On the asymptotic behavior of spectral functions of elliptic operators

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§ 1. Introduction

Let $\Omega$ be an open set in real space $\mathbb{R}^n$ with generic point $x=(x_1, \ldots, x_n)$. We denote by $\alpha=(\alpha_1, \ldots, \alpha_n)$ a multi-index of length $|\alpha|=\alpha_1+\cdots+\alpha_n$ and use the notations

$$D^\alpha=D_1^{\alpha_1} \cdots D_n^{\alpha_n}, \quad D_\xi=-\sqrt{-1} \partial/\partial x.$$

Let $\mathbb{A}=\mathbb{A}(x,D)=\sum_{|\alpha| \leq 2m} a_\alpha(x)D^\alpha$ be a linear differential operator of order $2m$ with $C^\infty$ coefficients in $\Omega$. We assume that $\mathbb{A}$ is a positive elliptic operator and that it is formally self-adjoint. That is, we assume that

$$\mathbb{A} u = (\mathbb{A} u, v) = (u, \mathbb{A} v) \quad \text{for any} \quad u, v \in C_0^\infty(\Omega).$$

We denote by $\mathbb{A}$ a self-adjoint realization of $\mathbb{A}$ in $L^2(\Omega)$. Assume that the self-adjoint realization is positive and let $\{E_t\}$ be its spectral resolution:

$$\mathbb{A} = \int_0^\infty t dE_t \quad (c>0).$$

It is well known that $E_t$ is an integral operator:

$$E_tf = \int_\Omega e(x,y,t)f(y)dy \quad f \in L^2(\Omega)$$

with a kernel called the spectral function of $\mathbb{A}$.

Spectral functions were studied by many writers. In particular Hörmander [5] proved that in the general situation described above,

$$e(x, x, t) = c(x)t^{n/2m} + O(t^{(n-1)/2m}) \quad \text{as} \quad t \to \infty,$$

$$c(x) = (2\pi)^{-n} \int_{\mathbb{S}^{2m}(x,t) < 1} d\xi,$$
Here the $O$-estimate holds uniformly in $x$ in any compact subset of $\Omega$. When $\Omega$ is bounded, using Hörmander's result, J. Brüning [4] derived some global version of (1.1) for a class of uniformly elliptic operators. That is,

\begin{equation}
|e(x, t)-c(x)t^{n/2m}| \leq C\delta(x)^{-1}t^{(n-1)/2m}
\end{equation}

for any $x \in \Omega$ and $t > 1$. Here and in what follows we write

$$\delta(x) = \min \{1, \text{dis}(x, \partial\Omega)\}.$$

The purpose of this paper is to derive (1.3), using a method that is different from Brüning's, for a general class of uniformly elliptic operators. This method is a modification of Hörmander's. In our result we do not require the boundness of $\Omega$ and the following inclusion relation which were assumed and essentialy used in [4]:

$$D(A^k) \subseteq H_{2m}^k(\Omega)$$

for some $k$ such that $2mk > n$. Moreover the result of this paper plays an important role in [13]. In [13] the author improved the results of Maruo & Tanabe [9] and Maruo [10] on the eigenvalue distribution of operators associated with strongly elliptic sesquilinear forms.

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§2. Main theorem

We denote by $B^\omega(\Omega)$ the class of functions $u \in C^\omega(\Omega)$ such that $u$ and all its derivatives are bounded on $\Omega$. We consider a positive self-adjoint operator $A$ on $L^2(\Omega)$ and assume that the following assumptions hold.

(i) The domain of definition $D(A) \supset C^\omega_0(\Omega)$ and there exists a constant $c > 0$ such that $(Au, u) \geq c\|u\|^2$ for any $u$ in $D(A)$.

(ii) For any $u \in C^\omega_0(\Omega)$, $Au = A(x, D)u$ where $A(x, D)$ is some uniformly elliptic differential operator of order $2m$ with coefficients in $B^\omega(\Omega)$. That is,

\begin{align}
A(x, D) &= \sum_{|\alpha| = 2m} a_{\alpha}(x)D^\alpha, \quad a_{\alpha} \in B^\omega(\Omega), \\
A(x, D) &\geq C^0\|\xi\|^{2m} \quad \text{for any } x \in \Omega, \xi \in \mathbb{R}^n
\end{align}

where $C^0$ is some positive constant.

(iii) Green's functions $G(x, \gamma)$ of $A$ (that is, kernels of resolvents $(A-\lambda)^{-1}$) satisfy the following estimates:
Asymptotic behavior of spectral functions of elliptic operators

If for any $x, y \in \Omega$ and $\lambda \in \Lambda$ where $C_1, c_1$ are constants and $\Lambda = \{ \lambda : \theta \leq \arg \lambda \leq 2\pi - \theta, |\lambda| \geq C_3 \}$ for some $\theta \in (0, \pi/2)$ and $C_3 > 0$.

**REMARK 2.1.** The assumption (iii) is satisfied, when $D(A) = \{ u \in H^2(\Omega) : B_j(x, D)u = 0$ on $\partial \Omega, j = 1, \ldots, m \}$ and the system $A(x, D) \{ B_j(x, D) \}_{j=1}^m$ are regular in the sense of S. Agmon. (See H. Tanabe [12].) We may assume $C_3 = 0$ adding some positive constant to $A$ if necessary.

Under the assumptions (i), (ii) and (iii) we have the following:

**MAIN THEOREM.** Let $e(x, y, t)$ be the spectral function of $A$. Then there exists a constant $C$ independent of $x$ and $t$ such that

\[
|G(x, y)| \leq C_1 |x-y|^{2m-n} e^{-c_1 |x-y|/|\lambda|^{1/2m}} \quad \text{if} \quad 2m > n,
\]

\[
|G(x, y)| \leq C_1 |x-y|^{2m-n} e^{-c_2 |x-y|/|\lambda|^{1/2m}} \quad \text{if} \quad 2m < n,
\]

\[
|G(x, y)| \leq C_1 (1 + \log^+ (|\lambda|^{-1/2m} |x-y|^{1/2m})) e^{-c_2 |x-y|/|\lambda|^{1/2m}} \quad \text{if} \quad 2m = n
\]

for any $x, y \in \Omega$ and $\lambda \in \Lambda$ where $C_1, c_1$ are constants and $\Lambda = \{ \lambda : \theta \leq \arg \lambda \leq 2\pi - \theta, |\lambda| \geq C_3 \}$ for some $\theta \in (0, \pi/2)$ and $C_3 > 0$.

Suppose $A$ has a compact resolvent, so that spectrum $A$ consists of a discrete set of eigenvalues. Let $\{ \lambda_j \}$ be the sequence of eigenvalues, each repeated according to its multiplicity. Let $N(t) = \sum_{j \leq t} 1$. Then it is well known that

\[
N(t) = \int_{\Omega} e(x, x, t) dx.
\]

We assume that there exists a constant such that

\[
\int_{\Omega_\varepsilon} \delta(x)^{-1} dx \leq C |\log \varepsilon|,
\]

\[
\int_{\partial \Omega_\varepsilon} dx \leq C \varepsilon
\]

where $\Omega_\varepsilon = \{ x \in \Omega : \delta(x) > \varepsilon \}$.

Then, from the main theorem we have the following:

**COROLLARY.** In the situation stated above the following asymptotic formula for $N(t)$ holds as $t \to \infty$:

\[
N(t) = t^{n/2m} + O(t^{(n-1)/2m} \log t)
\]

where
Remark 2.2. We note that the constant $C$ of the main theorem depends only on $n, m, \theta, C_0, C_1, C_2, C_3$ and $C_4$ where $C_1, C_2$ are constants such that $|e(x, y, t)| \leq C_1 t^{n/2}$ for $x, y \in \Omega$, $t > 1$ and $|D_\alpha a_\alpha(x)| \leq C_2$ for $x \in \Omega$.

Remark 2.3. We note that in this paper we use one and the same symbol $C$ in order to denote positive constants which may differ from each other. When we specify the dependence of such a constant on a parameter, say $m$, we denote it by $C_m$.

§ 3. Construction of a parametrix

In this section we shall construct a parametrix for $(A-\lambda)^{-1}$. We set

\begin{align}
(a_{zm}(x, \xi, \lambda) &= \sum_{|\alpha|=2m} a_\alpha(x) \xi^\alpha - \lambda = \bar{A}_{zm}(x, \xi) - \lambda, \\
(a_{zm-\ell}(x, \xi) &= \sum_{|\alpha|=2m-\ell} a_\alpha(x) \xi^\alpha \quad \text{for } 0 \leq \ell \leq 2m-1.
\end{align}

Let $\lambda$ be a complex number which is not on the positive real axis and $d(\lambda)$ be the distance from the point $\lambda$ to the positive real axis. Roughly speaking, a parametrix $B(\lambda)$ is a pseudo-differential operator formally satisfying

\[(\bar{A}(x, D) - \lambda) \cdot B(\lambda) = 1, \quad B(\lambda) = \sum_{j=0}^{\infty} b_{-2m-\ell}.
\]

According to the formula for the product of symbols, we have

\begin{align}
b_{-2m} a_{zm} &= 1, \\
b_{-2m-\ell} a_{zm} + \sum \partial_\alpha a_{zm-k} D_\alpha b_{-2m-\ell} &= 0, \\
j > 0 \text{ with the sum taken for } \ell + k + |\alpha| = j.
\end{align}

Then,

\begin{align}
b_{-2m-j} &= \sum_{k=0}^{j} \frac{\gamma_{k,j}(x, \xi)}{(\bar{A}_{zm}(x, \xi) - \lambda)^{j+1}} \\
j \geq 0
\end{align}

with the $\gamma_{k,j}$ polynomials in $a_{zm}, \ldots, a_{zm-j}$ and their derivatives of order $\leq j$.  

Lemma 3.1. There exist constants $C_{\alpha\beta}$ such that

\begin{align}
|D_\alpha \partial_\beta b_{-2m-j}(x, \xi, \lambda)| \\
\leq C_{\alpha\beta} \left( \sum_{k=0}^{j} \left( \frac{1 + |\xi|}{|\xi| + |\lambda|} \right)^{2m(j+1)} \right)^{2m(j+1)} (1 + |\xi|)^{-2m-j-|\beta|}
\end{align}

for any $x \in \Omega, \xi \in \mathbb{R}^n$ and $\lambda \in \Lambda$. 

Asymptotic behavior of spectral functions of elliptic operators

PROOF. We note that \(|\bar{A}_{m}(x, \xi) - \lambda| \geq C(\|\xi\| + |\lambda|^{1/2m})^{2m}\) for any \(x \in \Omega, \xi \in \mathbb{R}^n\) and \(\lambda \in \Lambda\). Using (3.4), by induction on \(j\) we get (3.6)

Now, we set for any positive integer \(N\)

\[
(3.7)\quad b_{(N)}(x, \xi, \lambda) = \sum_{j=1}^{N+1} b_{-2m-j},
\]

\[
(3.8)\quad c_{(N)}(x, \xi, \lambda) = e^{-i \langle x, \xi \rangle} (\bar{A}(x, D) - \lambda)(e^{i \langle x, \xi \rangle} b_{(N)}(x, \xi, \lambda)) - 1,
\]

where \(\langle x, \xi \rangle = x_1 \xi_1 + \cdots + x_n \xi_n\). For \(u \in C_{0}^{\infty}(\Omega)\) we set

\[
(3.9)\quad B_N(\lambda) u = \int b_{(N)}(x, \xi, \lambda) \hat{u}(\xi) e^{i \langle x, \xi \rangle} d\xi,
\]

\[
(3.10)\quad C_N(\lambda) u = \int c_{(N)}(x, \xi, \lambda) \hat{u}(\xi) e^{i \langle x, \xi \rangle} d\xi.
\]

Here and in what follows we denote by \(\hat{u}(\xi)\) the Fourier transform, that is,

\[
\hat{u}(\xi) = \int e^{i \langle x, \xi \rangle} u(x) dx \quad \text{and} \quad d\xi = (2\pi)^{-n} d\xi.
\]

From (3.7) (3.8) we have

\[
(3.11)\quad C_N(\lambda) u = (\bar{A}(x, D) - \lambda) B_N(\lambda) u - u.
\]

LEMMA 3.2. We get the following equality and estimate:

\[
(3.12)\quad c_{(N)}(x, \xi, \lambda) = \sum_{|\alpha| + \beta + \ell > N-1, \ell \leq N-1, k \leq 2m, |\alpha| \leq 2m - k} \frac{1}{\alpha!} \partial_{\xi}^\alpha a_{2m-k} D^\beta b_{2m-\ell},
\]

with the sum taken for \(|\alpha| + k + \ell > N-1, \ell \leq N-1, k \leq 2m, |\alpha| \leq 2m - k, \quad |

\[
\mathcal{D}_{\alpha, \beta} c_{(N)}(x, \xi, \lambda) \leq C_{\beta'} \sum_{|r|} \left\{ \frac{2^{2m}|\xi| + 2|\xi| + |r|}{|\xi| + |\lambda|^{1/2m}} \right\} (1 + |\xi|)^{-k - |\alpha| - \ell - |r|}
\]

for any \(x \in \Omega, \xi \in \mathbb{R}^n\) and \(\lambda \in \Lambda\), with the sum taken as in (3.12).

