Residues and characteristic classes for projective foliations

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(Received July 26, 1979)

Introduction

In this paper we shall study projective foliations with singularities. (The precise definition is given in Section 4.) Our object of the paper is twofold; that is, to prove residue formulas for these singular foliations on the one hand, and to explain the relation of the residues to secondary characteristic classes for projective foliations on the other.

In our previous papers [NS, NT], for Pontrjagin classes of the normal bundles of projective foliations (in fact, for those of more general family of F -foliations) we proved a vanishing theorem, a refinement of a well-known theorem of Bott [B1], from the viewpoint of the theory of Cartan connections and prolongations of the normal frame bundles of foliations. In Section 2 of this paper, we first present another proof of this vanishing theorem which employs neither the prolonged normal frame bundles nor Cartan connections (Theorem 2.1). Based on this theorem, in Section 4 we define the residue for each compact connected component of the singular set of a projective foliation with singularities (Definition 4.3). Then in Sections 5 and 7, following the general theory due to Baum-Bott [BB], we prove two types of residue formulas for projective foliations with singularities satisfying certain dimension conditions on the singular sets (Theorem 5.2 and Theorem 7.1). These results constitute the first part of the paper.

In the second part of the paper, we focus our study on constructing a continuous variation of secondary characteristic classes, defined for instance by Kamber-Tondeur [KT] and Morita [M], for projective foliations. In particular, we prove in Section 9 that there exists an epimorphism:

$$H_{2m-1}(BPF_{2m-1}; \mathbb{Z}) \longrightarrow R^{d(m)} \longrightarrow 0,$$

where BPF_{2m-1} is the classifying space for projective foliations of codimension $2m-1$ and $d(m)$ denotes the dimension of secondary characteristic classes $H^{2m-1}(PWO_{2m-1})$ for these foliations (Theorem 9.1).

^{*)} Supported partially by the Sakkokai Foundation.

In Baum-Bott [BB] and Vishik [V], residue formulas in the same spirit have been proved for holomorphic foliations with singularities. The same method based on the theory of residues has been used to construct continuous variations of secondary characteristic classes for holomorphic foliations in Bott [B2, B5] and Baum-Bott [BB], for smooth foliations in Heitsch [He 2] and for Riemannian foliations in Lazarov-Pasternack [LP 2].

The paper is divided into 10 sections with the following titles:

- § 1. Preliminaries
- § 2. The strong vanishing theorem
- § 3. Exact sequences and connections
- § 4. Residue existence theorem
- § 5. Residue formula I
- § 6. A reduction theorem
- § 7. Residue formula II
- § 8. Characteristic classes for projective foliations
- § 9. Continuous variation
- § 10. Rigidity of residues

This paper may be considered as a continuation of our study in [NS, NT] and some of the results have been announced in [N].

I would like to express my sincere thanks to Professors S. Morita, H. Sato and M. Takeuchi for their valuable suggestions and to Professors T. Ochiai and A. Poor for helpful discussions during the preparation of this paper.

§ 1. Preliminaries

This section is devoted to a brief survey of the relevant facts from the theory of projective structures and the Chern-Weil construction of characteristic classes. For details we refer the reader to Kobayashi [K] and Kobayashi-Nomizu [KN].

Let M be a smooth manifold of dimension n and $P(M)$ be the linear frame bundle of M . $P(M)$ is a principal bundle over M with group $GL(n, \mathbf{R})$ the general linear group. We can define projective structures on M by projective equivalence classes of torsionfree connections on $P(M)$ in the following manner.

Let $\mathfrak{sl}(n+1, \mathbf{R})$ be the Lie algebra of the special linear group $SL(n+1, \mathbf{R})$. Note that $\mathfrak{sl}(n+1, \mathbf{R})$ has the structure of a graded Lie algebra defined by the graded decomposition $\mathfrak{sl}(n+1, \mathbf{R}) = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$ into subspaces given by

$$\mathfrak{g}_1 = \left\{ \begin{pmatrix} 0 & \xi \\ 0 & 0 \end{pmatrix}; \xi \in \mathbf{R}^n \right\},$$

$$(1.1) \quad \begin{aligned} \mathfrak{g}_0 &= \left\{ \begin{pmatrix} A & 0 \\ 0 & a \end{pmatrix}; A \in \mathfrak{gl}(n, \mathbf{R}), \text{trace } A + a = 0 \right\}, \\ \mathfrak{g}_1 &= \left\{ \begin{pmatrix} 0 & 0 \\ u & 0 \end{pmatrix}; u \in \mathbf{R}^n \right\}. \end{aligned}$$

Then, with convention $\mathfrak{g}_{-2} = \mathfrak{g}_2 = 0$, the graded decomposition (1.1) satisfies

$$(1.2) \quad [\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j} \quad \text{for } i, j = -1, 0, 1.$$

Furthermore we can identify $\mathfrak{g}_0$ with $\mathfrak{gl}(n, \mathbf{R})$, the Lie algebra of $GL(n, \mathbf{R})$, by the following isomorphisms:

$$(1.3) \quad \mathfrak{g}_0 \ni \begin{pmatrix} A & 0 \\ a & 0 \end{pmatrix} \longmapsto A - aI_n \in \mathfrak{gl}(n, \mathbf{R}),$$

$$(1.4) \quad \mathfrak{gl}(n, \mathbf{R}) \ni B \longmapsto \begin{pmatrix} A & 0 \\ 0 & a \end{pmatrix} \in \mathfrak{g}_0,$$

where $A = B - (\text{trace } B/(n+1))I_n$, $a = -\text{trace } B/(n+1)$ and I_n denotes the identity matrix of degree n .

Let ω be a connection on the principal bundle $P(M)$, which is a $\mathfrak{gl}(n, \mathbf{R})$ -valued 1-form on $P(M)$. Let θ be the canonical form on $P(M)$. Identify $\mathfrak{gl}(n, \mathbf{R})$ with $\mathfrak{g}_0$ by (1.4), and $\mathbf{R}^n$ with $\mathfrak{g}_{-1}$ in a natural fashion. We thereby consider the connection ω as a $\mathfrak{g}_0$ -valued form and the canonical form θ as a $\mathfrak{g}_1$ -valued form. (In order to understand the reason why θ is regarded as a $\mathfrak{g}_1$ -valued form, one has to consider Cartan connections in second order G -structures. See [K, M].)

In terms of the Lie algebra structure on $\mathfrak{gl}(n+1, \mathbf{R}) = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$, the condition that ω be torsionfree can be expressed by

$$(1.5) \quad d\theta = -[\omega, \theta].$$

Let ω and ω' be two torsionfree connections on $P(M)$. Then ω and ω' are said to be *projectively equivalent* if there exists a $\mathfrak{g}_1$ -valued function η on $P(M)$ such that

$$(1.6) \quad \omega - \omega' = [\theta, \eta].$$

(Note that the left-hand side takes values in $\mathfrak{g}_0$ and the right-hand side in $[\mathfrak{g}_{-1}, \mathfrak{g}_1] \subset \mathfrak{g}_0$.) A *projective structure* on M is by definition a projective equivalence class of torsionfree connections on $P(M)$. It follows from (1.1)–(1.4) that the relation (1.6) can be rewritten as follows:

$$\omega - \omega' = \theta\eta + (\eta\theta)I_n,$$

or more explicitly, in terms of a natural basis,

$$\omega_j^i - {}'\omega_j^i = \theta^i \eta_j + (\sum_k \theta^k \eta_k) \delta_j^i,$$

which is a reformulation of the classical formula (cf. [Y]):

$$(1.6)' \quad \Gamma_{jk}^i - {}'\Gamma_{jk}^i = \delta_k^i \psi_j + \delta_j^i \psi_k,$$

Γ_{jk}^i and ${}'\Gamma_{jk}^i$ denoting the Christoffel's symbols of ω and ${}'\omega$.

Let M be a smooth manifold with a torsionfree connection ω on $P(M)$. Then ω defines naturally a projective structure on M , the projective equivalence class containing ω of torsionfree connections on $P(M)$. A diffeomorphism $\gamma: M \rightarrow M$ is called a *projective diffeomorphism* of M if γ preserves the projective structure defined by ω , that is, if ω and $\gamma^{i*}\omega$ are projectively equivalent, where $\gamma^i: P(M) \rightarrow P(M)$ denotes the bundle isomorphism induced by γ . Thus γ is a projective diffeomorphism of M if and only if there exists a $\mathfrak{g}_1$ -valued function η on $P(M)$ such that

$$(1.7) \quad \omega - \gamma^{i*}\omega = [\theta, \eta].$$

γ is called an affine diffeomorphism of M if, in particular, γ preserves ω : $\omega = \gamma^{i*}\omega$. It should be remarked that projective diffeomorphisms preserve, up to parametrization, the system of geodesics defined by ω , while affine diffeomorphisms preserve it together with affine parameters.

Let E be a smooth vector bundle, or simply vector bundle, of rank r over a smooth manifold M . As usual, we denote by $C^\infty(E)$ the space of smooth sections of E . Let $P(E)$ denote the frame bundle of E .¹⁾ $P(E)$ is a principal bundle over M with group $GL(r, \mathbf{R})$. Recall that for each connection ω on the principal bundle $P(E)$ there corresponds uniquely, in a standard manner, a connection ∇ on the vector bundle E and vice versa. ∇ is often referred to as the covariant derivative of ω and ω as the connection form of ∇ .

Let $p(E)$ denote the total Pontrjagin class of E in the de Rham cohomology $H^*(M; \mathbf{R})$ so that

$$p(E) = 1 + p_1(E) + p_2(E) + \cdots + p_{[r/2]}(E),$$

$p_i(E)$ denoting the i -th Pontrjagin class of E in $H^{4i}(M; \mathbf{R})$. The Chern-Weil construction of characteristic classes then gives a recipe for each $p_i(E)$ by the curvature of a connection on E in the following way.

We denote by $I(GL(r, \mathbf{R})) = \sum_{k \geq 0} I^k(GL(r, \mathbf{R}))$ the graded commutative algebra of $\text{Ad}(GL(r, \mathbf{R}))$ -invariant polynomials on the Lie algebra $\mathfrak{gl}(r, \mathbf{R})$ of $GL(r, \mathbf{R})$. $I^k(GL(r, \mathbf{R}))$ is the subspace of $I(GL(r, \mathbf{R}))$ consisting of homogeneous polynomials of degree k . As an algebra $I(GL(r, \mathbf{R})) = \mathbf{R}[c_1, \dots, c_r]$ a

¹⁾ The frame bundle $P(TM)$ of the tangent bundle TM of M is nothing but the linear frame bundle $P(M)$ of M , that is, $P(M) = P(TM)$.

polynomial algebra on r generators, the Chern polynomials $c_1, \dots, c_r$. Each c_i is a polynomial of degree i and they are defined by the equation

$$(1.8) \quad \det (tI_r - (1/2\pi)A) = \sum_{i=0}^r t^{r-i} c_i(A)$$

for $A \in \mathfrak{gl}(r, \mathbf{R})$. It should be noted that throughout this paper we often identify $I(GL(r, \mathbf{R}))$ with the graded algebra of $\text{Ad}(GL(r, \mathbf{R}))$ -invariant symmetric multilinear mappings on $\mathfrak{gl}(r, \mathbf{R})$, since they are naturally isomorphic. Thus, if $f \in I^k(GL(r, \mathbf{R}))$, we are free to regard it as an $\text{Ad}(GL(r, \mathbf{R}))$ -invariant symmetric multilinear mapping $\mathfrak{gl}(r, \mathbf{R}) \times \dots \times \mathfrak{gl}(r, \mathbf{R}) \rightarrow \mathbf{R}$ of degree k .

Let Ω be the curvature form of ω , that is,

$$\Omega = d\omega + (1/2)[\omega, \omega] = d\omega + \omega \wedge \omega.$$

For each $f \in I^k(GL(r, \mathbf{R}))$, consider the $2k$ -form $f(\Omega)$ on $P(E)$. Then $f(\Omega)$ is a $GL(r, \mathbf{R})$ -invariant horizontal form on $P(E)$ and hence is pushed down to a $2k$ -form $\bar{f}(\Omega)$ on M . It follows from the second Bianchi identity

$$d\Omega = [\Omega, \omega]$$

that $\bar{f}(\Omega)$ is a closed $2k$ -form. Thus we obtain a mapping

$$(1.9) \quad w: I(GL(r, \mathbf{R})) \longrightarrow H^*(M; \mathbf{R})$$

defined by

$$w(f) = [\bar{f}(\Omega)],$$

where $[\cdot]$ denotes cohomology class in $H^*(M; \mathbf{R})$. It is evident that w is an algebra homomorphism, and we call it the Chern-Weil homomorphism. Then the Chern-Weil construction of characteristic classes asserts that w is independent of the choice of a connection and a recipe for Pontrjagin classes $p_i(E)$ is given as follows:

$$(1.10) \quad p_i(E) = w(c_{2i}), \quad \text{and}$$

$$(1.11) \quad w(c_{2i+1}) = 0.$$

Let ω and ω' be two connections on $P(E)$. For each $t \in [0, 1]$, define the connection $\omega_t = t\omega + (1-t)\omega'$. Then the fact that the Chern-Weil homomorphism w is independent of the choice of a connection is a consequence of the well-known homotopy formula for characteristic forms:

$$(1.12) \quad f(\Omega) - f(\Omega') = d\Delta_f(\omega, \omega'),$$

where $f \in I^k(GL(r, \mathbf{R}))$ and

$$(1.13) \quad \Delta_f(\omega, ' \omega) = k \int_0^1 f(\omega - ' \omega, \Omega_t, \dots, \Omega_t) dt,$$

Ω_t denoting the curvature form of ω_t . Note that $\Delta_f(\omega, ' \omega)$ is a $GL(r, \mathbf{R})$ -invariant horizontal form on $P(E)$ and hence is pushed down to a $(2k-1)$ -form on M .

Recall the inclusion $\mathfrak{gl}(n, \mathbf{R}) \rightarrow \mathfrak{g}_0 \subset \mathfrak{sl}(n+1, \mathbf{R})$ defined by (1.4). Let $f \in I^k(SL(n+1, \mathbf{R}))$. If we restrict the invariant polynomial f to the subalgebra $\mathfrak{g}_0$ of $\mathfrak{sl}(n+1, \mathbf{R})$, then we obtain an element of $I^k(GL(n, \mathbf{R}))$. Thus we have an algebra homomorphism, called the restriction homomorphism,

$$(1.14) \quad r: I(SL(n+1, \mathbf{R})) \longrightarrow I(GL(n, \mathbf{R})).$$

Let $I(SL(n+1, \mathbf{R}), GL(n, \mathbf{R}))$ denote the image $r(I(SL(n+1, \mathbf{R})))$, which will play an important role in our study of characteristic classes. $I(SL(n+1, \mathbf{R}), GL(n, \mathbf{R}))$ is the subalgebra of $I(GL(n, \mathbf{R}))$ consisting of those elements which can be extended to $\text{Ad}(SL(n+1, \mathbf{R}))$ -invariant polynomials on $\mathfrak{sl}(n+1, \mathbf{R})$.

Let c_i now denote the Chern polynomials in $I(SL(n+1, \mathbf{R}))$, those polynomials defined by the equation (1.8) for $A \in \mathfrak{sl}(n+1, \mathbf{R})$ and with the degree $n+1$ instead of r . Note that c_1 always vanishes in this case. It is then known that as an algebra $I(SL(n+1, \mathbf{R})) = \mathbf{R}[c_2, \dots, c_{n+1}]$, a polynomial algebra on n generators $c_2, \dots, c_{n+1}$. Denote by $'c_i$ the image $r(c_i)$ under the restriction homomorphism (1.14). Then the relationship between the elements $'c_i$ and c_i in $I(GL(n, \mathbf{R}))$ is given by the following lemma, whose proof is immediate.

LEMMA 1.1. *If $A \in \mathfrak{gl}(n, \mathbf{R})$, then*

$$(1.15) \quad 'c_k(A) = c_k(A) + \sum_{j=0}^{k-1} A_{jk} c_1(A)^{k-j} c_j(A),$$

where A_{jk} denotes the factor $(-1/(n+1))^{k-j} \binom{n+1-j}{k-j}$. In particular, $'c_1 = 0$.

§ 2. The strong vanishing theorem

In this section we prove a vanishing theorem for Pontrjagin classes of the normal bundles of projective foliations. This gives a refinement of the well-known vanishing theorem of Bott [B1, B2] and will be called for contrast the strong vanishing theorem. In our previous papers [NS, NT], we prove it for more general family of Γ -foliations by making use of Cartan connections on the prolonged normal frame bundles of these foliations. See [NS, NT] for details. We present here another proof of the strong vanishing theorem which needs neither the prolonged normal frame bundles nor Cartan connections.

Let M be a smooth manifold. Projective foliations on M are then defined in the following manner.

DEFINITION 2.1. A *projective foliation* $\mathcal{F} = \{U_\alpha, f_\alpha, \gamma_{\beta\alpha}^x, \nabla_\alpha\}$ of codimension q on M is defined by the following data:

- (i) An open covering $\{U_\alpha\}_\alpha$ of M .
- (ii) For each index α , a smooth submersion $f_\alpha: U_\alpha \rightarrow (\mathbf{R}^q, \nabla_\alpha)$ of U_α to a Euclidean q -space $\mathbf{R}^q$ endowed with a torsionfree connection ∇_α .
- (iii) For each $x \in U_\alpha \cap U_\beta$, there exists a local projective diffeomorphism $\gamma_{\beta\alpha}^x$ from a neighborhood of $f_\alpha(x)$ in $(\mathbf{R}^q, \nabla_\alpha)$ to a neighborhood of $f_\beta(x)$ in $(\mathbf{R}^q, \nabla_\beta)$ satisfying $f_\beta = \gamma_{\beta\alpha}^x \circ f_\alpha$ in some neighborhood of x .

Examples of these foliations are given in [NS].

Note that an affine diffeomorphism from $(\mathbf{R}^q, \nabla_\alpha)$ to $(\mathbf{R}^q, \nabla_\beta)$ is a projective diffeomorphism. In particular, an isometry between Riemannian manifolds $(\mathbf{R}^q, g_\alpha)$ and $(\mathbf{R}^q, g_\beta)$ is a projective diffeomorphism with respect to the Levi-Civita connections ∇_α and ∇_β defined respectively by the metrics g_α and g_β . Corresponding to these projective diffeomorphisms we get restricted families of projective foliations, which are referred to as *affine foliations* and *Riemannian foliations* respectively.

In Definition 2.1, if we drop the condition that the Euclidean q -space $(\mathbf{R}^q, \nabla_\alpha)$ be endowed with a torsionfree connection ∇_α and require the $\{\gamma_{\beta\alpha}^x\}$ only to be local diffeomorphisms of $\mathbf{R}^q$, then we recover the definition of smooth foliations of codimension q (cf. Bott-Heafliger [BH]).

Let $\mathcal{F} = \{U_\alpha, f_\alpha, \gamma_{\beta\alpha}^x, \nabla_\alpha\}$ be a projective foliation of codimension q on a smooth manifold M . The fibers of each submersion f_α are then pieced together to define the leaves of the foliation $\mathcal{F}$. The kernels of the differentials df_α of submersions f_α constitute the *tangent bundle* $\tau(\mathcal{F})$ of $\mathcal{F}$, an integrable subbundle along the leaves of the tangent bundle TM of M . The quotient bundle $\nu(\mathcal{F}) = TM/\tau(\mathcal{F})$ is called the *normal bundle* of $\mathcal{F}$. Let $P(\mathcal{F})$ denote the frame bundle of $\nu(\mathcal{F})$, which we call the *normal frame bundle* of $\mathcal{F}$. Each submersion $f_\alpha: U_\alpha \rightarrow (\mathbf{R}^q, \nabla_\alpha)$ then induces the following bundle homomorphisms:

$$\begin{aligned} f_\alpha^! &: \nu(\mathcal{F})|U_\alpha \longrightarrow T\mathbf{R}^q, \\ f_\alpha^! &: P(\mathcal{F})|U_\alpha \longrightarrow P(\mathbf{R}^q), \end{aligned}$$

where $T\mathbf{R}^q$ is the tangent bundle of $\mathbf{R}^q$ and $P(\mathbf{R}^q)$ is the linear frame bundle of $\mathbf{R}^q$.

Let θ be the canonical form on $P(\mathbf{R}^q)$. Then the local 1-forms $\{f_\alpha^!*\theta; f_\alpha \in \mathcal{F}\}$, each of which is defined on $P(\mathcal{F})|U_\alpha$, define a 1-form on $P(\mathcal{F})$. In fact, it is verified that $f_\alpha^!*\theta = f_\beta^!*\theta$ on $P(\mathcal{F})|U_\alpha \cap U_\beta$, for θ is invariant under local

diffeomorphisms, in particular under $\gamma_{\beta\alpha}^x \in \mathcal{F}$. We denote by the same letter θ the 1-form on $P(\mathcal{F})$ obtained in this manner and call it the *canonical form* on $P(\mathcal{F})$.

By pulling back the given connections ∇_α on TR^q and the corresponding connections ω_α on $P(R^q)$ by the bundle homomorphisms $f_\alpha^!$, we can define connections ∇^b and ω^b on the bundles $\nu(\mathcal{F})$ and $P(\mathcal{F})$ respectively by

$$\begin{aligned}\nabla^b &= \sum_\alpha \psi_\alpha \cdot f_\alpha^! \nabla_\alpha, \\ \omega^b &= \sum_\alpha \pi^* \psi_\alpha \cdot f_\alpha^{!*} \omega_\alpha,\end{aligned}$$

where $\{\psi_\alpha\}_\alpha$ is a partition of unity subordinate to $\{U_\alpha\}_\alpha$ and $\pi: P(\mathcal{F}) \rightarrow M$ denotes the projection.

Note that ω^b is the connection form of ∇^b . The following lemmas then illustrate the fundamental features of these connections.

LEMMA 2.1. *If $X \in C^\infty(\tau(\mathcal{F}))$ and $Y \in C^\infty(TM)$, then*

$$(2.1) \quad \nabla_X^b \rho(Y) = \rho([X, Y]),$$

where $\rho: TM \rightarrow \nu(\mathcal{F})$ denotes the canonical projection.

The proof of this lemma is straightforward and will be given in Appendix. Compare the property (2.1) with the definition of basic connections in Bott [B2].

LEMMA 2.2. *The connection ω^b is torsionfree in the sense that*

$$(2.2) \quad d\theta + [\omega^b, \theta] = 0 \quad \text{on } P(\mathcal{F}).$$

PROOF. Note that for each index α we have

$$d\theta + [\omega_\alpha, \theta] = 0 \quad \text{on } P(R^q),$$

since ω_α is torsionfree by assumption. Hence by definition of ω^b we have on $P(\mathcal{F})$

$$\begin{aligned}d\theta &= \sum_\alpha \pi^* \psi_\alpha \cdot f_\alpha^{!*} d\theta \\ &= - \sum_\alpha \pi^* \psi_\alpha \cdot f_\alpha^{!*} [\omega_\alpha, \theta] \\ &= - [\sum_\alpha \pi^* \psi_\alpha \cdot f_\alpha^{!*} \omega_\alpha, \theta] \\ &= - [\omega^b, \theta].\end{aligned} \quad \text{q.e.d.}$$

In the following, for brevity, let $\cdot|U_\alpha$ denote the restriction of the quantity $\cdot$ on $P(\mathcal{F})$ to $P(\mathcal{F})|U_\alpha$. Then the local connection forms $\omega^b|U_\alpha$ and $f_\alpha^{!*}\omega_\alpha$ on $P(\mathcal{F})|U_\alpha$ are projectively equivalent in the following sense.

LEMMA 2.3. *For each index α there exists a $\mathfrak{g}_1$ -valued function η_α :*

$P(\mathcal{F})|_{U_\alpha \rightarrow \mathfrak{g}_1}$ such that

$$(2.3) \quad \omega^b|_{U_\alpha} - f_\alpha^{!*} \omega_\alpha = [\theta|_{U_\alpha}, \eta_\alpha] \quad \text{on } P(\mathcal{F})|_{U_\alpha}.$$

PROOF. Since $\gamma_{\beta\alpha}^x \in \mathcal{F}$ is a local projective diffeomorphism, it follows from (1.7) that there exists a $\mathfrak{g}_1$ -valued function $\eta_{\alpha\beta}: P(R^q) \rightarrow \mathfrak{g}_1$ such that

$$\gamma_{\beta\alpha}^{x*} \omega_\beta = \omega_\alpha + [\theta, \eta_{\alpha\beta}].$$

Hence on $P(\mathcal{F})|_{U_\alpha \cap U_\beta}$

$$\begin{aligned} f_\beta^{!*} \omega_\beta &= f_\alpha^{!*} \circ \gamma_{\beta\alpha}^{x*} \omega_\beta \\ &= f_\alpha^{!*} \omega_\alpha + [f_\alpha^{!*} \theta, f_\alpha^{!*} \eta_{\alpha\beta}]. \end{aligned}$$

Define a $\mathfrak{g}_1$ -valued function η_α on $P(\mathcal{F})|_{U_\alpha}$ by

$$\eta_\alpha = \sum_\beta \pi^*(\psi_\beta|_{U_\alpha}) \cdot f_\alpha^{!*} \eta_{\alpha\beta}.$$

Then on $P(\mathcal{F})|_{U_\alpha}$ we have

$$\begin{aligned} \omega^b|_{U_\alpha} &= (\sum_\beta \pi^* \psi_\beta \cdot f_\beta^{!*} \omega_\beta)|_{U_\alpha} \\ &= \sum_\beta \pi^*(\psi_\beta|_{U_\alpha}) \cdot (f_\beta^{!*} \omega_\beta|_{U_\alpha}) \\ &= \sum_\beta \pi^*(\psi_\beta|_{U_\alpha}) \cdot (f_\alpha^{!*} \omega_\alpha|_{U_\alpha}) + \pi^*(\psi_\beta|_{U_\alpha}) \cdot ([f_\alpha^{!*} \theta, f_\alpha^{!*} \eta_{\alpha\beta}]|_{U_\alpha}) \\ &= f_\alpha^{!*} \omega_\alpha + [\theta|_{U_\alpha}, \eta_\alpha]. \end{aligned} \quad \text{q.e.d.}$$

With these understood, we put the following definition.

