On isoparametric hypersurfaces in the Lorentzian space forms*

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The purpose of the present paper is to extend the basic identity concerning constant principal curvatures of an isoparametric hypersurface in E. Cartan's theory (see [1] and [2]) to the case of a spacelike hypersurface in any one of the standard Lorentzian space forms $M^{n+1}(c)$ of constant sectional curvature c .

A connected space-like hypersurface M^n in $M^{n+1}(c)$ is said to be *isoparametric* if its shape operator A has constant eigenvalues (principal curvatures). If $a_1, \dots, a_p$ are all the distinct constant principal curvatures with multiplicities $\nu_1, \dots, \nu_p$, respectively then we prove

THEOREM. *Let $p \geq 2$. For each $i, 1 \leq i \leq p$, we have*

$$\sum_{j \neq i} \nu_j \frac{-c + a_i a_j}{a_i - a_j} = 0.$$

We must note that this identity differs from the classical case in that the sectional curvature c of the ambient space appears with the minus sign in front. We present here a proof based on the consideration of focal varieties which works also in the classical case as mentioned in [2].

The identity above serves as the starting point for the study of isoparametric hypersurfaces in the Lorentzian space forms. As E. Cartan showed, the identity implies that $p \leq 2$ if $-c \leq 0$, that is, if $c \geq 0$. Thus in L^{n+1} and $S_1^{n+1}(c)$, $c > 0$, an isoparametric hypersurface has at most two distinct constant principal curvatures. It is in $H_1^{n+1}(c)$, $c < 0$, that we can hope to find an isoparametric hypersurface with more than two distinct constant principal curvatures.

§1. Geodesics in Lorentzian space forms

We summarize the known facts about the Lorentzian space forms we

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shall be working with. Let R_s^m be the m -dimensional real vector space R^m with an inner product of signature $(s, m-s)$ given by

$$\langle x, y \rangle = - \sum_{i=1}^s x_i y_i + \sum_{j=s+1}^m x_j y_j$$

for $x = (x_1, \dots, x_m)$ and $y = (y_1, \dots, y_m)$.

The flat Lorentz space L^{n+1} is R_1^{n+1} regarded as an affine space with parallel Lorentzian inner product $\langle, \rangle$. The sectional curvature is 0 and the geodesics are straight lines.

For a given positive number c , the standard space form $S_1^n(c)$ is the hypersurface

$$\{x \in L^{n+1}; \langle x, x \rangle = r^2\}, \quad r = \frac{1}{\sqrt{c}}$$

with the induced Lorentzian metric. It has constant sectional curvature c .

Let $x_0 \in S_1^n(c)$ and let X_0 be a tangent vector at x_0 (so that $\langle x_0, X_0 \rangle = 0$).

If $\langle X_0, X_0 \rangle = 1$, then the (space-like) geodesic with initial point x_0 and initial tangent vector X_0 is given by

$$x_t = \cos\left(\frac{t}{r}\right)x_0 + \sin\left(\frac{t}{r}\right)rX_0,$$

where t is the affine parameter (arc length) of the geodesic.

If $\langle X_0, X_0 \rangle = -1$, then the (time-like) geodesic with initial point x_0 and initial tangent vector X_0 is given by

$$x_t = \cosh\left(\frac{t}{r}\right)x_0 + \sinh\left(\frac{t}{r}\right)rX_0,$$

where t is the affine parameter (proper time) of the geodesic.

If $\langle X_0, X_0 \rangle = 0$, then the (null) geodesic with initial point x_0 and initial tangent vector X_0 is given by

$$x_t = x_0 + tX_0,$$

where t is an affine parameter.

For a given negative number c , the standard space form $H_1^n(c)$ is the hypersurface

$$\{x \in R_2^{n+1}; \langle x, x \rangle = -r^2\}, \quad r = \frac{1}{\sqrt{-c}},$$

with the induced Lorentzian metric. It has constant sectional curvature c .

Let $x_0 \in H_1^n(c)$ and let X_0 be a tangent vector at x_0 .

If $\langle X_0, X_0 \rangle = 1$, then the (space-like) geodesic with initial conditions (x_0, X_0) is given by

$$x_t = \cosh\left(\frac{t}{r}\right)x_0 + \sinh\left(\frac{t}{r}\right)rX_0,$$

where t is the affine parameter (arc length).