PROOF. With the use of (3.3), (3.4), (3.7) and (3.8), a direct calculation shows (3.12). Then, from (3.6) we get (3.13).

We take a function \(\psi\) in \(C_{0}^{\infty}(\Omega)\) such that

\[
(3.14)\quad \psi(\xi) = \begin{cases} 0 & \text{if } |\xi| < 1, \\ 1 & \text{if } |\xi| > 2 \end{cases}
\]

which will be fixed in what follows. Then we set for \(x \in \Omega, u \in C_{0}^{\infty}(\Omega)\)

\[
(3.15)\quad \tilde{B}_N(\lambda) u = \int b_{(N)}(x, \xi, \lambda) \psi(\xi) \hat{u}(\xi) e^{i \langle x, \xi \rangle} d\xi.
\]
§ 4. Fractional powers

It is well known that for $0 < s < 1$ the fractional powers of the positive self-adjoint operator $A$ are defined by

\begin{equation}
A^s = \int_{\mathbb{R}} t^s dE_t = \frac{A}{2\pi i} \int_{\Gamma} \lambda^{s-1}(A-\lambda)^{-1} d\lambda
\end{equation}

where \{E_t\} is the spectral resolution of $A$ and $\Gamma$ is an oriented curve lying in the resolvent set of $A$ consisting of two segments:

\{re^{i\theta} + c_0/2: 0 \leq r < \infty\} \quad \text{and} \quad \{re^{-i\theta} + c_0/2: 0 < r < \infty\}

with $0 < \theta < \pi/2$. We orient $\Gamma$ such that it runs from $\infty e^{i\theta} + c_0/2$ to $\infty e^{-i\theta} + c_0/2$.

Now, we define the pseudo-differential operators by

\begin{equation}
P^s_\lambda u = \tilde{A}(x, D) \frac{1}{2\pi i} \int_{\Gamma} \lambda^{s-1} \tilde{B}_s(\lambda) u d\lambda \quad \text{for } u \in C^\infty_0(\Omega).
\end{equation}

For $|\xi| > 1$, $\lambda \in \Gamma$ we have the estimate $|\tilde{A}_s(x, \xi) - \lambda| > C$. Then Fubini’s theorem and Cauchy’s integral formula show that

\begin{equation}
\frac{1}{2\pi i} \int_{\Gamma} \lambda^{s-1} \left( \frac{\hat{\tau}_x(x, \xi)}{(\tilde{A}_s(x, \xi) - \lambda)^{k+1}} \psi(\delta(x')\xi') \hat{u}(\xi) e^{it(x, \xi)} d\xi \right) d\lambda
\end{equation}

\begin{equation}
= \binom{s-1}{k} \int (\tilde{A}_s(x, \xi))^{s-1-k} \psi(\delta(x')\xi') \gamma_{k, s}(x, \xi) \hat{u}(\xi) e^{it(x, \xi)} d\xi
\end{equation}

where $\binom{s-1}{k}$ is the binomial coefficient. Hence from (3.5), (3.7) and (4.3) we see that there exist $p_s^*(x, \xi) \in C^\infty(\Omega \times \mathbb{R}^n)$ such that

\begin{equation}
|D_\xi^\alpha \hat{p}_s^*(x, \xi) \leq C_{s\delta}(1 + |\xi|)^{2ms - |\beta|},
\end{equation}

\begin{equation}
|D_\xi^\beta \{p_s^*(x, \xi) - (\tilde{A}_s(x, \xi)) \psi(\delta(x')\xi)\} | \leq C_{s\delta}(1 + |\xi|)^{2ms - 1 - |\beta|}
\end{equation}

for any $x \in \Omega$, $\xi \in \mathbb{R}^n$ and for $u \in C^\infty_0(\Omega)$

\begin{equation}
P^s_\lambda u = \int p^*_s(x, \xi) \hat{u}(\xi) e^{it(x, \xi)} d\xi.
\end{equation}

§ 5. Construction of an asymptotic solution

In this section we follow the method developed by Hörmander [5] in constructing an asymptotic solution of $D_t + P^s_\lambda 2m$; however, we need some modification since we must construct it near the boundary of $\Omega$.

In the following we write
Asymptotic behavior of spectral functions of elliptic operators

I. Phase function $S(t, x, y, \xi)$.

Following Hörmander [5], we set

$$S(t, x, y, \xi) = \varphi(x, y, \xi) - t \cdot a(y, \xi)$$

where $\varphi$ satisfies the following conditions:

(i) $a(x, P_x \varphi(x, y, \xi)) = a(y, \xi)$,

(ii) $\varphi(x, y, \xi) = 0$ when $\langle x - y, \xi \rangle = 0$,

(iii) $P_x \varphi(x, y, \xi)_{|x-y} = \xi$,

(iv) $\varphi(x, y, \xi) = |\xi| \varphi(x, y, \xi/|\xi|)$

where $P_x = (\partial/\partial x_1, \ldots, \partial/\partial x_n)$. For $x^0 \in \Omega$ we write $B(x^0) = \{x : |x - x^0| < d^0 \vartheta(x^0)\}$. The classical method of characteristics shows that for sufficiently small $d^0$ we can construct such a function $\varphi$ in $C^\infty(B(x^0) \times B(x^0) \times (\mathbb{R}^n - \{0\}))$. Then, we have the estimates

$$|D^\alpha \varphi(x, y, \xi)| \leq C_{\alpha \beta} |\xi|^{1-|\beta|}$$

for any $x, y \in B(x^0)$ and $\xi \in \mathbb{R}^n - \{0\}$ where $C_{\alpha \beta}$ are constants independent of $\vartheta(x^0)$.

II. For $s \in [0, 1]$ we set

$$\varphi_s(x, y, \xi) = \langle x - y, \xi \rangle + s[\varphi(x, y, \xi) - \langle x - y, \xi \rangle]$$

**Lemma 5.1.** For $s \in [0, 1]$ there exist $b^s_{j, k}(x, y, \xi) \in C^\infty(B(x^0) \times B(x^0) \times (\mathbb{R}^n - \{0\}))$, $(j, k = 1, \ldots, n)$ such that

$$|D^\beta \varphi_s(x, y, \xi)| \leq C_{\alpha \beta} |\xi|^{1-|\beta|}$$

for any $x, y \in B(x^0)$, $\xi \in \mathbb{R}^n - \{0\}$ where $C_{\alpha \beta}$ are constants independent of $\vartheta(x^0)$, $s$ and for $j = 1, \ldots, n$

$$x_j - y_j = \sum_{k=1}^n b^s_{j, k}(\partial/\partial \xi_k) \varphi_s$$

on $B(x^0) \times B(x^0) \times (\mathbb{R}^n - \{0\})$.

**Proof.** From (iii), (iv) of (5.1), using Taylor’s formula, we have

$$\varphi(x, y, \xi) = \langle x - y, \xi \rangle + \sum_{j=1}^n (x_j - y_j) \varphi_j(x, y, \xi)$$

where
\[ \varphi_j(x, y, \xi) = 2 \sum_{k=1}^n \frac{(x_k - y_k)}{(1 + \delta_{j,k})} \int_0^1 (1 - \tau) \frac{\partial^2 \varphi(y + \tau(x-y), y, \xi)}{\partial x_j \partial x_k} d\tau \]

\( (\delta_{j,k}: \text{Kronecker's delta}) . \) Hence, we have for \( j = 1, \ldots, n \)

\[ (\partial/\partial \xi_j) \varphi_s = x_j - y_j + s \sum_{k=1}^n (\partial/\partial \xi_k) \varphi_k \cdot (x_k - y_k). \]

That is,

\[ \begin{pmatrix}
  (\partial/\partial \xi_1) \varphi_s \\
  \vdots \\
  (\partial/\partial \xi_n) \varphi_s
\end{pmatrix} = \begin{pmatrix}
  1 & 0 \\
  0 & 1
\end{pmatrix} + s \begin{pmatrix}
  (\partial/\partial \xi_1) \varphi_s \\
  \vdots \\
  (\partial/\partial \xi_n) \varphi_s
\end{pmatrix} \begin{pmatrix}
  x_1 - y_1 \\
  \vdots \\
  x_n - y_n
\end{pmatrix}. \]

We note that \( |(\partial/\partial \xi_j) \varphi_s(x, y, \xi)| \leq C|x - y| \leq 2Cd^\theta. \) Hence, we may assume that the matrix \( I_n + s((\partial/\partial \xi_j) \varphi_s) \) has an inverse. That is, there exist \( b_{jk}(x, y, \xi) \in C^\infty(B(x^0) \times B(x^0) \times (R^n - \{0\})) \) which satisfy (5.4). Then, immediately we obtain (5.3). \( \text{q.e.d.} \)

We note that for \( u \in C_0^\infty(B(x^0)) \)

\[ u(x) = \int \int u(y)e^{i(x-y, \xi)}dyd\xi \]

\[ = \lim_{\epsilon \to 0} \int \int u(y)\chi(\epsilon \xi)e^{i(x-y, \xi)}dyd\xi \]

where \( \chi \in C_0^\infty(R^n) \) such that \( 0 \leq \chi \leq 1, \chi \equiv 1 \) for \( |\xi| < 1/2, \chi \equiv 0 \) for \( |\xi| > 2/3 \) which will be fixed in what follows. Now, we write:

\[ \int \int u(y)\chi(\epsilon \xi)e^{i(x-y, \xi)}dyd\xi = \int \int u(y)\chi(\epsilon \xi)\psi(\partial(x^0)\xi)e^{i(x-y, \xi)}dyd\xi 
+ \int \int u(y)\chi(\epsilon \xi)(1 - \psi(\partial(x^0)\xi))e^{i(x-y, \xi)}dyd\xi. \]

We consider

\[ \int \chi(\epsilon \xi)\psi(\partial(x^0)\xi)e^{i\varphi(x, y, \xi)}d\xi. \]

Taylor's formula shows that for any positive integer \( N \)

\[ \int \chi(\epsilon \xi)\psi(\partial(x^0)\xi)e^{i(x-y, \xi)}d\xi 
= \sum_{k=0}^{N-1} \frac{(-i)^k}{k!} \int \chi(\epsilon \xi)\psi(\partial(x^0)\xi)\{\varphi(x, y, \xi) - \langle x - y, \xi \rangle \}^k e^{i\varphi(x, y, \xi)}d\xi 
+ (-i)^N \int \chi(\epsilon \xi)\psi(\partial(x^0)\xi)\varphi(x - y, \xi)^N e^{i\varphi(x, y, \xi)}d\xi \frac{s^{N-1}}{(N-1)!}ds. \]

Noting that
we can integrate by parts after the replacement \((\partial \varphi / \partial x_j) e^{i\omega x_j} = -i(\partial / \partial x_j) e^{i\omega x_j}\) and if we replace one factor \(x_j - y_j\) by the expression (5.4) we can reduce the order by one unit without affecting the other factors \(x_j - y_j\). Repeating this argument \(2k\) times, we see that there exist \(I_k^i, J_k^i, L_k^i \in C^\infty(B(x^i) \times B(x^i) \times (\mathbb{R}^n - \{0\}))\) such that

\[
\int \chi(\varepsilon \xi) \psi(\partial(x^i)\xi)(\varphi - \langle x - y, \xi \rangle)^{\varepsilon} e^{i\omega x^i} d\xi
\]

\[
= \int I_k^i(x, y, \xi) \psi(\partial(x^i)\xi) \chi(\varepsilon \xi) e^{i\omega x^i} d\xi
\]

\[
+ \int J_k^i(x, y, \xi) \chi(\varepsilon \xi) e^{i\omega x^i} d\xi
\]

\[
+ \int L_k^i(x, y, \xi) e^{i\omega x^i} d\xi
\]

where \(I_k^i\) is the sum of terms which do not contain the derivatives of \(\psi(\partial(x^i)\xi), \chi(\varepsilon \xi), J_k^i\) is the sum of terms which contain at least one derivative of \(\psi(\partial(x^i)\xi)\) but not the derivatives of \(\chi(\varepsilon \xi)\) and \(L_k^i\) is the sum of terms which contain at least one derivative of \(\chi(\varepsilon \xi)\). From (iv) of (5.1) and (5.3), we obtain

\[
|D_z^x J_k^i(x, y, \xi)| \leq C_k |\xi|^{-k},
\]

