Tensor products for monotone complete $C^*$-algebras, I

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Introduction

Kaplansky [14, 15, 16] singled out a class of $C^*$-algebras, called $AW^*$-algebras, which imitates the $W^*$-algebras (=von Neumann algebras) algebraically, and carried out much of the theory of $W^*$-algebras for this class of $C^*$-algebras. However there does not seem to be a satisfactory $AW^*$-version of the $W^*$-tensor product of $W^*$-algebras except for a few special cases. For instance an $\mathfrak{H}$-homogeneous type I $AW^*$-algebra $B$ can be regarded as a sort of an $\mathfrak{H} \times \mathfrak{H}$ matrix algebra over its center $A$ [15, 16]. Hence $B$ is a certain "tensor product" of the commutative $AW^*$-algebra $A$ and the type I $W^*$-factor $B(K)$ with $K$ an $\mathfrak{H}$-dimensional Hilbert space. Moreover Berberian [1] showed that the tensor product $A \otimes M_n$ of any $AW^*$-algebra $A$ and the algebra $M_n$ of all $n \times n$ matrices over $\mathbb{C}$ is also an $AW^*$-algebra.

In this paper we work on a subclass of $AW^*$-algebras—the monotone complete $C^*$-algebras—and introduce a tensor product, written $A \otimes M$, of a monotone complete $C^*$-algebra $A$ and a $W^*$-algebra $M$ which is a monotone complete $C^*$-algebra uniquely determined by $A$ and $M$. The algebra $A \otimes M$ is called the monotone complete tensor product of $A$ and $M$ and satisfies the following properties (Theorem 4.2):

(a) $A \otimes M$ is $W^*$ or non $W^*$ according as $A$ is $W^*$ or non $W^*$, and in the former case it is the usual $W^*$-tensor product of $A$ and $M$.

(b) $A \otimes M$ contains $A \otimes 1$ and $1 \otimes M$ as monotone closed $C^*$-subalgebras and it is the monotone closure of the algebraic tensor product $A \odot M$ (i.e., the smallest monotone closed $C^*$-subalgebra, containing $A \odot M$, of $A \otimes M$).

(c) If $A$ (resp. $M$) is a monotone closed $C^*$-subalgebra (resp. $W^*$-subalgebra) of a monotone complete $C^*$-algebra $B$ (resp. $W^*$-algebra $N$), then $A \otimes M$ is the monotone closure of $A \odot M$ in $B \otimes N$.

Moreover the properties (b) and (c) characterize uniquely the operation $\otimes$ defined for each pair of a monotone complete $C^*$-algebra and a $W^*$-algebra.

In Section 2 the regular monotone completion $(A \otimes B(K))^-$, in the sense of [11], of the minimal $C^*$-tensor product $A \otimes B(K)$ of a monotone complete
C*-algebra $A$ and a type I $W^*$-factor $B(K)$ is characterized as the monotone complete C*-algebra of $\mathbb{K} \times \mathbb{K}$ matrices over $A$ with $\mathbb{K} = \dim K$. Following an idea by Tomiyama [21, 22], we define in Section 3 an operator system $\mathcal{F}(V, W)$ on $H \otimes K$, called the Fubini product of $V$ and $W$, for operator systems $V \subset B(H)$ and $W \subset B(K)$. In Section 4 we use $\mathcal{F}(A, M)$ to construct $A \otimes M$ and extend some results known for the $W^*$-tensor products to the case of the monotone complete tensor products. As a consequence the existence is shown of a pair of non *-isomorphic $\alpha$-finite monotone complete non $W^*$, $AW^*$-factors of type III (Theorem 4.9). First examples of non $W^*$, $AW^*$-factors were given by Dyer [7] and Takenouchi [20] independently, and they were shown to be $\alpha$-finite, monotone complete and of type III by Saitô [19]. As applications of the results in the preceding sections we investigate in Section 6 the regular monotone completion of a hereditary C*-subalgebra of a C*-algebra (resp. the minimal $C^*$-tensor product of two C*-algebras).

§1. Notation and preliminaries

Throughout the paper (excluding a part of Section 6) C*-algebras to be considered are unital, their C*-subalgebras contain the same units as they do, and the notation and terminology in [9, 10, 11] are used. For two C*-algebras $A$ and $B$, $A \otimes B$, $A \hat{\otimes} B$ and $A \odot B$ denote the algebraic tensor product, the minimal C*-tensor product and the $W^*$-tensor product (if $A$ and $B$ are both $W^*$-algebras) of $A$ and $B$ respectively.

Each C*-algebra $A$ has a unique regular monotone completion $\overline{A}$ (resp. injective envelope $I(A)$) with $A \subset A \subset I(A)$, and the inclusion maps $A \hookrightarrow \overline{A} \hookrightarrow I(A)$ are normal [9, 11]. For C*-algebras $A$ and $B$ the maps $A \ni x \mapsto x \otimes 1 \in A \otimes B$ and $B \ni y \mapsto 1 \otimes y \in A \otimes B$ are normal [11, Proposition 4.1].

Now we modify slightly the convergence of nets in a monotone complete C*-algebra defined by Kadison-Pedersen [12]. Let $A$ be a C*-algebra. If an increasing net $\{x_\alpha\}$ in $A_{s.a.}$ has a supremum $x$ in $A_{s.a.}$, then we write $x_\alpha \nearrow x$ (O) or $-x_\alpha \searrow -x$ (O). A net $\{x_\alpha\}$ in $A$ order-converges to an $x \in A$, written $x_\alpha \rightarrow x$ (O) or $O$-lim $x_\alpha = x$, if there are bounded nets $\{x(j)_\alpha\}$, $\{x'(j)_\alpha\}$ in $A_{s.a.}$ and elements $x(j) \in A_{s.a.}$, such that

$$0 \leq x^{(j)}_\alpha - x^{(j)} \leq x^{(j)}_\alpha - 0$$

and

$$\sum_{j=0}^3 i^j x^{(j)}_\alpha = x$$

and

$$\sum_{j=0}^3 i^j x^{(j)} = x$$

It is immediate to see that the $x$, called the order limit of $\{x_\alpha\}$, does not depend on the special choice of $\{x^{(j)}_\alpha\}$, $\{x^{(j)}_\alpha\}$ and $x^{(j)}$ and that if $A$ is a commutative $AW^*$-algebra, then $x_\alpha \rightarrow x$ (O) in $A$ in the above sense if and only if $\{x_\alpha\}$ order-converges to $x$ in the sense of Widom [23].
Suppose for a while that $A$ acts on a Hilbert space $H$. Then there is a completely positive projection $\phi$ on $B(H)$ so that $A \subseteq \operatorname{Im} \phi$ and we may identify $I(A)$ with $\operatorname{Im} \phi$ equipped with the order, involution and norm induced by those of $B(H)$ and the multiplication given by $x \cdot y = \phi(xy)$. If $x_\alpha \to x$ (O) in $A$, then $x_\alpha \to x$ (O) in $I(A)$ and $\phi(\text{s-lim } x_\alpha) = x$, where $\text{s-lim } x_\alpha$ denotes the strong limit of $x_\alpha$ in $B(H)$ (cf. [21, the proof of Theorem 7.1]).

