inclusion relation in Theorem 2 is interpreted as

$$"[U_1 \otimes U_2: V] \leq [V|H; \chi_{\mu_1}(\chi_{\mu_2})^{w_0}] \text{ for every irreducible representation } V."$$

This in turn is equivalent to the assertion in Theorem 3.

Now, before explaining the contents of the subsequent sections, we remark that Theorem 1 is a consequence of Theorem 2. In fact, H in Theorem 2 contains T , and the restriction on T of the character $\chi_{\mu_1}(\chi_{\mu_2})^{w_0}$ of H is the one denoted by the same symbol in Theorem 1.

In the next section, § 2, we prove Theorem 2. In § 3, we give certain inequalities for multiplicities and for the Kostant's partition function (Theorem 4) by interpreting the inclusion relation in Theorem 1. In the last section, § 4, we give some examples for $G=U(n)$ which support the "only if" part of the conjecture, and we see that it is true for $U(3)$.

§ 2. Proof of Theorem 2

To prove Theorem 2, we may assume that G is semisimple. Further, if G is semisimple, replacing it by its universal covering group if necessary, we may also assume that G is simply connected. Assume them from now on.

Let $\mathfrak{g}_c$ and $\mathfrak{t}_c$ be the complexification of $\mathfrak{g}$ and $\mathfrak{t}$ respectively. We fix an order in the set of roots of $(\mathfrak{g}_c, \mathfrak{t}_c)$. Let $\mathfrak{n}$ be a maximal nilpotent subalgebra of $\mathfrak{g}_c$ spanned by the root subspaces $\mathfrak{g}^\alpha$ with $\alpha < 0$, and put $\mathfrak{b} = \mathfrak{t}_c + \mathfrak{n}$, a Borel subalgebra. Let G_c be a simply connected Lie group with Lie algebra $\mathfrak{g}_c$, then the analytic subgroup corresponding to $\mathfrak{g}$ is naturally isomorphic to G . We identify them. Let B and N be the analytic subgroups corresponding to $\mathfrak{b}$ and $\mathfrak{n}$ respectively.

Let $\sqrt{-1}\mu$ with $\mu \in \mathfrak{t} \cong \mathfrak{t}^*$ be a dominant integral form. Then the complexification $\mathfrak{g}_c(\mu)$ of $\mathfrak{g}(\mu)$ is spanned by $\mathfrak{t}_c$ and $\mathfrak{g}^\alpha$ with α satisfying $s_\alpha \mu = \mu$, where $s_\alpha \in W$ denotes the reflexion corresponding to α . Let $G_c(\mu)$ be the analytic subgroup of G_c corresponding to $\mathfrak{g}_c(\mu)$, and put $P(\mu) = G_c(\mu)N$. Then, since μ is dominant, $P(\mu)$ is a parabolic subgroup of G_c containing B , which corresponds canonically to the set of simple roots α such that $s_\alpha \mu = \mu$.

To prove Theorem 2, we use the following two facts.

- (1) Every irreducible representation of G can be extended to a holomorphic representation of G_c .
- (2) The subset Bw_0B contains a non-void open subset of G_c .

NOTE 1. In reality, Bw_0B is itself open and dense in G_c . This can be seen from the Bruhat decomposition of G_c into B -double cosets and the closure relation among them. However, (2) is proved easily as follows. Put

$N_+ = w_0 N w_0^{-1}$, then it is a subgroup corresponding to the nilpotent subalgebra spanned by $\mathfrak{g}^\alpha$'s with $\alpha > 0$. Let T_c be the Cartan subgroup of G_c corresponding to $\mathfrak{t}_c$. Then $Bw_0 Bw_0^{-1} = NT_c N_+$. From this, (2) follows.

Let U be an irreducible representation of G with highest weight $\sqrt{-1}\mu$ on a vector space E . Denote again by U the holomorphically extended representation of G_c from U , and also again by χ_μ the holomorphically extended character of $G_c(\mu)$ from χ_μ . Let E^* be the dual space of E . Then, for $v \in E$, $v^* \in E^*$,

$$(2.1) \quad (U(g)v, v^*) = (v, U^*(g^{-1})v^*) \quad (g \in G_c),$$

where $(\cdot, \cdot)$ denotes the pairing of E and E^* . Let $v_0^* \neq 0$ be a lowest weight vector for U^* . The lowest weight of U^* is $-\sqrt{-1}\mu$. Therefore

$$(2.2) \quad U^*(h)v_0^* = \chi_\mu(h)^{-1}v_0^* \quad (h \in G_c(\mu)), \quad U^*(n)v_0^* = v_0^* \quad (n \in N).$$