(5.7)

\[
|D_z^x J_k^i(x, y, \xi)| \leq C_k \sum_{i=1}^{2k} |\xi|^{-k+i} \partial(x^i)^i
\]

for any \(x, y \in B(x^i)\) and \(\xi \in \mathbb{R}^n - \{0\}\) where \(C_k\) are constants independent of \(\partial(x^i)\) and \(s\). Moreover, noting that \(\psi(\partial(x^i)\xi) = 1\) if \(|\xi| > 2\partial(x^i)^{-1}\), we see that

\[
J_k^i(x, y, \xi) = 0 \quad \text{if } |\xi| > 2\partial(x^i)^{-1}.
\]

(5.9)

Since \(L_k^i\) contain derivatives of \(\chi(\varepsilon \xi)\), we see that as \(\varepsilon \to 0\)

\[
\int L_k^i(x, y, \xi) u(y) e^{i\omega y} dy d\xi \to 0 \quad \text{in } C^\infty(B(x^i))
\]

uniformly in \(s \in [0, 1]\). (See Hörmander [6], Kumano-go [7]). We set

\[
I(x, y, \xi) = 1 + \sum_{k=1}^{N-1} \frac{(-\partial)^k}{k!} I_k^i(x, y, \xi),
\]

(5.11)

\[
J(x, y, \xi) = \sum_{k=1}^{N-1} \frac{(-\partial)^k}{k!} J_k^i(x, y, \xi),
\]

(5.12)
Hence we obtain the equality:

\[
\int \int \chi(\varepsilon \xi)u(y)e^{i(x-y,t)}dyd\xi = \int \int \chi(\varepsilon \xi)I(x, y, \xi)\psi(\delta(x^0)\xi)e^{i\varphi(x,y,t)}dyd\xi + K^*u
\]

where

\[
K^*u = \int \int \chi(\varepsilon \xi)(1 - \psi(\delta(x^0)\xi))e^{i(x-y,t)}u(y)dyd\xi
\]

\[
+ \int \int J(x, y, \xi)\chi(\varepsilon \xi)e^{i\varphi(x,y,t)}u(y)dyd\xi
\]

\[
+ \int \int L^*(x, y, \xi)e^{i\varphi(x,y,t)}u(y)dyd\xi
\]

\[
+ (-i)^N \iint I_{\varepsilon}^*(x, y, \xi)\chi(\varepsilon \xi)e^{i\varphi(x,y,t)}u(y)dyd\xi - \frac{s^{N-1}}{(N-1)!}ds
\]

\[
+ (-i)^N \iint J_{\varepsilon}^*(x, y, \xi)\chi(\varepsilon \xi)e^{i\varphi(x,y,t)}u(y)dyd\xi - \frac{s^{N-1}}{(N-1)!}ds
\]

\[
+ (-i)^N \iint L^*_{\varepsilon}(x, y, \xi)e^{i\varphi(x,y,t)}u(y)dyd\xi - \frac{s^{N-1}}{(N-1)!}ds.
\]

III. We shall derive a product formula concerning pseudo-differential operators and Fourier integral operators. In general, for real number \( \ell \) we denote by \( S^\ell(\Omega) \) the class of functions \( p(x, \xi) \in C^\infty(\Omega \times R^n) \) such that for any compact subset \( K \) of \( \Omega \) and any multi-index \( \alpha, \beta \)

\[
|D^\alpha_\xi D^\beta x p(x, \xi)| \leq C_{\alpha, \beta, K}(1 + |\xi|)^{\ell - |\beta|}
\]
on \( K \times R^n \) for a constant \( C_{\alpha, \beta, K} \). For \( q(x, y, \xi) \in S^\ell(\Omega \times \Omega) \) and \( \phi(x, y, \xi) \in S^\ell(\Omega \times \Omega) \) we define a Fourier integral operator by

\[
Q^*u = \int \int q(x, y, \xi)\chi(\varepsilon \xi)e^{i\varphi(x,y,t)}u(y)dyd\xi
\]

for \( u \in C^\infty_0(\Omega) \) where \( \chi \) is the same function as in (5.14). It is well known that for \( p(x, \xi) \in S^\ell(\Omega) \) the pseudo-differential operator is defined by

\[
Pu = \int p(x, \xi)\tilde{u}(\xi)e^{i(x,t)}d\xi.
\]

We can form the composition of the two operators, if we insert a factor \( f \in C^\infty_0(\Omega) \) between them. Then we obtain the following formula for any positive integer \( N \)
Asymptotic behavior of spectral functions of elliptic operators

\begin{equation}
(5.16) \quad P^i Q^j u = \int \int r(x, z, \eta) \chi(\eta) u(z) e^{i(\xi, \eta)} d\eta d\xi,
\end{equation}

\begin{align}
(5.17) \quad r(x, z, \eta) &= \sum_{|a| \leq N} \frac{1}{a!} P^{(a)}(x, \Gamma z \phi(x, z, \eta)) \Gamma_z [f(y) q(y, z, \eta)] e^{i(\xi, \eta)} |_{y = x} \\
&\quad + N \sum_{|a| \leq N} \int \int \frac{(1 - \tau)^{N-1}}{a!} P^{(a)}(x, \Gamma z \phi(x, z, \eta) + \tau \xi) \xi f(y) q(y, z, \eta) \\
&\quad \times e^{i(\xi, \eta)} |_{y = x} + \xi d\tau d\eta d\xi
\end{align}

where

\begin{equation}
(5.18) \quad h(y, x, z, \eta) = \phi(y, z, \eta) - \phi(x, z, \eta) - \langle y - x, \Gamma z \phi(x, z, \eta) \rangle.
\end{equation}

Here and in what follows, \( \partial_i p(x, \xi) \) is denoted by \( p^{(i)}(x, \xi) \). Particularly, in the case of \( N = 3 \), we have

\begin{align}
r(x, z, \eta) &= p(x, \Gamma z \phi(x, z, \eta)) f(x) q(x, z, \eta) \\
&\quad + \sum_{j=1}^n P^{(j)}(x, \Gamma z \phi(x, z, \eta)) \Gamma_{z_j} [f(x) q(x, z, \eta)] \\
&\quad + \frac{1}{2} \sum_{j,k=1}^n P^{(j,k)}(x, \Gamma z \phi(x, z, \eta)) f(x) q(x, z, \eta)(\partial_x / \partial z_j \partial z_k) \phi(x, z, \eta) \\
&\quad + \frac{1}{2} \sum_{j,k=1}^n P^{(j,k)}(x, \Gamma z \phi(x, z, \eta)) \Gamma_{z_j} \Gamma_{z_k} [f(x) q(x, z, \eta)] \\
&\quad + 3 \sum_{|a| = 3} \frac{1}{a!} \int \int \frac{(1 - \tau)^{N-1}}{a!} P^{(a)}(x, \Gamma z \phi(x, z, \eta) + \tau \xi) \xi f(y) q(y, z, \eta) \\
&\quad \times e^{i(\xi, \eta)} |_{y = x} + \xi d\tau d\eta d\xi.
\end{align}

Here and in what follows, we write \( P^{(j)}(x, \xi) = (\partial / \partial z_j) p(x, \xi) \) and \( P^{(j,k)}(x, \xi) = (\partial / \partial z_j \partial z_k) p(x, \xi) \).

IV. We consider the following hyperbolic partial differential equations.

\begin{equation}
(5.19) \quad \begin{cases}
D_i q_i + \sum_{j=1}^n a^{(j)}(x, \Gamma z \phi(x, z, \eta)) \Gamma_{z_i} q_i + g q_i = 0, \\
q_0(0, x, z, \eta) = \phi_0 \left( \frac{x - x_0}{\delta(x)} \right) I(x, z, \eta) \psi(\delta(x) \eta)
\end{cases}
\end{equation}

where \( \phi_0 \in C_0^\infty(\mathbb{R}^n) \) such that \( \phi_0(0) = 1, \phi_0 = 0 \) for \( |x| > 1/4 \) and

\begin{equation}
(5.20) \quad \begin{cases}
g(x, z, \eta) = \frac{1}{2} \sum_{j=1}^n a^{(j,k)}(x, \Gamma z \phi(x, z, \eta))(\partial_x / \partial z_j \partial z_k) \phi(x, z, \eta) \\
+ \tilde{p}(x, \Gamma z \phi(x, z, \eta)), \\
\tilde{p}(x, \xi) = p(x, \xi) - a(x, \xi).
\end{cases}
\end{equation}
For any positive integer $\nu$ we can determine $q_{-\nu}$ inductively by

\begin{equation}
\left \{ \begin{array}{l}
D_q q_{-\nu} + \sum_{j=1}^{n} a^{(j)}(x, \varphi(x, z, \eta))D_{x_j} q_{-\nu} + g q_{-\nu} + Tq_{-\nu} = 0, \\
q_{-\nu}(0, x, z, \eta) = 0
\end{array} \right.
\end{equation}

where

\begin{equation}
Tq_{-\nu} = \frac{1}{2} \sum_{j, k=1}^{n} p^{(j, k)}_{\nu}(x, \varphi(x, z, \eta))D_{x_j} D_{x_k} q_{-\nu} + \sum_{j=1}^{n} \frac{1}{\alpha!} p^{(j)}_{\nu}(x, \varphi(x, z, \eta))D_{\varphi} q_{-\nu}(t, y, z, \eta) e^{i\varphi(y, x, z, \eta)}|_{y=x} \\
+ \sum_{j=1}^{n} \tilde{p}^{(j)}_{\nu}(x, \varphi(x, z, \eta))D_{\varphi} q_{-\nu} + \frac{1}{2} \sum_{j, k=1}^{n} \tilde{p}^{(j, k)}_{\nu}(x, \varphi(x, z, \eta))\partial_{x_j} \partial_{x_k} \varphi(x, z, \eta) q_{-\nu}.
\end{equation}

Here and in what follows, we write

\begin{equation}
h(y, x, z, \eta) = \varphi(y, z, \eta) - \varphi(x, z, \eta) - \langle y - x, \varphi(x, z, \eta) \rangle.
\end{equation}

We note that there exists a constant independent of $x^0$, $z$, $\eta$ such that

\[ \sup_{x \in B(x^0)} \sum_{j=1}^{n} |a^{(j)}(x, \varphi(x, z, \eta))| \leq C. \]

Hence, the theory of hyperbolic equations shows that for a sufficiently small $t^0$, there exist the solutions $q_{-\nu}(t, x, z, \eta)$ on $(-\delta(x^0)t^0, \delta(x^0)t^0)$ such that

\begin{equation}
q_{-\nu}(t, x, z, \eta) \in C^0_0 \left( \left\{ x : |x - x^0| < \frac{d^0}{2} \delta(x^0) \right\} \right)
\end{equation}

for any $t \in (-\delta(x^0)t^0, \delta(x^0)t^0)$, $x \in B(x^0)$, $\eta \in R^\nu - \{0\}$.

V. We shall estimate $q_{-\nu}$.

**Lemma 5.2.** We get the following equalities and estimates:

\[ q_{-\nu}(t, x, z, \eta) = 0 \quad \text{if} \quad |\eta| < \delta(x^0)^{-1} \quad (\nu = 0, 1, 2, \ldots), \]

\begin{equation}
|D_{\partial_\eta} q_{-\nu}(t, x, z, \eta)| \leq C |\eta|^{-\delta(x^0)^{-\nu - |\eta| - k}}
\end{equation}

($\nu = 0, 1, 2, \ldots$), for any $t \in (-\delta(x^0)t^0, \delta(x^0)t^0)$, $x, z \in B(x^0)$, $\eta \in R^\nu - \{0\}$.

For the proof of Lemma 5.2 we shall use the following lemmas:

**Lemma 5.3** (Haar's inequality). We set $D = \{(t, x) \in R^\nu : 0 \leq t \leq L, |x_k| \leq L - Mt, (k = 1, \ldots, n)$ where $L, M$ are positive constants). Let $f$ be a real
valued function contained in $C^1(D)$ and suppose that $|f(0,x)| \leq C$ for $|x_k| \leq L$ $(k=1, \ldots, n)$ and $|\partial f(t,x)| \leq M \sum_{k=1}^n |\partial_x f(t,x)| + V|f(t,x)| + W$, for any $(t,x) \in D$ where $C, V, W$ are positive constants. Then the following estimate holds:

$$|f(t,x)| \leq C e^{rt} + \frac{W}{V} (e^{rt} - 1)$$

for any $(t,x) \in D$.