DEFINITION 2.2. Let ∇^b be a connection on $\nu(\mathcal{F})$ and ω^b be the corresponding connection on $P(\mathcal{F})$. We call ∇^b a *basic connection* on $\nu(\mathcal{F})$ and ω^b a *basic connection* on $P(\mathcal{F})$ if they enjoy the properties (2.1), (2.2) and (2.3) in Lemmas 2.1, 2.2 and 2.3.

In the above argument, we have proved

LEMMA 2.4. For each projective foliation $\mathcal{F}$, there exist basic connections ∇^b on $\nu(\mathcal{F})$ and ω^b on $P(\mathcal{F})$.

Now we can state the first variant of our strong vanishing theorem.

THEOREM 2.1 (Strong vanishing theorem I). Let $\mathcal{F} = \{U_\alpha, f_\alpha, \gamma_{\beta\alpha}^x, \nabla_\alpha\}$ be a projective foliation of codimension q on a smooth manifold M . Let ω^b be a basic connection on the normal frame bundle $P(\mathcal{F})$ of $\mathcal{F}$ and Ω^b denote the curvature form of ω^b . Then

$$(2.4) \quad f(\Omega^b) = 0$$

for all $f \in I^k(SL(q+1, \mathbf{R}), GL(q, \mathbf{R}))$ with $k > q/2$.

We give the proof of this theorem by a series of lemmas.

LEMMA 2.5. *Let $f \in I^k(SL(q+1, \mathbf{R}), GL(q, \mathbf{R}))$. Let Ω^b and Ω_α denote the curvature forms of ω^b and ω_α respectively. Then we have*

$$(2.5) \quad f(\Omega^b) = f(f_\alpha^{!*} \Omega_\alpha) \quad \text{on } P(\mathcal{F})|U_\alpha.$$

PROOF. For simplicity denote $f_\alpha^{!*} \omega_\alpha$ by ω_α^* and let Ω_α^* denote the curvature form of ω_α^* . Ω_α^* is identical with $f_\alpha^{!*} \Omega_\alpha$ by naturality of curvature forms. Hence, to prove (2.5) it suffices to show

$$(2.5)' \quad f(\Omega^b) = f(\Omega_\alpha^*) \quad \text{on } P(\mathcal{F})|U_\alpha.$$

For $t \in [0, 1]$, define $\omega_t = t\omega^b + (1-t)\omega_\alpha^*$. Then by the homotopy formula (1.12) we have

$$f(\Omega^b) - f(\Omega_\alpha^*) = d\Delta_f(\omega^b, \omega_\alpha^*),$$

where

$$\Delta_f(\omega^b, \omega_\alpha^*) = k \int_0^1 f(\omega^b - \omega_\alpha^*, \Omega_t, \dots, \Omega_t) dt$$

and Ω_t denotes the curvature form of ω_t . Then (2.5)' follows from the following lemma.

LEMMA 2.6. *Under the assumption of Lemma 2.5,*

$$(2.6) \quad f(\omega^b - \omega_\alpha^*, \Omega_t, \dots, \Omega_t) = 0 \quad \text{on } P(\mathcal{F})|U_\alpha.$$

PROOF. First note that for Ω_t we have the so-called first Bianchi identity

$$(2.7) \quad [\Omega_t, \theta] = 0.$$

In fact, from the Jacobi identity for the Lie algebra $\mathfrak{sl}(q+1, \mathbf{R}) = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$ it follows that

$$[\omega_t, [\omega_t, \theta]] + [\omega_t, [\theta, \omega_t]] + [\theta, [\omega_t, \omega_t]] = 0,$$

or

$$(2.8) \quad 2[\omega_t, [\omega_t, \theta]] + [\theta, [\omega_t, \omega_t]] = 0.$$

On the other hand, since ω^b and ω_α are torsionfree, it follows from Lemma 2.2 together with the fact

$$d\theta = -[\omega_\alpha^*, \theta]$$

that

$$d\theta = -[\omega_t, \theta].$$

Hence we obtain by exterior differentiation

$$\begin{aligned} 0 &= -[d\omega_t, \theta] + [\omega_t, d\theta] \\ &= (1/2)[[\omega_t, \omega_t], \theta] - [\Omega_t, \theta] - [\omega_t, [\omega_t, \theta]] \\ &= -[\Omega_t, \theta], \end{aligned}$$

on account of (2.8).

Denote by the same letter f an element of $I^k(SL(q+1, \mathbf{R}))$ which gives rise to $f \in I^k(SL(q+1, \mathbf{R}), GL(q, \mathbf{R}))$. Since $f \in I^k(SL(q+1, \mathbf{R}))$ is invariant by $SL(q+1, \mathbf{R})$, we have

$$\begin{aligned} &f([Y, X_1], X_2, \dots, X_k) + f(X_1, [Y, X_2], \dots, X_k) \\ &+ \dots + f(X_1, X_2, \dots, [Y, X_k]) = 0 \end{aligned}$$

for $Y, X_1, \dots, X_k \in \mathfrak{sl}(q+1, \mathbf{R})$. Then, setting $Y = \theta$, $X_1 = \eta_\alpha$ and $X_2 = \dots = X_k = \Omega_t$, we get on $P(\mathcal{F})|U_\alpha$

$$\begin{aligned} &f([\theta, \eta_\alpha], \Omega_t, \dots, \Omega_t) + f(\eta_\alpha, [\theta, \Omega_t], \dots, \Omega_t) \\ &+ \dots + f(\eta_\alpha, \Omega_t, \dots, [\theta, \Omega_t]) = 0. \end{aligned}$$

Hence by (2.7) we obtain

$$f([\theta, \eta_\alpha], \Omega_t, \dots, \Omega_t) = 0 \quad \text{on } P(\mathcal{F})|U_\alpha.$$

On the other hand, we have

$$\omega^b - \omega_\alpha^* = [\theta, \eta_\alpha] \quad \text{on } P(\mathcal{F})|U_\alpha,$$

since ω^b and ω_α^* are projectively equivalent by Lemma 2.4. (2.6) is now evident. q.e.d.

The following lemma then completes the proof of Theorem 2.1.

LEMMA 2.7. *Under the assumption of Theorem 2.1, if $f \in I^k(SL(q+1, \mathbf{R}), GL(q, \mathbf{R}))$ with $k > q/2$, then*

$$(2.4) \quad f(\Omega^b) = 0.$$

PROOF. It suffices to prove

$$f(\Omega^b) = 0 \quad \text{on } P(\mathcal{F})|U_\alpha$$

for each index α .

Since $f \in I^k(SL(q+1, \mathbf{R}), GL(q, \mathbf{R}))$, it follows from (2.5) that

$$(2.9) \quad f(\Omega^b) = f(f_a^{!*} \Omega_a) = f_a^{!*} f(\Omega_a) \quad \text{on } P(\mathcal{F})|U_a.$$

Note here that $f(\Omega^b)$ and $f(\Omega_a)$ are $GL(q, \mathbf{R})$ -invariant horizontal $2k$ -forms on $P(\mathcal{F})$ and $P(\mathbf{R}^q)$ respectively. Hence $f(\Omega^b)$ is pushed down to a $2k$ -form $\bar{f}(\Omega^b)$ on M and $f(\Omega_a)$ to a $2k$ -form $\bar{f}(\Omega_a)$ on $\mathbf{R}^q$. Then it follows from (2.9) that

$$(2.10) \quad \bar{f}(\Omega^b) = f_a^{!*} \bar{f}(\Omega_a) \quad \text{on } U_a.$$

Since $k > q/2$, $\bar{f}(\Omega_a)$ vanishes for dimensional reason, and hence by (2.10) so does $\bar{f}(\Omega^b)$. Consequently we obtain

$$f(\Omega^b) = \pi^* \bar{f}(\Omega^b) = 0 \quad \text{on } P(\mathcal{F})|U_a.$$

This completes the proof.

For $\mathcal{F}$ a projective foliation on M , let $\text{Pont}^*(\nu(\mathcal{F}))$ denote the subalgebra of $H^*(M; \mathbf{R})$ generated by the real Pontrjagin classes of $\nu(\mathcal{F})$. Then the second variant of our strong vanishing theorem may be stated in the following

THEOREM 2.2 (Strong vanishing theorem II). *Let $\mathcal{F} = \{U_\alpha, f_\alpha, \gamma_{\beta\alpha}^x, \nabla_\alpha\}$ be a projective foliation of codimension q on a smooth manifold M . Let $\text{Pont}^k(\nu(\mathcal{F}))$ denote the k -dimensional homogeneous part of $\text{Pont}^*(\nu(\mathcal{F}))$. Then*

$$(2.11) \quad \text{Pont}^k(\nu(\mathcal{F})) = 0 \quad \text{for } k > q.$$

PROOF. This follows from Theorem 2.1, the first variant, as follows.

Let ω^b be the basic connection on $P(\mathcal{F})$ and Ω^b denote the curvature form of ω^b . Take Chern polynomials $c_i \in I^i(GL(q, \mathbf{R}))$ and consider, as in the proof of Theorem 2.1, the closed $2i$ -forms $\bar{c}_i(\Omega^b)$ on M defined by $2i$ -forms $c_i(\Omega^b)$ on $P(\mathcal{F})$. Then the Chern-Weil construction of characteristic classes asserts that $\text{Pont}^*(\nu(\mathcal{F}))$ is generated by

$$p_i(\nu(\mathcal{F})) = [\bar{c}_{2i}(\Omega^b)] \in H^{4i}(M; \mathbf{R})$$

for $i = 1, \dots, [q/2]$.

Denote by the same letter c_i the i -th Chern polynomial in $I^i(SL(q+1, \mathbf{R}))$ as in § 1 and let $'c$ denote the image $r(c_i)$ under the restriction homomorphism (1.14) (with q instead of n). Then it follows from Lemma 1.1 that

$$(2.12) \quad c_k(\Omega^b) = 'c_k(\Omega^b) - \sum_{j=0}^{k-1} A_{jk} c_1(\Omega^b)^{k-j} c_j(\Omega^b),$$

where A_{jk} stands for the factor $(-1/(q+1))^{k-j} \binom{q+1-j}{k-j}$. Since $[c_i(\Omega^b)] = 0$ for odd i by (1.11), it follows from (2.12) that for each k

$$[\bar{c}_k(\Omega^b)] = ['\bar{c}_k(\Omega^b)].$$

Since $'c_k \in I^k(SL(q+1, \mathbf{R}), GL(q, \mathbf{R}))$ by definition, it is now easily verified that (2.11) is implied by (2.4) of Theorem 2.1. This completes the proof.

REMARK 2.1. It should be remarked that, after suitable modifications, the lemmas in this section are also valid for Γ -foliations studied in our previous papers [NS, NT] and in consequence provide another proof of the Main Theorem in [NT]. For the necessary modifications on the terminology and the notation, see Kobayashi-Ochiai [KO] for example. See also Abe [A] in this connection.

To close this section, we make a notational convention which will be used throughout the rest of this paper.

In general, let E be a vector bundle of rank r over a smooth manifold M and $P(E)$ be the frame bundle of E . Let ω be a connection on $P(E)$ and ∇ be the connection on E which corresponds to ω . Denote by Ω the curvature form of ω and let $f \in I^k(GL(r, \mathbf{R}))$. Then, as in the preceding argument, we obtain a $2k$ -form $f(\Omega)$ on $P(E)$ and a $2k$ -form $\bar{f}(\Omega)$ on M which is induced by $f(\Omega)$. In the following, we often identify these two forms $f(\Omega)$ and $\bar{f}(\Omega)$ without explicit reference. Thus we are free to regard Ω as the curvature matrix of ∇ with respect to a local frame of E and $f(\Omega)$ as a closed $2k$ -form on M .

§ 3. Exact sequences and connections

For later use some well-known facts with relevant modifications about connections and exact sequences of vector bundles are collected in this section. For a general treatment, see Baum-Bott [BB].

First we recall some fundamental definitions. So let M be a smooth manifold of dimension n , and set $m = [n/4]$. If E is a vector bundle of rank r over M , let $p(E)$ denote the total Pontrjagin class of E in $H^*(M; \mathbf{R})$ so that

$$p(E) = 1 + p_1(E) + \cdots + p_m(E),$$

where, if $[r/2] < m$, $p_i(E) = 0$ whenever $[r/2] < i \leq m$. Note that in the algebra $H^*(M; \mathbf{R})$, $p(E)$ is invertible.

If E_0 and E_1 are two vector bundles over M , then the total Pontrjagin class of the virtual bundle $E_0 - E_1$ is defined by

$$p(E_0 - E_1) = p(E_0)/p(E_1).$$

Thus the Pontrjagin classes $p_1(E_0 - E_1), \dots, p_m(E_0 - E_1)$ are determined by

$$\begin{aligned} p_i(E_0 - E_1) &\in H^{4i}(M; \mathbf{R}), \\ p(E_0)/p(E_1) &= 1 + p_1(E_0 - E_1) + \cdots + p_m(E_0 - E_1). \end{aligned}$$

Let $\varphi \in \mathbf{R}[Y_1, \dots, Y_m]$ be a polynomial in indeterminates $Y_1, \dots, Y_m$ with

real coefficients. We call φ *sign-symmetric*¹⁾ if φ is symmetric and also is invariant under arbitrary change of signs in the Y_i . Note that if φ is sign-symmetric, then the degree of φ is even.

Let $\varphi \in R[Y_1, \dots, Y_m]$ be a sign-symmetric homogeneous polynomial of degree $2l$, $l \leq m$. Denote by σ_i the i -th elementary symmetric function $\sigma_i(Y_1^2, \dots, Y_m^2)$ in the Y_i^2 . Then there exists a unique polynomial $\tilde{\varphi}$ such that

$$\varphi = \tilde{\varphi}(\sigma_1, \dots, \sigma_l).$$

For E a vector bundle over M , we define an element $\varphi(E) \in H^{2l}(M; R)$ by

$$\varphi(E) = \tilde{\varphi}(p_1(E), \dots, p_l(E)).$$

If y_i denote the formal roots of the total Pontrjagin class $p(E)$ of E , that is,

$$p(E) = \prod_{j=1}^m (1 + y_j^2),$$

then $\varphi(E)$ is alternatively defined by

$$\varphi(E) = \varphi(y_1, \dots, y_m).$$

In the same way we can define $\varphi(E_0 - E_1)$ for the virtual bundle $E_0 - E_1$ of two vector bundles E_0, E_1 over M .

Now, let $\mathcal{F}$ be a projective foliation of codimension q on M . As in § 2, let $\tau(\mathcal{F})$ and $\nu(\mathcal{F})$ denote the tangent and the normal bundle of $\mathcal{F}$ respectively. $\nu(\mathcal{F})$ is by definition the quotient bundle $TM/\tau(\mathcal{F})$. Thus we have the following exact sequence of vector bundles over M :

$$(3.1) \quad 0 \longrightarrow \tau(\mathcal{F}) \xrightarrow{i} TM \xrightarrow{\rho} \nu(\mathcal{F}) \longrightarrow 0,$$

where i is the inclusion of $\tau(\mathcal{F})$ into the tangent bundle TM of M and ρ denotes the canonical projection.

Recall that on $\nu(\mathcal{F})$ we have a canonical connection ∇^b , a basic connection on $\nu(\mathcal{F})$ defined in Definition 2.2. Let ∇^1 be a metric connection on $\tau(\mathcal{F})$. ∇^1 is by definition a connection which preserves (infinitesimally) a given metric on $\tau(\mathcal{F})$ (cf. Bott [B2]). From these connections ∇^1 and ∇^b we can construct on TM the direct sum connection ∇^0 of ∇^1 and ∇^b in the following manner. Choose a splitting

$$(3.2) \quad j: \nu(\mathcal{F}) \longrightarrow TM$$

of the exact sequence (3.1) and identify TM with the direct sum of the subbundles $i(\tau(\mathcal{F}))$ and $j(\nu(\mathcal{F}))$: $TM = i(\tau(\mathcal{F})) \oplus j(\nu(\mathcal{F}))$. Then ∇^0 is defined by the formula

¹⁾ Our use of the term "sign-symmetric" is different from that in [BC].

$$(3.3) \quad \nabla^0(i \circ s_1 + j \circ s_2) = i(\nabla^1 s_1) + j(\nabla^b s_2),$$

where $s_1 \in C^\infty(\tau(\mathcal{F}))$ and $s_2 \in C^\infty(\nu(\mathcal{F}))$. It should be noted that these connections $(\nabla^1, \nabla^0, \nabla^b)$ are “compatible”, in the sense of Baum-Bott [BB], with the exact sequence (3.1).

For later use, we will revise, in terms of connections $(\nabla^1, \nabla^0, \nabla^b)$, the Chern-Weil construction of characteristic classes in the following fashion.

In what follows, c_i always denotes the i -th Chern polynomial in $I^i(GL(*, \mathbf{R}))$, where the degree $*$ is determined according to the context. Note also that a splitting j of (3.2) is chosen and fixed. Thus $\nu(\mathcal{F})$ is identified with the subbundle $j(\nu(\mathcal{F}))$ of TM .

(A) Let Ω^b be the curvature form, or rather the curvature matrix, of a basic connection ∇^b . Ω^b takes values in $\mathfrak{gl}(q, \mathbf{R})$. For Ω^b , define $\tilde{\Omega}^b$ by

$$(3.4) \quad \tilde{\Omega}^b = \left(\begin{array}{c|c} \Omega^b + (2\pi c_1(\Omega^b)/(q+1))I_q & 0 \\ \hline 0 & 2\pi c_1(\Omega^b)/(q+1) \end{array} \right).$$

Note that $\tilde{\Omega}^b$ takes values in $\mathfrak{gl}(q+1, \mathbf{R})$ (cf. (1.4)). It is an easy matter (cf. Lemma 1.1) to verify that

$$c_k(\tilde{\Omega}^b) = c_k(\Omega^b) + \sum_{j=0}^{k-1} A_{jk} c_1(\Omega^b)^{k-j} c_j(\Omega^b),$$

where A_{jk} stands for the factor $(-1/(q+1))^{k-j} \binom{q+1-j}{k-j}$. Then it follows from the Chern-Weil theory that

$$(3.5) \quad dc_k(\tilde{\Omega}^b) = 0,$$

$$(3.6) \quad p_k(\nu(\mathcal{F})) = [c_{2k}(\tilde{\Omega}^b)],$$

$$(3.7) \quad [c_{2k+1}(\tilde{\Omega}^b)] = 0,$$

for $[c_1(\Omega^b)] = 0$.

(B) Let Ω^0 be the curvature form of the direct sum connection ∇^0 , which takes values in $\mathfrak{gl}(n, \mathbf{R})$. For Ω^0 , define $\tilde{\Omega}^0$ by

$$(3.8) \quad \tilde{\Omega}^0 = \left(\begin{array}{c|c} \Omega^0 + (2\pi c_1(\Omega^0)/(q+1))I_n & 0 \\ \hline 0 & 2\pi c_1(\Omega^0)/(q+1) \end{array} \right).$$

It is then verified that

$$c_k(\tilde{\Omega}^0) = c_k(\Omega^0) + \sum_{j=0}^{k-1} B_{jk} c_1(\Omega^0)^{k-j} c_j(\Omega^0),$$

where B_{jk} stands for the factor $(-1/(q+1))^{k-j} \binom{n+1-j}{k-j}$. In consequence we have

$$(3.9) \quad dc_k(\tilde{\Omega}^0) = 0,$$

$$(3.10) \quad p_k(TM) = [c_{2k}(\tilde{\Omega}^0)],$$

$$(3.11) \quad [c_{2k+1}(\tilde{\Omega}^0)] = 0.$$

(C) Let Ω^1 be the curvature form of ∇^1 . Since ∇^1 is a metric connection on $\tau(\mathcal{F})$, Ω^1 is skew-symmetric with respect to a given metric on $\tau(\mathcal{F})$. Thus Ω^1 takes values in $\mathfrak{o}(n-q)$, the Lie algebra of the orthogonal group $O(n-q)$. For Ω^1 , define $\tilde{\Omega}^1$ by

$$(3.12) \quad \tilde{\Omega}^1 = (\Omega^1 + (2\pi c_1(\Omega^0)/(q+1))I_{n-q}).$$

It is then verified that

$$c_k(\tilde{\Omega}^1) = c_k(\Omega^1) + \sum_{j=0}^{k-1} C_{jk} c_1(\Omega^0)^{k-j} c_j(\Omega^1),$$

C_{jk} denoting the factor $(-1/(q+1))^{k-j} \binom{n-q-j}{k-j}$.

In consequence we have

$$(3.13) \quad dc_k(\tilde{\Omega}^1) = 0,$$

$$(3.14) \quad p_k(\tau(\mathcal{F})) = [c_{2k}(\tilde{\Omega}^1)],$$

$$(3.15) \quad [c_{2k+1}(\tilde{\Omega}^1)] = 0.$$

(D) For Ω^0 and Ω^1 , we define $2i$ -forms $c_i(\tilde{\Omega}^0 | \tilde{\Omega}^1)$ by

$$(3.16) \quad \begin{aligned} \det(I_{n+1} - (1/2\pi)\tilde{\Omega}^0) \cdot (\det(I_{n-q} - (1/2\pi)\tilde{\Omega}^1))^{-1} \\ = 1 + c_1(\tilde{\Omega}^0 | \tilde{\Omega}^1) + \cdots + c_{2m}(\tilde{\Omega}^0 | \tilde{\Omega}^1). \end{aligned}$$

Note that it follows from (3.9) and (3.13) that each $c_i(\tilde{\Omega}^0 | \tilde{\Omega}^1)$ is a closed form.

Let $\varphi \in R[Y_1, \dots, Y_m]$ be a sign-symmetric homogeneous polynomial of degree $2l$, $l \leq m$. Set $\varphi = \tilde{\varphi}(\sigma_1, \dots, \sigma_l)$. We define $4l$ -form $\varphi(\tilde{\Omega}^0 | \tilde{\Omega}^1)$ by

$$\varphi(\tilde{\Omega}^0 | \tilde{\Omega}^1) = \tilde{\varphi}(c_2(\tilde{\Omega}^0 | \tilde{\Omega}^1), \dots, c_{2l}(\tilde{\Omega}^0 | \tilde{\Omega}^1)).$$

In the same way, for φ and Ω^b , we define $\varphi(\tilde{\Omega}^b)$ by

$$\varphi(\tilde{\Omega}^b) = \tilde{\varphi}(c_2(\tilde{\Omega}^b), \dots, c_{2l}(\tilde{\Omega}^b)).$$

Then we have the following

LEMMA 3.1. *Notations being as above,*

$$(3.17) \quad \varphi(\tilde{\Omega}^b) = \varphi(\tilde{\Omega}^0 | \tilde{\Omega}^1).$$

PROOF. To prove (3.17) it suffices to show

$$c_{2k}(\tilde{\Omega}^b) = c_{2k}(\tilde{\Omega}^0 | \tilde{\Omega}^1)$$

for each k , that is, to show

$$(3.18) \quad \det(I_{q+1} - (1/2\pi)\tilde{\Omega}^b) = \det(I_{n+1} - (1/2\pi)\tilde{\Omega}^0) \cdot (\det(I_{n-q} - (1/2\pi)\tilde{\Omega}^1))^{-1}.$$

As ∇^0 is the direct sum connection of ∇^1 and ∇^b , the curvature form Ω^0 of ∇^0 decomposes naturally into

$$\Omega^0 = \left(\begin{array}{c|c} \Omega^1 & 0 \\ \hline 0 & \Omega^b \end{array} \right).$$

Note that $c_1(\Omega^1) = 0$ and hence $c_1(\Omega^0) = c_1(\Omega^b)$, for Ω^1 takes values in $\mathfrak{o}(n-q)$. It follows from these and the definition that

$$\tilde{\Omega}^0 = \left(\begin{array}{c|c} \tilde{\Omega}^1 & 0 \\ \hline 0 & \tilde{\Omega}^b \end{array} \right),$$

from which (3.18) is immediate.

q.e.d.

§ 4. Residue existence theorem

In this section we first define, in a very primitive form, projective foliations with singularities. Our goal is to prove for these a residue existence theorem of Baum-Bott type. See [BB] for their theory. Our theorem here is just an existence theorem and follows from the strong vanishing theorem obtained in § 2. Explicit formulas for residues in terms of local information near singularities will be given later in certain cases, which will be the subject of § 5 and § 7.

Throughout this section, let M always denote a smooth manifold of dimension n . For E a vector bundle over M , E_x denotes the fiber of E over the point $x \in M$. If $f: E \rightarrow F$ is a vector-bundle homomorphism of E to F over M , we denote by $f_x: E_x \rightarrow F_x$ the linear homomorphism which f induces between the fibers over $x \in M$. Projective foliations with singularities on M are then defined in the following manner.