**Lemma 1.1.** If $x_\alpha \to x$ (O) in $A$, then there is a subnet $\{x_\beta\}$ of $\{x_\alpha\}$ and an $\tilde{x} \in B(H)$ such that $x_\beta \to \tilde{x}$ weakly in $B(H)$ and $\phi(\tilde{x}) = x$.

**Proof.** Let $\{x_\alpha^{(j)}\}, \{x'_\alpha^{(j)}\}$ and $x^{(j)}$ be as above. By the weak compactness of the unit ball of $B(H)$ we can choose subnets $\{x_\beta^{(j)}\}, \{x'_\beta^{(j)}\}$ and elements $\tilde{x}^{(j)}, \tilde{x}'^{(j)} \in B(H)$ so that $x_\beta^{(j)} \to \tilde{x}^{(j)}$ and $x'_\beta^{(j)} \to \tilde{x}'^{(j)}$ weakly in $B(H)$. Since $x_\beta^{(j)} \to 0$ (O) in $A$, the above remark implies that $\phi(\tilde{x}^{(j)}) = 0$. On the other hand it follows from $0 \leq x_\alpha^{(j)} - x^{(j)} \leq x^{(j)}$ that $0 \leq \tilde{x}^{(j)} - x^{(j)} \leq x^{(j)}$, hence that $0 \leq \phi(\tilde{x}^{(j)} - x^{(j)}) \leq \phi(x^{(j)}) = 0$ and $\phi(\tilde{x}'^{(j)}) = x^{(j)}$. Thus $x_\beta = \sum_{j=0}^\infty i'x_\beta^{(j)} \to \sum_{j=0}^\infty i'\tilde{x}^{(j)} = \tilde{x}$, weakly in $B(H)$ and $\phi(\tilde{x}) = x$. q.e.d.

The basic properties of the order convergence are stated as follows:

**Lemma 1.2.** If $x_\alpha \to x$ (O) and $y_\alpha \to y$ (O) in $A$, then we have:

(i) $x_\alpha + y_\alpha \to x + y$ (O);
(ii) $ax_\alpha \to ax$ (O) for all $a, b \in A$;
(iii) $x_\alpha y_\alpha \to xy$ (O);
(iv) $x_\alpha \leq y_\alpha$ for all $\alpha$ implies $x \leq y$;
(v) $\|x\| = \limsup \|x_\alpha\|$. q.e.d.

**Proof.** To see (i) and (ii) modify the argument in [12, Lemma 2.1].

(iii) By (i) and (ii) we need only show that if $0 \leq x_\alpha \leq x_\alpha^{(j)} \to 0$ (O) and $0 \leq y_\alpha \leq y_\alpha^{(j)} \to 0$ (O), then $x_\alpha y_\alpha \to 0$ (O). We may assume that $\|x_\alpha\| \leq 1$ and $\|y_\alpha\| \leq 1$ for all $\alpha$. Then $(y_\alpha + i'x_\alpha)^*(y_\alpha + i'x_\alpha) \leq \sum_{j=0}^\infty i'(y_\alpha + i'x_\alpha)^*(y_\alpha + i'x_\alpha) \to 0$ (O), $j = 0, 1, 2, 3$; hence $4x_\alpha y_\alpha = \sum_{j=0}^\infty i'(y_\alpha + i'x_\alpha)^*(y_\alpha + i'x_\alpha) \to 0$ (O).

(iv) By (i) we need only show that $0 \leq x_\alpha \to x$ (O) implies $x \geq 0$. But Lemma 1.1 shows $x_\beta \to \tilde{x}$ weakly in $B(H)$ and $\phi(\tilde{x}) = x$ for some subnet $\{x_\beta\}$ of $\{x_\alpha\}$ and $\tilde{x} \in B(H)$. Then $\tilde{x} \geq 0$ and so $x = \phi(\tilde{x}) \geq 0$.

(v) Let $\{x_\beta\}$ and $\tilde{x}$ be as in (iv). Then $\|x\| = \|\phi(\tilde{x})\| \leq \|\tilde{x}\| \leq \limsup \|x_\beta\| = \limsup \|x_\alpha\|$. q.e.d.

Given a family $\{x_\alpha\}_{\alpha \in I}$ in $A$ we define the order convergence of the series with the $\alpha$th term $x_\alpha$ as that of the net $\{\sum_{\alpha \in I'} x_\alpha : I' \subseteq I \text{ finite}\}$, and we write $s = \lim_{\alpha} x_\alpha$ if $s$ is the order limit of the net.

**Definition 1.3.** A family $\{x_\alpha\}_{\alpha \in I}$ in $A$ is a $\ell^\infty$-family if the net $\{\sum_{\alpha \in I'} x_\alpha \phi : I' \subseteq I \text{ finite}\}$ is bounded. For a cardinal number $\aleph$ the $C^*$-algebra $A$ is
\( \mathfrak{A} \)-additively complete if \( O-\sum_{a \in I} x^*_a x_a \) exists for each \( \mathfrak{A} \)-family \( \{ x_a \}_{a \in I} \) in \( A \) with \( I \) (the cardinality of \( I \)) \( \leq \aleph_0 \), and it is additively complete if it is \( \mathfrak{A} \)-additively complete for each \( \mathfrak{A} \). When \( A \) is a \( \mathfrak{B}^* \)-subalgebra of another \( \mathfrak{B}^* \)-algebra \( B \), \( A \) is \( \mathfrak{B}^* \)-additively closed in \( B \) if, whenever \( O-\sum_{a \in I} x^*_a x_a \) exists in \( B \) for a \( \mathfrak{B}^* \)-family \( \{ x_a \}_{a \in I} \) in \( A \) with \( I \leq \aleph_0 \), then it is in \( A \).

Clearly monotone completeness implies additive completeness. Although the validity of the converse implication is not obvious, we see:

**Proposition 1.4.** An additively complete \( \mathfrak{B}^* \)-algebra is an \( \mathfrak{A} \mathfrak{W}^* \)-algebra.