**Lemma 5.4.** Let $v_p = v_p(t, x, \delta, \eta)$ $(p = 1, \ldots, s)$ be solutions of the following equations:

$$\partial_t v_p = \sum_{j=1}^n a_j(x, \delta, \eta) \partial_{x_j} v_p + \sum_{q=1}^s b_{pq}(x, \delta, \eta) v_q + f_p(t, x, \delta, \eta)$$

on $0 \leq t \leq \delta t_0$, $|x_k| \leq d \delta - Mt$ $(k=1, \ldots, n)$

where $\delta \in (0,1)$, $\eta \in \mathbb{R}^n - \{0\}$ are parameters, $d, t_0, M$ are positive constants and $a_j$ (real valued), $b_{pq}, f_p$ are $C^\infty$-functions such that

$$|a_j(x, \delta, \eta)| \leq M,$$  

$$|\partial^\alpha a_j(x, \delta, \eta)| \leq C (|\alpha| > 0),$$

where $\nu$ is some positive integer. Assume that

$$|\partial^\alpha v_p(0, x, \delta, \eta)| \leq C |\eta|^{-\nu - |\alpha|}$$

$p = 1, \ldots, s$

then we have

$$|\partial^\alpha \partial_t v_p(t, x, \delta, \eta)| \leq C |\eta|^{-\nu - |\alpha| - k}$$

$p = 1, \ldots, s$

on $0 \leq t \leq \delta t_0$, $|x_k| \leq d \delta - Mt$ $(k=1, \ldots, n)$.

**Proof.** The case of $|\alpha| = 0, k = 0$: From (2) we have

$$\partial_t \left( \sum_{p=1}^s v_p \overline{v_p} \right) = \sum_{p=1}^s \left( \partial_t v_p \overline{v_p} + v_p \partial_t \overline{v_p} \right)

= -\sum_{p=1}^s \sum_{j=1}^n a_j(\partial_{x_j} v_p \overline{v_p} + v_p \partial_{x_j} \overline{v_p})$$

$$- \sum_{p,q=1}^s (b_{pq} v_q \overline{v_p} + v_p \overline{b_{pq} v_q})$$

$$- \sum_{p=1}^s (f_p \overline{v_p} + v_p \overline{f_p}).$$

Using Schwarz’s inequality, we see that there exist constants $V, W$ such that

$$|\partial_t \left( \sum_{p=1}^s v_p \overline{v_p} \right)| \leq M \sum_{j=1}^n \left| \partial_{x_j} \left( \sum_{p=1}^s v_p \overline{v_p} \right) \right| + V \delta^{-1} \left| \sum_{p=1}^s v_p \overline{v_p} \right| + W \delta \left| \sum_{p=1}^s f_p \overline{f_p} \right|.$$
Hence, from Lemma 5.3 we have

$$\left( \sum_{p=1}^{s} v_p \widetilde{v}_p \right) \leq C |\eta|^{-\nu} \delta^{-\nu}.$$  

From (##), we have for $k=1, \cdots, n$

$$\frac{\partial_{x_k} v_p}{\partial z_k} = \sum_{j=1}^{n} a_j \partial_{x_j}(\partial_{z_k} v_p) + \sum_{q=1}^{s} b_q \partial_{z_k} v_q$$

Hence, in the same way as for $v_p$, we can estimate $\partial_{z_k} v_p$ ($p=1, \cdots, s$, $k=1, \cdots, n$). From the above argument, it is clear that by induction on the order of derivatives of $v_p$, we can prove the present lemma.

**Proof of Lemma 5.2.** We prove (5.25) by induction on $v$.

In the case of $v=0$. We note that

$$q_0(0, x, z, \eta) = \phi_0 \left( \frac{x-x_0}{d(x')} \right) I(x, z, \eta) \psi(\delta(x') \eta) = 0$$

if $|\eta| < \delta(x')^{-1}$. Hence the uniqueness of solution shows that if $|\eta| < \delta(x')^{-1}$, $q_0(t, x, z, \eta) = 0$ on $(-\delta(x')^\alpha, \delta(x')^\alpha)$. Applying Lemma 5.4 to $q_0$, we have

$$|D^\alpha q_0(t, x, z, \eta)| \leq C \delta(x')^{-|\eta|^{-\alpha}}$$  

(5.26)

Assume now the Lemma 5.2 has been proved for some $v-1$, we shall show that it holds for $v$. By the induction assumptions, we have

$$|D^\alpha q_v(t, x, z, \eta)| \leq C |\eta|^{-v} \delta(x')^{-|\eta|^{-\alpha}}$$  

(5.27)

and $Tq_{v-1}(t, x, z, \eta) = 0$ if $|\eta| < \delta(x')^{-1}$. Hence, using (5.27) we get (5.25) in the same way as for the proof for $q_0$. q.e.d.

VI. Now we set $q(t, x, z, \eta) = \sum_{v=0}^{\infty} q_v(t, x, z, \eta)$ and for $u \in C_0^\infty(B(x'))$

$$Q^i(t)u = \int q(t, x, z, \eta) \gamma(\epsilon \eta) e^{i \epsilon \eta(x, z, \eta)} - (1 + \alpha(2, \eta) u(z) dzd\eta.$$  

From (5.16), (5.22) we get

$$Q^i(t)u = (D_i + P_\alpha) R^i(t)u,$$

$$R^i(t)u = \int r(t, x, z, \eta) \gamma(\epsilon \eta) e^{i \epsilon \eta(x, z, \eta)} - (1 + \alpha(2, \eta) u(z) dzd\eta.$$  

where
Asymptotic behavior of spectral functions of elliptic operators

Let \( r(t, x, z, \eta) \)

\[
= \sum_{\nu=0}^{N-1} N \sum_{|z|=N} \frac{1}{\alpha!} \int \int \int \left( 1 - \tau \right)^{N-1} p^0_{\nu}(x, \mathcal{F}_x \phi(x, z, \eta) + \tau \xi) \xi^q \eta (t, x, z, \eta) \times e^{i \phi(y, x, z, \eta) + i \xi^q \eta (t, x, z, \eta)} d\tau dy d\xi
\]

\[
+ \frac{1}{2} \sum_{j=1}^{n} p^0_{\nu,j}(x, \mathcal{F}_x \phi(x, z, \eta)) D_{z_j} D_{z_k} q_N(t, x, z, \eta)
\]

\[
(5.29)
\]

\[
+ 3 \sum_{\nu=1}^{N} \frac{1}{\alpha!} \int \int \int \left( 1 - \tau \right)^{N-1} p^0_{\nu}(x, \mathcal{F}_x \phi(x, z, \eta) + \tau \xi) \xi^q \eta (t, x, z, \eta) \times e^{i \phi(y, x, z, \eta) + i \xi^q \eta (t, x, z, \eta)} d\tau dy d\xi
\]

\[
+ \sum_{j=1}^{n} \hat{p}^0_{\nu,j}(x, \mathcal{F}_x \phi(x, z, \eta)) D_{z_j} q_N(t, x, z, \eta)
\]

\[
+ \frac{1}{2} \sum_{j=1}^{n} \hat{p}^0_{\nu,j}(x, \mathcal{F}_x \phi(x, z, \eta)) \partial_{z_j} \mathcal{F}_x \phi(x, z, \eta) q_N(t, x, z, \eta).
\]

We take a function \( \phi_3 \in C^0_0(\Omega) \) such that \( \phi_3 \equiv 1 \) for \( |x| < 1/2 \), \( \phi_3 \equiv 0 \) for \( |x| > 3/4 \) and we write

\[
\phi_{x,0,z}(x) = \phi_3 \left( \frac{x-x^0}{d^0(\delta(x^0))} \right).
\]

Then, noting \( \phi_{x,0,z} Q^i(t)u = Q^i(t)u \), we have

\[
(5.30)
\]

\[
(D + A^{1/2m}) Q^i(t)u = \phi_{x,0,z} (D + P_N) Q^i(t)u + (A^{1/2m} - \phi_{x,0,z} P_N) Q^i(t)u
\]

\[
= \phi_{x,0,z} R(t)u + (A^{1/2m} - \phi_{x,0,z} P_N) Q^i(t)u.
\]

Let \( \{ E_i \} \) be the spectral resolution of \( A \). Now we set

\[
(5.31)
\]

\[
\hat{E}(t) = \int_0^\infty e^{-tu} dE_{it}.
\]

Then from (5.30) we have

\[
Q^i(t)u = \hat{E}(t) Q^i(0)u + \int_0^t \hat{E}(t-s)(A^{1/2m} - \phi_{x,0,z} P_N) Q^i(s)uds
\]

\[
+ \int_0^t \hat{E}(t-s) \phi_{x,0,z} R^i(s)uds.
\]

We note that

\[
Q^i(0)u = \int \phi_{x,1}(x) I(x, z, \eta) \psi(\delta(x^0) \eta) \eta u(z) e^{i \phi(x, z, \eta)} dxd\eta.
\]

Here and in what follows we write

\[
\phi_{x,1}(x) = \phi_1 \left( \frac{x-x^0}{d^0(\delta(x^0))} \right).
\]
Hence from (5.14) we have

\[ Q'(0)u = \int \phi_{x,z_1}(x) \chi(\eta) \alpha(u(z)e^{i(x-z \cdot \eta)}d\eta - \phi_{x,z_1}(x)K'u. \]

Combining (5.32), (5.33) we have

\[ Q'(t)u - \dot{E}(t)id'u = -\dot{E}(t)\phi_{x,z_1}K'u + \int_0^t \dot{E}(t-s)(A^{1/2m} - \phi_{x,z_1}P_N)Q'(s)uds \]
\[ + \int_0^t \dot{E}(t-s)\phi_{x,z_1}R'(s)uds \]

where we write

\[ id'u = \int \phi_{x,z_1}(x) \chi(\eta) \alpha(u(z)e^{i(x-z \cdot \eta)}d\eta. \]

We take a function \( \rho \) which belongs to \( \mathcal{S}(\mathbb{R}') \) and satisfies the following properties:

\[ \rho(s) > 0 \quad \text{for any } s \in \mathbb{R}', \]
\[ \rho(0) = \int \rho(s)ds = 1, \quad \text{supp } \rho \subset (-\ell, \ell). \]

Here \( \mathcal{S}(\mathbb{R}') \) is the Schwartz space of rapidly decreasing functions. We multiply (5.34) by \( \rho(t/\delta(x^0))e^{it\ell} \) and integrate with respect to \( t \). Then we obtain

\[ \int \rho(t/\delta(x^0))Q'(t)ue^{it\ell}dt - \int \rho(t/\delta(x^0))\dot{E}(t)id'uue^{it\ell}dt \]
\[ = \int \rho(t/\delta(x^0)) \int_0^t \dot{E}(t-s)(A^{1/2m} - \phi_{x,z_1}P_N)Q'(s)uds e^{i\ell t}dt \]
\[ + \int \rho(t/\delta(x^0)) \int_0^t \dot{E}(t-s)\phi_{x,z_1}R'(s)uds e^{i\ell t}dt \]
\[ - \int \rho(t/\delta(x^0))\dot{E}(t)\phi_{x,z_1}K'ue^{i\ell t}dt. \]

§ 6. Estimates of the asymptotic solution

In this section we shall estimate the terms of (5.36).