DEFINITION 4.1. A projective foliation $\mathcal{F}^* = \{\mathcal{F}, E, f_E, S\}$ of codimension q with singularities on M is defined by the following data:

- (i) A vector bundle E of rank $n-q$ over M and a vector-bundle homomorphism $f_E: E \rightarrow TM$ into the tangent bundle TM of M .
- (ii) A projective foliation $\mathcal{F}$ of codimension q on $M-S$ (cf. Definition 2.1).
- (iii) S is the closed subset of M where the linear homomorphism $(f_E)_x:$

$E_x \rightarrow TM_x$ induced by f_E has nontrivial kernel, that is,

$$S = \{x \in M; \text{Kernel } (f_E)_x \neq \{0\}\},$$

and on $M - S$, $f_E(E)$ coincides with the tangent bundle $\tau(\mathcal{F})$ of $\mathcal{F}$:

$$f_E(E)|_{M-S} = \tau(\mathcal{F}).$$

The set S is called the *singular set* of $\mathcal{F}^*$. Denote by $\nu(\mathcal{F}^*)$ the virtual bundle $TM - E$. $\nu(\mathcal{F}^*)$ will be referred to as the *virtual normal bundle* of $\mathcal{F}^*$. If we take the $\mathcal{F}$ in Definition 4.1 (ii) as an affine or a Riemannian foliation, then $\mathcal{F}^* = \{\mathcal{F}, E, f_E, S\}$ is called an *affine* or a *Riemannian foliation with singularities* respectively.

EXAMPLE 4.1. Let G be a Lie group and $\psi: G \times M \rightarrow M$ be a left action of G on a smooth manifold M . Denote by $\mathfrak{g}$ the Lie algebra of G and consider the trivial vector bundle E over M defined by the projection $\mathfrak{g} \times M \rightarrow M$. Identifying $\mathfrak{g}$ with the tangent space TG_e of G at the identity e , we obtain the vector-bundle homomorphism $f_E: E \rightarrow TM$ defined by

$$f_E(v, x) = d\psi_{(e, x)}(v, 0), \quad (v, 0) \in TG_e \oplus TM_x.$$

Set $F = f_E(E)$ and $S = \{x \in M; \text{Kernel } (f_E)_x \neq \{0\}\}$. Note that each $v \in \mathfrak{g}$ induces a smooth vector field $f_E(v)$ on M defined by $f_E(v)(x) = f_E(v, x)$ for $x \in M$, and the relation

$$[f_E(v), f_E(w)] = f_E([v, w])$$

is valid for $v, w \in \mathfrak{g}$. It then follows that on $M - S$, F is an involutive subbundle of TM and hence defines a smooth foliation $\mathcal{F}$ on $M - S$.

Assume that G is a compact Lie group. Then it is known that $\mathcal{F}$ is a Riemannian foliation (cf. Pasternack [P]). The quadruple $\mathcal{F}^* = \{\mathcal{F}, E, f_E, S\}$ then defines a Riemannian foliation with singular set S on M .

Let $\mathcal{F}^* = \{\mathcal{F}, E, f_E, S\}$ be a projective foliation of codimension q with singularities on M . Set $F = f_E(E)$. On $M - S$ the normal bundle $\nu(\mathcal{F})$ of the projective foliation $\mathcal{F}$ is by definition the quotient bundle TM/F . Hence on $M - S$, we have the following exact sequence:

$$(4.1) \quad 0 \longrightarrow E \xrightarrow{f_E} TM \xrightarrow{\rho} \nu(\mathcal{F}) \longrightarrow 0,$$

ρ denoting the canonical projection. Let $j: \nu(\mathcal{F}) \rightarrow TM$ be a splitting of the exact sequence (4.1). As in § 3, consider connections $(\nabla^1, \nabla^0, \nabla^b)$ compatible with the exact sequence (4.1):

$$\nabla^b = \text{a basic connection on } \nu(\mathcal{F}),$$

∇^1 = a metric connection on $E|_{M-S}$,
 ∇^0 = the direct sum connection on $TM|_{M-S}$ of ∇^1 and ∇^b
 with respect to j , which is defined by the formula (3.3)
 (with E and f_E instead of $\tau(\mathcal{F})$ and i respectively).

DEFINITION 4.2. A triple of connections $(\nabla^1, \nabla^0, \nabla^b)$ is said to be *fitted* to $\mathcal{F}^* = \{\mathcal{F}, E, f_E, S\}$ if

- (i) ∇^1 is a metric connection on E ,
- (ii) ∇^0 is a connection on TM ,
- (iii) ∇^b is a basic connection on $\nu(\mathcal{F})$ (Note that this is defined only on $M-S$), and
- (iv) there exists a closed subset V of M , called the *support* of $(\nabla^1, \nabla^0, \nabla^b)$, such that S is contained in the interior of V and on $M-V$, $(\nabla^1, \nabla^0, \nabla^b)$ is compatible with the exact sequence (4.1).

REMARK 4.1. Note that given $\mathcal{F}^* = \{\mathcal{F}, E, f_E, S\}$ and given any closed subset V of M with S contained in the interior, a partition of unity argument then shows that there always exist connections $(\nabla^1, \nabla^0, \nabla^b)$ fitted to $\mathcal{F}^*$ and with support V .

Let $(\nabla^1, \nabla^0, \nabla^b)$ be connections fitted to $\mathcal{F}^*$. Let Ω^1, Ω^0 and Ω^b denote the curvature forms of ∇^1, ∇^0 and ∇^b respectively. Denote by $c_k(\tilde{\Omega}^0|\tilde{\Omega}^1)$ the $2k$ -form defined by (3.16) in §3. Then we have the following

LEMMA 4.1. Let $p_k(\nu(\mathcal{F}^*))$ be the k -th Pontrjagin class of the virtual normal bundle $\nu(\mathcal{F}^*) = TM - E$ of $\mathcal{F}^*$. Then

$$(4.3) \quad p_k(\nu(\mathcal{F}^*)) = [c_{2k}(\tilde{\Omega}^0|\tilde{\Omega}^1)].$$

PROOF. To verify (4.3) it suffices to prove

$$[\det(I_{n+1} - (1/2\pi)\tilde{\Omega}^0)] \cdot [(\det(I_{n-q} - (1/2\pi)\tilde{\Omega}^1))^{-1}] = p(TM)p(E)^{-1}.$$

But, by virtue of (3.8)–(3.15), this is evident. q.e.d.

We are now in a position to prove the proposition which asserts that the residues we are going to define are well-defined.

PROPOSITION 4.1. Let $\mathcal{F}^* = \{\mathcal{F}, E, f_E, S\}$ be a projective foliation of codimension q with singularities on M . Let Z be a connected component of the singular set S of $\mathcal{F}^*$. Assume that Z is compact, and let U be an open set of M such that $U \cap S = Z$. Set $m = [n/4]$ where $n = \dim M$. Let $\varphi \in R[Y_1, \dots, Y_m]$ be a sign-symmetric homogeneous polynomial of degree $2l$, where $q < 4l \leq n$. Assume that connections $(\nabla^1, \nabla^0, \nabla^b)$ and $(\nabla'^1, \nabla'^0, \nabla'^b)$ are both fitted to $\mathcal{F}^*$. Denote

by $(\Omega^1, \Omega^0, \Omega^b)$ and $(\Omega^b, \Omega^0, \Omega^1)$ the corresponding curvature forms. On U consider the $4l$ -forms $\varphi(\tilde{\Omega}^0|\tilde{\Omega}^1)$ and $\varphi(\tilde{\Omega}^0|\tilde{\Omega}^1)$ defined as in § 3. Then the following hold.

- (1) Both $\varphi(\tilde{\Omega}^0|\tilde{\Omega}^1)|U$ and $\varphi(\tilde{\Omega}^0|\tilde{\Omega}^1)|U$ have compact supports.
- (2) There exists a $(4l-1)$ -form ω on U with compact support such that

$$(4.4) \quad d\omega = \varphi(\tilde{\Omega}^0|\tilde{\Omega}^1) - \varphi(\tilde{\Omega}^0|\tilde{\Omega}^1).$$

PROOF. (1) Given connections $(\nabla^1, \nabla^0, \nabla^b)$ fitted to $\mathcal{F}^*$, let V be their support. We may assume that V is compact and is contained in U . Then according to Lemma 3.1

$$\varphi(\tilde{\Omega}^b) = \varphi(\tilde{\Omega}^0|\tilde{\Omega}^1) \quad \text{on } U - V.$$

Note that there is a unique $\phi \in I^{2l}(SL(q+1, \mathbf{R}), GL(q, \mathbf{R}))$ such that $\phi(\Omega^b) = \varphi(\tilde{\Omega}^b)$ (cf. (A) in § 3). Then by Theorem 2.1

$$\varphi(\tilde{\Omega}^b) = 0 \quad \text{on } U - V,$$

since $2l > q/2$ by assumption. This proves (1) for $\varphi(\tilde{\Omega}^0|\tilde{\Omega}^1)$. The proof for $\varphi(\tilde{\Omega}^0|\tilde{\Omega}^1)$ proceeds in parallel.

(2) To prove (4.4) we may assume that on $U - V$, both $(\nabla^1, \nabla^0, \nabla^b)$ and $(\nabla^1, \nabla^0, \nabla^b)$ are compatible with the exact sequence (4.1):

$$0 \longrightarrow E \longrightarrow TM \longrightarrow \nu(\mathcal{F}) \longrightarrow 0.$$

Define $\tilde{U}, \tilde{V}, \tilde{Z}$ by

$$\begin{aligned} \tilde{U} &= U \times [0, 1], \\ \tilde{V} &= V \times [0, 1], \\ \tilde{Z} &= Z \times [0, 1]. \end{aligned}$$

Let $\pi: \tilde{U} \rightarrow U$ and $t: \tilde{U} \rightarrow [0, 1]$ be the projections. Define the inclusions $i_0: U \rightarrow \tilde{U}$ and $i_1: U \rightarrow \tilde{U}$ respectively by $i_0(x) = (x, 0)$ and $i_1(x) = (x, 1)$ for $x \in U$. On $\tilde{U}$ there are the pull-back bundles π^*E and π^*TM . On $\tilde{U} - \tilde{Z}$ there are the pull-back bundle $\pi^*\nu(\mathcal{F})$, with the pull-back connections $\pi^*\nabla^b$ and $\pi^*\nabla^b$, and the pull-back exact sequence:

$$(4.5) \quad 0 \longrightarrow \pi^*E \longrightarrow \pi^*TM \longrightarrow \pi^*\nu(\mathcal{F}) \longrightarrow 0.$$

Suppose ∇^1 (resp. ∇^1) is a metric connection on E with respect to a given metric g (resp. g), and ∇^0 (resp. ∇^0) is the direct sum connection of ∇^1 (resp. ∇^1) and ∇^b (resp. ∇^b) with respect to a given splitting j (resp. j) of (4.1).

With these data, define a triple of connections $(\nabla^1, \nabla^0, \nabla^b)$ on the pull-back bundles as follows. Let ∇^1 be a metric connection on π^*E with respect to a metric g such that $i_0^*(\nabla^1) = \nabla^1$, $i_0^*(g) = g$ and $i_1^*(\nabla^1) = \nabla^1$, $i_1^*(g) = g$, for

the set of metrics on $\pi^1 E$ is a convex set and so any two metric connections on $\pi^1 E$ are smoothly homotopic through metric connections. Let ${}^*\mathcal{V}^b$ be a connection on $\pi^1 \nu(\mathcal{F})$ defined by

$$(4.6) \quad {}^*\mathcal{V}^b = t\pi^1 \mathcal{V}^b + (1-t)\pi^1 \mathcal{V}^b.$$

On $\tilde{U} - \tilde{Z}$ choose a splitting ${}^*j: \pi^1 \nu(\mathcal{F}) \rightarrow \pi^1 TM$ of (4.5) such that ${}^*j|_{i_0(U)} = j$ and ${}^*j|_{i_1(U)} = j$, and define on $\pi^1 TM|_{\tilde{U} - \tilde{Z}}$ the direct sum connection of ${}^*\mathcal{V}^1$ and ${}^*\mathcal{V}^b$ with respect to the splitting *j . Then extend it, by a partition of unity argument, to a connection ${}^*\mathcal{V}^0$ on $\pi^1 TM$ in such a way that they agree on $\tilde{U} - \tilde{V}$, $i_0^1({}^*\mathcal{V}^0) = \mathcal{V}^0$ and $i_1^1({}^*\mathcal{V}^0) = \mathcal{V}^0$ (cf. Remark 4.1).

Denote by $({}^*\Omega^1, {}^*\Omega^0, {}^*\Omega^b)$ the corresponding curvature forms and define $\varphi({}^*\tilde{\Omega}^0|{}^*\tilde{\Omega}^1)$, a 4l-form on $\tilde{U}$, as in § 3. Then it is immediate from the definition that

$$\begin{aligned} i_0^*(\varphi({}^*\tilde{\Omega}^0|{}^*\tilde{\Omega}^1)) &= \varphi(\tilde{\Omega}^0|\tilde{\Omega}^1), \\ i_1^*(\varphi({}^*\tilde{\Omega}^0|{}^*\tilde{\Omega}^1)) &= \varphi(\tilde{\Omega}^0|\tilde{\Omega}^1). \end{aligned}$$

Hence, by the homotopy invariance of cohomology with compact supports, in order to prove (4.4) it suffices to prove

$$\varphi({}^*\tilde{\Omega}^0|{}^*\tilde{\Omega}^1) = 0 \quad \text{on } \tilde{U} - \tilde{V}.$$

Now note that on $\tilde{U} - \tilde{Z}$ the connections $({}^*\mathcal{V}^1, {}^*\mathcal{V}^0, {}^*\mathcal{V}^b)$ constructed above are compatible with the pull-back exact sequence (4.5). Hence it follows from Lemma 3.1 that

$$\varphi({}^*\tilde{\Omega}^0|{}^*\tilde{\Omega}^1) = \varphi({}^*\tilde{\Omega}^b) \quad \text{on } \tilde{U} - \tilde{V}.$$

On the other hand, by the following lemma, we have $\varphi({}^*\tilde{\Omega}^b) = 0$ on $\tilde{U} - \tilde{Z}$. This completes the proof.

LEMMA 4.2. *Under the assumption of Proposition 4.1,*

$$(4.7) \quad \varphi({}^*\tilde{\Omega}^b) = 0 \quad \text{on } \tilde{U} - \tilde{Z}.$$

PROOF. Let $\mathcal{F} = \{U_\alpha, f_\alpha, \gamma_{\beta\alpha}^x, \mathcal{V}_\alpha\}$ be the given projective foliation of codimension q on $U - Z$. Then, on $\tilde{U} - \tilde{Z}$, we have the foliation $\tilde{\mathcal{F}} = \{\pi^{-1}(U_\alpha), f_\alpha \circ \pi, \gamma_{\beta\alpha}^x, \mathcal{V}_\alpha\}$ induced from $\mathcal{F}$ by the projection $\pi: \tilde{U} \rightarrow U$. By definition, $\tilde{\mathcal{F}}$ is a projective foliation of codimension q on $\tilde{U} - \tilde{Z}$ and the normal bundle $\nu(\tilde{\mathcal{F}})$ of $\tilde{\mathcal{F}}$ is isomorphic with $\pi^1 \nu(\mathcal{F})$.

Now note that ${}^*\mathcal{V}^b$ is a basic connection on $\nu(\tilde{\mathcal{F}}) \cong \pi^1 \nu(\mathcal{F})$, which is readily seen from (4.6) and Definition 2.2. Then (4.7) follows from Theorem 2.1, since the degree of $\varphi > q/2$. q.e.d.

The setup and the notation in Proposition 4.1 will be continued in the

sequel. Furthermore, from now on we always assume that *for Z compact there exists an open set U of M such that $Z \subset U$ and Z is a deformation retract of U* . This is the case, for instance, when S is a finite union of closed submanifolds of M . (In fact, we are mainly concerned with these cases in the subsequent sections.)

Let $H_c^*(U; \mathbf{R})$ denote the cohomology of U with compact supports and real coefficients. Then there are isomorphisms:

$$H_c^j(U; \mathbf{R}) \xrightarrow{\cong} H_{n-j}(U; \mathbf{R}) \xleftarrow{\cong} H_{n-j}(Z; \mathbf{R}).$$

The isomorphism $\alpha: H_c^j(U; \mathbf{R}) \rightarrow H_{n-j}(U; \mathbf{R})$ is the usual Poincaré duality isomorphism. The isomorphism $i_*: H_{n-j}(Z; \mathbf{R}) \rightarrow H_{n-j}(U; \mathbf{R})$ is given by the inclusion $i: Z \rightarrow U$, for Z is a deformation retract of U . Thus a closed j -form on U with compact support determines an element of $H_{n-j}(Z; \mathbf{R})$. Set $\beta = i_*^{-1} \circ \alpha$.

With these understood, we put the following

DEFINITION 4.3. In the situation of Proposition 4.1, we define the φ -residue of $\mathcal{F}^*$ on Z , denoted by $\text{Res}_\varphi(\mathcal{F}^*, Z)$, to be the element

$$\beta([\varphi(\tilde{\Omega}^0 | \tilde{\Omega}^1) | U]) \in H_{n-4l}(Z; \mathbf{R}).$$

REMARKS 4.2. 1) By virtue of Proposition 4.1(2), $\text{Res}_\varphi(\mathcal{F}^*, Z)$ depends only on φ , Z and $\mathcal{F}^*$. In fact, it does not depend on the choice of connections $(\nabla^1, \nabla^0, \nabla^b)$ fitted to $\mathcal{F}^*$.

2) For $\text{Res}_\varphi(\mathcal{F}^*, Z)$ only the local structure of $\mathcal{F}^*$ near Z is relevant. In fact, $\text{Res}_\varphi(\mathcal{F}^*, Z)$ is independent of the choice of the supports V in Definition 4.2 (iv) with $Z \subset V \subset U$ (cf. Remark 4.1).

Recall that if M is compact and oriented, there is the homomorphism μ_* :

$$(4.8) \quad \mu_*: H_{n-j}(Z; \mathbf{R}) \rightarrow H^j(M; \mathbf{R}).$$

Indeed $\mu_* = \alpha \circ i_*$, where $i_*: H_{n-j}(Z; \mathbf{R}) \rightarrow H_{n-j}(M; \mathbf{R})$ is induced by the inclusion $i: Z \rightarrow M$, and $\alpha: H_{n-j}(M; \mathbf{R}) \rightarrow H^j(M; \mathbf{R})$ is the Poincaré duality isomorphism.

We then arrive at the main theorem of this section.

THEOREM 4.1 (Residue existence theorem). *Let M be an oriented smooth manifold of dimension n and $\mathcal{F}^* = \{\mathcal{F}, E, f_E, S\}$ be a projective foliation of codimension q with singularities on M . Let $\nu(\mathcal{F}^*)$ denote the virtual normal bundle of $\mathcal{F}^*$, and set $m = [n/4]$. Let $\varphi \in \mathbf{R}[Y_1, \dots, Y_m]$ be a sign-symmetric homogeneous polynomial of degree $2l$, where $q < 4l \leq n$. Let Z be a connected component of the singular set S , and assume that Z is compact. Then there exists a homology class*

$$\text{Res}_\varphi(\mathcal{F}^*, Z) \in H_{n-4l}(Z; \mathbb{R}),$$

the φ -residue of $\mathcal{F}^*$ on Z , such that

- (1) $\text{Res}_\varphi(\mathcal{F}^*, Z)$ depends only on the local structure of $\mathcal{F}$ near Z , and
- (2) if M is compact, then

$$(4.9) \quad \varphi(\nu(\mathcal{F}^*)) = \sum_Z \mu_* \text{Res}_\varphi(\mathcal{F}^*, Z).$$

In (4.9) the sum is taken over all the connected components of S . (If M is compact, then clearly every connected component of S is compact.)

PROOF. (1) From Remarks 4.2 it is readily seen that $\text{Res}_\varphi(\mathcal{F}^*, Z)$ depends only on φ , Z and the local structure of $\mathcal{F}^*$ near Z , and among the local structure of $\mathcal{F}^*$ that of $\mathcal{F}$ near Z is relevant.

(2) Let $Z_1, \dots, Z_r$ be the connected components of S . Since M is compact, each Z_i is compact. Choose open subsets $U_1, \dots, U_r$ of M such that $Z_i \subset U_i$, $U_i \cap U_j = \emptyset$ if $i \neq j$ and Z_i is a deformation retract of U_i . (Note that we have assumed that this is possible.) Set $U = \bigcup_{i=1}^r U_i$.

Let $(\nabla^1, \nabla^0, \nabla^b)$ be connections fitted to $\mathcal{F}^*$ with support contained in U , and denote by $(\Omega^1, \Omega^0, \Omega^b)$ the corresponding curvature forms. It then follows from Lemma 3.1 together with Theorem 2.1 that

$$\varphi(\tilde{\Omega}^0 | \tilde{\Omega}^1) = 0 \quad \text{on } M - U.$$

Let ω_i be the $4l$ -form on M with compact support such that

$$\omega_i|_{U_i} = \varphi(\tilde{\Omega}^0 | \tilde{\Omega}^1)|_{U_i}$$

and ω_i vanishes on $M - U_i$. Then

$$\varphi(\tilde{\Omega}^0 | \tilde{\Omega}^1) = \omega_1 + \dots + \omega_r.$$

Note that $\text{Res}_\varphi(\mathcal{F}^*, Z_i) = \beta([\omega_i])$ by definition. Then we have from definition (4.8) of μ_* that

$$(4.10) \quad \mu_* \text{Res}_\varphi(\mathcal{F}^*, Z) = [\omega_i].$$

On the other hand, according to Lemma 4.1 and the definition,

$$\varphi(\nu(\mathcal{F}^*)) = [\varphi(\tilde{\Omega}^0 | \tilde{\Omega}^1)].$$

It is now immediate from these facts that

$$\varphi(\nu(\mathcal{F}^*)) = \sum_Z \mu_* \text{Res}_\varphi(\mathcal{F}^*, Z_i).$$

This completes the proof of (4.9).

§ 5. Residue formula I

Recall that the residue existence theorem in the preceding section is just an existence theorem and does not give an explicit formula for $\text{Res}_\varphi(\mathcal{F}^*, Z)$ in terms of local information near Z . The aim of this section is thus to obtain such an explicit residue formula. We first achieve this in the simplest case (Theorem 5.1), that is, when Z is an isolated point and $\mathcal{F}^*$ is one-dimensional. Our formula contains as a special case the well-known Bott residue formula for Killing vector fields (Theorem 5.2). A more general case will be studied in § 7.

Let M be an oriented smooth manifold of dimension $n=4m$. Let $\mathcal{F}^* = \{\mathcal{F}, E, f_E, S\}$ be a projective foliation with singularities on M . Suppose that $\mathcal{F}^*$ is one-dimensional, that is, of codimension $n-1$, and the singular set S consists of isolated points, say $S = \bigcup_i p_i$. Note that for each p_i there is the usual identification $H_0(p_i; \mathbf{R}) = \mathbf{R}$.

Let $\varphi \in \mathbf{R}[Y_1, \dots, Y_m]$ be a sign-symmetric homogeneous polynomial of degree $2m$. Then from Definition 4.3 we have

$$\text{Res}_\varphi(\mathcal{F}^*, p_i) \in H_0(p_i; \mathbf{R}) = \mathbf{R}.$$

Let $(\nabla^1, \nabla^0, \nabla^b)$ be connections fitted to $\mathcal{F}^*$ with support $V = \bigcup_i V_i$. Denote by $(\Omega^1, \Omega^0, \Omega^b)$ the corresponding curvature forms. Then it follows from (4.8) and (4.10) that

$$(5.1) \quad \text{Res}_\varphi(\mathcal{F}^*, p_i) = \int_{U_i} \varphi(\tilde{\Omega}^0 | \tilde{\Omega}^1),$$

where U_i is an open set of M such that $p_i \in V_i \subset U_i$ and p_i is a deformation retract of U_i .

The remainder of this section is devoted to computing (5.1) in terms of the local structure of $\mathcal{F}^*$ around p_i .

As the first step, we remark that since the codimension q of $\mathcal{F}^*$ is $n-1$, $\varphi(\tilde{\Omega}^0 | \tilde{\Omega}^1)$, the n -form in the integrand, is reduced to a much simpler one. In fact, recall that E is a line bundle and hence the 2-forms $\Omega^1, \tilde{\Omega}^1$ under consideration take values in $\mathfrak{gl}(1; \mathbf{R})$. Let c_i denote the i -th Chern polynomial in $I(GL(n, \mathbf{R}))$ as before. Then we have

LEMMA 5.1. Define $\hat{\Omega}^0$ by

$$\hat{\Omega}^0 = (\Omega^0 + (2\pi/n)c_1(\Omega^0)I_n).$$

(1) Then $\hat{\Omega}^0$ takes values in $\mathfrak{gl}(n, \mathbf{R})$ and

$$c_k(\hat{\Omega}^0) = c_k(\Omega^0) + \sum_{j=0}^{k-1} (-1/n)^{k-j} \binom{n-j}{k-j} c_1(\Omega^0)^{k-j} c_j(\Omega^0),$$

hence, in particular,

$$\begin{aligned} dc_k(\hat{\Omega}^0) &= 0, \\ p_k(TM) &= [c_{2k}(\hat{\Omega}^0)], \\ [c_{2k+1}(\hat{\Omega}_0)] &= 0. \end{aligned}$$

$$(2) \quad c_k(\hat{\Omega}^0) = c_k(\tilde{\Omega}^0 | \tilde{\Omega}^1) \text{ for each } k.$$

PROOF. (1) is immediate (cf. § 3).