If $\langle X_0, X_0 \rangle = -1$, then the (time-like) geodesic with initial conditions (x_0, X_0) is given by

$$x_t = \cos\left(\frac{t}{r}\right)x_0 + \sin\left(\frac{t}{r}\right)rX_0,$$

where t is the affine parameter (proper time).

If $\langle X_0, X_0 \rangle = 0$, then the (null) geodesic with initial conditions (x_0, X_0) is given by

$$x_t = x_0 + tX_0,$$

where t is an affine parameter.

§ 2. Focal varieties

In a Lorentzian space form $M_1^{n+1}(c)$ we consider a connected space-like hypersurface M^n . For a field of time-like unit normals ξ , the corresponding shape operator A of M^n is given by

$$\tilde{\nabla}_X \xi = -AX,$$

where X is a tangent vector to M and $\tilde{\nabla}$ denotes covariant differentiation in $M_1^{n+1}(c)$. We suppose that M^n is isoparametric with constant principal curvatures $\alpha_1, \dots, \alpha_p$ of multiplicities $\nu_1, \dots, \nu_p$, respectively.

As in the classical case (see [1], [2]) we may define the focal varieties of M . (All the considerations here will be done locally.)

For each $x \in M$, we denote by $\text{Exp}_x t\xi_x$ the time-like geodesic with initial conditions (x, ξ_x) .

If $c=0$, let α_i be any non-zero constant principal curvature. We define $f_i : M \rightarrow L^{n+1}$ by

$$f_i(x) = \text{Exp}_x \frac{1}{\alpha_i} \xi_x = x + \frac{1}{\alpha_i} \xi_x.$$

If $c>0$, let $\alpha_i = (1/r) \coth(t_i/r)$, $r = 1/\sqrt{c}$, and define $f_i : M \rightarrow S_1^{n+1}(c)$ by

$$f_i(x) = \text{Exp}_x t_i \xi_i = \cosh\left(\frac{t_i}{r}\right)x + \sinh\left(\frac{t_i}{r}\right)r\xi_x.$$

If $c < 0$, let $a_i = (1/r) \cot(t_i/r)$, $0 < t_i/r < \pi$, $r = 1/\sqrt{-c}$, and define $f_i: M \rightarrow H_1^{n+1}(c)$ by

$$f_i(x) = \text{Exp}_x t_i \xi_x = \cos\left(\frac{t_i}{r}\right)x + \sin\left(\frac{t_i}{r}\right)r\xi_x.$$

In each case, it turns out that the differential $(f_i)_*$ is 0 on $T_i(x) = \{X \in T_x(M); A_i X = a_i X\}$ and injective on the subspace $\sum_{j \neq i} T_j(x)$ of $T_x(M)$.

We may therefore consider $V_i = f_i(M)$ as an $(n - \nu_i)$ -dimensional submanifold of $M_1^{n+1}(c)$, which we call the focal variety of M . It will also turn out that the number

$$\sum_{j \neq i} \nu_j \frac{-c + a_i a_j}{a_i - a_j}$$

is equal to the trace of the shape operator for V_i corresponding to the unit normal in the direction of the geodesic $\text{Exp}_x(t\xi_x)$ as it comes into the point $f_i(x)$. Thus, if we can prove that the mean curvature vector of V_i is 0, that is, V_i is a "maximal" submanifold of $M_1^{n+1}(c)$, then we have the basic identity. We shall give the detail of these arguments in the case $c < 0$, because it is the most interesting case as we explained in the introduction.

§ 3. Proof for $H_1^{n+1}(c)$

Now let M^n be a spacelike hypersurface in $H_1^{n+1}(c)$ with distinct constant principal curvatures $a_1, \dots, a_p$ with multiplicities $\nu_1, \dots, \nu_p$, respectively, where $p \geq 2$. We let

$$(1) \quad a_i = \frac{1}{r} \cot\left(\frac{t_i}{r}\right), \quad r = \frac{1}{\sqrt{-c}}, \quad 0 < \frac{t_i}{r} < \pi.$$

Consider the mapping

$$x \in M \rightarrow f_i(x) = \cos\left(\frac{t_i}{r}\right)x + \sin\left(\frac{t_i}{r}\right)r\xi_x \in V_i$$

as in the preceding section.