I. We note that

\[ \int \rho(t/\delta(x^0))Q'(t)ue^{i\ell t}dt \]
\[ = \int \rho(t/\delta(x^0)) \left\{ \int q(t, x, z, \eta)\chi(\eta)e^{i(x-z \cdot \eta)}d\eta \right\} u(z)dzd\eta \]
\[ = \int \left\{ \int q(z, \eta) \left\{ \int \rho(t/\delta(x^0))q(t, x, z, \eta)e^{i\ell t}dt e^{i(x-z \cdot \eta)}d\eta \right\} \right\} u(z)ds. \]
Lemma 6.1. There exists a constant $C$ independent of $\delta(x^0)$ such that

\[
\int \left| \int \delta(t/\delta(x^0)) q(t, x, z, \gamma) e^{i(t-t_0)(z, \gamma)} \, dt \right| \, d\gamma 
\leq C \left\{ \sum_{k=1}^{n} \delta(x^0)^{-k+1} |\lambda|^{n-k} \right\}.
\]

Proof. We set

\[
I(\gamma) = \int \delta(t/\delta(x^0)) q_{-}(t, x, z, \gamma) e^{i(t-t_0)(z, \gamma)} \, dt.
\]

Noting that $D \delta(t/\delta(x^0)) = (\lambda - a(z, \gamma)) e^{i(t-t_0)(z, \gamma)}$, we integrate by parts with respect to $t$. Then for any integer $j \geq 0$ we have

\[
I(\gamma) = \frac{(-1)^j}{(\lambda - a(z, \gamma))^j} \int D^j \delta(t/\delta(x^0)) q_{-}(t, x, z, \gamma) e^{i(t-t_0)(z, \gamma)} \, dt.
\]

From Lemma 5.2, noting that $\text{supp} \, \delta(t/\delta(x^0)) \subset (-\delta(x^0)^{-j}, \delta(x^0)^{-j})$, we have that $I(\gamma) = 0$ if $|\gamma| < \delta(x^0)^{-j}$ and

\[
|I(\gamma)| \leq \frac{C}{|\lambda - a(z, \gamma)|^j} |\gamma|^{-j} \delta(x^0)^{-j+1}.
\]

Hence we obtain for $j \geq n+1$

\[
\int |I(\gamma)| \, d\gamma \leq C \int_{|\gamma| > \delta(x^0)^{-1}} \frac{|\gamma|^{-j} \delta(x^0)^{-j+1}}{1 + \delta(x^0)^{-j} |\lambda - a(z, \gamma)|^j} \, d\gamma 
\leq C \int \frac{\delta(x^0)}{1 + \delta(x^0)^{-j} |\lambda - a(z, \gamma)|^j} \, d\gamma.
\]

Now we set $m(z, \gamma) = m(\gamma; a(z, \gamma) \leq \sigma)$. Here and in what follows, $m$ denotes the Lebesgue measure. Nothing that $d\bar{m}(z, \sigma) = \frac{\sigma^{n-1}}{n} \, d\sigma$ we have

\[
\int |I(\gamma)| \, d\gamma \leq C \delta(x^0) \int_{0}^{\infty} \frac{\sigma^{n-1}}{1 + \delta(x^0)^{-j} |\lambda - \sigma|^j} \, d\sigma 
\leq C \delta(x^0) \int_{0}^{\infty} \frac{(|\lambda| + \sigma)^{n-1}}{1 + \delta(x^0)^{-j} |\lambda - \sigma|^j} \, d\sigma 
\leq C \delta(x^0) \sum_{k=1}^{n} |\lambda|^{-k} \delta(x^0)^{-j+1} \int_{0}^{\infty} \frac{\sigma^{n-1}}{1 + \sigma^j} \, d\sigma 
\leq C \left\{ \sum_{k=1}^{n} |\lambda|^{-k} \delta(x^0)^{-k+1} \right\}.
\]

q.e.d.

From Lemma 6.1, noting $\chi(0) = 1$, we have
\[
\lim_{\varepsilon \to 0} \int \rho(t/\varepsilon(x^0))Q'(t)ue^{it\xi}dt
\]
(6.2)
\[= \int \left\{ \int \int \rho(t/\varepsilon(x^0))q(t, x, z, \gamma) e^{it(x-z, \gamma)} dt e^{i\xi(x,z,\gamma)} d\gamma \right\} u(z)dz.\]

II. Noting that as $\varepsilon \to 0$ id$^\star$u $\to \phi_{2\pi,1}u$ in $C_0^\infty(B(x^0))$, we have
\[
\lim_{\varepsilon \to 0} \int \rho(t/\varepsilon(x^0))\dot{E}(t)id^\star ue^{it\xi}dt=\int \rho(t/\varepsilon(x^0))\dot{E}(t)\phi_{2\pi,1}ue^{it\xi}dt.
\]

We note that the following equality holds:
\[
\int \rho(t/\varepsilon(x^0))\dot{E}(t)\phi_{2\pi,1}ue^{it\xi}dt
\]
(6.3)
\[= \int 2\pi \left\{ \int \delta(x^0)\rho(\delta(x^0)(\lambda-\mu))de(x, z, \mu_\infty) \right\} \phi_{2\pi,1}(z)u(z)dz.
\]

III. We shall prove two lemmas which we make frequent use of in what follows.

**Lemma 6.2.** Let $p(x, \xi) \in S^{m'}(\Omega)$ such that $p(x, \xi)=0$ for $|\xi|<\delta(x^0)^{-1}$ and
\[
|D^\alpha_x \partial^\beta_\xi p(x, \xi)| \leq C_{m'} \sum_{\ell=0}^L (1+|\xi|)^{m'-\ell-1}(|\delta(x^0)|^{-\ell-1}|\sigma|)
\]
where $L$ is some positive integer. We set for any integer $N' \geq 1$
\[
U[p, q_{-\gamma}, N'](t, x, z, \gamma)
\]
(6.4)
\[= \frac{1}{\alpha!} \int \int \int \int (1-\tau)^{N'-1} p^{(\alpha)}(x, F_\xi \rho + \tau \xi)^\sigma q_{-\gamma}(t, y, z, \gamma)
\]
\[\times e^{i\xi(y-x, z, \gamma) + i\xi(y-x, t, \gamma)} d\xi dy d\gamma.
\]

and
\[
U[p, q_{-\gamma}, 0]=\int \int p(x, \xi)q_{-\gamma}(t, y, z, \gamma)e^{i\xi(y-x, z, \gamma) - i\xi(y-x, t, \gamma)} dy d\gamma.
\]

Then we get the following equality and estimates:
\[
U[p, q_{-\gamma}, N'](t, x, z, \gamma)=0 \quad \text{if } |\gamma|<\delta(x^0)^{-1}
\]
(6.6)
\[|D^\xi_x U[p, q_{-\gamma}, N'](t, x, z, \gamma)| \leq C_{m'} |\gamma|^{m'-N'/2-1} \delta(x^0)^{-N'/2-1}.
\]

(N$'=0, 1, \cdots$). Particularly in the case of $N'=3, m'=1$ we get
\[
|D^\xi_x U[p, q_{-\gamma}, 3]| \leq C_{m'} |\gamma|^{-1-\delta(x^0)^{-2-1}m'}.
\]

**Proof.** We follow the method of Kumano-go [7]. From Lemma 5.2 (6.6) follows immediately. We note that
Asymptotic behavior of spectral functions of elliptic operators

\[ U[p, q\ldots, N'] - U[p, q\ldots, 0] \]

\[ = - \sum_{|\alpha| \leq N'} \frac{1}{\alpha!} p^{(\alpha)}(x, F_\tau \varphi) D_\tau[q\ldots(t, y, z, \eta)e^{it\langle y, z, \eta \rangle}]_{\eta = \tau}. \]

Now we set

\[ \Phi = (\Phi_1, \ldots, \Phi_n) = \xi - \int_0^\infty F_\tau \varphi(y + \tau(x - y), z, \eta) d\tau \]

and \( \Phi = (\Phi_1, \ldots, \Phi_n) = (\Phi_1 + \langle x - y, \partial_{\eta} \Phi \rangle, \ldots, \Phi_n + \langle x - y, \partial_{\eta_n} \Phi \rangle). \) Then we have

\[ \varphi(y, x, \eta) - \varphi(x, z, \eta) + \langle x - y, \xi \rangle = \langle x - y, \Phi \rangle. \]

We note that there exists a constant \( C \) such that for any \( x, z \in B(x_0), \eta \in \mathbb{R}^n, \) \( |F_\tau \varphi(x, z, \eta)| \leq C \). Hence we have

\[ |\Phi - \xi - F_\tau \varphi(x, \eta)| \leq C|x - y| + |\eta|, \quad |\Phi + \Phi| \leq C|x - y| + |\eta|. \]

For sufficiently small \( d \), these inequalities show that

\[ |\Phi(y, x, \xi, \eta)| \leq C(|\xi| + |\eta|) \]

for \( x, y, z \in B(x_0) = \{x: |x - x_0| < d, \varphi(x')\} \), \( x, \eta \in \mathbb{R}^n \) such that \( |\Phi - F_\tau \varphi(x, \eta)| \geq C_0|\eta|/4 \). We take a function \( \zeta \in C_0^\infty(\mathbb{R}^n) \) such that \( \zeta(\xi) \equiv 1 \) for \( |\xi| < 1/2, \zeta(\xi) \equiv 0 \) for \( |\xi| > 1 \). Now we set \( \zeta = 1 - \zeta(2(\xi - F_\tau \varphi(x, \eta))/C_0|\eta|) \) and \( \xi_0 = 1 - \zeta_0 \). Then we have

\[ U[p, q\ldots, 0] = \int \int p(x, \xi)\zeta_0 q\ldots(t, y, z, \eta)e^{it\langle y, z, \eta \rangle} d\xi d\eta \]

\[ + \int \int p(x, \xi)\zeta_0 q\ldots(t, y, z, \eta)e^{it\langle y, z, \eta \rangle} d\xi d\eta \]

\[ = I_1 + I_2. \]

Set \( M_y = -i|\Phi|^2 \sum_{j=1}^n \Phi_j \partial_j \partial_y \). Noting that \( M_y e^{it\langle y, x \rangle} = e^{it\langle y, x \rangle} \), we have for any integer \( k \geq 0 \) by integration by parts

\[ I_1 = \int \int p(x, \xi)\zeta_0 ((M_y)^{k} q\ldots(t, \xi) e^{it\langle y, x \rangle} d\xi d\eta, \]

where \( M_y \) denotes the transposed operator of \( M_y \). We note that \( |\Phi| \geq C(|\xi| + |\eta|) \) on \( supp \zeta_0 \) and that \( m \) \( supp q\ldots(t, \xi, z, \eta) \leq C \delta(x)^n \). Hence from Lemma 5.2 we have

\[ |D_\xi I_1| \leq C \sum_{l=0}^k \sum_{j=0}^{l} \int_{\mathcal{E}} (1 + |\xi|)^{m-l} \delta(x)^{j-l} |\xi|^{j-l} d\xi \]

where \( \mathcal{E} = \{\xi: |\xi| > \delta(x)^{-1}\} \). We note that for sufficiently large \( k \)
\[
\int_{\mathbb{R}} \left(1 + |\xi|^{m'} + |\eta| + |\xi|^{m'} + k + j + \ell + n\right) d\xi
\leq \begin{cases} 
C |\eta|^{-k + f + m' - \ell + n} & \text{if } m' - \ell \geq 0, \\
C |\eta|^{-k + f + n} \delta(x')^{f - m'} & \text{if } m' - \ell < 0.
\end{cases}
\]

Hence, noting (6.6), we get if \( k \) is sufficiently large
\[
(6.9) \quad |D^k I| \leq C |\eta|^{m' - N'/2 + j} \delta(x')^{-N'/2 - |\alpha|}.
\]

We set \( \theta_{\alpha} = 1 - \zeta(|\eta|^{1/2} |x - y|) \) and \( \theta_0 = 1 - \theta_{\alpha} \). Then we have
\[
I_5 = \iint p(x, \xi)\xi_\alpha \theta_{\alpha} q_\alpha(t, y, z, \eta)e^{i(x-y}\phi} e^{-i\xi \xi} dyd\xi
\]
\[
+ \iint p(x, \xi)\xi_\alpha \theta_{\alpha} q_\alpha(t, y, z, \eta)e^{i(x-y}\phi} e^{-i\xi \xi} dyd\xi
\]
\[
= I_5 + I_6.
\]

Noting that \( -|x-y|^{2} A_i e^{i(x-y)\phi} = e^{i(x-y)\phi}(A_i = \sum_{j=1}^{n}(\partial_j \xi)) \), we have, for any integer \( k \geq 0 \), by integration by parts
\[
I_5 = \iint A_i^k p(x, \xi)\xi_\alpha \theta_{\alpha} q_\alpha(t, y, z, \eta)|x-y|^{-2k} e^{i(x-y)\phi} dyd\xi.
\]

We note that \( |x-y|^{-1} \leq 2 |\eta|^{1/2} \) and \( c_0 |\gamma|/2 < |\xi| < C |\eta| \) on \( \text{supp } \xi \theta_{\alpha} \). Hence we have
\[
|D^k I| \leq C \sum_{k=0}^{n} \sum_{j=1}^{n} \left(1 + |\xi|^{m'} - N'/2 + j\right) \delta(x')^{-N'/2 - |\alpha|}.
\]