(2) By the definition of $c_k(\hat{\Omega}^0)$ and $c_k(\tilde{\Omega}^0 | \tilde{\Omega}^1)$ it suffices to prove

$$(5.2) \quad \det(I_n - (1/2\pi)\hat{\Omega}^0) = \det(I_{n+1} - (1/2\pi)\tilde{\Omega}^0) \cdot \det(1 - (1/2\pi)\tilde{\Omega}^1)^{-1}.$$

Since $(\nabla^1, \nabla^0, \nabla^b)$ is fitted to $\mathcal{F}^*$, it follows that $\Omega^1 = (0) \in \mathfrak{o}(1)$ and hence by (3.12)

$$\tilde{\Omega}^1 = ((2\pi/n)c_1(\Omega^0)) \in \mathfrak{gl}(1, \mathbf{R}).$$

Therefore from (3.8) we have

$$\tilde{\Omega}^0 = \left(\begin{array}{c|c} \hat{\Omega}^0 & 0 \\ \hline 0 & \tilde{\Omega}^1 \end{array} \right),$$

which implies (5.2).

q.e.d.

As in § 3, for φ of degree $2m$, set $\varphi = \tilde{\varphi}(\sigma_1, \dots, \sigma_m)$ and define $\varphi(\hat{\Omega}^0)$ by

$$\varphi(\hat{\Omega}^0) = \tilde{\varphi}(c_2(\hat{\Omega}^0), \dots, c_{2m}(\hat{\Omega}^0)).$$

Then, according to Lemma 5.1(2), we can replace $\varphi(\tilde{\Omega}^0 | \tilde{\Omega}^1)$ in (5.1) by a more tractable form $\varphi(\hat{\Omega}^0)$, that is,

$$\text{Res}_\varphi(\mathcal{F}^*, p_i) = \int_{U_i} \varphi(\hat{\Omega}^0).$$

Note here that there is a unique element $\hat{\varphi} \in I^{2m}(GL(n, \mathbf{R}))$ such that

$$(5.3) \quad \hat{\varphi}(\Omega^0) = \varphi(\hat{\Omega}^0).$$

In consequence (5.1) is reduced to

$$(5.4) \quad \text{Res}_\varphi(\mathcal{F}^*, p_i) = \int_{U_i} \hat{\varphi}(\Omega^0).$$

Before proceeding, we fix some notation and terminology.

On $M-S$ we have the exact sequence

$$(5.5) \quad 0 \longrightarrow E \xrightarrow{f_E} TM \xrightarrow{\rho} \nu(\mathcal{F}) \longrightarrow 0$$

of vector bundles as in (4.1). Set $F=f_E(E)$. Given a Riemannian metric g on M , let

$$(5.6) \quad j: \nu(\mathcal{F}) \longrightarrow TM$$

denote the splitting of the exact sequence (5.5) defined by g . Thus on $M-S$, j gives rise to an orthogonal decomposition $TM=F \oplus j(\nu(\mathcal{F}))$ of TM . In the following, we fix a Riemannian metric g on M and hence the splitting j of (5.6) defined by g once and for all.

Set $U=\bigcup_i U_i$. On U the line bundle E under consideration may be assumed to be trivial. So we regard $E|U$ as the trivial bundle $U \times \mathbf{R} \rightarrow U$ over U defined by the projection. Define a cross section $s: U \rightarrow U \times \mathbf{R}$ of $E|U$ by $s(x)=(x, 1)$ for $x \in U$. We may further assume that with respect to a metric on E the norm $\|s\|$ of s is one: $\|s\|=1$. Define a vector field X on U by

$$(5.7) \quad X=f_E \circ s: U \longrightarrow TM.$$

Then X is a smooth vector field along the leaves of $\mathcal{F}$ and $\text{Zero}(X)$, the set of zeros of X , coincides with the singular set S of $\mathcal{F}^*$.

Consider the Lie differentiation $\mathcal{L}_X$ with respect to X on M . $\mathcal{L}_X$ is a differential operator on the tensor fields on M which has order 0 at each zero of X . In particular, for each $p \in \text{Zero}(X)$, $\mathcal{L}_X$ induces a linear endomorphism L_p on the tangent space TM_p of M at p :

$$L_p = \mathcal{L}_X|_{TM_p}: TM_p \longrightarrow TM_p, \quad p \in \text{Zero}(X).$$

X is called nondegenerate if L_p is nonsingular at each zero p of X .

Suppose L_p is nonsingular and let $\phi \in I(GL(n, \mathbf{R}))$ be an invariant polynomial. We then define real numbers $\chi^{-1}(*L_p)$ and $\phi(L_p^*)$ in the following manner.

Let $\{e_i\}$ be a positively-oriented orthonormal basis of TM_p relative to g and $(x^i)=(x^1, \dots, x^n)$ be the normal coordinate system centered at p such that $\partial/\partial x^i=e_i$ for each i . Let $L_p=(A_{ij})$ be the matrix representation of L_p with respect to $\{e_i\}$, which is defined by $L_p(e_i)=\sum_j A_{ij}e_j$ and is a nonsingular matrix by assumption, and decompose it into the skew-symmetric part $*A=(*A_{ij})$ and the symmetric part $A^*=(A_{ij}^*)$ so that $A_{ij}=*A_{ij}+A_{ij}^*$, $*A_{ij}=-*A_{ji}$ and $A_{ij}^*=A_{ji}^*$ for each i, j . Then, for $*A=(*A_{ij})$ a skew-symmetric matrix, the Pfaffian $\text{Pf}(*A)$ of $*A$ is defined by

$$(5.8) \quad \text{Pf}(*A) = \frac{1}{2^{2m}(2m)!} \sum \varepsilon_{i_1 \dots i_{4m}} *A_{i_1 i_2} \dots *A_{i_{4m-1} i_{4m}},$$

where $\varepsilon_{i_1 \dots i_{4m}}$ is the sign of the permutation $(1, \dots, 4m) \rightarrow (i_1, \dots, i_{4m})$ and the sum is taken over all the permutations. Note that $\text{Pf}(*A)$ is the "square

root", determined by the orientation of M , of the determinant $\det (*A)$ of $*A$ so that we have $\det (*A) = (\text{Pf} (*A))^2$ (cf. [KN, vol. II]). We then define

$$(5.9) \quad \chi^{-1}(*L_p) = 2^{2m-1}(2m-1)! \text{Pf} (*A) \cdot \det (A)^{-1}.$$

Next, let $(z_j) = (z_1, \dots, z_n)$ be a local coordinate system around p defined by $z_j = \sum_i A_{ij} x^i$ and set $\|z\|^2 = \sum_j (z_j)^2$. For each $z = (z_j)$ and $A^* = (A_{ij}^*)$ a symmetric matrix, we set $A^*(z, z) = \sum_{k,l=1}^n A_{kl}^* z_k z_l$ and let $B^*(z)$ denote the matrix $(b_{ij}^*(z))$ with components

$$(5.10) \quad b_{ij}^*(z) = 2\|z\|^2 \left(\sum_{k=1}^n A_{ik}^* z_k \right) z_j - 2A^*(z, z) z_i z_j.$$

We then define $\phi(L_p^*)$, a "characteristic number" of L_p , by

$$(5.11) \quad \phi(L_p^*) = \int_{\|z\|=1} \phi(L_p - B^*(z)) v(z),$$

where $v(z)$ is the volume form of the unit sphere $\|z\|=1$:

$$v(z) = \sum_i (-1)^{i+1} dz_1 \wedge \dots \wedge dz_{i-1} \wedge z_i dz_{i+1} \wedge \dots \wedge dz_n.$$

With these conventions understood, we put the following

DEFINITION 5.1. Let $\mathcal{F}^* = \{\mathcal{F}, E, f_E, S\}$ be a projective foliation with singularities on M . Assume that $\mathcal{F}^*$ is one-dimensional and S consists of isolated points. Then the singular set S of $\mathcal{F}^*$ is said to be *nondegenerate* if the local vector field X defined by (5.7) is nondegenerate.

It is easily seen that the nondegeneracy of S in Definition 5.1 is independent of the choice of the local vector field X .

We now come to the main theorem of this section.

THEOREM 5.1. Let $\mathcal{F}^* = \{\mathcal{F}, E, f_E, S\}$ be a one-dimensional projective foliation with singularities on an oriented smooth manifold M of dimension $4m$. Suppose the singular set S consists of isolated points and is nondegenerate. Let $\varphi \in R[Y_1, \dots, Y_m]$ be a sign-symmetric homogeneous polynomial of degree $2m$. Then for each $p \in S$

$$(5.12) \quad \text{Res}_\varphi(\mathcal{F}^*, p) = \chi^{-1}(*L_p) \cdot \phi(L_p^*),$$

L_p , $\chi(*L_p)^{-1}$ and $\phi(L_p^*)$ being defined as above.

The proof of this theorem is done by the following series of lemmas.

From now on we denote p_i by p , V_i by V and U_i by U , since we are now concerned with individual $p_i \in S$. Furthermore, to distinguish from $\mathcal{V}^0$, we denote by $\mathcal{V}^B$ the direct sum connection of $\mathcal{V}^1$ and $\mathcal{V}^0$ defined on $TM|U - \{p\}$

with respect to the splitting j . Note that by the definition of ∇^B and ∇^0 we have $\nabla^B = \nabla^0$ on $TM|U - V$.

Let ω^B and ω^0 denote the connection forms of ∇^B and ∇^0 respectively. For each $t \in [0, 1]$, define the connection $\nabla_t = t\nabla^B + (1-t)\nabla^0$ on $TM|U - \{p\}$ and denote its curvature form by Ω_t . For $\hat{\phi} \in I^{2m}(GL(n, \mathbf{R}))$, define a $(4m-1)$ -form $A_{\hat{\phi}}(\omega^B, \omega^0)$ on $U - \{p\}$ by

$$A_{\hat{\phi}}(\omega^B, \omega^0) = 2m \int_0^1 \hat{\phi}(\omega^B - \omega^0, \Omega_t, \dots, \Omega_t) dt.$$

Then we have

LEMMA 5.2. (1) On $U - \{p\}$,

$$\hat{\phi}(\Omega^0) = -dA_{\hat{\phi}}(\omega^B, \omega^0).$$

(2) $A_{\hat{\phi}}(\omega^B, \omega^0) = 0$ on $U - V$.

PROOF. (1) Let Ω^B denote the curvature form of ω^B and define $\hat{\Omega}^B$ as in Lemma 5.1 (with Ω^B instead of Ω^0). Then by the homotopy formula (1.12) we have

$$dA_{\hat{\phi}}(\omega^B, \omega^0) = \hat{\phi}(\Omega^B) - \hat{\phi}(\Omega^0).$$

Hence it suffices to prove that $\hat{\phi}(\Omega^B)$ vanishes on $U - \{p\}$. Note that due to the definition Ω^B naturally decomposes into

$$\Omega^B = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \Omega^b \end{array} \right).$$

It is then immediately seen that as curvature matrices $\hat{\Omega}^B$ is conjugate to $\tilde{\Omega}^b$. Hence we have

$$\hat{\phi}(\Omega^B) = \varphi(\hat{\Omega}^B) = \varphi(\tilde{\Omega}^b) \quad \text{on } U - \{p\}.$$

But by Theorem 2.1, $\varphi(\tilde{\Omega}^b) = 0$ on $U - \{p\}$, for φ is of degree $2m$. This proves (1).

(2) Since $\omega^B - \omega^0 = 0$ on $U - V$, this is evident. q.e.d.

For $\varepsilon > 0$ small, let $B_\varepsilon(p)$ denote the ε -neighborhood of p , relative to some Riemannian metric on U . Let $S_\varepsilon(p)$ denote the boundary $\partial B_\varepsilon(p)$ of $B_\varepsilon(p)$.

In view of (5.4),

$$\text{Res}_\varphi(\mathcal{F}^*, p) = \int_U \hat{\phi}(\Omega^0).$$

Applying Stokes' formula and Lemma 5.2 (1), we then have

$$\begin{aligned}
\text{Res}_\varphi(\mathcal{F}^*, p) &= \lim_{\varepsilon \rightarrow 0} \int_{U - B_\varepsilon(p)} \hat{\phi}(\Omega^0) \\
&= - \lim_{\varepsilon \rightarrow 0} \int_{U - B_\varepsilon(p)} d\Delta_\phi(\omega^B, \omega^0) \\
&= \lim_{\varepsilon \rightarrow 0} \int_{S_\varepsilon(p)} \Delta_\phi(\omega^B, \omega^0) + \int_{\partial U} \Delta_\phi(\omega^B, \omega^0).
\end{aligned}$$

Since $\Delta_\phi(\omega^B, \omega^0) = 0$ on the boundary ∂U of U by Lemma 5.2(2), it follows that

$$(5.13) \quad \text{Res}_\varphi(\mathcal{F}^*, p) = \lim_{\varepsilon \rightarrow 0} \int_{S_\varepsilon(p)} \Delta_\phi(\omega^B, \omega^0).$$

We are now interested in computing the right-hand side of (5.13).

We define a 1-form π_1 on $U - \{p\}$ by

$$\pi_1(Y) = g(X, Y)/g(X, X)$$

for $Y \in C^\infty(TM|U - \{p\})$. According to the direct sum decomposition $TM = F \oplus j(\nu(\mathcal{F}))$, we may write $Y \in C^\infty(TM|U - \{p\})$ as

$$Y = \pi_1(Y)X + \pi_2(Y).$$

Note that $\pi_2 = j \circ \rho$ by definition. Then we have

LEMMA 5.3. *Let $Y, Z \in C^\infty(TM|U - \{p\})$. Then*

$$(5.14) \quad \nabla_Z^B Y = (Z \cdot \pi_1(Y))X + j(\nabla_Z^b \rho(Y)),$$

$$\begin{aligned}
(5.15) \quad \nabla_Z^B Y - \nabla_Z^0 Y &= \pi_1(Z)([X, Y] - \nabla_X^0 Y) \\
&\quad + \pi_1(Z)(X \cdot \pi_1(Y) - \pi_1([X, Y]))X \\
&\quad + j(\nabla_{\pi_2(Z)}^b \rho(Y)) - \nabla_{\pi_2(Z)}^0 \pi_2(Y) - \pi_1(Y) \nabla_{\pi_2(Z)}^0 X.
\end{aligned}$$

PROOF. Recall that by (5.7) $X = f_E \circ s$, where s is a cross section of $E|U$ with norm $\|s\| = 1$. Since ∇^1 is a metric connection on E , we have

$$(5.16) \quad \nabla^1 s = 0.$$

Decompose $Y \in C^\infty(TM|U - \{p\})$ into $Y = \pi_1(Y)X + \pi_2(Y)$. Then from the definition of ∇^B and (5.16) it follows that

$$\begin{aligned}
\nabla_Z^B Y &= (Z \cdot \pi_1(Y))X + \pi_1(Y) \nabla_Z^B X + \nabla_Z^B \pi_2(Y) \\
&= (Z \cdot \pi_1(Y))X + j(\nabla_Z^b \rho(Y)).
\end{aligned}$$

This proves (5.14), which illustrates ∇^B .

To prove (5.15) we first note the formula:

$$(5.17) \quad \nabla_{\pi_1(Z)X}^b \rho(Y) = \pi_1(Z)\rho([X, Y]).$$

This follows from (2.1) of Lemma 2.1, since ∇^b is a basic connection and $\pi_1(Z)X \in C^\infty(\tau(\mathcal{F}))$. Then it is easy to verify from (5.14) and (5.17) that

$$\begin{aligned}\nabla_Z^b Y &= \pi_1(Z)[X, Y] - \pi_1(Z)\pi_1([X, Y])X \\ &\quad + (Z \cdot \pi_1(Y))X + j(\nabla_{\pi_2(Z)}^b \rho(Y)).\end{aligned}$$

On the other hand, we may write

$$\begin{aligned}\nabla_Z^0 Y &= \pi_1(Z)\nabla_Z^0 Y + (\pi_2(Z) \cdot \pi_1(Y))X \\ &\quad + \pi_1(Y)\nabla_{\pi_2(Z)}^0 X + \nabla_{\pi_2(Z)}^0 \pi_2(Y).\end{aligned}$$

From these (5.15) is now immediate. q.e.d.

The following lemma, which is more or less well-known, provides a useful device for an explicit evaluation of the limit in (5.13).

LEMMA 5.4. *Let $\mathbf{R}^n$ be a Euclidean n -space with an inner product $\langle \cdot, \cdot \rangle$. Let ψ_1 be a k -form on $\mathbf{R}^n$ and ψ_2 be an $(n-k-1)$ -form on $\mathbf{R}^n - \{0\}$. Suppose there exist a nonnegative constant K and an integer $t < n-1$ such that*

$$|\psi_2(v_1, \dots, v_{n-k-1})| \leq K \|x\|^{-t}$$

for any unit tangent vectors $v_1, \dots, v_{n-k-1}$ at $x \in \mathbf{R}^n - \{0\}$. Then

$$\lim_{\varepsilon \rightarrow 0} \int_{S_\varepsilon} \psi_1 \wedge \psi_2 = 0,$$

where $S_\varepsilon = \{x \in \mathbf{R}^n; \|x\|^2 = \langle x, x \rangle = \varepsilon^2\}$.

The proof of the lemma is an easy exercise so that we omit it.

Associated with the local vector field X on U , we define a smooth function l_p on U by

$$l_p(x) = g(X(x), X(x)) \quad \text{for } x \in U.$$

Remark the following. Owing to the nondegeneracy of X the hessian of l_p at p is positive definite. Hence by a lemma of Morse there exists a local coordinate system $(y^1, \dots, y^n)$ around p such that $l_p(y) = \sum_{i=1}^n (y^i)^2$. By means of this local coordinate system, we may identify U , taking smaller if necessary, with an open set of a Euclidean n -space $\mathbf{R}^n$ with canonical inner product.

Let A_X denote the differential operator defined by

$$A_X Y = \mathcal{L}_X Y - \nabla_X^0 Y \quad \text{for } Y \in C^\infty(TM|U).$$

Note that A_X is a tensor field of type $(1, 1)$ on U and so induces a linear endomorphism $A_X(x)$ on each tangent space TM_x of M at $x \in U$.

We define a tensor field B_x of type $(1, 1)$ on $U - \{p\}$ by the formula:

$$B_x Y = (\mathcal{L}_x \pi_1)(Y) \cdot X \quad \text{for } Y \in C^\infty(TM|U - \{p\}),$$

where $\mathcal{L}_x \pi_1$ denotes the Lie derivative of the 1-form π_1 with respect to X , that is, $(\mathcal{L}_x \pi_1)(Y) = X \cdot \pi_1(Y) - \pi_1([X, Y])$. B_x also induces a linear endomorphism $B_x(x)$ on each tangent space TM_x of M at $x \in U - \{p\}$.

With these understood, we have

LEMMA 5.5. *Notations being as above,*

$$(5.18) \quad \lim_{\varepsilon \rightarrow 0} \int_{l_p = \varepsilon^2} \Delta_{\hat{\phi}}(\omega^B, \omega^0) = \lim_{\varepsilon \rightarrow 0} \int_{l_p = \varepsilon^2} \hat{\phi}(A_x + B_x)\pi_1 \wedge (d\pi_1)^{2m-1},$$

where the integral is taken over the sphere $l_p(y) = \varepsilon^2$ in U .

PROOF. Recall that by definition

$$(5.19) \quad \lim_{\varepsilon \rightarrow 0} \int_{l_p = \varepsilon^2} \Delta_{\hat{\phi}}(\omega^B, \omega^0) = \lim_{\varepsilon \rightarrow 0} \int_{l_p = \varepsilon^2} \left(2m \int_0^1 \hat{\phi}(\omega^B - \omega^0, \Omega_t, \dots, \Omega_t) dt \right).$$

In view of Lemma 5.4 and the remark before this lemma, we will study the "growth with respect to the function l_p " of $\omega^B - \omega^0$ and Ω_t around p . Set $\sigma = \omega^B - \omega^0$.

First observe the equation (5.15) and note the following: The 1-form π_1 has growth of order $l_p^{-1/2}$, for $\pi_1(Z) = (1/\|X\|)g(X/\|X\|, Z)$, and it follows from the definition of basic connections, for instance from (2.2), that the term $j(\nabla_{\pi_2(Z)}^b \rho(Y))$ has an effect of growth of order 0. Then it is not difficult to see that for σ only the term $\pi_1 \otimes (A_x + B_x)$ gives rise to growth of order $l_p^{-1/2}$.

On the other hand, it follows immediately from the definition that for each fixed $t \in [0, 1]$

$$\Omega_t = \Omega^0 + t d\sigma + t\sigma \wedge \omega^0 + t\omega^0 \wedge \sigma + t^2 \sigma \wedge \sigma,$$

from which it is easily observed that for Ω_t only the term $td\pi_1 \otimes (A_x + B_x)$ gives rise to growth of order l_p^{-1} . Combining these we finally see, by virtue of Lemma 5.4, that all the terms except

$$\begin{aligned} & 2m \int_0^1 \hat{\phi}(\pi_1 \otimes (A_x + B_x), td\pi_1 \otimes (A_x + B_x), \dots, td\pi_1 \otimes (A_x + B_x)) dt \\ & = \hat{\phi}(A_x + B_x)\pi_1 \wedge (d\pi_1)^{2m-1} \end{aligned}$$

disappear as $\varepsilon \rightarrow 0$ in the limit of (5.19). This proves (5.18). q.e.d.

Due to Lemma 5.5 and (5.13) the problem is further reduced to that of computing

$$\text{Res}_\varphi(\mathcal{F}^*, p) = \lim_{\varepsilon \rightarrow 0} \int_{l_p = \varepsilon^2} \hat{\phi}(A_X + B_X) \pi_1 \wedge (d\pi_1)^{2m-1}.$$

For this purpose, let ξ be a 1-form dual to X with respect to the Riemannian metric g : $\xi(\cdot) = g(X, \cdot)$. Then we have

$$\pi_1 = \xi / \|\xi\|^2$$

so that

$$\begin{aligned} \pi_1 \wedge (d\pi_1)^{2m-1} &= \xi \wedge (d\xi)^{2m-1} / \|\xi\|^{4m}, \\ \mathcal{L}_X \pi_1 &= (\|\xi\|^2 \mathcal{L}_X \xi - (\mathcal{L}_X(\|\xi\|^2) \xi) / \|\xi\|^4. \end{aligned}$$

At $p \in S = \text{Zero}(X)$, choose a positively-oriented orthonormal basis $\{e_i\}$ of TM_p and let $L_p = (A_{ij})$ be the matrix representation of L_p with respect to this basis as before. It should be noted that L_p is identical with $A_X(p)$ as linear endomorphisms on TM_p . Let $(x^i) = (x^1, \dots, x^n)$ be the normal coordinate system (with respect to g) centered at p such that $\partial/\partial x^i = e_i$ for each i .

We consider the Taylor expansions of X , ξ and $d\xi$ at p with respect to (x^i) . Then, since (x^i) is a normal coordinate system, we have the formulas:

$$\begin{aligned} (5.20) \quad X &= -\sum A_{ij} x^i \partial/\partial x^j + [2], \\ \xi &= -\sum A_{ij} x^i dx^j + [2], \\ d\xi &= -\sum A_{ij} dx^i \wedge dx^j + [2], \end{aligned}$$

where $[2]$ indicates terms with total degree in x and dx greater than 2. We set

$$\begin{aligned} (5.21) \quad \alpha &= -\sum A_{ij} x^i dx^j, \\ \beta &= -\sum A_{ij} dx^i \wedge dx^j, \\ z_j &= \sum A_{ij} x^i, \\ \|\mathbf{z}\|^2 &= \sum_{i=1}^n (z_i)^2. \end{aligned}$$

Note that α and β formally approximate ξ and $d\xi$ to the first order respectively, and $(z_j) = (z_1, \dots, z_n)$ defines a local coordinate system about p , for (A_{ij}) is a nonsingular matrix.

For the matrix $A = (A_{ij})$ we denote the skew-symmetric part by ${}^*A = ({}^*A_{ij})$ and the symmetric part by $A^* = (A_{ij}^*)$ respectively so that $A_{ij} = {}^*A_{ij} + A_{ij}^*$. We then set

$$A^*(z, z) = \sum_{k,l=1}^n A_{kl}^* z_k z_l,$$

and denote by $B^*(z) = (b_{ij}^*(z))$ the matrix whose components are given by (5.10):

$$b_{ij}^*(z) = 2\|z\|^2(\sum_{k=1}^n A_{ik}^* z_k) z_j - 2A^*(z, z) z_i z_j.$$

With these understood, we may state our final reduction.

LEMMA 5.6. *Notations being as above,*

$$(5.22) \quad \lim_{\varepsilon \rightarrow 0} \int_{l_p = \varepsilon^2} \hat{\phi}(A_X + B_X) \pi_1 \wedge (d\pi_1)^{2m-1} = \int_{\|z\|=1} \hat{\phi}(L_p - B^*(z)) \alpha \wedge \beta^{2m-1}.$$

PROOF. This follows from Proposition 3.1 of Bott [B4].