The differential $(f_i)_*$ maps $X \in T_i(x)$ into

$$(2) \quad \left(\cos \frac{t_i}{r}\right)X + \sin\left(\frac{t_i}{r}\right)(-rAX) = \left(\cos \frac{t_i}{r}\right)X + \left(\sin \frac{t_i}{r}\right)\left(\cot \frac{t_i}{r}\right)X = 0.$$

If $X \in T_j(x)$ with $j \neq i$, then

$$(3) \quad \begin{aligned} (f_i)_*(X) &= \left(\cos \frac{t_i}{r} \right) X - \left(\sin \frac{t_i}{r} \right) \left(\cot \frac{t_j}{r} \right) X \\ &= \left(\cos \frac{t_i}{r} - \sin \frac{t_i}{r} \cot \frac{t_j}{r} \right) X \end{aligned}$$

which is a non-zero multiple of X . This shows $(f_i)_*$ is injective on $\sum_{j \neq i} T_j(x)$.

It follows that the tangent space $T_p(V_i)$ at $p = f_i(x)$ may be considered as $(f_i)_*(\sum_{j \neq i} T_j(x)) = \sum_{j \neq i} T_j(x)$ by virtue of (3).

The geodesic

$$(4) \quad \text{Exp}_x t \xi_x = \cos\left(\frac{t}{r}\right)x + \sin\left(\frac{t}{r}\right)r\xi_x$$

has the tangent vector

$$(5) \quad \xi'_x = -\sin\left(\frac{t_i}{r}\right)\frac{x}{r} + \cos\left(\frac{t_i}{r}\right)\xi_x$$

at the point $p = f_i(x)$.

Obviously ξ'_x is normal to V_i at p .

If we translate any vector $Z \in T_i(x)$ to p (in R_3^{n+2}) then it can be considered as a tangent vector at p to H_1^{n+1} which is normal to V_i , as well as to ξ'_x . It follows that the normal space $N_p(V_i)$ of V_i in H_1^{n+1} at p , which has dimension $\nu_0 + 1$, is spanned by ξ'_x and all $Z \in T_i(x)$. We now compute the trace of the shape operator of V_i for $B_{\xi'_x}$ and B_Z , $Z \in T_i(x)$.

Denoting by D the covariant differentiation in R_3^n , we have from (5)

$$D_x \xi'_x = -\sin\left(\frac{t_i}{r}\right)\frac{X}{r} + \cos\left(\frac{t_i}{r}\right)D_x \xi_x.$$

Since $D_x \xi_x = \tilde{\nabla}_x \xi_x = -AX$, we get for $X \in T_j(x)$

$$\begin{aligned} D_x \xi'_x &= -\sin\left(\frac{t_i}{r}\right)\frac{X}{r} - \cos\left(\frac{t_i}{r}\right)\frac{\cot(t_j/r)X}{r} \\ &= \left(-\sin \frac{t_i}{r} - \cos \frac{t_i}{r} \cot \frac{t_j}{r}\right)\frac{X}{r}, \end{aligned}$$

which is going to be equal to $(f_i)_*(-B_{\xi'_x}X)$. By virtue of (3) and (1) we get

$$\begin{aligned} B_{\xi'_x}X &= \frac{\sin(t_i/r) + \cos(t_i/r)\cot(t_j/r)}{\cos(t_i/r) - \sin(t_i/r)\cot(t_j/r)} \frac{X}{r} = \frac{1 + \cot(t_i/r)\cot(t_j/r)}{\cot(t_i/r) - \cot(t_j/r)} \frac{X}{r} \\ &= \frac{1 + ra_i ra_j}{ra_i - ra_j} \frac{X}{r} = \frac{1/r^2 + a_i a_j}{a_i - a_j} X \end{aligned}$$

i.e.

$$B_{\xi'_x} X = \frac{-c + a_i a_j}{a_i - a_j} X \quad \text{for } X \in T_j(x).$$

Thus

$$(6) \quad \text{trace } B_{\xi'_x} = \sum_{j \neq i} \nu_j \frac{-c + a_i a_j}{a_i - a_j}.$$

To compute trace B_Z for $Z \in T_i(x)$ regarded as a normal vector to V_i , we may extend Z to a vector field belonging to the distribution T_i in a neighborhood of x and set

$$(f_i)_*(B_Z X) = -\tilde{\nabla}_X Z, \quad X \in T_j(x)$$

where Z on the right-hand side is a field of normal vectors to V_i in a neighborhood of x .