Noting (6.6), we get for sufficiently large \( k \)
\[
|D^k I| \leq C |\eta|^{m' - N'/2 + j} \delta(x')^{-N'/2 - |\alpha|}.
\]

We note that
\[
I_4 = \iint p(x, F_{x}\phi + \xi)\xi_\alpha \theta_{\alpha} (2\xi/c_0 |\eta|) q_\alpha e^{i(x-y)\phi} dyd\xi.
\]

Taylor's formula shows that for any integer \( N'' > N' \)
\[
I_4 = \sum_{|\alpha| < N''} \frac{1}{\gamma!} \iint p^{(\gamma)}(x, F_{x}\phi)\theta_{\alpha} \xi_\alpha (2\xi/c_0 |\eta|) \xi_\gamma q_\alpha e^{i(x-y)\phi} dyd\xi
\]
\[
+ N' \sum_{|\alpha| = N''} \frac{1}{\gamma!} \iint (1 - \tau)^{N'' - 1} p^{(\gamma)}(x, F_{x}\phi + \tau\xi)\theta_{\alpha} \xi_\alpha (2\xi/c_0 |\eta|) \xi_\gamma q_\alpha e^{i(x-y)\phi} dyd\xi d\tau.
\]

Asymptotic behavior of spectral functions of elliptic operators

Noting that \( \theta_{\varphi}|_{y=x}=1 \) and \( \partial^\alpha_{\varphi} \theta_{\varphi}|_{y=x}=0 \) (\( \alpha \neq 0 \)), we have

\[
I_\varphi(x, \varphi|_{y=x}) = \frac{1}{\gamma!} \int \left. \left( x + \varphi|_{y=x} \right) \right|_{y=x} d\tau.
\]

From (6.7) we have

\[
U[p, q_{y}, N'] - I_1 + I_3 + \sum_{|r|<N''} \left( I_3 - I_2 \right) + \sum_{|r|=N''} \frac{1}{\gamma!} \int I_\varphi(\tau) d\tau
\]

\[
= \sum_{|r|<N''} \left( I_3 - I_2 \right) + \sum_{|r|=N''} \frac{1}{\gamma!} \int I_\varphi(\tau) d\tau.
\]

Hence we have

\[
I_{\varphi} = \int p_{\varphi}(x, \varphi|_{y=x}) |\xi|^{-2k} e^{-(2\xi/c_0 |\eta|)} (\xi - \xi_{e^{i\theta}}) e^{i \varphi(x, \varphi)} dy d\xi.
\]

Using

\[
\int_{|\xi| \geq |\eta|/4} |\xi|^{-2k + |r|} d\xi \leq C |\eta|^{-2k + |r| + n}
\]

and noting (6.6), we have for sufficiently large \( k \)

\[
|I_{\varphi}| \leq C |\eta|^{m'' - N'/2 - \frac{1}{2}} e^{(2\xi/c_0 |\eta|)} dy d\xi.
\]

Next we estimate

\[
I_{\varphi}(\tau) = \frac{1}{\gamma!} \int \left. p_{\varphi}(x, \varphi + \tau \xi) \xi (2\xi/c_0 |\eta|) D_{\varphi}(\varphi - \tau e^{i\theta}) e^{i \varphi(x, \varphi)} dy d\xi.
\]
We note that \(|x - y| \leq |\eta|^{-1/2}, c'_\eta|\eta|/2 \leq |L_0 + \sigma| \leq C|\eta|\) and \(|\xi| \leq c'_\eta|\eta|/2\) on \(\text{supp} \zeta(2\xi/c'_\eta|\eta|)\). Hence, noting that \(|h(y, x, z, \eta) \leq C|\eta|^{1/2}\), we have
\[
|I^{(\gamma)}(\tau)| \leq C \sum_{l=0}^L \sum_{j=0}^{N''} |\eta|^{m'-\ell-\nu-N'/2-j/2} d(x')^{\nu-\ell-\nu-j+n} \int \frac{d\xi}{|\xi|^{c'_\eta|\eta|/2}}.
\]
Noting (6.6), we have for sufficiently large \(N''\)
\[
|I^{(\gamma)}(\tau)| \leq C|\eta|^{m'-N'/2-\nu} \delta(x')^{-N'/2-\nu}.
\]
In the same way, we have
\[
|D_\xi I^{(\gamma)}(\tau)| \leq C|\eta|^{m'-N'/2-\nu} \delta(x')^{-N'/2-\nu-|z|}.
\]
Finally we estimate
\[
J_{(\gamma)} = \kappa^{(\gamma)}(x, L_0 + \sigma) D_\gamma^\nu q_{(\nu)}(t, y, z, \eta) e^{\kappa t(y, z, \eta)}
\]
Since \(D_\gamma^\nu h(y, x, z, \eta) = 0\) \((|\gamma| = 1)\), we have
\[
|J_{(\gamma)}| \leq C \sum_{l=0}^L \sum_{j=0}^{[j/2]} |\eta|^{m'-\ell-\nu-\ell+j/2} d(x')^{-\ell-\nu-\ell+j} \delta(x')^{-\ell-\nu-\ell+j}.
\]
where we denote by \([j/2]\) the largest integer which is not larger than \(j/2\). Hence we get
\[
|J_{(\gamma)}| \leq C|\eta|^{m'-\ell-\nu-\ell+j/2} d(x')^{-\ell-\nu-\ell+j} \delta(x')^{-\ell-\nu-\ell+j}.
\]
In the same way, we get
\[
|D_\xi J_{(\gamma)}| \leq C|\eta|^{m'-\ell-\nu-\ell+j/2} d(x')^{-\ell-\nu-\ell+j} \delta(x')^{-\ell-\nu-\ell+j}.
\]
Thus the lemma has been proved completely.

**Lemma 6.3.** Set

\[
V(t, x, z, \eta) = \int \{1 - \psi(\delta(x')\xi)) q_{(\nu)}(t, y, z, \eta) e^{\kappa t(y, z, \eta)} + (z - y, \xi) dyd\xi.
\]
Then we have that
\[
V(t, x, z, \eta) = 0 \quad \text{if} \quad |\eta| < \delta(x')^{-1},
\]
and for any integer \(N' \geq 0\)
\[
|D_\xi V(t, x, z, \eta)| \leq C|\eta|^{-N'-\delta(x')^{-N'-\nu-|z|}}.
\]

**Proof.** (6.10) is obvious. Set \(L_\eta = -i|\partial \varphi(y, z, \eta)|^{-2} \sum_{j=1}^n (\partial \varphi(y, z, \eta) \partial y_j \varphi(y, z, \eta) \partial_j \varphi(y, z, \eta)\). Noting that
Asymptotic behavior of spectral functions of elliptic operators

\[ L_\nu e^{i\psi(y, x, r)} = e^{i\psi(y, x, r)} \]

we have, for any integer \( k \geq 0 \), by integration by parts

\[ V = \int \{ 1 - \psi(\partial(\gamma, \xi))(L_\nu^t)^k \{ q_{-} e^{i\psi(x, y, \xi)} \} e^{i\psi(y, x, r)} dyd\xi \]

where \( L_\nu^t \) denotes the transposed operator of \( L_\nu \). Hence we get (6.11). q.e.d.

IV. We denote by \( k^0 \) the smallest integer which is not smaller than \( n/2m \). Now we set

\[ \hat{E}_1(t) = \int_{\alpha(x^0)}^{\beta(x^0)} e^{-t\xi} dE_{\rho_0^m}, \]
\[ \hat{E}_2(t) = \int_{\alpha(x^0)}^{\beta(x^0)} (e^{-t\xi}/\mu^{2m}) dE_{\rho_0^m}. \]

Then we have for \( u \in D(A^{k_0}) \)

(6.12) \[ \hat{E}(t)u = \hat{E}_1(t)u + \hat{E}_2(t) A^{k_0} u. \]

On the other hand, it is well known that \( |e(x, y, \mu)| \leq C\mu^{n/2m} \) for \( x, y \in \Omega, \mu > 0 \). Hence we have

(6.13) \[ |e(x, y, \mu^{2m})| \leq C\mu^n. \]

From (6.13) we see that \( \hat{E}_1(t), \hat{E}_2(t) \) are integral operators with kernels \( \hat{E}_1(t, x, y), \hat{E}_2(t, x, y) \) such that

(6.14) \[ \hat{E}_1(t, x, y) = \int_{\alpha(x^0)}^{\beta(x^0)} e^{-t\xi} d\hat{e}(x, y, \mu^{2m}), \]
(6.15) \[ \hat{E}_2(t, x, y) = \int_{\alpha(x^0)}^{\beta(x^0)} (e^{-t\xi}/\mu^{2m}) d\hat{e}(x, y, \mu^{2m}) \]

and for any \( x, y \in \Omega \)

(6.16) \[ |\hat{E}_1(t, x, y)| \leq C\hat{\delta}(x^0)^{-n}, \]
(6.17) \[ |\hat{E}_2(t, x, y)| \leq C\hat{\delta}(x^0)^{2m+1-n}. \]

Now we consider

\[ \int \{ 1 - \psi(\partial(\gamma, \xi))(L_\nu^t)^k \{ q_{-} e^{i\psi(x, y, \xi)} \} e^{i\psi(y, x, r)} dyd\xi \]

Since \( \hat{\phi}_{x^0, z} R'(s)u \in C_0^\infty(\Omega) \subset D(A^{k_0}) \), we have

(6.18) \[ \hat{E}(t-s)\hat{\phi}_{x^0, z} R'(s)u = \hat{E}_1(t-s)\hat{\phi}_{x^0, z} R'(s)u + \hat{E}_2(t-s)\tilde{A}(x, D)^{k_0}\hat{\phi}_{x^0, z} R'(s)u. \]
LEMMA 6.4. There exist kernels $K_{1,i}^{(t,i)}(x, z), K_{2,i}^{(t,i)}(x, z)$ such that for $x \in \Omega, z \in B(x')$, $t, s \in (-\delta(x')e^t, \delta(x')e^t)$

$$|K_{j,i}^{(t,i)}(x, z)| \leq C\delta(x')^{-n-1} \quad (j = 1, 2)$$

and we have for $u \in C^o_0(B(x'))$

$$\lim_{t \to 0} E_i(t - s)\phi_{x_0, z}^i R^i(s)u = \int K_{1,i}^{(t,i)}(x, z)u(z)dz,$$

$$\lim_{t \to 0} E_i(t - s)\phi_{x_0, z}^i R^i(s)u = \int K_{2,i}^{(t,i)}(x, z)u(z)dz.$$

PROOF. Applying Lemma 6.2 to (5.29), we have that $r(s, x, z) = 0$ if $|\eta| < \delta(x')^{-1}$ and

$$|D^*_r(s, x, z, \eta)| \leq C|\eta|^{1-N/2} \delta(x')^{-N/2 - |\eta|}.$$

Hence for sufficiently large $N$ we have

$$(6.19) \quad \int |D^*_r(s, x, z, \eta)| |\eta|^t \, d\eta \leq C\delta(x')^{-n-1 - |\eta| - t}.$$

From (6.16), (6.17) and (6.19), noting that $\bar{m} \sup \phi_{x_0, z} \leq \delta(x')^n$, we obtain the present lemma.