First note that the function l_p is a “neighborhood function of p ” in the sense of Bott. The Taylor expansion of l_p at p breaks up as

$$l_p = l_p^{(2)} + l_p^{(3)} + \dots$$

into components $l_p^{(k)}$ of degree k in x , and

$$l_p^{(2)} = \|z\|^2$$

due to (5.20) and (5.21). Set

$$C_X = \|\xi\|^4 A_X + (\|\xi\|^2 \mathcal{L}_X \xi - (\mathcal{L}_X \|\xi\|^2) \xi) \otimes X,$$

a smooth tensor field of type $(1, 1)$ on U , and define

$$\mu = \hat{\phi}(C_X) \xi \wedge (d\xi)^{2m-1},$$

a smooth $(n-1)$ -form on U . We note that the Taylor expansions of C_X and μ at p break up as

$$\begin{aligned} C_X &= C_X^{(4)} + C_X^{(5)} + \dots, \\ \mu &= \mu^{(3n)} + \mu^{(3n+1)} + \dots \end{aligned}$$

into components $C_X^{(k)}$ of degree k in x and $\mu^{(k)}$ of degree k in x and dx . Then, by a straightforward calculation, it follows from (5.10), (5.20) and (5.21) that

$$\begin{aligned} C_X^{(4)} &= \sum_{i,j} (\|z\|^4 A_{ij} - b_{ij}^*(z)) dx^i \otimes \partial/\partial x^j, \\ \mu^{(3n)} &= \hat{\phi}(C_X^{(4)}) \alpha \wedge \beta^{2m-1}. \end{aligned}$$

From these it is verified by Proposition 3.1 of Bott [B4] that

$$\begin{aligned} &\lim_{\varepsilon \rightarrow 0} \int_{l_p = \varepsilon^2} \hat{\phi}(A_X + B_X) \pi_1 \wedge (d\pi_1)^{2m-1} \\ &= \lim_{\varepsilon \rightarrow 0} \int_{l_p = \varepsilon^2} \mu / l_p^{2m} = \int_{l_p^{(2)} = 1} \mu^{(3n)} \\ &= \int_{\|z\|=1} \hat{\phi}(L_p - B^*(z)) \alpha \wedge \beta^{2m-1}. \end{aligned}$$

This completes the proof.

In consequence, by virtue of (5.22) we know that

$$(5.23) \quad \text{Res}_\varphi(\mathcal{F}^*, p) = \int_{\|z\|=1} \hat{\phi}(L_p - B^*(z)) \alpha \wedge \beta^{2m-1}.$$

We now decompose $\alpha \wedge \beta^{2m-1}$ into the terms

$$\begin{aligned} A &= (\sum * A_{ij} x^i dx^j) \wedge (\sum * A_{kl} dx^k \wedge dx^l)^{2m-1}, \\ B &= (\sum A_{ij}^* x^i dx^j) \wedge (\sum * A_{kl} dx^k \wedge dx^l)^{2m-1}, \end{aligned}$$

so that $\alpha \wedge \beta^{2m-1} = A + B$. Note that B is an exact form:

$$B = (1/2) d((\sum A_{ij}^* x^i x^j) \wedge (\sum * A_{kl} dx^k \wedge dx^l)^{2m-1}).$$

On the other hand, it follows from definition (5.8) of $\text{Pf}(*A)$ that

$$A = 2^{2m-1} (2m-1)! \text{Pf}(*A) v(x),$$

where $v(x) = \sum_i (-1)^{i+1} dx^1 \wedge \cdots \wedge dx^{i-1} \wedge x^i dx^{i+1} \wedge \cdots \wedge dx^n$. Since $v(x) = \det(A)^{-1} v(z)$, we finally conclude from (5.9), (5.11) and (5.23) that

$$\text{Res}_\varphi(\mathcal{F}^*, p) = \chi^{-1}(*L_p) \cdot \hat{\phi}(L_p^*).$$

This completes the final step and the proof of Theorem 5.1. q.e.d.

Now we combine Theorem 5.1 with Theorem 4.1. Then the following residue formula is immediate.

THEOREM 5.2 (Residue formula I). *Let M be a compact oriented smooth manifold of dimension $4m$. Let $\mathcal{F}^* = \{\mathcal{F}, E, f_E, S\}$ be a one-dimensional projective foliation with singularities on M . Assume that the singular set S consists of isolated (hence a finite number of) points and is nondegenerate. Let $\varphi \in \mathbf{R}[Y_1, \dots, Y_m]$ be a sign-symmetric homogeneous polynomial of degree $2m$. Consider the characteristic class $\varphi(\nu(\mathcal{F}^*))$ of $\nu(\mathcal{F}^*)$, the virtual normal bundle of $\mathcal{F}^*$, defined by φ . For each $p \in S$, define L_p , $\chi^{-1}(*L_p)$ and $\hat{\phi}(L_p^*)$ as in Theorem 5.1. Then*

$$(5.24) \quad \varphi(\nu(\mathcal{F}^*)) [M] = \sum_{p \in S} \text{Res}_\varphi(\mathcal{F}^*, p),$$

where $[M]$ denotes the fundamental homology class of M and $\text{Res}_\varphi(\mathcal{F}^*, p)$, the φ -residue of $\mathcal{F}^*$ at p , is given by

$$(5.25) \quad \text{Res}_\varphi(\mathcal{F}^*, p) = \chi^{-1}(*L_p) \cdot \hat{\phi}(L_p^*).$$

Remark 5.1. If, in particular, $\mathcal{F}^*$ in Theorem 5.2 is a one-dimensional Riemannian foliation with singularities, then the formula (5.25) is reduced to a much simpler one. In fact, if $\mathcal{F}^*$ is defined by a Killing vector field with

isolated zeros, then the residue formula (5.24) coincides with the well-known Bott residue formula in [B3].

To be more precise, let (M, g) be a compact oriented Riemannian manifold of dimension $4m$ and X be a Killing vector field or an infinitesimal isometry on X . Then by definition X induces a one-parameter transformation group of isometries on M and hence, as in Example 4.1, defines a one-dimensional Riemannian foliation $\mathcal{F}^* = \{\mathcal{F}, E, f_E, S\}$ with singularities on M . E is the trivial line bundle defined by the projection $M \times \mathbf{R} \rightarrow M$, and the vector-bundle homomorphism $f_E: E \rightarrow TM$ is defined by $f_E(x, r) = rX(x)$ for $(x, r) \in M \times \mathbf{R}$. Clearly $S = \text{Zero}(X)$ and S is nondegenerate. We can take as the splitting $j: \nu(\mathcal{F}) \rightarrow TM$ of (5.6) the one defined by the Riemannian metric g . Furthermore X itself is a candidate for the local vector field X of (5.7).

Let π_1 be the 1-form on $M - S$ defined by $\pi_1(Y) = g(X, Y)/g(X, X)$ for $Y \in C^\infty(TM|_{M-S})$, and at each $p \in S$ consider the linear endomorphism L_p on TM_p defined by the Lie differentiation $\mathcal{L}_X$. Then, since X is a Killing vector field on (M, g) , $\mathcal{L}_X \pi_1 = 0$ and $L_p = \mathcal{L}_X|_{TM_p}$ is skew-symmetric with respect to g : $L_p = {}^*L_p$. Hence in (5.25) we have

$$\chi^{-1}({}^*L_p) = 2^{2m-1}(2m-1)! \text{Pf}(L_p)^{-1}$$

and, noting that $\int_{\|z\|=1} \nu(z) = 2\pi^{2m}/(2m-1)!$,

$$\hat{\phi}(L_p^*) = 2\pi^{2m}/(2m-1)! \phi(L_p),$$

because $B^*(z) = 0$ and $c_i(L_p) = (1/2\pi) \text{trace } L_p = 0$ in the definition of $\hat{\phi}(L_p^*)$. On the other hand, since E is a trivial bundle,

$$\phi(\nu(\mathcal{F}^*)) = \phi(TM - E) = \phi(TM).$$

In consequence, the residue formula (5.24) reads as follows:

$$\phi(TM)[M] = \sum_{p \in \text{Zero}(X)} (2\pi)^{2m} \text{Pf}(L_p)^{-1} \cdot \phi(L_p).$$

This is nothing but the Bott residue formula for Killing vector fields with isolated zeros.

In Baum-Cheeger [BC] the formula has been extended to the case where $\text{Zero}(X)$ contains submanifolds of higher dimension.

§ 6. A reduction theorem

In this section we prove, in a very primitive form, a theorem which reduces a foliation singularity satisfying certain conditions to the pull-back via a smooth submersion of a vector field singularity. The theorem is

essentially due to Baum [Ba]. An application of this reduction theorem will be given in § 7.

Let M be a smooth manifold of dimension n and $\mathcal{F}^* = \{\mathcal{F}, E, f_E, S\}$ be a projective foliation of codimension q with singularities on M as in § 4. Set $F = f_E(E)$. Recall that on $M - S$, F is the tangent bundle $\tau(\mathcal{F})$ of $\mathcal{F}$, which is an integrable subbundle of the tangent bundle TM of M . For each $x \in M$ we denote by E_x the fiber of E over x and set $F_x = f_E(E_x)$. Let D be an imbedded disc in M , called a *section*, which is transversal to the leaves of the foliation $\mathcal{F}$. Then note that on D there is cut out a smooth foliation $\mathcal{F}$ from $\mathcal{F}$.

A point $p \in S$ will be said to be *regular* if in a neighborhood of p , S is a smooth submanifold of M and hence there is the tangent space TS_p of S at p .

With these understood, we put the following

DEFINITION 6.1. $\mathcal{F}^* = \{\mathcal{F}, E, f_E, S\}$ is said to be of *general type* if it satisfies the following conditions:

(i) (dimension condition) For each regular point $p \in S$, $\dim TS_p = r - 1$ where $r = n - q$ is the rank of E .

(ii) (tangency condition) F is tangent to S , that is, for each regular point $p \in S$, $F_p = TS_p$.

(iii) (section property condition) Let $p \in S$ be a regular point and D be an $(n - r)$ -dimensional section transversal to S at p and to the leaves of $\mathcal{F}$. Then the one-dimensional foliation $\mathcal{F}$ on $D - \{p\}$ induced from $\mathcal{F}$ is locally generated by a smooth vector field X on D with isolated zero p , that is, each leaf of $\mathcal{F}$ is an orbit of X .

For example, in the situation of Theorem 5.2, $\mathcal{F}^*$ is of general type.

REMARK 6.1. In connection with the tangency condition in Definition 6.1, the following property is known in general. If $\mathcal{F}^*$ is full, then at each regular point $p \in S$, F_p is contained in TS_p : $F_p \subset TS_p$. Here $\mathcal{F}^*$ is said to be *full* if it satisfies the condition: Let U be an open subset of M on which there is given a smooth vector field X such that $X(x) \in F_x$ for each $x \in U \cap (M - S)$. Then for all $x \in U$, $X(x) \in F_x$.

For the proof, see Baum [Ba, p. 366].

The aim of this section is to prove the following

THEOREM 6.1. Let $\mathcal{F}^* = \{\mathcal{F}, E, f_E, S\}$ be a projective foliation of codimension q with singularities on a smooth manifold M . Suppose that $\mathcal{F}^*$ is of general type and let $p \in S$ be a regular point. Then in a neighborhood of p in M , $\mathcal{F}^*$ is the "pull-back" via a smooth submersion of a one-dimensional projective foliation $\mathcal{F}^*$ with an isolated singularity. (The precise statement is given in the proof.)

PROOF. Since $\mathcal{F}^*$ is of general type and $p \in S$ is a regular point, there exist an open neighborhood U of p and a local coordinate system $w^1, \dots, w^n$ on U centered at p , that is $w^1(p) = \dots = w^n(p) = 0$, such that

$$U \cap S = \{x \in U; w^r(x) = \dots = w^n(x) = 0\},$$

where $n = \dim M$ and $r = \text{rank } E$. Furthermore, according to the tangency condition for $\mathcal{F}^*$ we have $F_x = TS_x$ for $x \in U \cap S$, since each $x \in U \cap S$ is a regular point. Hence, for each $x \in U \cap S$, $\{\partial/\partial w^1(x), \dots, \partial/\partial w^{r-1}(x)\}$ is a vector-space basis of F_x . Therefore, taking U smaller if necessary, it may be assumed that on U there exist C^∞ vector fields $v_1, \dots, v_{r-1}$ such that:

$$(6.1) \quad \begin{aligned} &\text{For each } x \in U \cap S, \quad v_i(x) = \partial/\partial w^i(x), \quad i=1, \dots, r-1; \\ &\text{For each } x \in U - S, \quad v_i(x) \in F_x, \quad i=1, \dots, r-1; \\ &\text{For each } x \in U, \quad \{v_1(x), \dots, v_{r-1}(x), \partial/\partial w^r(x), \dots, \partial/\partial w^n(x)\} \\ &\quad \text{is a vector-space basis of } TM_x. \end{aligned}$$

Given $x \in U \cap S$, define a smooth vector field V_x on U by

$$V_x = w^1(x)v_1 + w^2(x)v_2 + \dots + w^{r-1}(x)v_{r-1}.$$

For $s \in \mathbb{R}$ and $x \in U \cap S$, let $\{f_{x,s}\}$ be the flow generated by V_x . For $\varepsilon > 0$, set

$$U_\varepsilon = \{x \in U; |w^i(x)| < \varepsilon, i=1, \dots, n\}.$$

Then we may choose $\varepsilon > 0$ such that for $x \in U \cap S$ and $|s| < 2$, $\{f_{x,s}\}$ is defined on U_ε : $f_{x,s}: U_\varepsilon \rightarrow U$. Set $S_\varepsilon = U_\varepsilon \cap S$. For each $x \in S_\varepsilon$, define $\psi_x: U_\varepsilon \rightarrow U$ by $\psi_x = f_{x,1}$. It follows from (6.1) that $\psi_x(p) = x$ for each $x \in S_\varepsilon$ and that $\psi_x(q)$ is in the leaf of the foliation $\mathcal{F}$ through q provided $x \in S_\varepsilon$ and $q \in U_\varepsilon - S_\varepsilon$.

Let $D \subset U_\varepsilon$ be

$$D = \{q \in U_\varepsilon; w^1(q) = \dots = w^{r-1}(q) = 0\}.$$

For $x \in S_\varepsilon$, set $D_x = \psi_x(D)$. We then define U_p by

$$(6.2) \quad U_p = \bigcup_{x \in S_\varepsilon} D_x.$$

It may be assumed that $\varepsilon > 0$ has been chosen small enough so that the following are valid: U_p is an open subset of M . For each $x \in S_\varepsilon$, D_x intersects S_ε only at x and is transversal to S_ε . If x and y are distinct points of S_ε , then $D_x \cap D_y = \emptyset$.

Define smooth mappings $\sigma: U_p \rightarrow S$ and $\eta: U_p \rightarrow D$ as follows: For $y \in U_p$,

$\sigma(y)$ is the unique point in S_i with $y \in D_{\sigma(y)}$. Then $\eta(y)$ is determined by $\eta(y) = \psi_{\sigma(y)}^{-1}$. We then define a local coordinate system $(z^1, \dots, z^n)$ on U_p by

$$(6.3) \quad \begin{aligned} z^i(y) &= w^i(\sigma(y)), \quad i=1, \dots, r-1, \quad y \in U_p, \\ z^i(y) &= w^i(\eta(y)), \quad i=r, \dots, n, \quad y \in U_p. \end{aligned}$$

Note that the local coordinate system $(z^1, \dots, z^n)$ has the property:

$$\partial/\partial z^i(x) \in F_x \quad \text{for } i=1, \dots, r-1 \quad \text{and for all } x \in U_p.$$

Recall that D is an imbedded disc of dimension $(n-r+1)$ and, by virtue of (6.1), $D-\{p\}$ intersects the leaves of $\mathcal{F}$ transversely. Note that $\mathcal{F}$ induces on $D-\{p\}$ a one-dimensional smooth foliation $\bar{\mathcal{F}}$. The tangent bundle $\bar{F} = \tau(\bar{\mathcal{F}})$ of $\bar{\mathcal{F}}$ is described as follows: Denote by TD the tangent bundle of D and for each $x \in D-\{p\}$, set $\bar{F}_x = F_x \cap TD_x$. Then $\bar{F} = \bigcup_{x \in D-\{p\}} \bar{F}_x$. According to the section property condition for $\mathcal{F}^*$, we may assume that $\bar{\mathcal{F}}$ is generated by a C^∞ vector field $X \in C^\infty(TD)$ on D with isolated zero p .

Our first assertion is that there exist a line bundle $\bar{E}$ over D and a vector-bundle homomorphism $f_E: \bar{E} \rightarrow TD$ such that $f_E(\bar{E}|D-\{p\}) = \bar{F}$ and $f_E(\bar{E}_p) = \{0\}$. In fact, let $\bar{E}$ be a trivial line bundle $D \times \mathbb{R} \rightarrow D$ over D defined by the projection. Define a cross section $s: D \rightarrow \bar{E}$ by $s(q) = (q, 1)$ for $q \in D$. Then the vector-bundle homomorphism $f_E: \bar{E} \rightarrow TD$ is given by $f_E(s(q)) = X(q)$ for $q \in D$.

Our second assertion is that $\bar{\mathcal{F}}^* = \{\bar{\mathcal{F}}, \bar{E}, f_E, p\}$ defines a one-dimensional projective foliation on D with singularity p . To verify this, we have only to prove that $\bar{\mathcal{F}}$ is a projective foliation of codimension $n-r$ on $D-\{p\}$. So let $\mathcal{F} = \{U_\alpha, f_\alpha, \gamma_{\beta\alpha}^x, \nabla_\alpha\}$ be the defining data for the given projective foliation on $M-S$. Let $i: D \rightarrow U_p$ be the inclusion and define $\bar{U}_\alpha$ and $\bar{f}_\alpha$ by

$$(6.4) \quad \begin{aligned} \bar{U}_\alpha &= D \cap i^{-1}(U_p), \\ \bar{f}_\alpha &= f_\alpha \circ i: \bar{U}_\alpha \longrightarrow (R^{n-r}, \nabla_\alpha). \end{aligned}$$

Then $\bar{f}_\alpha$ is a smooth submersion, for $i: D \rightarrow U_p$ is transversal to the leaves of $\mathcal{F}$. Moreover, for each $x \in \bar{U}_\alpha \cap \bar{U}_\beta$, we have

$$(6.5) \quad \bar{f}_\beta = f_\beta \circ i = \gamma_{\beta\alpha}^x \circ f_\alpha \circ i = \gamma_{\beta\alpha}^x \circ \bar{f}_\alpha$$

around x . It is also clear that the leaves of $\bar{\mathcal{F}}$ are obtained by glueing together the fibers of each submersion $\bar{f}_\alpha$. Combining these together, it follows that $\bar{\mathcal{F}} = \{\bar{U}_\alpha, \bar{f}_\alpha, \gamma_{\beta\alpha}^x, \nabla_\alpha\}$ is a projective foliation of codimension $n-r$ on $D-\{p\}$.

Define a smooth submersion

$$(6.6) \quad f_\pi: U_p \longrightarrow D$$

by $f_\pi(q) = (z^r(q), \dots, z^n(q))$ for $q \in U_p$. We are now in a position to prove that on U_p , $\mathcal{F}^*$ is the pull-back via the submersion f_π of $\mathcal{F}^*$ in the following sense. First decompose the tangent bundle $TM|_{U_p}$ into $TM|_{U_p} = J_1 \oplus J_2$, where J_1 is the subbundle spanned by $\{\partial/\partial z^1, \dots, \partial/\partial z^{r-1}\}$ and J_2 is the one spanned by $\{\partial/\partial z^r, \dots, \partial/\partial z^n\}$. Then J_2 is naturally isomorphic to the pull-back bundle $f_\pi^! TD$ of TD by f_π and hence a vector-bundle homomorphism

$$f_\pi^!: TM|_{U_p} \longrightarrow TD$$

is naturally defined. Choose a frame $\{e_1, \dots, e_r\}$ of $E|_{U_p}$ such that $f_E(e_i) = \partial/\partial z^i$ for $i=1, \dots, r-1$, and decompose $E|_{U_p}$ into $E|_{U_p} = E_1 \oplus E_2$, where E_1 is the subbundle spanned by $\{e_1, \dots, e_{r-1}\}$ and E_2 is the sub-line bundle spanned by $\{e_r\}$. Then it follows immediately that a vector-bundle homomorphism

$$f_\pi^!: E|_{U_p} \longrightarrow \bar{E}$$

is naturally (and well-) defined in such a way that the following diagram is commutative:

$$(6.7) \quad \begin{array}{ccc} E|_{U_p} & \xrightarrow{f_E} & TM|_{U_p} \\ f_\pi^! \downarrow & & \downarrow f_\pi^! \\ \bar{E} & \xrightarrow{f_{\bar{E}}} & TD \end{array}$$

This completes our final assertion that on U_p , $\mathcal{F}^* = \{\mathcal{F}, E, f_E, S\}$ is the pull-back via the smooth submersion $f_\pi: U_p \rightarrow D$ of the one-dimensional projective foliation $\mathcal{F}^* = \{\mathcal{F}, \bar{E}, f_{\bar{E}}, p\}$ on D with an isolated singularity p . q.e.d.

§ 7. Residue formula II

In this section we combine, following Baum and Bott [BB], Theorem 6.1 with Theorem 4.1 and Theorem 5.1 to obtain an explicit residue formula for projective foliations with singularities $\mathcal{F}^*$ of general type (cf. Definition 6.1).

In fact, we prove the following

THEOREM 7.1 (Residue formula II). *Let M be an oriented smooth manifold of dimension n . Let $\mathcal{F}^* = \{\mathcal{F}, E, f_E, S\}$ be a projective foliation of codimension $n-r$ with singularities on M . Assume that $\mathcal{F}^*$ is of general type*

and each connected component Z_i of the singular set S is a compact orientable smooth submanifold of dimension $r-1$: $S = \bigcup_i Z_i$. If $n-r+1=4l$, then let $\varphi \in \mathbf{R}[Y_1, \dots, Y_m]$, $m=[n/4]$, be a sign-symmetric homogeneous polynomial of degree $2l$. Then the following hold.

(1) The φ -residue $\text{Res}_\varphi(\mathcal{F}^*, Z_i) \in H_{r-1}(Z_i; \mathbf{R})$ of $\mathcal{F}^*$ on Z_i is given by

$$(7.1) \quad \text{Res}_\varphi(\mathcal{F}^*, Z_i) = \text{Res}_\varphi(\mathcal{F}^*, p_i)[Z_i],$$

where p_i is a point of Z_i , $\mathcal{F}^*$ is a one-dimensional projective foliation with an isolated singularity p_i as in Theorem 6.1 and $[Z_i]$ denotes the fundamental homology class of Z_i .

(2) If M is compact, then

$$(7.2) \quad \varphi(\nu(\mathcal{F}^*)) = \sum_i \mu_* \text{Res}_\varphi(\mathcal{F}^*, Z_i),$$

notations being as in Theorem 4.1.

REMARKS 7.1. 1) Note that since $\mathcal{F}^*$ is of general type, it is assumed in Definition 6.1 that the dimension of S is one less than the dimension of the leaves of $\mathcal{F}$.

2) The right-hand side of (7.1) can be computed explicitly by Theorem 5.1 applied to $\mathcal{F}^*$. In particular, if we suppose $r=1$, then Theorem 7.1 is reduced to Theorem 5.2 (Residue formula I).

3) It should be remarked that $\deg \varphi$, the degree of φ , is assumed in Theorem 7.1 to be $2l$ with $n-r+1=4l$, while Theorem 4.1 (Residue existence theorem) asserts that there exist φ -residues $\text{Res}_\varphi(\mathcal{F}^*, Z_i)$ for each sign-symmetric homogeneous polynomial φ with $n-r < 2 \deg \varphi \leq n$. See also § 10.

PROOF. (2) is immediate from Theorem 4.1 (2). Thus we have only to prove (1).

Since each $[Z_i] \in H_{r-1}(Z_i; \mathbf{R}) \cong \mathbf{R}$ is a generator, there exist real numbers $\lambda_i \in \mathbf{R}$ such that

$$(7.3) \quad \text{Res}_\varphi(\mathcal{F}^*, p_i) = \lambda_i [Z_i].$$

Hence to verify (7.1) it suffices to prove

$$(7.4) \quad \lambda_i = \text{Res}_\varphi(\mathcal{F}^*, p_i).$$

In what follows we omit the index i for Z_i , λ_i , p_i and so forth, as we are mainly concerned with each individual component Z_i of S .

Let U be an open subset of M such that $Z \subset U$ and Z is a deformation retract of U . As in § 4, on $U-Z$ we have the exact sequence:

$$(7.5) \quad 0 \longrightarrow E \xrightarrow{f_E} TM \xrightarrow{\rho} \nu(\mathcal{F}) \longrightarrow 0.$$

Let $(\nabla^1, \nabla^0, \nabla^b)$ be a triple of connections fitted to $\mathcal{F}^*$ with support V , sufficiently small, containing S (cf. Definition 4.2). Denote the corresponding curvature forms by $(\Omega^1, \Omega^0, \Omega^b)$. Define a closed $4l$ -form $\varphi(\tilde{\Omega}^0|\tilde{\Omega}^1)$ with compact support in V as in § 3. Recall then that by Definition 4.3, $\varphi(\tilde{\Omega}^0|\tilde{\Omega}^1)|U$ determines the φ -residue $\text{Res}_\varphi(\mathcal{F}^*, Z)$.