Consider the space D_μ of holomorphic functions on G_c spanned by

$$f_v(g) = (v, U^*(g^{-1})v_0^*) = (U(g)v, v_0^*) \quad (g \in G_c),$$

for $v \in E$. Then for $f \in D_\mu$,

$$(2.3) \quad f(pg) = \chi_\mu(p)f(g) \quad (p \in P(\mu)),$$

where χ_μ denotes the character of $P(\mu)$ extended from $G_c(\mu)$ by putting $\chi_\mu(n) = 1$ for $n \in N$. The space E is isomorphic to D_μ , and the representation U on E is transformed to $\tilde{U}$ on D_μ as

$$(2.4) \quad \tilde{U}(g_0)f(g) = f(gg_0) \quad (g_0, g \in G_c).$$

Thus we get the following

LEMMA 2.1. *Let U be an irreducible representation of G with highest weight $\sqrt{-1}\mu$. Then the holomorphically extended one is contained in the representation by right translations on the space of holomorphic functions on G satisfying (2.3).*

Now let U_1, U_2 be as in Theorem 2. Consider the tensor product $\tilde{U}_1 \otimes \tilde{U}_2$ of the representations $\tilde{U}_1$ and $\tilde{U}_2$ of G_c . It is given on the space $D = D_{\mu_1} \otimes D_{\mu_2}$ as follows: for $\varphi \in D$,

$$(2.5) \quad (\tilde{U}_1 \otimes \tilde{U}_2)(g_0)\varphi(g_1, g_2) = \varphi(g_1g_0, g_2g_0) \quad (g_0 \in G_c).$$

On the other hand, $\varphi \in D$ is a holomorphic function on $G_c \times G_c$ satisfying

$$(2.6) \quad \varphi(p_1g_1, p_2g_2) = \chi_{\mu_1}(p_1)\chi_{\mu_2}(p_2)\varphi(g_1, g_2) \quad (g_1, g_2 \in G_c, p_1 \in P(\mu_1), p_2 \in P(\mu_2)).$$

Let Δ be the diagonal $\{(g, g); g \in G_c\}$ in $G_c \times G_c$, then $\Delta \cong G_c$. Consider the double coset $\Omega = (P(\mu_1) \times P(\mu_2))(1, w_0)\Delta$ in $G_c \times G_c$, where 1 and w_0 in W must be replaced by their representatives in $N_o(T)$. Then we see from (2) that Ω contains a non-void open subset. Therefore the map $\varphi \mapsto \varphi_\Omega = \varphi|_\Omega$ of D is 1-1, and it follows from (2.6) that

$$(2.7) \quad \varphi_\Omega(p_1 g, p_2 g_{w_0} g) = \chi_{\mu_1}(p_1) \chi_{\mu_2}(p_2) \varphi_\Omega(g, g_{w_0} g) \quad (g \in G_c, p_1 \in P(\mu_1), p_2 \in P(\mu_2)).$$

Put $\psi(g) = \varphi_\Omega(g, g_{w_0} g) = \varphi(g, g_{w_0} g)$, then ψ is holomorphic on G_c . Note that $g_{w_0} p g_{w_0}^{-1} \in P(\mu_2)$ is equivalent to $p \in g_{w_0}^{-1} P(\mu_2) g_{w_0} = P(\mu_2)^{w_0} = G_c(\mu_2)^{w_0} N^+$ because $w_0^2 = 1$. Since $P(\mu_1) \cap P(\mu_2)^{w_0} \supset G_c(\mu_1) \cap G_c(\mu_2)^{w_0}$, we get from (2.7)

$$(2.8) \quad \psi(hg) = \chi_{\mu_1}(h) (\chi_{\mu_2})^{w_0}(h) \psi(g) \quad (g \in G_c, h \in G_c(\mu_1) \cap G_c(\mu_2)^{w_0}).$$

Put $f = \psi|_G$. Since ψ is holomorphic, the map $\psi \mapsto f$ is 1-1. Moreover, since $G \cap G_c(\mu_1) \cap G_c(\mu_2)^{w_0} \supset G(\mu_1) \cap G(\mu_2)^{w_0}$, f satisfies that

$$(2.9) \quad f(hg) = \chi_{\mu_1}(h) (\chi_{\mu_2})^{w_0}(h) f(g) \quad (g \in G_c, h \in G(\mu_1) \cap G(\mu_2)^{w_0}).$$

Thus the image $\mathcal{E}$ of the space D under $\varphi \mapsto \psi \mapsto f$ is a subspace of continuous functions on G satisfying (2.9). The representation $\tilde{U}_1 \otimes \tilde{U}_2$ of G_c on D is transformed after the restriction on G into that on $\mathcal{E}$ by right translations.