**Proof.** Let \( A \) be additively complete. For the proof we have to show that (a) any orthogonal family \( \{ p_a \} \) of projections in \( A \) has a supremum, written \( \vee p_a \), in the set of all projections of \( A \) and (b) any maximal abelian \( \mathfrak{B}^* \)-subalgebra \( C \) of \( A \) is generated by its projections. But by hypothesis \( O-\sum p_a \) exists in \( A_{a,a} \) and it coincides with \( \vee p_a \) by [11, Lemma 3.11]. The algebra \( C \), being maximal abelian, is the fixed point algebra under the inner \( \mathfrak{B}^* \)-automorphisms of \( A \) implemented by all unitaries in \( C \), and so it is also additively complete and \( \mathfrak{A} \mathfrak{W}^* \). Hence (b) follows. q.e.d.

**Lemma 1.5.** Let \( \{ x_a \}_{a \in I} \) and \( \{ y_a \}_{a \in I} \) with \( I \leq \aleph_0 \) be \( \mathfrak{A}^* \)-families in an \( \mathfrak{A} \)-additively complete \( \mathfrak{B}^* \)-algebra \( A \).

(i) \( \{ x_a + y_b \} \) is a \( \mathfrak{A}^* \)-family for all \( a, b \in A \).

(ii) \( O-\sum_{a \in I} y^*_a x_a \) exists in \( A \); in more detail,

\[
4 \left( O-\sum_{a \in I} y^*_a x_a \right) = \sum_{i < \aleph_0} i' O-\sum_{a \in I} (x_a + i'y_a)^* (x_a + i'y_a).
\]

**Proof.** (i) follows from the inequality \( \sum_{a \in I'} (x_a + y_b)^* (x_a + y_b) \leq 2[a^*(\sum_{a \in I'} x^*_a x_a) a + b^*(\sum_{a \in I'} y^*_a y_a) b] \) with \( I' \subset I \) finite. (ii) follows from the fact that the right hand side in the above equality exists by (i). q.e.d.

**§ 2. Matrix units in a \( \mathfrak{B}^* \)-algebra**

By a family of matrix units in a \( \mathfrak{B}^* \)-algebra \( B \) we mean a family \( \{ e_{a,b} \}_{a,b \in I} \) of partial isometries in \( B \) such that \( e_{a,b} e_{a',b'} = \delta_{a,a'} e_{a,b'} \) for all \( \alpha, \alpha' \), \( \beta, \beta' \) and \( O-\sum e_{a,a}=1 \), where \( \delta_{a,a} \) denotes the Kronecker's symbol. Then \( B \) can be viewed as an algebra of \( \aleph_0 \times \aleph_0 \) matrices over \( A=e_{a,a}, Be_{a,a} \) where \( \aleph_0 = I \) and \( \alpha_i \in I \) is fixed. In fact, to each \( x \in B \) there corresponds a matrix \( x=[x_{a,b}]_{a,b \in I} \) over \( A \) given by \( x_{a,b}=e_{a,a} x e_{a,b} \) and the involution and multiplication in \( B \) are calculated in \( A \) as follows:

\[
(x^*)_{a,b} = x^*_{b,a}, \quad (xy)_{a,b} = O-\sum_{\lambda \in I} x_{a,\lambda} x_{\lambda,b}.
\]
the order sum being converging in $A$ by Lemma 1.2. We will show that $B$ with the above property exists uniquely for each $A$ and $\mathfrak{X}$ under the assumption of the monotone completeness of $A$ and $B$ (Theorem 2.5). A similar result for the $\mathfrak{X}$-additively complete case will be given in Corollary 3.13.

**Lemma 2.1.** Let $B$ be an $\mathfrak{X}$-additively complete C*-algebra and $\{p_a\}_{a \in I}$ an orthogonal family of projections in $B$ such that $O_\sum p_a = 1$ and $I \subseteq \mathfrak{X}$. Let $J$ be the family of all finite subsets of $I$ and put $q_r = \sum_{a \in r} p_a$ for $r \in J$. Suppose that $\{x_r\}_{r \in J}$ is a bounded net in $B$ such that $q_r x_r = x_r$ for all $r, r' \in J$ with $r \subseteq r'$. Then there is an $O$-lim $x_r = x$, say, in $B$ such that $q_r x_r = x_r$ for all $r \in J$ and $\|x\| = \sup \|x_r\|$.

**Proof.** For $a, b \in I$ and $r \in J$ with $a, b \not\in r$ put $x_{ab} = p_a x_r p_b$. Then $\{x_{ab}\}_{a \in I}$ with $b$ fixed is a $\mathfrak{E}$-family in $B$ since for each $r \in J$ with $a, b \not\in r$ we have $\sum_{a \in r} x_{ab} x_{ab} ^* \leq p_r x_r ^* p_r \leq \lambda^2$, where $\lambda = \sup \|x_r\|$. It follows from Lemma 1.5 and $p_r x_r = x_\beta$ that $O \sum_{a \in r} x_{ab} = y_r$, say, exists. Then $\{y_r\}_{r \in J}$ is also $\mathfrak{E}$-since for each $b \in I$ and $r, r' \in J$ with $b \in r \subseteq r'$ we have $q_r y_{r'} = x_r p_b q_{r'} (\sum_{a \in r'} y_{r'} p^*_a) q_{r'} = \sum_{a \in r} x_r p_b x_r ^* \leq \lambda^2$, and so $\sum_{a \in r} y_{r'} y_{r'} ^* \leq \lambda^2$ by Lemma 1.2. Since $y_r = (y_r ^*)^* p_r$, Lemma 1.5 implies the existence of $O \sum_{a \in r} y_r = x$, and we have $q_r x_r = O \sum_{a \in r} (q_r y_r) q_r = O \sum_{a \in r} (x_r p_b q_r = x_r q_r = x_r, x = O \lim q_r x_r = O \lim x_r$. Moreover $\|x\| = \sup \|x_r\|$ since $\|x\| = \sup \|x_r\|$ by Lemma 1.2. q.e.d.

**Lemma 2.2.** For $j = 1, 2$ let $B_j$ be an $\mathfrak{X}$-additively complete C*-algebra and $\{p_{ja}\}_{a \in I}$ an orthogonal family of projections in $B_j$ such that $O_\sum p_{ja} = 1$ and $I \subseteq \mathfrak{X}$. Let $J$ and $q_{j, r}$ be defined as in Lemma 2.1 and suppose that there are *-isomorphisms $\pi_r$ of $q_r B q_r$ onto $q_{j, r} B q_{j, r}$ such that $\pi_r |_{q_r B q_r} = \pi_r$ for all $r, r' \in J$ with $r \subseteq r'$. Then there is a *-isomorphism $\pi$ of $B_1$ onto $B_2$ such that $\pi |_{q_r B q_r} = \pi_r$ for all $r \in J$.