Choose a point $p \in Z$ arbitrarily and fix it. Note that p is a regular point, for Z is a submanifold of M . Let U_p be an open neighborhood of p in U defined as in (6.2) and $(z^1, \dots, z^n)$ be a local coordinate system on U_p centered at p defined as in (6.3). Let

$$D = \{x \in U_p; z^1(x) = \dots = z^{r-1}(x) = 0\}$$

be a section, an imbedded disk of dimension $(n-r+1)$ transversal to Z at p and to the leaves of $\mathcal{F}$. Then it follows from Theorem 6.1 that on D there exists a one-dimensional projective foliation $\mathcal{F}^* = \{\mathcal{F}, \bar{E}, f_{\bar{E}}, p\}$ with an isolated singularity p and on U_p , $\mathcal{F}^*$ is the pull-back of $\mathcal{F}^*$ via a smooth submersion $f_\pi: U_p \rightarrow D$ of (6.6). Let $i: D \rightarrow U$ be the inclusion. It may be assumed that U and U_p have been chosen so that i is proper. Thus i induces a homomorphism of cohomology with compact supports

$$i^*: H_c^*(U; \mathbf{R}) \rightarrow H_c^*(D; \mathbf{R}).$$

Consider the homomorphism $I_p: H_{r-1}(Z; \mathbf{R}) \rightarrow \mathbf{R}$ given by

$$H_{r-1}(Z; \mathbf{R}) \xrightarrow{\cong} H_c^{n-(r-1)}(U; \mathbf{R}) \xrightarrow{i^*} H_c^{n-(r-1)}(D; \mathbf{R}) \xrightarrow{\cong} \mathbf{R}.$$

Then from (7.3) we get

$$I_p(\text{Res}_\varphi(\mathcal{F}^*, Z)) = \lambda.$$

On the other hand, it follows from the definition of $\text{Res}_\varphi(\mathcal{F}^*, Z)$ and I_p that

$$\lambda = \int_D i^* \varphi(\tilde{\Omega}^0|\tilde{\Omega}^1).$$

Hence to prove (7.4) it suffices to show

$$(7.6) \quad \text{Res}_\varphi(\mathcal{F}^*, p) = \int_D i^* \varphi(\tilde{\Omega}^0|\tilde{\Omega}^1).$$

To this end, on D set

$$\dot{E} = i^! E, \quad \dot{T}M = i^! TM, \quad \dot{\nu} = i^! \nu(\mathcal{F}), \quad \bar{F} = f_{\bar{E}}(\bar{E}).$$

Then on $D - \{p\}$ we have an exact sequence:

$$(7.7) \quad 0 \longrightarrow \dot{E} \longrightarrow \dot{T}M \longrightarrow TD/\bar{F} \longrightarrow 0.$$

In fact, applying $i^!$ to (7.5) gives an exact sequence on $D - \{p\}$:

$$0 \longrightarrow \mathring{E} \xrightarrow{f_E} T\mathring{M} \xrightarrow{\mathring{\rho}} \mathring{\nu} \longrightarrow 0.$$

Recall now the decomposition $TM|_{U_p} = J_1 \oplus J_2$ of $TM|_{U_p}$ given in the proof of Theorem 6.1: J_1 is the subbundle spanned by $\{\partial/\partial z^1, \dots, \partial/\partial z^{r-1}\}$ and J_2 is that spanned by $\{\partial/\partial z^r, \dots, \partial/\partial z^n\}$. Set $J = i^! J_1$. Then $T\mathring{M}$ decomposes into

$$T\mathring{M} = J \oplus TD.$$

Corresponding to this decomposition, $\mathring{F} = f_E^*(\mathring{E})$ is decomposed into

$$\mathring{F} = J \oplus \bar{F}.$$

These direct sum decompositions then give rise to a projection

$$T\mathring{M} \longrightarrow TD \longrightarrow TD/\bar{F}.$$

Combining this with the above exact sequence, we obtain the desired exact sequence (7.7). Note also that $\mathring{\nu}$ is then naturally isomorphic with $T\mathring{M}/\mathring{F}$ as well as with $TD/\bar{F}$. Now let $D^1 = i^! V^1$ and $D^0 = i^! V^0$. Denote their curvature forms respectively by K^1 and K^0 , and define a closed $4l$ -form $\varphi(\tilde{K}^0|\tilde{K}^1)$ on D as in § 3. Then (7.6) will follow from

$$(7.8) \quad \text{Res}_p(\mathcal{F}^*, p) = \int_D \varphi(\tilde{K}^0|\tilde{K}^1).$$

In fact, note that $K^0 = i^* \Omega^0$ and $K^1 = i^* \Omega^1$, and hence

$$i^* \varphi(\tilde{\Omega}^0|\tilde{\Omega}^1) = \varphi(\tilde{K}^0|\tilde{K}^1).$$

To prove (7.8), consider the diagram

$$(7.9) \quad \begin{array}{ccccccc} & 0 & & 0 & & 0 & \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 & \longrightarrow & J & \xrightarrow{\cong} & J & \longrightarrow & 0 \longrightarrow 0 \\ & & \downarrow \iota_1 & & \downarrow \iota_0 & & \downarrow \\ 0 & \longrightarrow & \mathring{E} & \xrightarrow{f_E} & T\mathring{M} & \xrightarrow{\mathring{\rho}} & T\mathring{M}/\mathring{F} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \cong \\ 0 & \longrightarrow & \bar{E} & \xrightarrow{f_E} & TD & \xrightarrow{\bar{\rho}} & TD/\bar{F} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

of smooth vector bundles satisfying the following conditions:

- (a) The left column $0 \rightarrow J \xrightarrow{\iota_1} \mathring{E} \rightarrow \bar{E} \rightarrow 0$ is defined and exact on D (cf. (6.7)).
- (b) The middle column $0 \rightarrow J \xrightarrow{\iota_0} T\mathring{M} \rightarrow TD \rightarrow 0$ is defined and exact on D (cf. (6.7)).
- (c) The right column $0 \rightarrow T\mathring{M}/\mathring{F} \xrightarrow{\cong} TD/\bar{F} \rightarrow 0$ is defined and exact on $D - \{p\}$.
- (d) The top row $0 \rightarrow J \xrightarrow{\cong} J \rightarrow 0$ is defined and exact on D .
- (e) The middle row $0 \rightarrow \mathring{E} \rightarrow T\mathring{M} \rightarrow T\mathring{M}/\mathring{F} \rightarrow 0$ is defined and exact on $D - \{p\}$.
- (f) The bottom row $0 \rightarrow \bar{E} \rightarrow TD \rightarrow TD/\bar{F} \rightarrow 0$ is defined and exact on $D - \{p\}$.
- (g) The diagram is commutative on $D - \{p\}$.

Let ∇ be a metric connection on J . We may assume that the top row isomorphism $J \rightarrow J$ preserves ∇ . Let $(\nabla^1, \nabla^0, \nabla^b)$ be a triple of connections fitted to the foliation $\mathcal{F}^*$ on D : ∇^1 is a metric connection on $\bar{E}$, ∇^0 is a basic connection on $TD/\bar{F}$ and ∇^b is a connection on TD which is on $D - (D \cap V)$ the direct sum connection of ∇^1 and ∇^b . From these connections we will construct a triple of connections $(*\nabla^1, *\nabla^0, *\nabla^b)$ compatible with the exact sequence $0 \rightarrow \mathring{E} \rightarrow T\mathring{M} \rightarrow T\mathring{M}/\mathring{F} \rightarrow 0$ as follows: First note that there exist a subbundle F_1 of $\mathring{E}|_{D - \{p\}}$ and a subbundle F_0 of $T\mathring{M}|_{D - \{p\}}$ such that

$$\begin{aligned}\mathring{E}|_{D - \{p\}} &= \iota_1(J|_{D - \{p\}}) \oplus F_1, \\ T\mathring{M}|_{D - \{p\}} &= \iota_0(J|_{D - \{p\}}) \oplus F_0,\end{aligned}$$

and $f_x(F_1) \subset F_0$. Then extend F_1 and F_0 to all of D in such a way that the extensions $\tilde{F}_1$ and $\tilde{F}_0$ satisfy

$$\begin{aligned}\tilde{F}_1 &= F_1, & \tilde{F}_0 &= F_0 & \text{on } D - (D \cap V), \\ \mathring{E} &= \iota_1(J) \oplus \tilde{F}_1, & T\mathring{M} &= \iota_0(J) \oplus \tilde{F}_0.\end{aligned}$$

(See, for example, [BB, Lemma (4.45)].) Making use of these direct sum decompositions together with the left and the right column exact sequences in (7.9), define

- $*\nabla^1$ = the direct sum connection on $\mathring{E}$ of ∇ and ∇^1 ,
- $*\nabla^0$ = the direct sum connection on $T\mathring{M}$ of ∇ and ∇^0 ,
- $*\nabla^b$ = the connection on $T\mathring{M}/\mathring{F}$ induced from ∇^b .

Note that $*\nabla^1$ is a metric connection on $\mathring{E}$ with respect to a metric on $\mathring{E}$, and the triple of connections $(*\nabla^1, *\nabla^0, *\nabla^b)$ is "compatible" in the sense of Baum-Bott [BB] with the middle row exact sequence in (7.9) (cf. § 3). Denote by

$(*\Omega^1, *\Omega^0, *\Omega^b)$ the curvature forms corresponding to $(*\nabla^1, *\nabla^0, *\nabla^b)$. Define a closed $4l$ -form $\varphi(*\tilde{\Omega}^0|*\tilde{\Omega}^1)$ on D as in § 3. Then it follows that modulo an exact form with compact support in D

$$\varphi(\tilde{K}^0|\tilde{K}^1) \equiv \varphi(*\tilde{\Omega}^0|*\tilde{\Omega}^1).$$

This can be proved by the same argument with slight modifications as in the proof of Proposition 4.1. In consequence, the problem is now reduced to verifying

$$(7.10) \quad \text{Res}_\varphi(\mathcal{F}^*, p) = \int_D \varphi(*\tilde{\Omega}^0|*\tilde{\Omega}^1).$$

Now denote by $('\Omega^1, '\Omega^0, '\Omega^b)$ the curvature forms corresponding to the connections $('\nabla^1, '\nabla^0, '\nabla^b)$ mentioned above, which are fitted to $\mathcal{F}^*$. Define a closed $4l$ -form $\varphi(' \tilde{\Omega}^0|'\tilde{\Omega}^1)$ on D as in § 3, which has compact support in $D \cap V$. Since by Definition 4.3, $\varphi(' \tilde{\Omega}^0|'\tilde{\Omega}^1)$ determines $\text{Res}_\varphi(\mathcal{F}^*, p)$, (7.10) is proved by showing

$$c_i(' \tilde{\Omega}^0|'\tilde{\Omega}^1) = c_i(*\tilde{\Omega}^0|*\tilde{\Omega}^1)$$

for each Chern polynomial c_i . Thus it suffices to verify

$$(7.11) \quad \begin{aligned} & \det(I_{n-r+2} - (1/2\pi)' \tilde{\Omega}^0) \cdot (\det(I_2 - (1/2\pi)' \tilde{\Omega}^1))^{-1} \\ &= \det(I_{n+1} - (1/2\pi)*\tilde{\Omega}^0) \cdot (\det(I_{r+1} - (1/2\pi)*\tilde{\Omega}^1))^{-1}. \end{aligned}$$

To prove this, first note that the curvature form $*\Omega^0$ naturally decomposes into

$$*\Omega^0 = \left(\begin{array}{c|c} \Omega & 0 \\ \hline 0 & '\Omega^0 \end{array} \right).$$

Recall that ∇ is a metric connection on J , and hence its curvature form Ω takes values in $\mathfrak{o}(r-1)$. This implies that $c_1(\Omega) = 0$ and hence $c_1(*\Omega^0) = c_1(' \Omega^0)$. Therefore it follows from the definition of $*\tilde{\Omega}^0$ in § 3 that

$$*\tilde{\Omega}^0 = \left(\begin{array}{c|c} \Omega + (2\pi/(n-r+1))c_1(' \Omega^0)I_{r-1} & 0 \\ \hline 0 & '\tilde{\Omega}^0 \end{array} \right).$$

On the other hand, by the same reason, $*\Omega^1$ and $*\tilde{\Omega}^1$ naturally decompose respectively into

$$\begin{aligned} *\Omega^1 &= \left(\begin{array}{c|c} \Omega & 0 \\ \hline 0 & '\Omega^1 \end{array} \right), \\ *\tilde{\Omega}^1 &= \left(\begin{array}{c|c} \Omega + (2\pi/(n-r+1))c_1(' \Omega^0)I_{r-1} & 0 \\ \hline 0 & '\tilde{\Omega}^1 \end{array} \right). \end{aligned}$$

From these the relation (7.11) is now immediate. This completes the proof.

§ 8. Characteristic classes for projective foliations

Based on the strong vanishing theorem in § 2, we will define in this section characteristic classes for projective foliations. The general scheme for the construction has been developed, for instance, in Bernstein-Rozenfeld [BR], Bott-Haefliger [BH], Haefliger [H2] and Kamber-Tondeur [KT]. Our construction here follows closely Bott [B2].

A different and slightly more general construction of characteristic classes for projective foliations is given in Morita [M]. His construction makes use of Cartan connections on the prolonged normal frame bundles of these foliations, while ours does not. For Riemannian foliations, Lazarov-Pasternack [LP1] defines characteristic classes based on the strong vanishing theorem of Pasternack [P]. If one restricts our construction to Riemannian foliations, then it coincides with theirs modulo Euler class.

Our construction proceeds in the following fashion.

We denote by $\mathbf{R}[c_2, c_3, c_4, \dots, c_q]$ the polynomial algebra over $\mathbf{R}$ on the indicated variables $c_2, c_3, c_4, \dots, c_q$. We write $E(h_3, h_5, h_7, \dots, h_l)$ for the exterior algebra over $\mathbf{R}$ on the indicated variables $h_3, h_5, h_7, \dots, h_l$, where $l = 2[(q-1)/2] + 1$, the largest odd integer $\leq q$. Let PWO_q be the differential graded algebra

$$(8.1) \quad PWO_q = \mathbf{R}[c_2, c_3, c_4, \dots, c_q] / \{\varphi; \deg \varphi > q\} \otimes E(h_3, h_5, h_7, \dots, h_l),^{1)}$$

where the gradation and differential are defined as follows: Each c_i is assigned degree $2i$ and $\{\varphi; \deg \varphi > q\}$ denotes the ideal generated by the elements of degree greater than q . Each h_i is assigned degree $2i-1$. The anti-derivation d of degree 1 of PWO_q is defined on generators by

$$d(c_i \otimes 1) = 0, \quad d(1 \otimes h_i) = c_i \otimes 1.$$

For a smooth manifold M , we denote by $A^*(M; \mathbf{R})$ the de Rham complex of real-valued smooth forms on M , which is a differential graded algebra with respect to the canonical gradation and exterior differentiation d .

Let $\mathcal{F} = \{U_\alpha, f_\alpha, \gamma_{\beta\alpha}^x, \nabla_\alpha\}$ be a projective foliation of codimension q on M as in § 2. Let ∇^b be a basic connection on the normal bundle $\nu(\mathcal{F})$ of $\mathcal{F}$ (cf. Definition 2.2) and ∇^r be a metric connection on $\nu(\mathcal{F})$. Denote by ω^b and ω^r the corresponding connection forms, and by Ω^b and Ω^r the corresponding curvature forms. For each $t \in [0, 1]$, define the connection form ω_t by

$$\omega_t = t\omega^b + (1-t)\omega^r$$

1) Our definition of PWO_q is slightly different from that in [M].

and denote its curvature form by Ω_t . For an invariant polynomial $f \in I^k(GL(q, R))$, set

$$\Delta_f(\omega^b, \omega^r) = k \int_0^1 f(\omega^b - \omega^r, \Omega_t, \dots, \Omega_t) dt,$$

which is a $(2k-1)$ -form on M (cf. § 1).

Let $c_i \in I^i(GL(q, R))$ be the i -th Chern polynomial. Denote by the same letter c_i the i -th Chern polynomial in $I^i(SL(q+1, R))$ and let $'c_i$ denote the image $r(c_i)$ under the restriction homomorphism

$$r: I^i(SL(q+1, R)) \longrightarrow I^i(GL(q, R))$$

as in § 1. The relationship between c_i and $'c_i$ in $I(GL(q, R))$ is given by the formula (1.15) in Lemma 1.1 (with q instead of n). Note that $'c_1 = 0$.

For a given projective foliation $\mathcal{F}$ of codimension q on M , we define a homomorphism of graded algebras

$$\lambda_{\mathcal{F}}: R[c_2, c_3, \dots, c_q] \otimes E(h_3, h_5, \dots, h_l) \longrightarrow A^*(M; R)$$

by requiring on generators

$$\begin{aligned} \lambda_{\mathcal{F}}(c_i \otimes 1) &= 'c_i(\Omega^b), \\ \lambda_{\mathcal{F}}(1 \otimes h_i) &= \Delta_{'c_i}(\omega^b, \omega^r). \end{aligned}$$

Then the following lemma is immediate.

LEMMA 8.1. *The homomorphism $\lambda_{\mathcal{F}}$ induces a homomorphism of differential graded complexes*

$$(8.2) \quad \lambda_{\mathcal{F}}: PWO_q \longrightarrow \lambda^*(M; R).$$

In fact, note that according to the strong vanishing theorem (Theorem 2.1), $\lambda_{\mathcal{F}}$ annihilates all elements $\varphi \in R[c_2, c_3, \dots, c_q]$ with $\deg \varphi > q$. Since ∇^r is a metric connection on $\nu(\mathcal{F})$ and hence $'c_i(\Omega^r) = 0$ whenever i is odd, the fact that $\lambda_{\mathcal{F}}$ commutes with the differentials is a direct consequence of the homotopy formula

$$d\Delta_{'c_i}(\omega^b, \omega^r) = 'c_i(\Omega^b) - 'c_i(\Omega^r).$$

In view of Lemma 8.1, we now come to

DEFINITION 8.1. The homomorphism in cohomology

$$(8.3) \quad \lambda_{\mathcal{F}}^*: H^*(PWO_q) \longrightarrow H^*(M; R)$$

induced from $\lambda_{\mathcal{F}}$ of (8.2) is called the *characteristic classes for projective foliation $\mathcal{F}$* .

REMARKS 8.1. 1) Under the homomorphism $\lambda_{\mathcal{F}}^*$, each class $[c_{2i}] \in H^*(PWO_q)$ determined by c_{2i} is mapped into the i -th Pontrjagin class $p_i(\nu(\mathcal{F}))$ of $\nu(\mathcal{F})$: $\lambda_{\mathcal{F}}^*([c_{2i}]) = p_i(\nu(\mathcal{F}))$.

2) $\lambda_{\mathcal{F}}^*$ depends only on $\mathcal{F} = \{U_a, f_a, \gamma_{\beta a}^x, V_a\}$. In fact, recall that if V^b and V^b are basic connections then $tV^b + (1-t)V^b$ is also a basic connection for each $t \in [0, 1]$, and any two metric connections are homotopic through metric connections. Then, by a routine argument (see, for example, [B2, Proposition (10.13)]) based on the homotopy invariance of the de Rham cohomology $H^*(M; \mathbb{R})$ we can prove:

LEMMA 8.2. *The homomorphism $\lambda_{\mathcal{F}}^*$ of (8.3) is independent of the choices of a metric connection V^r and a basic connection V^b on $\nu(\mathcal{F})$.*

By the same analysis first done by Vey (see [He 1, G] for example), we can determine an additive basis of the cohomology $H^*(PWO_q)$ of the differential graded complex PWO_q as follows. Let $I = (i_1, i_2, \dots, i_s)$ and $J = (j_1, j_2, \dots, j_t)$ be s - and t -tuples of positive integers with $i_1 \leq i_2 \leq \dots \leq i_s$ and $j_1 < j_2 < \dots < j_t$. Set $|I| = i_1 + i_2 + \dots + i_s$. We define an element $c_I h_J$ of PWO_q by

$$c_I h_J = c_{i_1} c_{i_2} \dots c_{i_s} \otimes h_{j_1} \wedge h_{j_2} \wedge \dots h_{j_t}.$$

Then we have the following

THEOREM 8.1. *A basis of $H^*(PWO_q)$ is given by the classes determined by the elements $c_I h_J \in PWO_q$ satisfying one of the following conditions:*

- (1) $2(j_1 + |I|) > q$ and $j_1 \leq i_1$, where i_1 is the smallest odd integer in I .
- (2) $J = \emptyset$ and $i_1, \dots, i_s$ are all even integers.

For the proof, see [He 1, G, KT]. See also Morita [M]. The condition $2(j_1 + |I|) > q$ implies that $c_I h_J$ is a cocycle, while the condition $j_1 \leq i_1$ assures that the $c_I h_J$ are linearly independent.

The cocycles listed in Theorem 8.1 are referred to as the *basic cocycles*. As in the case of characteristic classes for smooth foliations (see Thurston [T]), the most remarkable feature of our characteristic classes is their behavior under deformations of projective foliations. In this respect, we will show in § 9 some of these classes can indeed vary continuously. In contrast, in this section we prove the rigidity of those classes in $H^*(PWO_q)$ with cohomology dimension greater than $q+2$.

To this end, we first define a differentiable one-parameter family of projective foliations in the following fashion. For a smooth manifold M and the closed unit interval $[0, 1]$, set $\tilde{M} = M \times [0, 1]$ and let $\pi: \tilde{M} \rightarrow M$, $t: \tilde{M} \rightarrow [0, 1]$ be the projections. Then we have on $\tilde{M}$ the pull-back bundle $\pi^* TM$ of the tangent bundle TM of M by π . For each $u \in [0, 1]$, define the inclusion $i_u: M \rightarrow$

$\tilde{M}$ by $i_u(x)=(x, u)$ for $x \in M$. Moreover, by the same letter π and i_u denote respectively the projection $\pi: \mathbf{R}^q \times [0, 1] \rightarrow \mathbf{R}^q$ and the inclusion $i_u: \mathbf{R}^q \rightarrow \mathbf{R}^q \times [0, 1]$ defined by $i_u(x)=(x, u)$ for $x \in \mathbf{R}^q$ and $u \in [0, 1]$. On $\mathbf{R}^q \times [0, 1]$ there is the pull-back bundle $\pi^!TR^q$ of the tangent bundle TR^q of $\mathbf{R}^q$ by π .

DEFINITION 8.2. A differentiable one-parameter family $\{\mathcal{F}_u\}$, $u \in [0, 1]$, of projective foliations of codimension q on M is defined by the following data:

- (i) For each $u \in [0, 1]$, $\mathcal{F}_u = \{U_\alpha, f_\alpha^u, \tau_{\beta\alpha}^{xu}, \nabla_\alpha^u\}$ is a projective foliation of codimension q on M (cf. Definition 2.1).
- (ii) For the tangent bundle $\tau(\mathcal{F}_u)$ of each $\mathcal{F}_u$, there exists a smooth subbundle $\tilde{F}$ of $\pi^!TM$ with $i_u^!(\tilde{F}) = \tau(\mathcal{F}_u)$ for each $u \in [0, 1]$.
- (iii) For the connections ∇_α^u on TR^q , there exists a connection D_α on $\pi^!(TR^q)$ with $i_u^!D_\alpha = \nabla_\alpha^u$ for each $u \in [0, 1]$.

REMARKS 8.2. 1) Due to the condition (ii) of Definition 8.2, the smooth submersions $f_\alpha^u: U_\alpha \rightarrow (\mathbf{R}^q, \nabla_\alpha^u)$ fit together to define a smooth submersion

$$\tilde{f}_\alpha: U_\alpha \times [0, 1] \rightarrow \mathbf{R}^q \times [0, 1]$$

with $\tilde{f}_\alpha \circ i_u = f_\alpha^u$ for each $u \in [0, 1]$.

2) Consider the quotient bundle $\tilde{\nu} = \pi^!(TM)/\tilde{F}$ on $\tilde{M}$. Then for each $u \in [0, 1]$ we have $i_u^!(\tilde{\nu}) = \nu(\mathcal{F}_u)$. In particular, all the normal bundles are isomorphic. Furthermore due to the condition (iii) of Definition 8.2 we have, by a partition of unity argument, a basic connection ∇_u^b on each normal bundle $\nu(\mathcal{F}_u)$ of $\mathcal{F}_u$ and a connection ${}^*\nabla^b$ on $\tilde{\nu}$, referred to as a basic connection on $\tilde{\nu}$, such that

$$i_u^!({}^*\nabla^b) = \nabla_u^b \quad \text{for each } u \in [0, 1].$$

With these understood, we have the following theorem which has been remarked also by Morita [M].

THEOREM 8.2. Let $\mathcal{F}_u$, $u \in [0, 1]$, be a differentiable one-parameter family of projective foliations of codimension q on M . Let $c_I h_J \in PWO_q$ be a basic cocycle. Then the cohomology class $[c_I h_J] \in H^*(PWO_q)$ is rigid in the sense that

$$\lambda_{\mathcal{F}_0}^*([c_I h_J]) = \lambda_{\mathcal{F}_1}^*([c_I h_J])$$

if $2(j_1 + |I|) > q + 2$ or $J = \phi$.