This proves that the tensor product $U_1 \otimes U_2$ is contained isomorphically in $\text{Ind}_H^G \chi_{\mu_1} (\chi_{\mu_2})^{w_0}$ with $H = G(\mu_1) \cap G(\mu_2)^{w_0}$. Thus the proof of Theorem 2 is now complete.

NOTE 2. Theorem 1 asserts that $U \otimes U^* \subset \text{Ind}_T^G 1_T$ for every irreducible representation U of G . On the other hand, we see from the Plancherel formula for G that any irreducible component of $\text{Ind}_T^G 1_T$ is contained in $U \otimes U^*$ for an infinitely many inequivalent U 's. In fact, consider the direct sum R of $U \otimes U^*$ over a set of representatives of all equivalent classes of irreducible representations. Then, by the Peter-Weyl's theorem on the regular representation of G , R is equivalent to a representation M on $L^2(G, dg)$ given by

$$(2.10) \quad (M(g_0)f)(g) = f(g_0^{-1}gg_0) \quad (f \in L^2(G, dg)),$$

where dg denotes the normalized Haar measure on G . Let us prove that M is equivalent to an infinite multiple of $\text{Ind}_T^G 1_T$. Every element g in G is expressed as

$$(2.11) \quad g = k^{-1}tk \quad \text{with } k \in G, t \in T.$$

Then the Weyl's integration formula says that for $f \in L^1(G, dg)$,

$$(2.12) \quad \int_G f(g) dg = \frac{1}{|W|} \int_T \int_{T \backslash G} f(k^{-1}tk) d\bar{k} |\Delta(t)|^2 dt,$$

where dt and $d\bar{k}$ ($\bar{k} = Tk$) denote the normalized invariant measures on T and $T \setminus G$ respectively, W is the Weyl group of G , and $\Delta(t)$ is a certain well-defined function on T . Let G' be the set of all regular elements of G , and T_1 a measurable section of T'/W in $T' = T \cap G'$. Then, the decomposition (2.11) of $g \in G'$ with $t \in T_1$ gives a measurable bijection of G' onto $(T \setminus G) \times T_1$ as $g \mapsto (\bar{k}, t)$, and by (2.12), $L^2(G, dg)$ is isomorphic to $\mathcal{H} = L^2(T_1, |\Delta(t)|^2 dt) \times L^2(T \setminus G, d\bar{k})$. The representation M on the former is transformed to the representation M_1 on the latter: for $\varphi \times \psi \in \mathcal{H}$,

$$M_1(g_\delta)(\varphi \times \psi)(t, \bar{k}) = \varphi(t)\psi(\bar{k}g_\delta).$$

This proves our assertion.

§ 3. Inequalities for multiplicities

For an integral form $\omega \in (\mathfrak{t}_c)^*$, denote by $P_K(\omega)$ the number of different expressions of ω in a non-negative integral linear combination of positive roots. Then $P_K(\omega)$ is called the Kostant's partition function. In this section, we prove an inequality for multiplicities and, translating it by means of P_K , another one for the function P_K .

Let ρ be half the sum of all positive roots of $(\mathfrak{g}_c, \mathfrak{t}_c)$. For a dominant integral form ω , let U_ω be an irreducible representation of G with highest weight ω . For any weight θ , denote by $m(\omega; \theta)$ the multiplicity of θ in U_ω . Then the Kostant's formula gives

$$(3.1) \quad m(\omega; \theta) = \sum_{w \in W} \text{sgn}(w) P_K(w(\omega + \rho) - (\theta + \rho)),$$

where $\text{sgn}(w)$ denotes the usual sign of $w \in W$. Further let λ, μ and ν be dominant integral forms, and put

$$m(\lambda, \mu, \nu) = [U_\lambda \otimes U_\mu : U_\nu].$$

Then we have the Steinberg's formula [1, Ch. VIII, § 9]

$$(3.2) \quad m(\lambda, \mu, \nu) = \sum_{w, w' \in W} \text{sgn}(ww') P_K(w(\lambda + \rho) + w'(\mu + \rho) - (\nu + 2\rho)).$$

Using Theorem 1, we get the following result.