We use

LEMMA. *Let Z be a vector field belonging to the distribution T_i . If $X \in T_j(x)$, $j \neq i$, then $\nabla_X Z$ is normal to $T_j(x)$.*

PROOF OF LEMMA. We extend X to a vector field belonging to the distribution T_j . From Codazzi's equation

$$(\nabla_X A)(Z) = (\nabla_Z A)(X)$$

we obtain

$$\sum_{k=1}^p (a_i - a_k) (\nabla_X Z)_k = \sum_{k=1}^p (a_j - a_k) (\nabla_Z X)_k,$$

where $(\)_k$ denotes the T_k -component. Taking the j -th component of each side, we get

$$(a_i - a_j) (\nabla_X Z)_j = 0.$$

Since $a_i \neq a_j$, we have $(\nabla_X Z)_j = 0$. Thus $\nabla_X Z \perp T_j(x)$.

We now take $Z \in T_i(x)$ and extend it to a vector field belonging to the distribution T_i . For $X \in T_j(x)$, we have

$$\tilde{\nabla}_X Z = \nabla_X Z,$$

which is normal to $T_j(x)$ by the lemma. Thus

$$(f_i)^*(-B_Z X) = \sum_{k \neq i, j} (\nabla_X Z)_k.$$

Hence

$$B_Z X = - \sum_{k \neq i, j} \frac{1}{\cos(t_i/r) - \sin(t_i/r) \cot(t_k/r)} (\nabla_X Z)_k \quad \text{for } X \in T_j(x)$$

by virtue of (3).

Since

$$\langle B_Z X, X \rangle = 0,$$

we have

$$(7) \quad \text{trace } B_Z = 0.$$

Let M_i be an integral manifold of the distribution T_i through the point x . We define a differentiable mapping g_i of M_i into $N_p(V_i)$ as follows (this is an idea of Münzer as mentioned in [2]).

First we remark that f_i maps M_i into the single point p , because $(f_i)_*$ vanishes on $T_i(y)$ for every point y as we saw before. For any $y \in M_i$, we define $g_i(y)$ to be the tangent vector at $p = f_i(y)$ of the geodesic $\text{Exp}_y t \xi_y = \cos(t/r)y + \sin(t/r)r\xi_y$. Thus

$$(8) \quad g_i(y) = -\sin \frac{t_i}{r} \frac{y}{r} + \cos \frac{t_i}{r} \xi_y.$$

This vector is normal to V_i just like ξ'_x in (5); indeed, $\xi'_x = g_i(x)$.

We compute the differential of g_i at x . For $X \in T_i(x)$ we have

$$\begin{aligned} (g_i)_*(X) &= -\sin \frac{t_i}{r} \frac{X}{r} + \cos \frac{t_i}{r} D_X \xi = -\sin \frac{t_i}{r} \frac{X}{r} + \cos \frac{t_i}{r} (-AX) \\ (9) \quad &= -\sin \frac{t_i}{r} \frac{X}{r} - \cos \frac{t_i}{r} \cot \frac{t_i}{r} \frac{X}{r} = -\frac{1}{\sin(t_i/r)} \frac{X}{r}. \end{aligned}$$

(9) shows that $(g_i)_*$ is injective.

We consider the linear function w on the normal space $N_p(V_i)$ given by

$$(10) \quad w(Z) = \text{trace } B_Z.$$

We shall show that w is identically 0; then (6) shows that the constant

$$q = \sum_{j \neq i} \nu_j \frac{-c + a_i a_j}{a_i - a_j}$$

is equal to 0.

Let H_p be the connected component containing ξ'_x of the set of all time-like unit vectors in $N_p(V_i)$. Then $g_i(M_i) \subset H_p$. Since $(g_i)_*$ is injective, we

see that $g_i(M_i)$ is an open subset of H_p . Since (6) is valid for every point of M_i , it follows that $w(Z)=q$ for all $Z \in H_p$. On the other hand, (7) shows that $w(Z)=0$ for all $Z \in T_i(x)$, considered as a subspace of $N_p(V_i)$.