PROOF. We may assume without loss of generality that we have one normal bundle $\nu(\mathcal{F}_0)$, one metric connection ∇^r on $\nu(\mathcal{F}_0)$ and a differentiable family of basic connections ∇_u^b on $\nu(\mathcal{F}_0)$, for all the normal bundles $\nu(\mathcal{F}_u)$ are isomorphic. Denote by ω^r and ω_u^b the connection forms of ∇^r and ∇_u^b respec-

tively. For each $u \in [0, 1]$, set $\omega_u^t = t\omega_u^b + (1-t)\omega^r$ and $\Omega_u^t = d\omega_u^t + (1/2)[\omega_u^t, \omega_u^t]$. For $I = (i_1, \dots, i_s)$ we denote $(i_1, \dots, \hat{i}_p, \dots, i_s)$, i_p being deleted, by I_p . J_p stands for the same meaning for $J = (j_1, \dots, j_t)$. Then by a formula of Heitsch [H1] we have modulo exact forms

$$\begin{aligned}
 (8.4) \quad & \partial/\partial u(\lambda_{\mathcal{F}_u}(c_I h_J)) \\
 & \equiv \sum_{p=1}^t j_p \lambda_{\mathcal{F}_u}(c_{I_p} h_{J_p}) \wedge 'c_{j_p}(\partial/\partial u(\omega_u^b), \Omega_u^b, \dots, \Omega_u^b) \\
 & \quad \pm \sum_{p=1}^s \sum_{r=1}^t i_p \lambda_{\mathcal{F}_u}(c_{I_p} h_{J_r}) \wedge 'c_{j_r}(\Omega_u^b, \dots, \Omega_u^b) \\
 & \quad \wedge 'c_{i_p}(\partial/\partial u(\omega_u^b), \Omega_u^b, \dots, \Omega_u^b).
 \end{aligned}$$

It then follows from the strong vanishing theorem (Theorem 2.1) that on the right-hand side of (8.4) each term in the first sum = 0 if $2(|I| + j_1 - 1) > q$ and each term in the second sum = 0 if $2(|I| - 1 + j_1) > q$. Consequently, if $2(j_1 + |I|) > q + 2$ or $J = \phi$, then

$$\partial/\partial u(\lambda_{\mathcal{F}_u}(c_I h_J)) \equiv 0.$$

This completes the proof.

To close this section, we briefly recall in our context the theory of classifying spaces for Γ -foliations or more precisely for topological groupoid structures. See Bott [B2] and Haefliger [H1] for details.

First we can define, generalizing Definition 2.1 of projective foliations of codimension q , projective Haefliger structures or $PI\Gamma_q$ -structures. Secondly, we have a natural notion of homotopy between $PI\Gamma_q$ -structures. Then a general theory of Haefliger [H1] yields a space $BPI\Gamma_q$, called the *classifying space for projective foliations* of codimension q , with the universal $PI\Gamma_q$ -structure $\mathcal{H}$ in such a way that given a paracompact topological space M there is a canonical one-to-one correspondence between the set of homotopy classes of mappings of M into $BPI\Gamma_q$ and the set of homotopy classes of $PI\Gamma_q$ -structures on M .

Let $\mathcal{F}$ be a given projective foliation of codimension q on a smooth manifold M , and let $f_{\mathcal{F}}: M \rightarrow BPI\Gamma_q$ classify $\mathcal{F}$, which is defined up to homotopy so that the pull-back $f_{\mathcal{F}}^* \mathcal{H}$ of $\mathcal{H}$ by $f_{\mathcal{F}}$ is homotopic to the $PI\Gamma_q$ -structure defined by $\mathcal{F}$. Remark that the characteristic classes

$$\lambda_{\mathcal{F}}^*: H^*(PWO_q) \longrightarrow H^*(M; \mathbf{R})$$

of $\mathcal{F}$ defined in Definition 8.1 is functorial, that is, natural with respect to foliation preserving mappings. Thus it induces a universal homomorphism

$$(8.5) \quad \lambda^*: H^*(PWO_q) \longrightarrow H^*(BPI\Gamma_q; \mathbf{R})$$

which makes the diagram

$$(8.6) \quad \begin{array}{ccc} H^*(PWO_q) & \xrightarrow{\lambda^*} & H^*(BPT_q; \mathbf{R}) \\ & \searrow \lambda_{\mathcal{F}}^* & \swarrow f_{\mathcal{F}}^* \\ & H^*(M; \mathbf{R}) & \end{array}$$

commute. The elements of $H^*(PWO_q)$ may be then regarded as the universal characteristic classes for projective foliations of codimension q . The basic cocycles of type (1) of Theorem 8.1 are often referred to as *secondary* or *exotic characteristic classes*, while those of type (2) of Theorem 8.1 are the Pontrjagin classes.

§ 9. Continuous variation

The object of this section is to explain the relation between residues of projective foliations with singularities and secondary characteristic classes for projective foliations. As a consequence, in particular, we prove the following theorem for the topology (the integral homology group and the homotopy group) of the classifying space BPT_q .

THEOREM 9.1. *Let BPT_{2m-1} be the classifying space for projective foliations of codimension $2m-1$ and $\lambda^*: H^*(PWO_{2m-1}) \rightarrow H^*(BPT_{2m-1}; \mathbf{R})$ be the universal homomorphism as in § 8. Then there exist natural epimorphisms*

$$(9.1) \quad \begin{aligned} H_{2m-1}(BPT_{2m-1}; \mathbf{Z}) &\longrightarrow \mathbf{R}^{d(m)} \longrightarrow 0, \\ \pi_{2m-1}(BPT_{2m-1}) &\longrightarrow \mathbf{R}^{d(m)} \longrightarrow 0, \end{aligned}$$

where $d(m) = \dim H^{2m-1}(PWO_{2m-1})$. In particular, these groups are not finitely generated.

The basic technique for the proof is to construct a continuous variation of secondary characteristic classes $H^{2m-1}(PWO_{2m-1})$ on the family of projective foliations of codimension $2m-1$ defined by expansive affine vector fields on $\mathbf{R}^{2m}$ with a single nondegenerate zero at the origin. The same method, which is in fact based on the theory of residues of foliations with singularities, has been used in Bott [B2] and Baum-Bott [BB] for holomorphic foliations, in Heitsch [He 2] for smooth foliations and in Lazarov-Pasternack [LP 2] for Riemannian foliations. A general point of view of this method can be found in Bott [B5].

First of all, from Bott [B3] we quote the notion of projectible vector field.

DEFINITION 9.1. Let X be a smooth vector field on a smooth manifold M . X is called *projectible* if there exists a smooth 1-form α , defined on $M - \text{Zero}(X)$, such that

$$(9.2) \quad a(X)=1 \quad \text{and} \quad \mathcal{L}_X a=0,$$

$\text{Zero}(X)$ and $\mathcal{L}_X$ denoting the set of zeros of X and the Lie differentiation with respect to X respectively.

For example, given a Riemannian manifold (M, g) , a Killing vector field X on M is projectible. In fact, define a smooth 1-form a on $M - \text{Zero}(X)$ by

$$a(Y) = g(X, Y)/g(X, X) \quad \text{for } Y \in C^\infty(TM|_{M - \text{Zero}(X)}).$$

Then it is easily seen that $a(X)=1$ and $\mathcal{L}_X a=0$, for $\mathcal{L}_X g=0$ by assumption (cf. Remark 5.1).

Now let (M, ∇) be a smooth manifold with a torsionfree connection ∇ . In general, a smooth vector field X on M is called a *projective vector field* or an infinitesimal projective transformation if it generates a local one-parameter transformation group of local projective diffeomorphisms of (M, ∇) . It is then classical that a smooth vector field X is projective vector field if and only if there exists a smooth 1-form b on M satisfying

$$(9.3) \quad (\mathcal{L}_X \nabla)_Y Z = b(Y)Z + b(Z)Y$$

for arbitrary $Y, Z \in C^\infty(TM)$. This can be seen easily from the formula (1.6) or its classical reformulation (1.6)' in § 1 (see, for example, Yano [Y]).

With these understood, we have the following

THEOREM 9.2. *Let (M, ∇) be a smooth manifold with a torsionfree connection ∇ and X be a projective vector field on M . Suppose X is projectible. Then X defines a one-dimensional projective foliation $\mathcal{F}_X$, whose leaves are given by the integral curves of X , on $M - \text{Zero}(X)$.*

The proof of this theorem will be given in Appendix.

In the remainder of this section we are in particular concerned with the following situation.

Let $(\mathbf{R}^{2m}, \langle, \rangle)$ be a Euclidean $2m$ -space with the standard flat metric $\langle, \rangle$ and the canonical coordinate system (x^i) . Let S^{2m-1} be the unit sphere in $\mathbf{R}^{2m}$ with center at the origin 0 of $\mathbf{R}^{2m}$; $S^{2m-1} = \{x \in \mathbf{R}^{2m}; \langle x, x \rangle = 1\}$. Let $A = (a_{ij}) \in GL(2m, \mathbf{R})$. We associate with A a smooth vector field X on $\mathbf{R}^{2m}$ defined by

$$(9.4) \quad X = \sum_{i,j} a_{ij} x^i \partial / \partial x^j,$$

which has a single nondegenerate zero at the origin $0 \in \mathbf{R}^{2m}$. Note that X is a (complete) affine vector field on $(\mathbf{R}^{2m}, \langle, \rangle)$ in the sense that it generates a one-parameter group of affine diffeomorphisms of $(\mathbf{R}^{2m}, \langle, \rangle)$ with respect to the Levi-Civita connection ∇ defined by $\langle, \rangle$. We call X *expansive* if the real part of each eigenvalue of the matrix $A = (a_{ij})$ is positive.

Then the following proposition is a direct consequence of Theorem 9.2.

PROPOSITION 9.1. *Let $(\mathbf{R}^{2m}, \langle, \rangle)$ and $X = \sum a_{ij}x^i\partial/\partial x^j$ be as above. Suppose X is expansive and set $M = \mathbf{R}^{2m} - \{0\}$. Then X defines*

- (1) *on M a projective foliation $\mathcal{F}_X$ of codimension $2m-1$, and hence*
- (2) *on $\mathbf{R}^{2m}$ a projective foliation $\mathcal{F}_X^*$ of codimension $2m-1$ with a single nondegenerate singularity at the origin $0 \in \mathbf{R}^{2m}$.*

PROOF. (1) Since an affine X is also a projective vector field, it suffices to show, on account of Theorem 9.2, that X is projectible.

Let $\{\psi_t\}$ be the flow generated by X . Since X is expansive, each orbit of $\{\psi_t\}$ intersects the unit sphere $S^{2m-1} \subset M = \mathbf{R}^{2m} - \{0\}$ transversely only at one point, say x , and $\lim_{t \rightarrow -\infty} \psi_t(x) = 0$ and $\lim_{t \rightarrow +\infty} \psi_t(x) = \infty$. Thus at each point $x \in S^{2m-1}$ the tangent space TR_x^{2m} of $\mathbf{R}^{2m}$ is spanned by the vector $X(x)$ and the tangent space TS_x^{2m-1} of S^{2m-1} . Define a linear form a_x on each TR_x^{2m} , $x \in S^{2m-1}$ by requiring $a_x(X(x)) = 1$ and $a_x = 0$ on TS_x^{2m-1} . Then extend a_x to a smooth 1-form a on M by the flow $\{\psi_t\}$, that is, define $a|_{TR_{\psi(x)}^{2m}} = (\psi_t^{-1})^*a_x$ for each $x \in S^{2m-1}$ and $t \in \mathbf{R}$. It is then immediate from definition that a satisfies $a(X) = 1$ and $\mathcal{L}_X a = 0$, which shows that X is projectible.

(2) This is now evident (cf. Remark 5.1).

q.e.d.

REMARK 9.1. It should be remarked that by modifying the argument in Appendix we can indeed prove that the vector field X in Proposition 9.1 defines an affine foliation $\mathcal{F}_X$ of codimension $2m-1$ on M .

By varying the vector field X of Proposition 9.1, we can obtain families of projective foliations of codimension $2m-1$. Our next goal is to evaluate on these families certain secondary characteristic classes for projective foliations defined in § 8.

To be precise, let $H^*(PWO_{2m-1})$ be the universal characteristic classes for projective foliations of codimension $2m-1$. Then by virtue of Theorem 8.1, an additive basis of $H^{2m-1}(PWO_{2m-1})$ is immediately determined as follows.

LEMMA 9.1. *A basis of $H^{2m-1}(PWO_{2m-1})$ is given by the classes determined by the basic cocycles*

$$c_i h_j = c_{i_1} \cdots c_{i_s} h_j$$

with $2|I| + 2j - 1 = 2m - 1$ and $j \leq$ the smallest odd integer in $I = (i_1, \dots, i_s)$.

Let $\mathcal{F}_X$ and $\mathcal{F}_X^*$ be the projective foliations of codimension $2m-1$ in Proposition 9.1; $\mathcal{F}_X$ on M without singularities and $\mathcal{F}_X^*$ on $\mathbf{R}^{2m}$ with a nondegenerate singularity at $0 \in \mathbf{R}^{2m}$. Let $\lambda_{\mathcal{F}_X}^*: H^*(PWO_{2m-1}) \rightarrow H^*(M; \mathbf{R})$ denote the characteristic classes for $\mathcal{F}_X$ (cf. Definition 8.1). Let $[c_i h_j] \in$

$H^{2m-1}(PWO_{2m-1})$, the universal characteristic class determined by the basic cocycle $c_i h_j \in PWO_{2m-1}$ as in Lemma 9.1. By regarding c_i as the i -th Chern polynomial, we may canonically identify $c_i c_j = d(c_i h_j) \in PWO_{2m-1}$ with an invariant polynomial of degree m : $c_i c_j \in I^m(GL(2m, \mathbf{R}))$. We then define $c_i c_j$ -residue $\text{Res}_{c_i c_j}(\mathcal{F}_X^*, 0)$ of $\mathcal{F}_X^*$ at $0 \in \mathbf{R}^{2m}$ by the same formula as (5.12) of Theorem 5.1.

REMARK 9.2. It should be remarked here that based on Theorem 2.1 (the strong vanishing theorem I) and the curvature description of characteristic classes of Chern-Weil as in §§ 1 and 3, we are in fact able to formulate and prove Theorem 4.1 (the residue existence theorem) for every symmetric (not necessarily assumed to be sign-symmetric) homogeneous polynomial $\varphi \in \mathbf{R}[Y_1, \dots, Y_m]$ of degree l with $q < 2l \leq n$, where q is the codimension of a projective foliation $\mathcal{F}^*$ with singularities on a smooth manifold M of dimension n and $m = [n/4]$. In this case, however, the left hand side of the formula (4.9) of Theorem 4.1, that is, the corresponding characteristic class $\varphi(\nu(\mathcal{F}^*))$ of the virtual normal bundle $\nu(\mathcal{F}^*)$ of $\mathcal{F}^*$, is trivial whenever the degree l of φ is odd, while the φ -residues $\text{Res}_\varphi(\mathcal{F}^*, Z)$ on the right hand side are not in general.

The same facts are also true for Theorem 5.2 (the residue formula I) as well as Theorem 7.1 (the residue formula II). This justifies the definition of $c_i c_j$ -residue $\text{Res}_{c_i c_j}(\mathcal{F}_X^*, 0)$ in the above.

With these understood, we have the following theorem which illustrates a relationship between residues and characteristic classes for projective foliations.

THEOREM 9.3. *Notations being as above,*

$$(9.5) \quad \lambda_{\mathcal{F}_X^*}^*([c_i h_j]) [S^{2m-1}] = \text{Res}_{c_i c_j}(\mathcal{F}_X^*, 0),$$

where $[S^{2m-1}] \in H_{2m-1}(M; \mathbf{Z})$ denotes the generator defined by the unit sphere $S^{2m-1} \subset M = \mathbf{R}^{2m} - \{0\}$.

PROOF. As in § 8, let $\lambda_{\mathcal{F}_X^*}: PWO_{2m-1} \rightarrow \mathcal{A}^*(M; \mathbf{R})$ be the homomorphism of differential graded complexes which induces $\lambda_{\mathcal{F}_X^*}^*$ in cohomology and let $'c_i$ denote the image of the i -th Chern polynomial $c_i \in I^i(SL(2m, \mathbf{R}))$ under the restriction homomorphism $r: I^i(SL(2m, \mathbf{R})) \rightarrow I^i(GL(2m-1, \mathbf{R}))$ (cf. Lemma 8.1.). Then it is immediate from the definition that

$$(9.6) \quad \begin{aligned} \lambda_{\mathcal{F}_X^*}^*([c_i h_j]) [S^{2m-1}] &= \int_{S^{2m-1}} \lambda_{\mathcal{F}_X^*}(c_i h_j) \\ &= \int_{S^{2m-1}} 'c_i(\Omega^b) \wedge \left(j \int_0^1 'c_j(\omega^b - \omega^r, \Omega_t, \dots, \Omega_t) dt \right). \end{aligned}$$

Here ω^b (resp. ω^r) is the connection form of a basic connection ∇^b (resp. a metric connection ∇^r) on the normal bundle $\nu(\mathcal{F}_X)$ of $\mathcal{F}_X$, Ω^b is the curvature form of ω^b , and Ω_t is the curvature form of the connection ω_t defined for each $t \in [0, 1]$ by $\omega_t = t\omega^b + (1-t)\omega^r$.

We are going to prove that the integral in (9.6) is equal to $\text{Res}_{c_{1c_j}}(\mathcal{F}_X^*, 0)$. To do this, first recall the exact sequence associated with $\mathcal{F}_X$:

$$0 \longrightarrow E \longrightarrow TM \longrightarrow \nu(\mathcal{F}_X) \longrightarrow 0,$$

where E is the trivial line bundle over M generated by X . With respect to the splitting defined by the metric $\langle \cdot, \cdot \rangle$ on $\mathbf{R}^{2m}$, identify TM with the direct sum of E and $\nu(\mathcal{F}_X)$. Let ∇^1 be a metric connection on E . We then define the direct sum connections ∇^B of ∇^1 and ∇^b , and ∇^R of ∇^1 and ∇^r . Remark that the matrix representations of ω^B and ω^R are, with a suitable choice of a local frame of TM , of the form:

$$(9.7) \quad \omega^B = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \omega^b \end{array} \right), \quad \omega^R = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \omega^r \end{array} \right)$$

Let Ω^B denote the curvature form of Ω^B and define $\hat{\Omega}^B$ as in Lemma 5.1, that is, $\hat{\Omega}^B = (\Omega^B + (2\pi/2m)c_1(\Omega^B)I_{2m})$. It then follows from (9.7) that $\hat{\Omega}^B$ takes values in $\mathfrak{sl}(2m, \mathbf{R})$ and in fact has the form

$$\hat{\Omega}^B = \left(\begin{array}{c|c} (2\pi/2m)c_1(\Omega^b) & 0 \\ \hline 0 & \Omega^b + (2\pi/2m)c_1(\Omega^b)I_{2m-1} \end{array} \right),$$

which implies by the definition that

$$'c_i(\Omega^b) = c_i(\hat{\Omega}^B) = \hat{c}_i(\Omega^B)$$

for each Chern polynomial $c_i \in I^i(SL(2m, \mathbf{R}))$. Next, for each $t \in [0, 1]$ define the connection $\omega_t^* = t\omega^B + (1-t)\omega^R$ and denote its curvature form by Ω_t^* . Then, by the same argument as above, it is verified that

$$'c_i(\omega^b - \omega^r, \Omega_t, \dots, \Omega_t) = \hat{c}_i(\omega^B - \omega^R, \Omega_t^*, \dots, \Omega_t^*)$$

for each Chern polynomial $c_i \in I^i(SL(2m, \mathbf{R}))$.

From these the problem is now reduced to that of calculating the integral

$$\begin{aligned} & \int_{S^{2m-1}} \hat{c}_I(\Omega^B) \wedge \left(j \int_0^1 \hat{c}_j(\omega^B - \omega^R, \Omega_t^*, \dots, \Omega_t^*) dt \right) \\ &= \int_{S^{2m-1}} \left(j \int_0^1 \hat{c}_I(\Omega^B) \wedge \hat{c}_j(\omega^B - \omega^R, \Omega_t^*, \dots, \Omega_t^*) dt \right). \end{aligned}$$

Since the integrand is a closed $(2m-1)$ -form on M , this may be rewritten as

$$(9.8) \quad \lim_{\varepsilon \rightarrow 0} \int_{S_\varepsilon^{2m-1}} \left(j \int_0^1 \hat{c}_I(\Omega^B) \wedge \hat{c}_J(\omega^B - \omega^R, \Omega_t^*, \dots, \Omega_t^*) dt \right),$$

the limit of the integrals over the spheres $S_\varepsilon^{2m-1} = \{x \in \mathbf{R}^{2m}; \langle x, x \rangle = \varepsilon^2\}$ homotopic to S^{2m-1} . Hence the proof is complete if we show that (9.8) is in fact identical with the $c_I c_J$ -residue $\text{Res}_{c_I c_J}(\mathcal{F}_X^*, 0)$ of $\mathcal{F}_X^*$ at the origin $0 \in \mathbf{R}^{2m}$. But this can be done by duplicating, with respect to the integrand in (9.8), the computations in § 5 such as those in Lemmas 5.3, 5.5, 5.6 and so forth. Indeed, essentially we need only to replace the connections ω^0 and ω_t therein by ω^R and ω_t^* in the present proof. In consequence (9.5) is proved. q.e.d.

Now we turn to the

PROOF OF THEOREM 9.1. First recall the data of the theorem. BPF_{2m-1} is the classifying space for projective foliations of codimension $2m-1$. $\lambda^*: H^*(PWO_{2m-1}) \rightarrow H^*(BPF_{2m-1}; \mathbf{R})$ is the universal characteristic homomorphism defined in § 8. $d(m)$ denotes the dimension of $H^{2m-1}(PWO_{2m-1})$. Then we want to prove the existence of epimorphisms

$$(9.1) \quad \begin{aligned} H_{2m-1}(BPF_{2m-1}; \mathbf{Z}) &\longrightarrow \mathbf{R}^{d(m)} \longrightarrow 0, \\ \pi_{2m-1}(BPF_{2m-1}) &\longrightarrow \mathbf{R}^{d(m)} \longrightarrow 0. \end{aligned}$$

To prove this, we proceed as follows. Let S denote the unit sphere $S^{2m-1} \subset \mathbf{R}^{2m}$ and denote by $[S]$ the fundamental homology class of S , which is a generator of $H_{2m-1}(S; \mathbf{Z})$. For a mapping $f: S \rightarrow BPF_{2m-1}$, denote by $[f]$ the homotopy class of f , which is an element of $\pi_{2m-1}(BPF_{2m-1}) = [S, BPF_{2m-1}]$ the set of homotopy classes of mappings $S \rightarrow BPF_{2m-1}$. Let $\phi_1, \dots, \phi_{d(m)}$ be a basis of $H^{2m-1}(PWO_{2m-1})$ given in Lemma 9.1. Then we define the mappings

$$(9.9) \quad \begin{aligned} \Phi: H_{2m-1}(BPF_{2m-1}; \mathbf{Z}) &\longrightarrow \mathbf{R}^{d(m)}, \\ \Psi: \pi_{2m-1}(BPF_{2m-1}) &\longrightarrow \mathbf{R}^{d(m)} \end{aligned}$$

respectively by

$$\begin{aligned} \Phi(\alpha) &= (\lambda^* \phi_1(\alpha), \dots, \lambda^* \phi_{d(m)}(\alpha)), \\ \Psi([f]) &= (f^*(\lambda^* \phi_1)[S], \dots, f^*(\lambda^* \phi_{d(m)}[S])) \end{aligned}$$

for $\alpha \in H_{2m-1}(BPF_{2m-1}; \mathbf{Z})$ and $[f] \in \pi_{2m-1}(BPF_{2m-1})$, where f^* denotes the homomorphism $f^*: H^*(BPF_{2m-1}; \mathbf{R}) \rightarrow H^*(S; \mathbf{R})$ induced by $f: S \rightarrow BPF_{2m-1}$.

We are going to prove that these mappings Φ and Ψ are surjective. To this end, as in Proposition 9.1, let $(\mathbf{R}^{2m}, \langle, \rangle)$ be a Euclidean $2m$ -space and $X = \sum a_{ij} x^i \partial / \partial x^j$ be an expansive affine vector field on $\mathbf{R}^{2m}$ with a single non-degenerate zero at the origin $0 \in \mathbf{R}^{2m}$. Let $M = \mathbf{R}^{2m} - \{0\}$. Then, according to Proposition 9.1, X defines on M a projective foliation $\mathcal{F}_X$ of codimension $2m$

-1 and on $\mathbf{R}^{2m}$ a projective foliation $\mathcal{F}_X^*$ of codimension $2m-1$ with a non-degenerate singularity at $0 \in \mathbf{R}^{2m}$. Let $f_X: M \rightarrow BPF_{2m-1}$ be a mapping, defined up to homotopy, which classifies the foliation $\mathcal{F}_X$ (cf. § 8). Since M has the homotopy type of the sphere $S^{2m-1} \subset M$, f_X then naturally gives rise to the elements:

$$\begin{aligned} (f_X)_*[S] &\in H_{2m-1}(BPF_{2m-1}; \mathbf{Z}) \\ [f_X] &\in \pi_{2m-1}(BPF_{2m-1}), \end{aligned}$$

where $(f_X)_*: H_*(S; \mathbf{Z}) \rightarrow H_*(BPF_{2m-1}; \mathbf{Z})$ stands for the homomorphism defined by f_X .