**Proof.** For each $x \in B_1$ and $r \in J$ put $y_r = \pi_r (q_r x q_r)$. Then $\{y_r\}_{r \in J}$ is bounded and $q_{j, r} y_r = y_r$ for all $r, r' \in J$ with $r \subseteq r'$. Hence Lemma 2.1 shows the existence of $O \lim y_r = \pi(x)$, say, such that $\|\pi(x)\| = \sup \|y_r\| = \sup \|q_r x q_r\| = \|x\|$. Thus $\pi$ is a linear isometry of $B_1$ into $B_2$ which, being unital, is positive. For an $x \in B_1$ and a fixed $r' \in J$ we have

$$\pi(x^*) q_{r'} \pi(x) = O \lim \pi_r (q_r x^* q_r) \pi_r (q_r x q_r) = \pi(x^*) \pi(x) \leq \pi(x^* x),$$

so that $\pi(x^*) \pi(x) \leq \pi(x^* x)$. The above argument applied to $\pi^{-1}$ implies that $\pi$ has the inverse $\pi^{-1}$ and that $\pi^{-1} (y^*) \pi^{-1} (y) \leq \pi^{-1} (y^* y)$ for $y \in B_2$. Hence $\pi(x^* x) = \pi(x^*) \pi(x)$ for $x \in B_1$, and $\pi$ is a *-isomorphism of $B_1$ onto $B_2$. q.e.d.

**Lemma 2.3.** For $j = 1, 2$ let $B_j$ be an $\mathfrak{X}$-additively complete C*-algebra which has a family $\{e_{j, a}\}_{a, b \in I}$ of matrix units such that $I = \mathfrak{X}$. If $\pi_0$ is a *-iso-
morphism of \( e_{1_{\infty \beta}}B e_{1_{\infty \beta}} \) onto \( e_{1_{\infty \beta}}B e_{1_{\infty \beta}} \) with \( \alpha_0 \in I \) fixed, then there is a unique *-isomorphism \( \pi \) of \( B \), onto \( B \), such that \( \pi e_{1_{\infty \beta}}B e_{1_{\infty \beta}} = \pi_0 \) and \( \pi(e_{1_{\infty \beta}}) = e_{2_{\infty \beta}} \) for all \( \alpha, \beta \in I \).

**Proof.** Let \( J \) and \( q_{\beta} \) be as above. The application of Lemma 2.2 to the maps \( \pi_\gamma: B q_{\gamma} \to B q_{\gamma} \) given by \( \pi_\gamma(x) = \sum_{\alpha, \beta \in I} e_{2_{\infty \beta}} \pi(e_{1_{\infty \alpha}} x e_{1_{\infty \alpha}}) e_{2_{\infty \beta}} \) implies the existence of \( \pi \), and the uniqueness of \( \pi \) is clear from the construction.

q.e.d.

The proof of the following lemma is quite similar to that of [11, Proposition 1.11], so we omit it.

**Lemma 2.4.** If \( C \) is a self-adjoint linear subspace, containing the unit, of a C*-algebra \( B \) such that the monotone closure \( m-cl_C \) \( C \) is \( B \) and if \( e \) is a projection of \( B \), then \( m-cl_B e Ce = e Be \).

**Theorem 2.5.** Let \( A \) be a monotone complete C*-algebra, \( \aleph \) a cardinal number and \( K \) an \( \aleph \)-dimensional Hilbert space. Then the regular monotone completion \( (A \otimes B(K))^- \) of \( A \otimes B(K) \) is a monotone complete C*-algebra \( B \) uniquely characterized by one of the following conditions:

(i) \( B \) contains the algebraic tensor product \( A \otimes B(K) \) as a *-subalgebra so that \( A \otimes 1 \) and \( 1 \otimes B(K) \) are monotone closed in \( B \) and \( B \) is the monotone closure of \( A \otimes B(K) \).

(ii) There is a family \( \{e_{a\beta}\}_{a, \beta \in I} \) of matrix units in \( B \) such that \( e_{a\beta} B e_{a\beta} \) is *-isomorphic to \( A \) and \( I = \aleph \).

In particular if \( A \) is a W*-algebra, then \( (A \otimes B(K))^- \) is *-isomorphic to the W*-tensor product \( A \otimes B(K) \).

**Proof.** It follows from the normality of the maps \( A \ni x \mapsto x \otimes 1 \in A \otimes B(K), \ B(K) \ni y \mapsto 1 \otimes y \in A \otimes B(K) \) and the inclusion map \( A \otimes B(K) \hookrightarrow (A \otimes B(K))^- \) that \( (A \otimes B(K))^- \) satisfies (i).

We show that for a monotone complete C*-algebra \( B \), (i) implies (ii). Let \( \{e_{a\beta}\}_{a, \beta \in I} \) be an orthonormal basis in \( K \) with \( I = \aleph \) and \( \{f_{a\beta}\}_{a, \beta \in I} \) the family of matrix units in \( B(K) \) defined by \( f_{a\beta} = (\eta, \epsilon_{a\beta}) \epsilon_{\gamma} \in K \). Putting \( e_{a\beta} = 1 \otimes f_{a\beta} \), we obtain a family of matrix units in \( B \) since \( 0 \leq e_{a\beta} \leq 1 \) by the monotone closedness of \( 1 \otimes B(K) \) in \( B \). Let \( C \) be the C*-subalgebra of \( B \) generated by \( A \otimes B(K) \). Then \( e_{a\beta} C e_{a\beta} = A \otimes f_{a\beta} \) is monotone closed in \( e_{a\beta} B e_{a\beta} \) since \( A \otimes 1 \) is monotone closed in \( B \). Moreover by Lemma 2.4 we have \( e_{a\beta} B e_{a\beta} = e_{a\beta} (m-cl_K C) e_{a\beta} = e_{a\beta} C e_{a\beta} = A \otimes f_{a\beta} \simeq A \).

Therefore \( (A \otimes B(K))^- \) satisfies both (i) and (ii). On the other hand a monotone complete C*-algebra \( B \) satisfying (ii) is *-isomorphic to \( (A \otimes B(K))^- \) by Lemma 2.3. The final assertion in the above statement is clear since both \( (A \otimes B(K))^- \) and \( A \otimes B(K) \) satisfy (ii). q.e.d.
Corollary 2.6. Let $A$ be a $C^*$-algebra, $\aleph$ a cardinal number and $K$ an $\aleph$-dimensional Hilbert space. If $B$ is a $C^*$-algebra which has a family $\{e_{ab}\}_{a,b \in I}$ of matrix units such that $e_{ab}Be_{ba}$ is $^*$-isomorphic to $A$ and $I = \aleph$, then $B$ is $^*$-isomorphic to $(A \otimes B(K))^-$.  

Proof. Note that $\{e_{ap}\}_{a,p \in I}$ and $B$ [resp. $\{1 \otimes f_{ap}\}_{a,p \in I}$ and $(A \otimes B(K))^-]$ satisfy the above (ii). q.e.d.