THEOREM 4. *Let λ, μ and ν be dominant integral forms. Then, $m(\lambda, \mu, \nu) \leq m(\lambda; \nu - \mu)$, or equivalently,*

$$(3.3) \quad - \sum_{\substack{w, w' \in W \\ w' \neq 1}} \text{sgn}(ww') P_K(w(\lambda + \rho) + w'(\mu + \rho) - (\nu + 2\rho)) \geq 0.$$

PROOF. By Theorem 1, we have for dominant integral forms λ_1, λ_2 and λ_3 ,

$$[U_{\lambda_1} \otimes U_{\lambda_2} : U_{\lambda_3}] \leq [\text{Ind}_T^G X_{\lambda_1 + w_0 \lambda_2} : U_{\lambda_3}],$$

where X_w denotes the character of T given by $\omega \in (\mathfrak{t}_c)^*$. Applying the Frobenius reciprocity theorem on the right hand side, we get

$$[U_{\lambda_1} \otimes U_{\lambda_2} : U_{\lambda_3}] \leq [U_{\lambda_3} | T : X_{\lambda_1 + w_0 \lambda_2}].$$

On the other hand, we have

$$[U_{\lambda_1} \otimes U_{\lambda_2} : U_{\lambda_3}] = [U_{\lambda_3}^* \otimes U_{\lambda_1} : U_{\lambda_2}^*], \quad [U_{\lambda_3} | T : X_{\lambda_1 + w_0 \lambda_2}] = [U_{\lambda_3}^* | T : X_{-\lambda_1 - w_0 \lambda_2}].$$

The highest weight of U_i^* is equal to $\lambda_i^* = -w_0 \lambda_i$. Therefore, putting $(\lambda, \mu, \nu) = (\lambda_3^*, \lambda_1, \lambda_2^*)$, we get $m(\lambda, \mu, \nu) \leq m(\lambda; \nu - \mu)$. Using (3.1), (3.2), this inequality is rewritten as (3.3). Q.E.D.

NOTE 3. The value in (3.3) is equal to $m(\lambda; \nu - \mu) - m(\lambda, \mu, \nu)$. In the case where $\mu = 0$ and $\lambda \neq \nu$, $m(\lambda, \mu, \nu) = 0$ and it gives a formula

$$m(\lambda; \nu) = - \sum_{w \in W, \neq 1} \text{sgn}(w) m(\lambda; \nu + \rho - w\rho).$$

This formula is well known [1, Ch. VIII, § 9], and so is the inequality in this case.

§ 4. Examples

Let $G = U(n)$, the unitary group of order n , and T the maximal torus of G of all diagonal elements. For $\lambda \in \mathfrak{t}$, the subgroup $G(\lambda)$ is conjugate under G to one of the following: for an ordered partition $\nu = (n_1, n_2, \dots, n_r)$ of n , put

$$(4.1) \quad H(\nu) = \begin{bmatrix} U(n_1) & & & 0 \\ & U(n_2) & & \\ & & \ddots & \\ 0 & & & U(n_r) \end{bmatrix}.$$

Let $A = (m_1, m_2, \dots, m_n)$ be an ordered set of integers. We define a function AA on T as follows. Let $\mathfrak{S}_n$ be the symmetric group of order n , and for $s = (s(1), s(2), \dots, s(n)) \in \mathfrak{S}_n$, put $sA = (m_{s(1)}, m_{s(2)}, \dots, m_{s(n)})$. Let $\delta = \text{diag}(\delta_1, \delta_2, \dots, \delta_n)$ be a diagonal matrix with diagonal elements $\delta_1, \delta_2, \dots, \delta_n$. Then we put $\delta^A = \delta_1^{m_1} \delta_2^{m_2} \dots \delta_n^{m_n}$, and

$$(4.2) \quad AA = \sum_{s \in \mathfrak{S}_n} \text{sgn}(s) \delta^{sA},$$

where $\text{sgn}(s)$ denotes the sign of s . Put $\rho = (n-1, n-2, \dots, 1, 0)$, and assume

$m_1 \geq m_2 \geq \dots \geq m_n$ for λ . Then the characters of irreducible representations of G are given on T by the formula

$$(4.3) \quad \pi_\lambda = A(\lambda + \rho)/A\rho.$$

Denote by $T(\lambda)$ a representation with character π_λ . The highest weight $\mu \in (\mathfrak{t}_e)^*$ of $T(\lambda)$ is given by $\mu(X) = m_1 x_1 + m_2 x_2 + \dots + m_n x_n$ for $X = \text{diag}(x_1, x_2, \dots, x_n) \in \mathfrak{t}_e$.