We will evaluate Φ and Ψ on these elements. Let $\lambda_{\mathcal{F}_X}^*: H^*(PWO_{2m-1}) \rightarrow H^*(M; \mathbf{R})$ be the characteristic classes for $\mathcal{F}_X$. Note that $\lambda_{\mathcal{F}_X}^*$ then factors through λ^* as in (8.6): $\lambda_{\mathcal{F}_X}^* = f_X^* \circ \lambda^*$. Recall that due to Lemma 9.1 each ϕ_i of the basis of $H^{2m-1}(PWO_{2m-1})$ has the form $[c_I h_j]$ ($c_I h_j = c_{i_1} \cdots c_{i_s} h_j$ with $2|I| + 2j - 1 = 2m - 1$ and $j \leq$ the smallest odd integer in $I = (i_1, \dots, i_s)$). Then it is immediate from (9.5) of Theorem 9.3 that

$$\begin{aligned} \Phi((f_X)_*[S]) &= \Psi([f_X]) \\ (9.10) \quad &= (\dots, f_X^* \circ \lambda^*(c_I h_j)[S], \dots) \\ &= (\dots, \text{Res}_{c_I c_j}(\mathcal{F}_X^*, 0), \dots). \end{aligned}$$

Now, let E denote the set of matrices $A \in GL(2m, \mathbf{R})$ such that the real parts of eigenvalues of A are positive, which is an open subset of $GL(2m, \mathbf{R})$. Then, by (9.4) each $A \in E$ defines an expansive vector field X on $\mathbf{R}^{2m}$ and hence gives rise to projective foliations $\mathcal{F}_X$, $\mathcal{F}_X^*$ of codimension $2m-1$ as above. Thus we can define a mapping $\tilde{\Phi}: E \rightarrow \mathbf{R}^{d(m)}$ by

$$\tilde{\Phi}(A) = \Phi((f_X)_*[S]).$$

It follows from (9.10) and the explicit formula (5.12) for the residues that $\tilde{\Phi}$ is a smooth mapping, in fact is a real analytic one.

Since Φ is a homomorphism of abelian groups and the image of Φ contains that of $\tilde{\Phi}$, in order to prove the surjectivity of Φ it suffices to show that $\tilde{\Phi}(E)$, the image of $\tilde{\Phi}$, generates $\mathbf{R}^{d(m)}$ additively. This can be seen as follows. Let N be a connected component of the regular values of $\tilde{\Phi}$, that is, those values of $\tilde{\Phi}$ where $\tilde{\Phi}$ has maximal rank. Then N is a smooth submanifold of $\mathbf{R}^{d(m)}$. Furthermore it is known that N is not contained in any affine hyperplane of $\mathbf{R}^{d(m)}$. In fact, this is a consequence of the linear independence on E of the invariant polynomials $\{c_I c_j\}$ in $I^*(GL(2m, \mathbf{R}))$ which define $c_I c_j$ -residues $\text{Res}_{c_I c_j}(\mathcal{F}_X^*, 0)$ of $\mathcal{F}_X^*$ at $0 \in \mathbf{R}^{2m}$ and correspond in canonical way to the basis $\{[c_I h_j]\}$ of $H^{2m-1}(PWO_{2m-1})$ (cf. Theorem 9.3). Then, by a mimicry of the argument in Appendix of [BB, p. 341], we see that N and hence $\tilde{\Phi}(E)$

generate $R^{d(m)}$ additively. This concludes the proof for Φ .

The proof for Ψ proceeds in parallel.

q.e.d.

REMARKS 9.3. 1) Theorem 9.1 is of significance only when the dimension $d(m)$ of $H^{2m-1}(PWO_{2m-1})$ is not zero. The smallest m for which $d(m) \neq 0$ is 3. In this case $d(3)=1$ and the class $[h_3] \in H^3(PWO_3)$ is a generator.

2) The surjectivity of the homomorphism Φ in Theorem 9.1 means that the secondary characteristic classes $H^{2m-1}(PWO_{2m-1})$ can all vary continuously and linearly independently.

3) It should be remarked that if we confine ourselves to Riemannian foliations, a restricted family of projective foliations, then we can not obtain a continuous variation of, for instance, the class $[h_m] \in H^{2m-1}(PWO_{2m-1})$ (cf. Lazarov-Pasternack [LP 2]).

§ 10. Rigidity of residues

Let $\mathcal{F}^*$ be a projective foliation of codimension q with singularities on a smooth manifold M of dimension n . The object of this section is to prove a rigidity property of φ -residues $\text{Res}_\varphi(\mathcal{F}^*, Z)$ of $\mathcal{F}^*$ when $q+1 < 2\deg \varphi \leq n$. Recall that when $2\deg \varphi = q+1$ the residue formula (5.10) of Theorem 5.1 suggests the possibility of continuous variation of φ -residues, which in fact turns out to be the case in § 8. In this respect, the rigidity theorem in this section illustrates that the two cases $2\deg \varphi = q+1$ and $2\deg \varphi > q+1$ are quite different.

To begin with we fix some notation. For a smooth manifold M and the closed unit interval $[0, 1]$, set $\tilde{M} = M \times [0, 1]$ and let $\pi: \tilde{M} \rightarrow M$, $t: \tilde{M} \rightarrow [0, 1]$ be the projections. For each $u \in [0, 1]$ define the inclusion $i_u: M \rightarrow \tilde{M}$ by $i_u(x) = (x, u)$ for $x \in M$. Moreover by the same letter π and i_u denote respectively the projection $\pi: \mathbf{R}^q \times [0, 1] \rightarrow \mathbf{R}^q$ and the inclusion $i_u: \mathbf{R}^q \rightarrow \mathbf{R}^q \times [0, 1]$ defined by $i_u(x) = (x, u)$ for $x \in \mathbf{R}^q$ and $u \in [0, 1]$.

Let $\{\mathcal{F}_u\}$, $u \in [0, 1]$ be a differentiable one-parameter family of projective foliations of codimension q on M (cf. Definition 8.2). It should be recalled that by definition (i) each $\mathcal{F}_u$ is a projective foliation of codimension q on M , (ii) for the tangent bundle $\tau(\mathcal{F}_u)$ of each $\mathcal{F}_u$ there exists a smooth subbundle $\tilde{F}$ of $\pi^!TM$, the pull-back bundle of the tangent bundle TM of M by π , with $i_u^!(\tilde{F}) = \tau(\mathcal{F}_u)$ and (iii) there exist a basic connection ∇_u on each normal bundle $\nu(\mathcal{F}_u)$ of $\mathcal{F}_u$ and a connection ${}^*\nabla$ on the quotient bundle $\tilde{\nu} = \pi^!TM/\tilde{F}$, referred to as a basic connection on $\tilde{\nu}$, such that $i_u^!({}^*\nabla) = \nabla_u$ for each $u \in [0, 1]$. (cf. Remarks 8.2 2)).

Then we have the following proposition which is fundamental to our proof of the rigidity theorem of residues.

PROPOSITION 10.1. *Let $\{\mathcal{F}_u\}$, $u \in [0, 1]$, be a differentiable one-parameter family of projective foliations of codimension q on M . Let ${}^*\mathcal{V}^b$ be a basic connection on the quotient bundle $\tilde{\nu} = \pi^! TM / \tilde{F}$ and ${}^*\Omega^b$ denote the curvature form of ${}^*\mathcal{V}^b$. Let $f \in I^k(SL(q+1, \mathbf{R}), GL(q, \mathbf{R}))$ and assume $2k > q+1$. Then*

$$(10.1) \quad f({}^*\Omega^b) = 0.$$

PROOF. Since the proof is essentially the same as that of Theorem 2.1 (the strong vanishing theorem I), we describe here only the matters which are worth mentioning.

Let $\mathcal{F}_u = \{U_\alpha, f_\alpha^u, \gamma_{\beta\alpha}^{xu}, \nabla_\alpha^u\}$ be the defining data for $\mathcal{F}_u$, a projective foliation of codimension q on M . Recall that we have on $\pi^! TR^q$, the pull-back bundle of the tangent bundle TR^q of $\mathbf{R}^q$ by π , a connection D_α with $i_u^! D_\alpha = \nabla_\alpha^u$ for each $u \in [0, 1]$ (Definition 8.2 (iii)), and there exists a smooth submersion

$$\tilde{f}_\alpha: U_\alpha \times [0, 1] \longrightarrow \mathbf{R}^q \times [0, 1]$$

with $\tilde{f}_\alpha \circ i_u = f_\alpha^u$ for each $u \in [0, 1]$ (Remarks 8.1 1)). Denote by Ω_α , Ω_u^b and Ω_α^u the curvature forms of the connection D_α on $\pi^! TR^q$, the basic connection $\mathcal{V}_u^b$ on $\nu(\mathcal{F}_u)$ and the torsionfree connection ∇_α^u on TR^q respectively. It then follows from Lemma 2.5 that for each $u \in [0, 1]$

$$f(\Omega_u^b) = f(f_\alpha^{u!} {}^*\Omega_\alpha^u) \quad \text{on } P(\mathcal{F}_u)|U_\alpha,$$

$P(\mathcal{F}_u)|U_\alpha$ being the frame bundle of $\nu(\mathcal{F}_u)$ restricted to U_α . Furthermore, noting the definition of $\tilde{f}_\alpha$ and D_α as well as ${}^*\mathcal{V}^b$, it is verified by the same argument as in the proof of Lemma 2.5 that

$$f({}^*\Omega^b) = f(\tilde{f}_\alpha^! {}^*\Omega_\alpha) \quad \text{on } P(\tilde{F})|U_\alpha \times [0, 1],$$

$P(\tilde{F})|U_\alpha \times [0, 1]$ being the frame bundle of the smooth subbundle $\tilde{F}$ of $\pi^! TM$ restricted to $U_\alpha \times [0, 1]$. Note that by the naturality of curvature forms

$$f(\tilde{f}_\alpha^! {}^*\Omega_\alpha) = \tilde{f}_\alpha^! f({}^*\Omega_\alpha).$$

Since $f({}^*\Omega_\alpha)$ may be pushed down to a $2k$ -form on $\mathbf{R}^q \times [0, 1]$, we have for dimensional reason that

$$f({}^*\Omega_\alpha) = 0 \quad \text{if } 2k > q+1.$$

Hence we know that

$$f({}^*\Omega^b) = \tilde{f}_\alpha^! f({}^*\Omega_\alpha) = 0 \quad \text{on } P(\tilde{F})|U_\alpha \times [0, 1]$$

for each index α . This proves (10.1).

In view of the fact that our rigidity theorem for residues is local in nature, we let U be an open subset of M , and set the following

DEFINITION 10.1. A differentiable one-parameter family $\{\mathcal{F}_u^*\}$, $u \in [0, 1]$, of projective foliations of codimension q with singular set Z on U is defined by the following data:

- (i) For each $u \in [0, 1]$, $\mathcal{F}_u^* = \{\mathcal{F}_u, E_u, f_{E_u}, Z\}$ is a projective foliation of codimension q with singular set Z on U .
- (ii) $\{\mathcal{F}_u^*\}$, $u \in [0, 1]$ defines a differentiable one-parameter family of projective foliations of codimension q on $U - Z$ (cf. Definition 8.2).
- (iii) There exists a smooth subbundle of the pull-back bundle $\pi^!(TM|U)$ on $\tilde{U} = U \times [0, 1]$ such that $i_u^!(\tilde{E}) = E_u$ for each $u \in [0, 1]$.

REMARK 10.1. Set $\tilde{Z} = Z \times [0, 1]$ and recall the quotient bundle $\tilde{\nu} = \pi^!TM/\tilde{F}$. Then on $\tilde{U} - \tilde{Z}$ we have the following exact sequence of vector bundles:

$$(10.2) \quad 0 \longrightarrow \tilde{E} \longrightarrow \pi^!TM|_{\tilde{U}} \longrightarrow \tilde{\nu} \longrightarrow 0,$$

which reduces for each $u \in [0, 1]$ to

$$0 \longrightarrow E_u \longrightarrow TM|_U \longrightarrow \nu(\mathcal{F}_u) \longrightarrow 0,$$

the exact sequence naturally associated with $\mathcal{F}_u$.

With these understood, our rigidity theorem for residues may be stated as follows.

THEOREM 10.1. Let M be a smooth manifold of dimension n and U be an open subset of M . Let $\{\mathcal{F}_u^*\}$, $u \in [0, 1]$, be a differentiable one-parameter family of projective foliations of codimension q with singular set Z on U . Assume that Z is compact and connected, and set $m = [n/4]$. Let $\varphi \in R[Y_1, \dots, Y_m]$ be a sign-symmetric homogeneous polynomial of degree $2l$, where $q+1 < 2l \leq n$. Then

$$(10.3) \quad \text{Res}_\varphi(\mathcal{F}_0^*, Z) = \text{Res}_\varphi(\mathcal{F}_1^*, Z),$$

$\text{Res}_\varphi(\mathcal{F}_0^*, Z)$ and $\text{Res}_\varphi(\mathcal{F}_1^*, Z)$ being the φ -residues of $\mathcal{F}_0^*$ and $\mathcal{F}_1^*$ on Z respectively.

PROOF. As in the proof of Proposition 4.1 and Theorem 4.1, let V be a compact subset of U such that $Z \subset V \subset U$ and Z is a deformation retract of V . (We have assumed the existence of such V .) Set $\tilde{V} = V \times [0, 1]$. Recall the exact sequence (10.2) on $\tilde{U} - \tilde{Z}$:

$$0 \longrightarrow \tilde{E} \longrightarrow \pi^!TM|_{\tilde{U}} \longrightarrow \tilde{\nu} \longrightarrow 0,$$

and choose a splitting j of it. Let ${}^*F^1$ be a metric connection on $\tilde{E}$ (defined over $\tilde{U}$) and ${}^*F^0$ be a basic connection on $\tilde{\nu}$ (defined over $\tilde{U} - \tilde{Z}$). Define the

direct sum connection $*\mathcal{V}^0$ on $\pi^!TM|_{\tilde{U}-\tilde{Z}}$ of $*\mathcal{V}^1$ and $*\mathcal{V}^b$ with respect to the splitting j . By a partition of unity argument, extend $*\mathcal{V}^0$ to a connection on $\pi^!TM|_{\tilde{U}}$, which we denote also by $*\mathcal{V}^0$, in such a way that they agree on $\pi^!TM|_{\tilde{U}-\tilde{V}}$ (cf. Remark 4.1).

Let $(*\Omega^1, *\Omega^0, *\Omega^b)$ denote the corresponding curvature forms of the triple of connections $(*\mathcal{V}^1, *\mathcal{V}^0, *\mathcal{V}^b)$, and define closed $4l$ -forms $\varphi(*\tilde{\Omega}^b)$ and $\varphi(*\tilde{\Omega}^0|*\tilde{\Omega}^1)$ as in § 3. Then, since on $\tilde{U}-\tilde{V}$ the connections $(*\mathcal{V}^1, *\mathcal{V}^0, *\mathcal{V}^b)$ are compatible with the exact sequence (10.2), it follows from Lemma 3.1 that

$$\varphi(*\tilde{\Omega}^b) = \varphi(*\tilde{\Omega}^0|*\tilde{\Omega}^1) \quad \text{on } \tilde{U}-\tilde{V}.$$

On the other hand, it follows from Proposition 10.1 that

$$\varphi(*\tilde{\Omega}^b) = 0 \quad \text{on } \tilde{U}-\tilde{Z},$$

for $q+1 < 2l \leq n$ by assumption. (Recall that there is a unique $\phi \in I^{2l}(SL(q+1, \mathbf{R}), GL(q, \mathbf{R}))$ such that $\phi(*\Omega^b) = \varphi(*\tilde{\Omega}^b)$.) In consequence, $\varphi(*\tilde{\Omega}^0|*\tilde{\Omega}^1)$ is a closed $4l$ -form on $\tilde{U}$ with compact support $\tilde{V}$ and hence defines an element

$$[\varphi(*\tilde{\Omega}^0|*\tilde{\Omega}^1)] \in H_c^{4l}(\tilde{U}; \mathbf{R})$$

of the cohomology of $\tilde{U}$ with compact supports.

Let $i_u: U \rightarrow \tilde{U}$ be the inclusion and define for each $u \in [0, 1]$ a triple of connections $(\mathcal{V}_u^1, \mathcal{V}_u^0, \mathcal{V}_u^b)$ by

$$\mathcal{V}_u^1 = i_u^!(*\mathcal{V}^1), \quad \mathcal{V}_u^0 = i_u^!(*\mathcal{V}^0), \quad \mathcal{V}_u^b = i_u^!(*\mathcal{V}^b).$$

Then it is immediate from the definition that $(\mathcal{V}_u^1, \mathcal{V}_u^0, \mathcal{V}_u^b)$ is fitted to $\mathcal{F}_u^*$ for each $u \in [0, 1]$ (cf. Definition 4.2). Denote by $(\Omega_u^1, \Omega_u^0, \Omega_u^b)$ the corresponding curvature forms and define a closed $4l$ -form $\varphi(\tilde{\Omega}_u^0|\tilde{\Omega}_u^1)$ on U as in § 3. Recall that by Definition 4.3 each φ -residue $\text{Res}_\varphi(\mathcal{F}_u^*, Z) \in H_{n-4l}(Z; \mathbf{R})$ of $\mathcal{F}_u^*$ on Z is determined by $\varphi(\tilde{\Omega}^0|\tilde{\Omega}^1)$ through the Poincaré isomorphism

$$H_c^{4l}(U; \mathbf{R}) \xrightarrow{\cong} H_{n-4l}(U; \mathbf{R}).$$

Therefore, in order to prove (10.3) it is sufficient to show that $\varphi(\tilde{\Omega}_0^0|\tilde{\Omega}_0^1)$ and $\varphi(\tilde{\Omega}_1^0|\tilde{\Omega}_1^1)$ give rise to the same element in $H_c^{4l}(U; \mathbf{R})$. But it follows from the naturality of curvature forms that

$$i_u^*\varphi(*\tilde{\Omega}^0|*\tilde{\Omega}^1) = \varphi(\tilde{\Omega}_u^0|\tilde{\Omega}_u^1)$$

for each $u \in [0, 1]$. Hence we have

$$[\varphi(\tilde{\Omega}_0^0|\tilde{\Omega}_0^1)] = [\varphi(\tilde{\Omega}_1^0|\tilde{\Omega}_1^1)] \in H_c^{4l}(U; \mathbf{R})$$

due to the homotopy invariance of cohomology with compact supports. This completes the proof.

Appendix

In this Appendix we present a proof of Lemma 2.1 in § 2 and that of Theorem 9.2 in § 9 in a computational manner.

1°. We first prove Lemma 2.1 briefly. So, let $\mathcal{F} = \{U_\alpha, f_\alpha, \gamma_{\beta\alpha}^x, \nabla_\alpha\}$ be a projective foliation of codimension q on a smooth manifold M of dimension n , ∇^b be the basic connection on the normal bundle $\nu(\mathcal{F}) = TM/\tau(\mathcal{F})$ of $\mathcal{F}$ and $\rho: TM \rightarrow \nu(\mathcal{F})$ be the projection. Taking U_α smaller if necessary, let $(x^1, \dots, x^n)$ be a Frobenius coordinate system on U_α so that the submersion $f_\alpha: U_\alpha \rightarrow \mathbf{R}^q$ is written as

$$f_\alpha: (x^1, \dots, x^q, x^{q+1}, \dots, x^n) \longrightarrow (x^1, \dots, x^q).$$

Thus $\rho(\partial/\partial x^i) = 0$ for $i = q+1, \dots, n$ and $\{\rho(\partial/\partial x^1), \dots, \rho(\partial/\partial x^q)\}$ defines a local frame of $\nu(\mathcal{F})|_{U_\alpha}$.

Now, let $X \in C^\infty(\tau(\mathcal{F}))$ and $Y = \sum_{i=1}^n Y^i \partial/\partial x^i \in C^\infty(TM)$. It then follows from Definition 2.2 of ∇^b that

$$\nabla_X^b \rho(\partial/\partial x^i) = 0 \quad \text{for } i = 1, \dots, q,$$

from which it is immediate that

$$\begin{aligned} \nabla_X^b \rho(Y) &= \sum_{i=1}^q dY^i(X) \rho(\partial/\partial x^i) \\ &= \rho([X, Y]). \end{aligned}$$

This completes the proof of (2.1).

2°. Recall the data of Theorem 9.2. (M, ∇) is a smooth manifold of dimension n with torsionfree connection ∇ . X is a projective vector field on M and $\text{Zero}(X)$ denotes the set of zeros of X . Then it is well-known that on $M - \text{Zero}(X)$, X defines a one-dimensional smooth foliation $\mathcal{F}_X$, whose leaves are given by the integral curves of X . We suppose X is projectible so that there exists a smooth 1-form α , defined on $M - \text{Zero}(X)$, such that $\alpha(X) = 1$ and $\mathcal{L}_X \alpha = 0$.

Let $\{U_\alpha\}_\alpha$ be an open covering of $M - \text{Zero}(X)$ consisting of distinguished neighborhoods of the foliation $\mathcal{F}_X$ and $(x_\alpha^1, \dots, x_\alpha^n)$ be a Frobenius coordinate system on U_α . Thus U_α is a cubic neighborhood on which $|x_\alpha^i(x)| < \varepsilon$ for $x \in U_\alpha$ and $i = 1, \dots, n$, and each leaf of $\mathcal{F}_X$ is determined by the conditions $x_i = \text{constant}$ for $i = 1, \dots, n-1$. We may assume, without loss of generality, $X = \partial/\partial x_\alpha^n$ on each U_α . Let $\mathbf{R}^{n-1}$ be a Euclidean $(n-1)$ -space with the canonical coordinate system $(x^1, \dots, x^{n-1})$. For each index α , we have a smooth submersion $f_\alpha: U_\alpha \rightarrow \mathbf{R}^{n-1}$ defined by

$$f_\alpha: U_\alpha \ni x \longrightarrow (x_\alpha^1(x), \dots, x_\alpha^{n-1}(x)) \in \mathbf{R}^{n-1}.$$

Furthermore, for each $x \in U_\alpha \cap U_\beta$ we have a local diffeomorphism $\gamma_{\beta\alpha}^x$ from

a neighborhood of $f_\alpha(x) \in \mathbf{R}^{n-1}$ into a neighborhood of $f_\beta(x) \in \mathbf{R}^{n-1}$ satisfying $f_\beta = \gamma_{\beta\alpha}^x \circ f_\alpha$ in some neighborhood of x . In fact, $\gamma_{\beta\alpha}^x$ is given by

$$(1) \quad \gamma_{\beta\alpha}^x: (x_\alpha^1(z), \dots, x_\alpha^{n-1}(z)) \longrightarrow (x_\beta^1(z), \dots, x_\beta^{n-1}(z))$$

for each z around x .

Let $\Gamma_{\alpha jk}^i$ denote the Christoffel's symbol, with respect to the Frobenius coordinate system (x_α^i) on U_α , of the given torsionfree connection ∇ on the tangent bundle TM of M :

$$\nabla_{\partial/\partial x_\alpha^j} \partial/\partial x_\alpha^k = \sum_{i=1}^n \Gamma_{\alpha jk}^i \partial/\partial x_\alpha^i.$$

Then, since $X = \partial/\partial x_\alpha^n$ is a projective vector field on (M, ∇) , it follows from (9.3) that

$$(2) \quad \begin{aligned} \Gamma_{\alpha jk}^i(x_\alpha^1, \dots, x_\alpha^n) &= \Gamma_{\alpha jk}^i(x_\alpha^1, \dots, x_\alpha^{n-1}, 0) \\ &\quad + \delta_k^i b_j(x_\alpha^1, \dots, x_\alpha^n) + \delta_j^i b_k(x_\alpha^1, \dots, x_\alpha^n), \end{aligned}$$

b_j denoting the components of a smooth 1-form b on U_α . Intuitively speaking, the relation (2) shows that $\Gamma_{\alpha jk}^i(x_\alpha^1, \dots, x_\alpha^n)$ and $\Gamma_{\alpha jk}^i(x_\alpha^1, \dots, x_\alpha^{n-1}, 0)$ are projectively equivalent.

With this in mind, for each index α we define a torsionfree connection ∇_α on the tangent bundle TR^{n-1} of $\mathbf{R}^{n-1}$ in the following fashion. First define local vector fields $v_{\alpha i}$ on U_α by

$$v_{\alpha i} = \partial/\partial x_\alpha^i - a(\partial/\partial x_\alpha^i) \partial/\partial x_\alpha^n$$

for $i=1, 2, \dots, n-1$. Then $\{v_{\alpha 1}, \dots, v_{\alpha n-1}, X\}$ defines a local frame of $TM|_{U_\alpha}$. It should be remarked that on $U_\alpha \cap U_\beta$, $v_{\alpha i}$ transforms by the law

$$(3) \quad v_{\alpha i} = \sum_{j=1}^{n-1} \partial x_\beta^j / \partial x_\alpha^i \cdot v_{\beta j}, \quad j=1, \dots, n-1,$$

for $a(X)=1$ by assumption. Now define a connection ∇_α on $TR^{n-1}|_{f_\alpha(U_\alpha)}$ by

$$(4) \quad \begin{aligned} (\nabla_{\alpha \partial/\partial x_\alpha^i} \partial/\partial x_\alpha^j)(x_\alpha^1, \dots, x_\alpha^{n-1}) \\ = \sum_{k=1}^{n-1} [dx_\alpha^k(\nabla_{v_{\alpha i}} v_{\alpha j})](x_\alpha^1, \dots, x_\alpha^{n-1}, 0) \cdot \partial/\partial x_\alpha^k(x_\alpha^1, \dots, x_\alpha^{n-1}) \end{aligned}$$

and extend it to a torsionfree connection ∇_α on TR^{n-1} . Then, by a straightforward but tedious calculation it follows from (2)–(4) together with the fact $\mathcal{L}_X a = 0$ that each local diffeomorphism $\gamma_{\beta\alpha}^x$ of (1) is in fact a projective diffeomorphism from $(\mathbf{R}^{n-1}, \nabla_\alpha)$ into $(\mathbf{R}^{n-1}, \nabla_\beta)$, that is, ∇_α and $\gamma_{\beta\alpha}^1 \nabla_\beta$ are projectively equivalent. It is now evident from Definition 2.1 that $\mathcal{F}_X = \{U_\alpha, f_\alpha, \gamma_{\alpha\beta}^x, \nabla_\alpha\}$ defines a projective foliation of codimension q on $M - \text{Zero}(X)$. This proves Theorem 9.2.

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