Corollary 2.7. Let $A$ be a monotone complete $C^*$-algebra, $K$ a Hilbert space and $A \otimes B(K)$ a $C^*$-tensor product of $A$ and $B(K)$. If the maps $A \ni x \mapsto x \otimes 1 \in A \otimes B(K)$ and $B(K) \ni y \mapsto 1 \otimes y \in A \otimes B(K)$ are normal, then $A \otimes B(K)$ is the minimal $C^*$-tensor product $A \otimes B(K)$.

Proof. Note that since the inclusion map $A \otimes B(K) \hookrightarrow (A \otimes B(K))^-$ is normal, $A \otimes 1$ and $1 \otimes B(K)$ are monotone closed in $(A \otimes B(K))^-$, hence that $(A \otimes B(K))^- \cong (A \otimes B(K))^-$. q.e.d.

§ 3. Fubini products of operator systems

An operator system is a self-adjoint linear subspace, containing the unit, of $B(H)$ with $H$ some Hilbert space, and it is injective if there is a completely positive projection of $B(H)$ onto it [4]. Each operator system $V \subset B(H)$ is contained in a minimal injective operator system $I(V) \subset B(H)$, called the injective envelope of $V$, which is unique up to unital complete order isomorphism [10].

Definition 3.1. An operator system $V \subset B(H)$ is a $C^*$-algebra (resp. $W^*$-algebra, monotone complete $C^*$-algebra, etc.) if it is unitaly completely order isomorphic to some $C^*$-algebra (resp. $W^*$-algebra, monotone complete $C^*$-algebra, etc.).

Note that an operator system is unitally completely order isomorphic to at most one $C^*$-algebra.

For future reference we record the following result due to Choi-Effros [4, Theorem 3.1] and Tomiyama [21, Theorem 7.1].

Lemma 3.2. Let $V$ and $W$ be operator systems with $V \subset W \subset B(H)$ such that there is a completely positive projection $\phi$ of $W$ onto $V$.

(i) If $W$ is a $C^*$-algebra, then $V$ is a $C^*$-algebra equipped with the order, involution, norm induced by those of $W$ and the multiplication given by $x \circ y = \phi(xy)$, $x, y \in V$, where $xy$ denotes the product of $x, y$ in the $C^*$-algebra $W$.

(ii) If, in addition, $W$ is monotone complete, then $V \ni x \mapsto x(O)$ in $W$ implies $x \mapsto \phi(x)(O)$ in $V$ and $V$ is monotone complete.
Lemma 3.2 means that each injective operator system is a monotone complete C*-algebra. Since the injective envelope of a C*-algebra [9] coincides with that as an operator system, an operator system $V$ is a C*-algebra (resp. monotone complete, additively complete C*-algebra) if and only if $V$ is a C*-subalgebra (resp. monotone closed, additively closed C*-subalgebra) of its injective envelope $I(V)$.

In the remainder of the paper we assume that operator systems are always norm closed and $K$ denotes a fixed Hilbert space with an orthonormal basis $\{e_a\}_{a \in I}$. As before $\{f_{\alpha \beta}\}_{\alpha, \beta \in I}$ denotes the family of matrix units in $B(K)$ given by $f_{\alpha \beta}(\eta) = (\eta_j \otimes e_{a})_{\alpha, \beta \in K}$. For another Hilbert space $H$ let $J_{\alpha}: H \rightarrow H \otimes K$ be the linear isometry given by $J_{\alpha} = \xi \otimes e_{\alpha}$. Then each $x \in B(H \otimes K)$ is written as the strong sum $x = \sum_{\alpha, \beta} x_{\alpha \beta} \otimes f_{\alpha \beta}$, where $x_{\alpha \beta} = J_{\alpha}^* x J_{\beta} \in B(H)$. For $f \in B(H)_\alpha$ (resp. $g \in B(K)_\alpha$) we denote by $f \otimes id_{B(K)}$ (resp. $id_{B(H)} \otimes g$) the right (resp. left) slice map, in the sense of Tomiyama [21], of $B(H \otimes K)$ into $B(K)$ (resp. $B(H)$).

**Definition 3.3.** For operator systems $V \subset B(H)$ and $W \subset B(K)$ let $V \otimes B(K)$ (resp. $B(H) \otimes W$) be the subspace of $B(H \otimes K)$ consisting of elements $x$ such that $(id_{B(H)} \otimes g)(x) \in V$ for all $g \in B(K)_\alpha$ (resp. $(f \otimes id_{B(K)})(x) \in W$ for all $f \in B(H)_\alpha$). We call the norm closure $V \otimes W$ of $V \circ W$ (resp. intersection $\mathcal{F}(V, W) = V \otimes B(K) \cap B(H) \otimes W$) the spatial tensor product (resp. Fubini product) of $V$ and $W$.

Tomiyama [21] observed that if $V$ and $W$ are W*-subalgebras of $B(H)$ and $B(K)$ respectively, then Tomita's commutation theorem implies $\mathcal{F}(V, W) = V \otimes W$.

**Lemma 3.4.** For an operator system $V \subset B(H)$ an $x \in B(H \otimes K)$ is in $V \otimes B(K)$ if and only if $x$ is written as $x = \sum_{\alpha, \beta} x_{\alpha \beta} \otimes f_{\alpha \beta}$ with $x_{\alpha \beta} \in V$ for all $\alpha, \beta$.

**Proof.** Denote by $\omega_{\gamma \gamma'}$ the element of $B(K)_\alpha$ given by $\omega_{\gamma \gamma'}(y) = (y \gamma_\alpha \gamma')$, $\gamma, \gamma' \in K$ and $y \in B(K)$. Since $(id_{B(H)} \otimes \omega_{\gamma \gamma'})(x) = x_{\gamma \gamma'}$, the necessity follows. Conversely suppose that $x = \sum_{\alpha, \beta} x_{\alpha \beta} \otimes f_{\alpha \beta}$ with $x_{\alpha \beta} \in V$ for all $\alpha, \beta$. Each $g \in B(K)_\alpha$ is the norm limit of a sequence $g_n \in B(K)_\alpha$ which is a finite linear combination of elements of the form $\omega_{\gamma, \gamma'}$. Then $V \ni (id_{B(H)} \otimes g_n)(x) \rightarrow (id_{B(H)} \otimes g)(x)$ in norm and $(id_{B(H)} \otimes g)(x) \in V$ since $V$ is norm closed. 