From Lemma 6.4, noting that $\sup \phi_{x_0, z} \leq \delta(x')^n$, we see that there exists a kernel $H_{i}^{(t,i)}(x, z)$ such that for $x \in \Omega, z \in B(x')$

$$(6.20) \quad |H_{i}^{(t,i)}(x, z)| \leq C\delta(x')^{-n-1}$$

and we have for $u \in C^o_0(B(x'))$

$$\lim_{t \to 0} \int \phi(t)\delta(x') \int_0^t E_i(t - s)\phi_{x_0, z}^i R^i(s)u \, ds \, e^{it} \, dt$$

$$(6.21) \quad = \int H_{i}^{(t,i)}(x, z)u(z)dz.$$

Next we consider

$$\int \phi(t)\delta(x') \int_0^t E_i(t - s)(A^{1/2m} - \phi_{x_0, z}^i P_N)Q^i(s)u \, ds \, e^{it} \, dt.$$

Noting that $(A^{1/2m} - \phi_{x_0, z}^i P_N)Q^i(s)u \in D(A^{1/2m})$, we have

$$E_i(t - s)(A^{1/2m} - \phi_{x_0, z}^i P_N)Q^i(s)u$$

$$(6.22) \quad = E_i(t - s)(A^{1/2m} - \phi_{x_0, z}^i P_N)Q^i(s)u + E_i(t - s)A^{1/2m} - \phi_{x_0, z}^i P_N)Q^i(s)u.$$

LEMMA 6.5. There exist kernels $K_{j,i}^{(t,i)}(x, z) \ (j = 3, 4)$ such that for $x \in \Omega, z \in B(x')$
Asymptotic behavior of spectral functions of elliptic operators

and we have for \( u \in C^\infty_0(B(x)) \)

\[
\lim_{s \to 0} \hat{E}_i(t-s)(A^{1/2m} - \phi_{x_0, z}P_{x_0})Q'(s)u = \int K^{(i,:)}_{i}(x, z)u(z)dz,
\]

\[
\lim_{s \to 0} \hat{E}_i(t-s)A^{2s}(A^{1/2m} - \phi_{x_0, z}P_{x_0})Q'(s)u = \int K^{(i,:)}_{i}(x, z)u(z)dz.
\]

For the proof of Lemma 6.5 we shall use the following:

**Lemma 6.6.** Let \( g(\cdot, z, \eta, \lambda), \phi(\cdot, z, \eta) \in C^\infty_0(B(x)) \) for \( z \in B(x), \eta \in \mathbb{R}^n - \{0\} \), \( \lambda \in \Gamma \) satisfy

(a) \( g(x, z, \eta, \lambda) = 0 \), if \( |\eta| < \delta(x^{-1}) \)

(b) \( |D^k_\eta g(x, z, \eta, \lambda)| \leq C |\eta|^{n - M} \delta(x^{-1})^{-M-|\eta|} \) for some integer \( M \), \( |D^k_\eta \phi(x, z, \eta)| \leq C |\eta| \)

and let \( T \) be the integral operator on \( L^1(\Omega) \) with the kernel \( T(x, z) \) such that

\( |T(x, z)| \leq C \delta(x)^{-1} \) for \( x, z \in \Omega \).

Now we set for \( u \in C^\infty_0(B(x)) \)

\[
G'(|\lambda|)u = \int \chi(2\eta)g(x, z, \eta, \lambda)e^{i\phi(x, z, \eta)}u(z)dz d\eta.
\]

Then, there exists a kernel \( K(x, z) \) such that for \( x \in \Omega, z \in B(x) \)

\[
|K(x, z)| \leq C \delta(x)^{-1 - 2m k - 1}
\]

and we have for \( u \in C^\infty_0(B(x)) \)

\[
\lim_{s \to 0} TA^{k+1} \int \lambda^{1/2m-1}(A - \lambda)^{-1}G'(\lambda)ud\lambda
\]

\[
= \int K(x, z)u(z)dz
\]

where \( M > n + 2m(k+1) \). Moreover we assume that

(c) \( |D^k_\eta g(x, z, \eta, \lambda)| \leq C |\lambda|^{-1} |\eta|^{-M} \delta(x)^{-1 - M - |\eta| - 2m} \).

Then we have

\[
\lim_{s \to 0} TA^{k} \int \lambda^{1/2m-1}G'(\lambda)ud\lambda
\]

\[
= \int \tilde{K}(x, z)u(z)dz
\]

where \( |\tilde{K}(x, z)| \leq C \delta(x)^{-1 - 2m k - 1} \) for \( x \in \Omega, z \in B(x) \).
Proof. Now we set \( \Gamma_1 = \Gamma \cap \{ \lambda \in \mathbb{C} : |\lambda| \leq \delta(x^0)^{-2m} \} \), \( \Gamma_2 = \Gamma \cap \{ \lambda \in \mathbb{C} : |\lambda| > \delta(x^0)^{-2m} \} \). Then we have

\[
T A^{k+1} \int_{\Gamma_1} \lambda^{1/m-1} (A - \lambda)^{-1} G(\lambda) u d\lambda = T \int_{\Gamma_1} \lambda^{1/m-1} \tilde{A} G(\lambda) u d\lambda + T \int_{\Gamma_1} \lambda^{1/m} (A - \lambda)^{-1} \tilde{A} G(\lambda) u d\lambda + T \int_{\Gamma_1} \lambda^{1/m-1} (A - \lambda)^{-1} \tilde{A}^{k+1} G(\lambda) u d\lambda = T_{0,t} u + T_{1,t} u + T_{2,t} u.
\]

We note that

\[
|T(x, y) \int \tilde{A}(y, D_y)^{\alpha} g(y, z, \eta, \lambda) e^{i\phi(p, y, \eta)} d\eta|
\]

(6.23)

\[
\leq C \delta(x^0)^{l+1} \sum_{p+q+1 = m(k-1)} \int_{|\eta| > \delta(x^0)^{-1}} |\eta|^{-p+q} \delta(x^0)^{-q} d\eta
\]

\[
\leq C \delta(x^0)^{l-2m(k-1)}.
\]

Hence, noting that

\[
\left| \int_{\Gamma_1} \lambda^{1/m-1} d\lambda \right| \leq C \delta(x^0)^{-1},
\]

we see that there exist a kernel \( K_0(x, z) \) such that for \( x \in \Omega, z \in B(x^0) \)

\[
|K_0(x, z)| \leq C \delta(x^0)^{l-2m(k-1)}
\]

and we have

\[
\lim_{t \to 0} T_{0,t} u = \int K_0(x, z) u(z) dz.
\]

From assumption (iii) we have

\[
\int |G(x, y)| dx \leq C |\lambda|^{-1}.
\]

Hence we get

(6.24) \[
\int_{\Omega} \int_{\Gamma_1} \lambda^{1/m} G(x, y) |d\lambda| dx \leq C \delta(x^0)^{-1},
\]

(6.25) \[
\int_{\Omega} \int_{\Gamma_1} \lambda^{1/m-1} G(x, y) |d\lambda| dx \leq C \delta(x^0)^{-1+2m}.
\]

Noting (6.24), (6.25), analogously we have

\[
\lim_{t \to 0} T_{1,t} u = \int K_1(x, z) u(z) dz,
\]
\[
\lim_{z \to 0^+} T_{z,U} \mu = \int K_j(x, z) u(z) dz
\]

where \( |K_j(x, z)| \leq C \delta(x')^{-1-2m-1} \) \((j=1, 2)\) for \(x \in \Omega, z \in B(x')\). In the same way, we get (**). \( \text{q.e.d.} \)

PROOF OF LEMMA 6.5. We prove only the case of \(j=4\). Now we set

\[
d_{(N)}(x, \xi, \lambda) = e^{-i(x', \xi)} \tilde{A}(x, D) \{ \phi_{x', \xi} b_{(N)}(x, \xi, \lambda) \psi(\bar{\xi}) \bar{\xi} e^{i(x', \xi)} \}
\]

and for \(v \in C_0^\infty(\Omega)\)

\[
D_N(\lambda)v = \int d_{(N)}(x, \xi, \lambda) \bar{\partial}(\xi) e^{i(x', \xi)} d\xi.
\]

Then we have

\[
(A^{1/2m} - \phi_{x', \xi} P_N) Q'(s) u = \frac{A}{2\pi i} \int \lambda^{1/2m-1} \{(A-\lambda)^{-1} - \phi_{x', \xi} \tilde{B}_N(\lambda)\} Q'(s) u d\lambda
\]

\[
- \frac{1}{2\pi i} \int \lambda^{1/2m-1} D_N(\lambda) Q'(s) u d\lambda.
\]

We note that for \(v \in C_0^\infty(\Omega)\)

\[
(A-\lambda)^{-1} v - \phi_{x', \xi} \tilde{B}_N(\lambda) v
= (A-\lambda)^{-1} \{v - (\tilde{A}(x, D) - \lambda) \phi_{x', \xi} \tilde{B}_N(\lambda) v\}
= (A-\lambda)^{-1} \{- \phi_{x', \xi} \tilde{C}_N(\lambda) v - D_N(\lambda) v + \phi_{x', \xi} (1 - \bar{\psi}) v + (1 - \phi_{x', \xi}) v\}
\]

where we write

\[
\tilde{C}_N(\lambda)v = \int c_{(N)}(x, \xi, \lambda) \psi(\bar{\xi}) \bar{\xi} e^{i(x', \xi)} d\xi,
\]

\[
(1 - \bar{\psi})v = \int \{1 - \psi(\bar{\xi}) \bar{\xi} e^{i(x', \xi)} d\xi.
\]

Hence we get

\[
E_{s}(t-s) A^{1/2m} (A^{1/2m} - \phi_{x', \xi} P_N) Q'(s) u
= -E_{s}(t-s) A^{1/2m} \frac{1}{2\pi i} \int \lambda^{1/2m-1} (A-\lambda)^{-1} \phi_{x', \xi} \tilde{C}_N(\lambda) Q'(s) u d\lambda
+ E_{s}(t-s) A^{1/2m} \frac{1}{2\pi i} \int \lambda^{1/2m-1} (A-\lambda)^{-1} \phi_{x', \xi} (1 - \bar{\psi}) Q'(s) u d\lambda
- E_{s}(t-s) A^{1/2m} \frac{1}{2\pi i} \int \lambda^{1/2m-1} (A-\lambda)^{-1} D_N(\lambda) Q'(s) u d\lambda
\]
Since \( \phi_{x_0^0} = 1 \) on \( \text{supp } q \), we have for any \( \alpha \)

\[
d^{(\alpha)}_\lambda(x, F_x \varphi(x, z, \gamma), \lambda)D^\alpha_{\gamma}[q(t, y, z, \gamma)e^{i\langle \varphi(x, y, t, \xi) \rangle}]|_{y=x} = 0.
\]

Hence we get for any integer \( N' \geq 0 \)

\[
D_\lambda(\lambda)Q^\alpha(s)u = \int \chi(z)U[D_\lambda(\lambda), q, N'](s, x, z, \gamma)e^{i\langle \varphi(x, y, t, \xi) \rangle - i\alpha(x, t, \xi)}u(z)dzd\eta
\]

where

\[
U[D_\lambda(\lambda), q, N'](s, x, z, \gamma) = N' \sum_{|\tau|=N'} \frac{1}{\tau!} \int \int_\mathbb{R} (1 - \tau)^{N'-1}d^{(\tau)}_\lambda(x, F_x \varphi + \tau \xi, \lambda)\xi^\alpha q(s, y, z, \gamma) \\
\times e^{i\lambda(y, z, t, \xi) + i(\xi - y, t, \xi)}dxdyd\xi.
\]

From Lemma 3.1 we have for any \( \alpha, \beta \)

\[
|D_\xi^\alpha d^{(\beta)}_\lambda(x, \xi, \lambda)| \leq \left\{ \begin{array}{ll}
C_{\alpha, \beta} \sum_{k=1}^{2m} |\xi|^{k-2-k} |\lambda|^{k-2-|\alpha|} \delta(x')^{-k-|\alpha|} \\
C_{\alpha, \beta} |\lambda|^{-1} \sum_{k=1}^{2m} |\xi|^{2m-k-2|\beta|} \delta(x')^{-k-|\alpha|}.
\end{array} \right.
\]

Applying Lemma 6.2 to \( U[D_\lambda(\lambda), q, N'] \), we get

\[
|D_\xi^\alpha U[D_\lambda(\lambda), q, N'](s, x, z, \gamma)| \leq \left\{ \begin{array}{ll}
C|\gamma|^{-N'/2} \delta(x')^{-N'/2 - |\alpha|} \\
C|\lambda|^{-1} |\gamma|^{2m-N'/2} \delta(x')^{-N'/2 - |\alpha|}.
\end{array} \right.
\]

From Lemma 3.2, 6.2 we have

\[
|D_\xi^\alpha U[\tilde{C}_\lambda(\lambda), q, 0](s, x, z, \gamma)| \leq C|\gamma|^{-N} \delta(x')^{-|\alpha|}.
\]

From Lemma 6.3 we see that \( (1 - \tilde{\varphi})Q^\alpha(s)u \) satisfies the conditions of Lemma 6.6. Hence, applying Lemma 6.6 to

\[
U[D_\lambda(\lambda), q, N'], U[\tilde{C}_\lambda(\lambda), q, 0], (1 - \tilde{\varphi})Q^\alpha(s)u
\]

respectively, we get the present lemma.