Let H_p be $H(\nu)$ with $\nu = (1, 1, \dots, 1, n-p)$, i.e., $n_1 = n_2 = \dots = n_p = 1$, $n_{p+1} = n-p$. Then we have

LEMMA 4.1. *A representation $T(\lambda)$ of G with parameter*

$$(4.4) \quad \lambda = (m_1, m_2, \dots, m_n), \quad m_1 \geq m_2 \geq \dots \geq m_n,$$

is of class 1 with respect to H_p , that is, $[T(\lambda): 1_{H_p}] \geq 1$, if and only if the following two conditions hold for λ .

- (1) $\sum_{1 \leq i \leq n} m_i = 0$,
- (2) *the number of positive m_i 's and that of negative m_i 's are both $\leq p$.*

PROOF. The group H_1 is isomorphic to $U(1) \times U(n-1)$, and the restriction of $T(\lambda)$ on H_1 is decomposed into irreducibles as

$$(4.5) \quad T(\lambda)|_{H_1} = \sum T\left(\sum_{i=1}^n m_i - \sum_{j=1}^{n-1} m'_j\right) \times T(m'_1, m'_2, \dots, m'_{n-1}),$$

where $(m'_1, m'_2, \dots, m'_{n-1})$ runs over all ordered set of integers satisfying

$$(4.6) \quad m_1 \geq m'_1 \geq m_2 \geq m'_2 \geq m_3 \geq \dots \geq m_{n-1} \geq m'_{n-1} \geq m_n,$$

and $T(m) \times T(m'_1, m'_2, \dots, m'_{n-1})$ denotes a product of a representation $T(m)$ of $U(1)$ given by $U(1) \ni t \mapsto t^m$, and $T(m'_1, m'_2, \dots, m'_{n-1})$ of $U(n-1)$. Using this fact inductively, we get the result. Q.E.D.

EXAMPLE 1. Let U be an irreducible representation of G with highest weight μ . Assume that $U \otimes U^* \subset \text{Ind}_{H_1}^G 1_{H_1}$, then $G(\mu)$ contains a conjugate of H_1 . This means that the conjecture is true in case $G(\lambda) = H_1$.

In fact, we know that $T(\lambda) \subset \text{Ind}_{H_1}^G 1_{H_1}$ if and only if the conditions (1), (2) hold for λ . Now let $U = T(\lambda)$. Then $U^* = T(\lambda^*)$ with $\lambda^* = (-m_n, -m_{n-1}, \dots, -m_1)$, whence $U \otimes U^*$ contains $T(\lambda + \lambda^*)$. Therefore, if $U \otimes U^* \subset \text{Ind}_{H_1}^G 1_{H_1}$, $\lambda + \lambda^*$ must satisfy the condition (2). This gives us $m_2 - m_{n-1} = 0$, and so λ is of the form

$$\lambda = (a+c, c, c, \dots, c, -b+c), \quad a, b \geq 0.$$

To prove that $a=0$ or $b=0$, we apply Exercise 18) in [1, Ch. VIII, § 7]. By

this exercise, we see that if $a, b > 0$, $T(\lambda) \otimes T(\lambda)^*$ contains $T(\lambda')$ with $\lambda' = \lambda + \lambda^* - (1, -1, 0, 0, \dots, 0) = (a+b-1, 1, 0, 0, \dots, 0, -a-b)$. However λ' does not satisfy the condition (2). This is a contradiction.

Note that when $a=0$, $G(\mu)$ contains a conjugate $H_1^{w_0}$ of H_1 .

We know that the representation on the space of symmetric tensors of degree q is $T(\lambda^q)$ with $\lambda^q = (q, 0, 0, \dots, 0)$, and that on the space of skew-symmetric tensors of degree q is $T(\lambda_q)$ with $\lambda_q = (1, 1, \dots, 1, 0, \dots, 0)$ (the number of 1 is equal to q).

LEMMA 4.2.