**Lemma 3.5.** Let $V_j \subset B(H)_j, j = 1, 2$, be operator systems.

(i) If $\phi: V_1 \rightarrow V_2$ is a unital completely positive map, then the map $\phi \otimes id_{B(K)}: V_1 \otimes B(K) \rightarrow V_2 \otimes B(K)$ defined by

\[
(\phi \otimes id_{B(K)})(s \sum x_{\alpha \beta} \otimes f_{\alpha \beta}) = s \sum \phi(x_{\alpha \beta}) \otimes f_{\alpha \beta}
\]

is a unique completely positive extension of $\phi \circ id_{B(K)}: V_1 \circ B(K) \rightarrow V_2 \circ B(K)$,
and satisfies

\[(3.2) \quad (\phi \otimes \text{id}_{B(K)})((1 \otimes b_1)x(1 \otimes b_2)) = (1 \otimes b_1)(\phi \otimes \text{id}_{B(K)})(x)(1 \otimes b_2)\]

for all \(x \in V_1 \otimes B(K)\) and \(b_1, b_2 \in B(K)\).

(ii) With \(\phi\) as in (i), \(\phi \otimes \text{id}_{B(K)}\) is a complete order isomorphism (resp. faithful on positive elements, \(\sigma\)-weakly continuous) if and only if \(\phi\) is a complete order isomorphism (resp. faithful on positive elements, \(\sigma\)-weakly continuous).

(iii) If \(\phi : V_1 \to V_2\) is a linear combination of unital completely positive maps (resp. \(f \in \mathcal{V}^*_1\)) and \(\phi \otimes \text{id}_{B(K)}\) (resp. \(f \otimes \text{id}_{B(K)} : V_1 \otimes B(K) \to B(K)\)) is defined as in (i) (resp. \((f \otimes \text{id}_{B(K)})(x) = \sum f(x_{\alpha})f_{a_{\beta}}\)), then for each \(g \in B(K)_*\),

\[(3.3) \quad \phi \circ (\text{id}_{B(H)} \otimes g) = (\text{id}_{B(H)} \otimes g) \circ (\phi \otimes \text{id}_{B(K)}),\]

(resp. \(f \circ (\text{id}_{B(H)} \otimes g) = g \circ (f \otimes \text{id}_{B(K)})\)).

**Proof.** (i) Since \(\sum_{\alpha \in I} f(x_{\alpha}) \otimes f_{a_{\beta}} \leq \sum_{\alpha \in I} \|x_{\alpha}\| \|f_{a_{\beta}}\| = \|x\|\) for each finite \(I \subseteq I\) and \(x \in V_1 \otimes B(K)\), \(s \sum f(x_{\alpha}) \otimes f_{a_{\beta}}\) defines an element of \(V_1 \otimes B(K)\) of norm \(\|x\|\); hence \(\phi \otimes \text{id}_{B(K)}\) is well-defined and contractive. Moreover, replacing \(K\) in the above argument by the direct sum of \(n\) copies of \(K\) \((n = 1, 2, \ldots)\) we see that it is completely positive. Thus it follows from \(\phi \otimes \text{id}_{B(K)}|_{\otimes B(K)} = \text{id}_{\otimes B(K)}\) and Choi [3, Theorem 3.1] that (3.2) holds. To see the uniqueness of \(\phi \otimes \text{id}_{B(K)}\) let \(\psi : V_1 \otimes B(K) \to V_2 \otimes B(K)\) be another completely positive extension of \(\phi \otimes \text{id}_{B(K)}\). For the same reason for \(\phi \otimes \text{id}_{B(K)}\), \(\psi\) satisfies (3.2) with \(\phi \otimes \text{id}_{B(K)}\) replaced by \(\phi\). Hence for \(x \in V_1 \otimes B(K)\) and \(\alpha, \beta \in I\) we have \(1 \otimes f_{a_{\beta}}\psi(x)(1 \otimes f_{a_{\beta}}) = \psi(1 \otimes f_{a_{\beta}})(1 \otimes f_{a_{\beta}}) = 1 \otimes f_{a_{\beta}}\psi(x)(1 \otimes f_{a_{\beta}}) = \phi(x_{\alpha}) \otimes f_{a_{\beta}} = 1 \otimes f_{a_{\beta}}\phi(x_{\alpha})\psi(x)(1 \otimes f_{a_{\beta}})\) and so \(\psi = \phi \otimes \text{id}_{B(K)}\).

(ii) is immediate from the definition of \(\phi \otimes \text{id}_{B(K)}\).

(iii) Since, as in the proof of Lemma 3.4, \(g \in B(K)_*\) is the norm limit of a sequence of finite linear combinations of the \(\omega_{x_{\alpha}a_{\beta}}\), we need only consider the case \(g = \omega_{x_{\alpha}a_{\beta}}\). Then \(\phi \circ (\text{id}_{B(H)} \otimes g)(x) = \phi(x_{\alpha}) = (\text{id}_{B(H)} \otimes g) \circ (\phi \otimes \text{id}_{B(K)})(x)\) for \(x \in V_1 \otimes B(K)\) and similarly for \(f \otimes \text{id}_{B(K)}\).

**LEMMA 3.6.** Let \(V \subseteq B(H)\) and \(W \subseteq B(K)\) be operator systems with \(W\) \(\sigma\)-weakly closed.

(i) \((f \otimes \text{id}_{B(K)})(\mathcal{F}(V, W)) \subseteq W\) for all \(f \in V^*\).

(ii) An \(x \in B(H \otimes K)\) is in \(\mathcal{F}(V, W)\) if and only if \((f \otimes \text{id}_{B(K)})(x) \in W\) and \((\text{id}_{B(H)} \otimes g)(x) \in V\) for all \(f \in B(H)^*\) and \(g \in B(K)_*\).

**Proof.** (i) If \(f_1 \in B(H)^*\) is an extension of \(f \in V^*\), then \(f_1 \otimes \text{id}_{B(K)}|_{\otimes B(K)} = f_1 \otimes \text{id}_{B(K)}\) on \(B(H)^* \times B(H)\) \(\sigma\)-weakly in \(B(K)\). Then \((f_1 \otimes \text{id}_{B(K)})(x) \in W\) for all \(x \in \mathcal{F}(V, W)\) and \((f_1 \otimes \text{id}_{B(K)})(x) = (f_1 \otimes \text{id}_{B(K)})(x)\) \(\sigma\)-weakly in \(B(K)\) since by (3.3), \(g \circ (f_1 \otimes \text{id}_{B(K)})(x) = (f_1 \otimes \text{id}_{B(K)})(x)\).
f \circ (\text{id}_{B(H)} \otimes g)(x) \to f \circ (\text{id}_{B(H)} \otimes g)(x) = g \circ (f \otimes \text{id}_{B(K)})(x) \text{ for all } g \in B(K)_s, \text{ so that } (f \otimes \text{id}_{B(K)})(x) \in W.

(ii) follows readily from (i). q.e.d.