From Lemma 6.5, noting that \( \text{supp } \delta(t/\delta(x')) \subset (-\delta(x')\ell', \delta(x')\ell') \), we see that there exists a kernel \( S^\alpha(x, z) \) such that

\[
|S^\alpha(x, z)| \leq C \delta(x')^{-n+1}
\]

for \( x \in Q, z \in B(x') \) and we have for \( u \in C_0(B(x')) \).
Finally we consider
\[
\int \rho(t/\theta(x')) \hat{E}(t) \phi_{x,0} K' u e^{it} dt.
\]
We note that
\[
\hat{E}(t) \phi_{x,0} K' u = \hat{E}_i(t) \phi_{x,0} K' u + \hat{E}_o(t) \hat{A}^0 \phi_{x,0} K' u.
\]

**Lemma 6.7.** There exist the kernels \( K'(x, z) \) \((j=5, 6)\) such that for \( x \in \Omega, z \in B(x') \)
\[
|K'_j(x, z)| \leq C \delta(x')^{-n} \quad (j=5, 6)
\]
and we have for \( u \in C_0^1(B(x')) \)
\[
\lim_{\epsilon \to 0} \int K'_i(x, z) u(z) dz = 0,
\]
\[
\lim_{\epsilon \to 0} \int K'_o(x, z) u(z) dz = 0.
\]

**Proof.** Using (5.7), (5.8), (5.9) and noting (5.10), we can prove the present lemma in the same way as for the proof of Lemma 6.4, 6.5.

From Lemma 6.7 we see that there exists a kernel \( H^{(j)}(x, z) \) such that for \( x \in \Omega, z \in B(x') \)
\[
|H^{(j)}(x, z)| \leq C \delta(x')^{-n+1}
\]
and we have for \( u \in C_0^1(B(x')) \)
\[
\lim_{\epsilon \to 0} \int \rho(t/\theta(x')) \hat{E}(t) \phi_{x,0} K' u e^{it} dt
\]
\[
= \int H^{(j)}(x, z) u(z) dz.
\]

§ 7. **Proof of the main theorem**

From (5.36), (6.2), (6.3), (6.21), (6.27) and (6.29) we have
\[
\int \left\{ 2\pi \int \rho(\theta(x')(\lambda - \mu)) \delta(x, z, \mu^2, \phi_{x,0}(z)) u(z) dz \right\} u(z) dz
\]
\[
- \int \left\{ \int \tilde{R}(x, z, \gamma, \lambda - a(z, \gamma)) e^{i\phi(x', z, \gamma)} dz \right\} u(z) dz
\]
where we write

\[ \tilde{R}(x, z, \eta, \lambda) = \int \hat{\rho}(t) \hat{q}(t, x, \eta) e^{i\mu t} dt. \]

Hence, from (6.20), (6.26), (6.28) we have

\[
2\pi \int \tilde{\rho}(\hat{x}) \rho(\tilde{\rho}(\lambda - \mu)) d\tilde{\rho}(x, z, \mu^{\infty}) \phi_{\phi, \lambda}(z) - \int \tilde{R}(x, z, \eta, \lambda - a(z, \eta)) e^{\psi(x, z, x, \eta)} d\eta \leq C \tilde{\rho}(x)^{-\frac{n+1}{2}}.
\]  

(7.1)

Using this estimate and Lemma 6.1, noting that \( \phi_{\phi, \lambda}(\hat{x}) = 1 \), we get

\[
\int \tilde{\rho}(\hat{x}) \rho(\tilde{\rho}(\lambda - \mu)) d\tilde{\rho}(x, x', \mu^{\infty}) \leq C \left\{ \sum_{k=1}^{\infty} \tilde{\rho}(x)^{-1} |\lambda^{n-k}| \right\}.
\]  

(7.2)

Noting that \( e(x, x', \mu^{\infty}) \) is a real non-negative non-decreasing function of \( \mu \), and taking into account the positivity of \( \rho \), from (7.2) we get

\[
|\tilde{e}(x', x, \lambda + 1/\tilde{\rho}(x')) - \tilde{e}(x', x, \lambda)| \leq C \left\{ \sum_{k=1}^{\infty} \tilde{\rho}(x)^{-k} |\lambda^{k-n}| \right\},
\]  

(7.3)

where we write \( \tilde{e}(x', x, \mu) = e(x', x, \mu^{\infty}) \). From (7.1) we have for \( \lambda > 0 \)

\[
\int_{0}^{1} 2\pi \int \tilde{\rho}(\hat{x}) \rho(\tilde{\rho}(\lambda' - \mu)) d\tilde{e}(x, x', \mu) d\lambda' - \int \tilde{R}(x, x', \eta, \lambda' = a(x, \eta)) d\eta d\lambda' \leq C \tilde{\rho}(x)^{-\frac{n+1}{2}} \lambda.
\]  

(7.4)

**Lemma 7.1.** For \( \lambda > 0 \) we get the following estimate:

\[
\int |\tilde{e}(x', x, \mu)| d\lambda' \leq C \left\{ \sum_{k=1}^{\infty} \tilde{\rho}(x)^{-k} |\lambda^{n-k}| \right\}.
\]  

(7.5)

**Proof.** We have by integration by parts

\[
\int \tilde{\rho}(x') \rho(\tilde{\rho}(\lambda' - \mu)) d\tilde{e}(x', x', \mu) d\lambda' = \int \tilde{\rho}(x') \rho(\tilde{\rho}(\lambda - \mu)) e(x', x', \mu) d\mu
\]  

(7.1)
Asymptotic behavior of spectral functions of elliptic operators

Noting that \(|\tilde{e}(x^0, x^0, \mu)| \leq C \mu^n\), we have

\[
\left| \int \tilde{e}(x^0) \rho(-\tilde{e}(x^0) \mu) \tilde{e}(x^0, x^0, \mu) d\mu \right| \leq C \delta(x^0)^{-n}.
\]

We note that

\[
\int \tilde{e}(x^0) \rho(\delta(x^0) (\lambda - \mu)) d\mu = 1.
\]

Hence we have

\[
\int \tilde{e}(x^0) \rho(\delta(x^0) (\lambda - \mu)) \tilde{e}(x^0, x^0, \mu) d\mu - \tilde{e}(x^0, x^0, \lambda)
\]

\[
= \int \tilde{e}(x^0) \rho(\delta(x^0) (\lambda - \mu)) \tilde{e}(x^0, x^0, \mu) - \tilde{e}(x^0, x^0, \lambda) d\mu
\]

\[
= \int \rho(\mu) (\tilde{e}(x^0, x^0, \lambda + \mu/\delta(x^0)) - \tilde{e}(x^0, x^0, \lambda)) d\mu.
\]

From (7.3) we have

\[
|\tilde{e}(x^0, x^0, \lambda + \mu/\delta(x^0)) - \tilde{e}(x^0, x^0, \lambda)|
\]

\[
\leq C \left\{ \sum_{k=1}^n \delta(x^0)^{-k} (\lambda + |\mu|/\delta(x^0))^{n-k} (|\mu| + 1) \right\}.
\]

Using this estimate, we get

\[
\left| \int \rho(\mu) (\tilde{e}(x^0, x^0, \lambda + \mu/\delta(x^0)) - \tilde{e}(x^0, x^0, \lambda)) d\mu \right|
\]

\[
\leq C \left\{ \sum_{k=1}^n \delta(x^0)^{-k} \lambda^{n-k} \right\}.
\]

q.e.d.

**Lemma 7.2.** For \(\lambda > 0\) we get the following estimate:

\[
\left| \int_0^\infty \int \tilde{R}(x^0, x^0, \eta, \lambda' - a(x^0, \eta)) d\eta d\lambda' - 2\pi \int_{a(x^0, \eta) < \lambda} I(x^0, x^0, \eta) \psi(\delta(x^0) \eta) d\eta \right|
\]

\[
\leq C \left\{ \sum_{k=1}^n \delta(x^0)^{-k} \lambda^{n-k} \right\}.
\]

**Proof.** We note that

\[
\int_{-\infty}^\infty \tilde{R}(x^0, x^0, \eta, \lambda' - a(x^0, \eta)) d\lambda' = 2\pi I(x^0, x^0, \eta) \psi(\delta(x^0) \eta).
\]

Hence we have
\[
\int_0^1 \int_0^1 \tilde{R}(x^0, x^0, \eta, \lambda - a(x^0, \eta)) d\eta d\lambda' = 2\pi \int_{a(x^0, \eta) < 1} I(x^0, x^0, \eta) \psi(\delta(x^0, \eta)) d\eta
\]
\[
- \int_{a(x^0, \eta) < 1} \int_0^1 \tilde{R}(x^0, x^0, \eta, \lambda - a(x^0, \eta)) d\lambda' d\eta
d\eta
\]
\[
- \int_{a(x^0, \eta) < 1} \int_{-\infty}^0 \tilde{R}(x^0, x^0, \eta, \lambda - a(x^0, \eta)) d\lambda' d\eta
\]
\[
+ \int_{a(x^0, \eta) > 1} \int_0^1 \tilde{R}(x^0, x^0, \eta, \lambda - a(x^0, \eta)) d\lambda' d\eta
\]
\[
= 2\pi \int_{a(x^0, \eta) < 1} I(x^0, x^0, \eta) \psi(\delta(x^0, \eta)) d\eta - I_1 - I_2 + I_3.
\]

From the proof of Lemma 6.1 we have for any integer \( k \geq 0 \)
\[
|\tilde{R}(x^0, x^0, \eta, \lambda')| \leq C \left\{ \sum_{v=0}^{q+1} |\eta|^{-v} \delta(x^0)^{-v+1} \right\} (1 + \delta(x^0) |\lambda'|)^{-q-1}.
\]

Using this estimate, we have for \( \lambda > a(x^0, \eta) \)
\[
\int_0^1 |\tilde{R}(x^0, x^0, \eta, \lambda - a(x^0, \eta))| d\lambda' \leq C \left\{ \sum_{v=0}^{q+1} |\eta|^{-v} \delta(x^0)^{-v} \right\} (1 + \delta(x^0) |\lambda - a(x^0, \eta)|)^{-q-1}.
\]

Noting that \( \tilde{R}(x^0, x^0, \eta, \lambda') = 0 \) if \( |\eta| < \delta(x^0)^{-1} \), we get
\[
|I_1| \leq C \int_{a(x^0, \eta) < 1} (1 + \delta(x^0) |\lambda - a(x^0, \eta)|)^{-q+1} d\eta.
\]

Hence, in the same way as for the proof of Lemma 6.1 we get
\[
(7.6) \quad |I_1| \leq C \left\{ \sum_{k=0}^{q} \delta(x^0)^{-k} \lambda^{n-k} \right\}.
\]

Analogously we get the estimates for \( I_2 \) and \( I_3 \), q.e.d.

**Lemma 7.3.** For \( \lambda > 0 \) we get the following estimate:

\[
\left| \int_{a(x^0, \eta) < 1} I(x^0, x^0, \eta) \psi(\delta(x^0, \eta)) d\eta - \int_{a(x^0, \eta) < 1} d\eta \right| \leq C \left\{ \sum_{k=0}^{q} \delta(x^0)^{-k} \lambda^{n-k} \right\}.
\]

**Proof.** From (5.11) we have

\[
\int_{a(x^0, \eta) < 1} I(x^0, x^0, \eta) \psi(\delta(x^0, \eta)) d\eta
\]
Asymptotic behavior of spectral functions of elliptic operators

\[ \int_{a(x_0, \eta) < 1} d\eta + \int_{a(x_0, \eta) < 1} \{ -1 + \psi(\partial(x^0)\eta) \} d\eta \\
+ \sum_{j=1}^{N-1} I_j(x^0, x^0, \eta) \psi(\partial(x^0)\eta) d\eta. \]

From (5.7) we have

\[ \left| \int_{a(x_0, \eta) < 1} I_j(x^0, x^0, \eta) \psi(\partial(x^0)\eta) d\eta \right| \leq C \lambda^{-j}. \]

Hence, noting that

\[ \left| \int_{a(x_0, \eta) < 1} \{ -1 + \psi(\partial(x^0)\eta) \} d\eta \right| \leq C \delta(x^0)^{-n}, \]

we have the present lemma.

From Lemma 7.1, 7.2, 7.3 we have for \( \lambda > 0 \)

\[ e(x^0, x^0, \lambda) - \int_{a(x^0, \eta) < 1} d\eta \leq C \left\{ \sum_{k=1}^{n} \delta(x^0)^{-k} \lambda^{n-k} \right\}. \]

Hence we get

\[ e(x^0, x^0, \lambda) - \lambda^{n/2m} \int_{a(x^0, \eta) < 1} d\eta \leq C \left\{ \sum_{k=1}^{n} \delta(x^0)^{-k} \lambda^{(n-k)/2m} \right\}. \]

From this estimate and \( |e(x^0, x^0, \lambda)| \leq C \lambda^{n/2m} \) we get the main theorem.

**Bibliography**


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