- (1) $T(\lambda^q) \otimes T(\lambda^q)^* = \sum_{r=0}^q T(r, 0, 0, \dots, 0, -r)$.
- (2) $T(\lambda_q) \otimes T(\lambda_q)^* = \sum_{r=0}^{\min(q, n-q)} T(\underbrace{1, 1, \dots, 1}_r, 0, \dots, 0, \underbrace{-1, \dots, -1}_r)$.

PROOF. Here we prove (2). Note first that if we put for λ , $S\lambda = \sum_{s \in \mathfrak{S}_n} \delta^{s\lambda}$, then

$$(4.7) \quad S\lambda \cdot \lambda\lambda' = \sum_{s \in \mathfrak{S}_n} \lambda(\lambda' + s\lambda).$$

On the other hand, $\pi_{\lambda_q} = A(\lambda_q + \rho)/A\rho$, and also $\pi_{\lambda_q} = N^{-1}S\lambda_q$ with $N = \#\{s \in \mathfrak{S}_n; s\lambda_q = \lambda_q\}$. Therefore the character of $T(\lambda_q) \otimes T(\lambda_q)^*$ is given by

$$\pi_{\lambda_q} \cdot \bar{\pi}_{\lambda_q} = N^{-1}A(\lambda_q + \rho)S(\lambda_q^*)/A\rho,$$

where $\lambda_q^* = (0, 0, \dots, 0, -1, -1, \dots, -1)$. Then by (4.7),

$$A(\lambda_q + \rho)S(\lambda_q^*) = \sum_{s \in \mathfrak{S}_n} A(\lambda_q + s\lambda_q^* + \rho).$$

Picking up non-zero terms in the right hand side, we get the formula (2).

Q.E.D.

EXAMPLE 2. From Lemmas 4.1 and 4.2, we see that $T(\lambda_q) \otimes T(\lambda_q)^* \subset \text{Ind}_{H_p}^G 1_{H_p}$ if and only if $\min(q, n-q) \leq p$. On the other hand, $\min(q, n-q) \leq p$ if and only if $G(\mu)$ contains a conjugate of H_p , where μ denotes the highest weight of $T(\lambda_q)$.

ADDED IN PROOF. Reexamining the proof of Theorem 2, we get the following amelioration of Theorem 3.

THEOREM 3'. For $i=1, 2$, let (U_i, E_i, μ_i) be as in Theorem 3, and let $v_1 \in E_1$ (resp. $v_2 \in E_2$) be a non-zero highest (resp. lowest) weight vector for U_1 (resp. for U_2). Then the vector $v = v_1 \otimes v_2$ in $E_1 \otimes E_2$ is a cyclic vector for $U_1 \otimes U_2$, and satisfies that

$$(U_1 \otimes U_2)(h)v = (\chi_{\mu_1}(\chi_{\mu_2}^{w_0}))(h)v \quad (h \in H = G(\mu_1) \cap G(\mu_2)^{w_0}).$$

PROOF. Let $v_i^* \in E_i^*$ be a non-zero highest weight vector in E_i^* for U_i^* . Then, in the proof of Theorem 2, the space D is spanned by the functions $\varphi = \varphi_{v'v''}$ with $v' \in E_1$, $v'' \in E_2$, where

$$\varphi_{v'v''}(g_1, g_2) = (U_1(g_1)v' \otimes U_2(g_2)v'', v_1^* \otimes v_2^*) \quad (g_1, g_2 \in G_c).$$

The correspondence $\varphi \mapsto f$ of D onto $\mathcal{E}$ induces a linear mapping of $E_1 \otimes E_2$ onto $\mathcal{E}$ given by $v' \otimes v'' \mapsto f$, where

$$f(g) = (U_1(g)v' \otimes U_2(g_{w_0}g)v'', v_1^* \otimes v_2^*) = (v' \otimes v'', (U_1 \otimes U_2)^*(g^{-1})v^*)$$

with $v^* = v_1^* \otimes U_2^*(g_{w_0}^{-1})v_2^* \in E_1^* \otimes E_2^*$. The injectivity of the map corresponds to that the vector v^* is cyclic for $(U_1 \otimes U_2)^*$. Note that $U_2^*(g_{w_0}^{-1})v_2^* \in E_2^*$ is a lowest weight vector for U_2^* . Then we get the assertion of the theorem for U_1^* , U_2^* . Q.E.D.

We know also that Prof. N. Tatsuuma gave an algebraic proof of Theorem 3' using the enveloping algebra of $\mathfrak{g}_c$.

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