DEFINITION 3.7. For operator systems $V \subset B(H)$, $W \subset B(K)$ with $W \sigma$-weakly closed and $f \in V^*$, $g \in W^*$ (= the restrictions to $W$ of elements of $B(K)_s$) the map $R_f = f \otimes \text{id}_{B(K)}|_{\mathcal{F}(V, W)} : \mathcal{F}(V, W) \to W$ (resp. $L_g = \text{id}_{B(H)} \otimes g|_{\mathcal{F}(V, W)} : \mathcal{F}(V, W) \to V$) is the right (resp. left) slice map on $\mathcal{F}(V, W)$. The element $\mathcal{F}(f, g)$ of $\mathcal{F}(V, W)^*$ defined by $\mathcal{F}(f, g)(x) = g(R_f(x)) = f(L_g(x))$ is the product functional of $f$ and $g$.

If the above $f$ (resp. $g$) is a state, then the map $x \mapsto 1 \otimes R_f(x)$ (resp. $x \mapsto L_g(x) \otimes 1$) is a completely positive projection of $\mathcal{F}(V, W)$ onto $1 \otimes W$ (resp. $V \otimes 1$).

LEMMA 3.8. Let $V_j \subset B(H_j)$, $j = 1, 2$, and $W \subset B(K)$ be operator systems and $\phi : V_1 \to V_2$ a unital completely positive map. If (i) $W$ is $\sigma$-weakly closed or if (ii) $V_1$ and $V_2$ are $\sigma$-weakly closed and $\phi$ is $\sigma$-weakly continuous, then $\phi \circ \text{id}_{B(K)}(\mathcal{F}(V_1, W)) \subset \mathcal{F}(V_2, W)$.

(iii) If, in addition to (i) or (ii), $V_2 \subset V_1 \subset B(H_1)$ and $\phi$ is a completely positive projection of $V_1$ onto $V_2$, then $\phi \circ \text{id}_{B(K)}|_{\mathcal{F}(V_1, W)}$ is a completely positive projection of $\mathcal{F}(V_1, W)$ onto $\mathcal{F}(V_2, W)$.

PROOF. We have $(\phi \circ \text{id}_{B(K)})(\mathcal{F}(V_1, W)) \subset (\phi \otimes \text{id}_{B(K)})(V_1 \otimes B(K)) \subset V_2 \otimes B(K)$ and from the construction, $(f \otimes \text{id}_{B(K)}) \circ (\phi \otimes \text{id}_{B(K)}) = f \circ \phi \circ \text{id}_{B(K)}$ on $V_1 \otimes B(K)$ for all $f \in B(H_2)^*$.

(i) By Lemma 3.6, $(f \otimes \text{id}_{B(K)}) \circ (\phi \otimes \text{id}_{B(K)})(\mathcal{F}(V_1, W)) = (f \circ \phi \otimes \text{id}_{B(K)})(\mathcal{F}(V_1, W)) \subset W$ for all $f \in B(H_2)_s$, so that $(\phi \otimes \text{id}_{B(K)})(\mathcal{F}(V_1, W)) \subset B(H_2) \otimes W$ and $(\phi \circ \text{id}_{B(K)})(\mathcal{F}(V_1, W)) \subset \mathcal{F}(V_2, W)$.

(ii) Since $f \circ \phi \in B(H_2)_s$ for $f \in B(H_2)_s$, $(f \circ \phi \otimes \text{id}_{B(K)})(\mathcal{F}(V_1, W)) \subset W$ and the argument as in (i) implies the asserted inclusion.

(iii) is clear.

q.e.d.

For $j = 1, 2$ let $V_j \subset B(H_j)$, $W_j \subset B(K_j)$ be operator systems and $\phi : V_1 \to V_2$, $\psi : W_1 \to W_2$ unital completely positive maps. We will show that the map $\phi \circ \psi : V_1 \circ V_2 \to W_1 \otimes W_2$ extends to a completely positive map of $\mathcal{F}(V_1, W_1)$ into $\mathcal{F}(V_2, W_2)$ under some additional hypotheses. By Lemma 3.5 we obtain completely positive maps $\phi \otimes \text{id}_{B(K)} : V_1 \otimes B(K) \to V_2 \otimes B(K)$, $\text{id}_{B(H)} \otimes \psi : B(H) \otimes W_1 \to B(H) \otimes W_2$ and so

$(\text{id}_{B(H)} \otimes \psi) \circ (\phi \otimes \text{id}_{B(K)})(V_1 \otimes B(K)) = (\phi \otimes \text{id}_{B(K)})(V_1 \otimes B(K)) \to (\text{id}_{B(H)} \otimes \psi)(V_1 \otimes W_1) = \phi \otimes \psi$. 

Obviously $(\text{id}_{B(H)} \otimes \psi) \circ (\phi \otimes \text{id}_{B(K)})(V_1 \otimes W_1) = (\phi \otimes \text{id}_{B(K)})(V_1 \otimes W_1) = \phi \otimes \psi$. 


say, is a unique completely positive map of $V_1 \otimes W_1$ into $V_2 \otimes W_2$ which extends $\phi \otimes \psi$. (This fact is implicit in Effros-Lance [8, Lemma 2.5].)

**Lemma 3.9.** Keep the above notation.

(i) If $\phi$ and $\psi$ are unital complete order isomorphisms, then $\phi \otimes \psi$ is a unital complete order isomorphism.

(ii) If $V_1$ and $W_2$ are $\sigma$-weakly closed, then

$$(\phi \otimes \text{id}_{B(K)}) \circ (\text{id}_{B(H)} \otimes \psi)(\mathcal{F}(V_1, W_2)) \subset \mathcal{F}(V_2, W_2).$$

(iii) If $W_1$ and $W_2$ are $\sigma$-weakly closed and $\psi$ is $\sigma$-weakly continuous, then

$$(\phi \otimes \text{id}_{B(K)}) \circ (\text{id}_{B(H)} \otimes \psi)(\mathcal{F}(V_1, W_1)) \subset \mathcal{F}(V_2, W_2).$$

**Proof.** (i) Note that $\phi^{-1} \otimes \psi^{-1}$ is the complete positive inverse of $\phi \otimes \psi$.

(ii) By Lemma 3.8 (i), $(\text{id}_{B(H)} \otimes \psi)(\mathcal{F}(V_1, W_2)) \subset \mathcal{F}(V_1, W_2)$ and $(\phi \otimes \text{id}_{B(K)}) \circ (\text{id}_{B(H)} \otimes \psi)(\mathcal{F}(V_1, W_2)) \subset \mathcal{F}(V_2, W_2)$.