Our assertion $w \equiv 0$ on $N_p(V_i)$ will follow from the following lemma which is easily proved.

LEMMA. *Let L^{m+1} be the Lorentzian vector space and let*

$$H^m = \{x = (x_1, x_2, \dots, x_{m+1}); \langle x, x \rangle = -1, x_1 > 0\}.$$

Suppose w is a linear function on L^{n+1} such that

$$(1) \quad w \equiv q, \text{ constant, on } H^m$$

$$(2) \quad w(e) = 0 \text{ for some space-like unit vector.}$$

Then $w \equiv 0$ on L^{m+1} .

§ 4. Examples

It is easy to determine all complete space-like umbilical hypersurfaces in L^{n+1} , $S_1^{n+1}(c)$, $c > 0$, and $H_1^{n+1}(c)$, $c < 0$.

We shall give a list of all complete space-like hypersurfaces with two distinct constant principal curvatures in each of the model space forms ($c = 0, 1, -1$). The shape operator is parallel.

1) In L^{n+1} , they are

$$H^k \left(-\frac{1}{r^2} \right) \times E^{n-k} = \{(x_1, x_2, \dots, x_{n+1}); -x_1^2 + x_2^2 + \dots + x_{n+1}^2 = -r^2\}$$

where $0 < k < n$.

For the time-like unit normal

$$\xi = -\left(\frac{x_1}{r}, \dots, \frac{x_{k+1}}{r}, 0, \dots, 0\right)$$

the principal curvatures are

$$\frac{1}{r} \quad \text{and} \quad 0$$

with multiplicities k and $n-k$, respectively.

2) In $S_1^{n+1}(1)$, they are

$$H^k \left(-\frac{1}{r^2} \right) \times S^{n-k} \left(\frac{1}{1+r^2} \right) = \{(x_1, \dots, x_{k+1}, \dots, x_{n+2});$$

$$-x_1^2 + \sum_{i=2}^{k+1} x_i^2 = -r^2 \quad \text{and thus} \quad \sum_{j=k+2}^{n+2} x_j^2 = 1+r^2\}$$

where

$$r > 0$$

and

$$0 < k < n.$$

For the time-like unit normal

$$\xi = - \left(\frac{\sqrt{1+r^2}}{r} x_1, \dots, \frac{\sqrt{1+r^2}}{r} x_k, \frac{r}{\sqrt{1+r^2}} x_{k+1}, \dots, \frac{r}{\sqrt{1+r^2}} x_{n+2} \right)$$

the principal curvatures are

$$\frac{\sqrt{1+r^2}}{r} \quad \text{and} \quad \frac{r}{\sqrt{1+r^2}}$$

of multiplicities k and $n-k$, respectively.

3) In $H_1^{n+1}(-1)$, they are

$$H^k \left(-\frac{1}{r^2} \right) \times H^{n-k} \left(\frac{1}{r^2-1} \right) = \left\{ (x_1, x_2, x_3, \dots, x_{k+2}, x_{k+3}, \dots, x_{n+2}); \right. \\ \left. -x_1^2 + \sum_{i=3}^{k+2} x_i^2 = -r^2 \quad \text{and thus} \quad -x_2^2 + \sum_{j=k+3}^{n+2} x_j^2 = r^2 - 1 \right\}$$

where

$$0 < k < n, \quad 0 < r < 1.$$

For the time-like unit normal

$$\xi = - \left(\frac{\sqrt{1-r^2}}{r} x_1, \frac{-r}{\sqrt{1-r^2}} x_2, \frac{\sqrt{1-r^2}}{r} x_3, \dots, \frac{\sqrt{1-r^2}}{r} x_{k+2}, \right. \\ \left. \frac{-r}{\sqrt{1-r^2}} x_{k+3}, \dots, \frac{-r}{\sqrt{1-r^2}} x_{n+2} \right)$$

the principal curvatures are

$$\frac{\sqrt{1-r^2}}{r} \quad \text{and} \quad \frac{-r}{\sqrt{1-r^2}}$$

of multiplicities k and $n-k$.

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