(iii) As in (ii) we apply Lemma 3.8 to obtain $(\phi \otimes \text{id}_{B(K)}) \circ (\text{id}_{B(H)} \otimes \psi)(\mathcal{F}(V_1, W_1)) \subset \mathcal{F}(V_2, W_2)$. By Lemma 3.5 (iii), $\phi \circ (\text{id}_{B(H)} \otimes g_2) \circ (\phi \otimes \text{id}_{B(K)}) \circ (\text{id}_{B(H)} \otimes g_1) \circ (\phi \otimes \text{id}_{B(K)}) \circ (\text{id}_{B(H)} \otimes \psi) = \text{id}_{B(H)} \otimes \psi$ on $V_1 \otimes \mathcal{B}(K_1)$ for $g_1 \in \mathcal{B}(K_1)$. If $g_2 \in \mathcal{B}(K_2)$, then $g_2 \circ \psi$ extends to a $g_1 \in \mathcal{B}(K_1)$ and $(\phi \otimes \text{id}_{B(K)}) \circ (\text{id}_{B(H)} \otimes g_2) \circ (\phi \otimes \text{id}_{B(K)}) = \text{id}_{B(H)} \otimes \psi$ on $\mathcal{F}(V_2, W_2)$.

The results similar to the above lemmas were proved by Nagisa and Tomiyama [17] when $V_j$ and $W_j$ are $W^*$-subalgebras, but $\phi$ and $\psi$ are not necessarily unital.

The following is an analogue of [21, Theorem 7.5].

**Proposition 3.10.** Given operator systems $V \subset B(H)$ and $W \subset B(K)$ with $W$ $\sigma$-weakly closed, $\mathcal{F}(V, W)$ is injective if and only if $V$ and $W$ are injective.
For a state $f \in V^*$ (resp. $g \in W_*$) the map $x \rightarrow 1 \otimes R_f(x)$ (resp. $x \rightarrow L_g(x) \otimes 1$) is a completely positive projection of $\mathcal{F}(V, W)$ onto $1 \otimes W$ (resp. $V \otimes 1$). Hence the necessity follows. Conversely if $V$ and $W$ are injective, then there are completely positive projections $\phi$ of $B(H)$ onto $V$ and $\psi$ of $B(K)$ onto $W$. Then $$(\phi \otimes \text{id}_{B(K)}) \circ (\text{id}_{B(H) \otimes B(K)}) \subset \mathcal{F}(V, W)$$ by Lemma 3.9 (ii) and clearly $$(\phi \otimes \text{id}_{B(K)}) \circ (\text{id}_{B(H) \otimes B(K)})|_{\mathcal{F}(V, W)} = \text{id}_{\mathcal{F}(V, W)}.$$ Thus $\mathcal{F}(V, W)$ is injective. q.e.d.

The injective envelope of the spatial tensor product of the form $V \otimes B(K)$ is determined as follows:

**Proposition 3.11.** Let $V \subseteq B(H)$ be an operator system and $I(V)$ its injective envelope with $V \subseteq I(V) \subseteq B(H)$. Then $I(V) \otimes B(K)$ is the injective envelope of $V \otimes B(K)$, i.e., $I(V) \otimes B(K) = I(V) \otimes B(K)$.

**Proof.** By Proposition 3.10, $I(V) \otimes B(K)$ is injective and contains $V \otimes B(K)$. Hence it suffices to show that the completely positive map $\phi: I(V) \otimes B(K) \rightarrow I(V) \otimes B(K)$ with $\phi |_{V \otimes B(K)} = \text{id}_{V \otimes B(K)}$ is necessarily $\text{id}_{I(V) \otimes B(K)}$. But it follows from [3, Theorem 3.1] that $\phi$ is a $(1 \otimes B(K))$-module homomorphism, hence that $\phi(I(V) \otimes 1) \subseteq I(V) \otimes B(K) \cap (1 \otimes B(K))' = I(V) \otimes B(K) \cap B(H) \otimes 1 = I(V) \otimes 1$. Then the map $\psi: I(V) \rightarrow I(V)$ given by $\phi(a \otimes 1) = \psi(a) \otimes 1$ is completely positive and $\psi \otimes \text{id}_{B(K)}: I(V) \otimes B(K) \rightarrow I(V) \otimes B(K)$ is a completely positive map with $\phi |_{I(V) \otimes B(K)} = \psi \otimes \text{id}_{B(K)} |_{I(V) \otimes B(K)}$, so that $\phi = \psi \otimes \text{id}_{B(K)}$ by Lemma 3.5 (i). On the other hand, $\psi = \text{id}_{I(V)}$ since $\psi |_V = \text{id}_V$; hence $\phi = \text{id}_{I(V) \otimes B(K)}$. q.e.d.

The next result is the main theorem of this section.

**Theorem 3.12.** Let $V \subseteq B(H)$ be an operator system, $M \subseteq B(K)$ a $W^*$-subalgebra, $\{\alpha_{s,t}\} \subseteq I$ the family of matrix units in $B(K)$ defined as above and $\dim K = \aleph_1$.

(i) $V \otimes B(K)$ is a $C^*$-algebra if and only if $V$ is an $\aleph_1$-additively complete $C^*$-algebra. In this case $V \otimes B(K)$ is also $\aleph_1$-additively complete and $\{1 \otimes \alpha_{s,t}\} \subseteq I$ is a family of matrix units in $V \otimes B(K)$.

(ii) $V \otimes B(K)$ is an additively complete $C^*$-algebra (resp. monotone complete $C^*$-algebra, $W^*$-algebra) if and only if $V$ is an additively complete $C^*$-algebra (resp. monotone complete $C^*$-algebra, $W^*$-algebra).

(iii) If $V$ is an additively (resp. monotone) complete $C^*$-algebra, then $\mathcal{F}(V, M)$, $V \otimes 1$ and $1 \otimes M$ are additively (resp. monotone) closed $C^*$-subalgebras of $V \otimes B(K)$.

**Proof.** Since $I(V)$ is an injective $C^*$-algebra, we replace $H$ by the representation space of a faithful $*$-representation of $I(V)$ to assume that $I(V)$ is a $C^*$-subalgebra of $B(H)$. We describe the multiplication $\circ$ in the
injective C*-algebra $I(V \otimes B(K)$ by means of that in $I(V)$. (Note that $I(V) \otimes B(K)$ need not be a C*-subalgebra of $B(H \otimes K)$. See Corollary 3.15 below.) For a completely positive projection $\phi$ of $B(H)$ onto $I(V)$, $\phi \otimes \text{id}_{B(K)}$ is a completely positive projection of $B(H \otimes K)$ onto $I(V) \otimes B(K)$, and for $x = s- \sum x_\alpha \otimes f_\alpha$, $y = s- \sum y_\beta \otimes f_\beta \in I(V) \otimes B(K)$ we have $x \circ y = (\phi \otimes \text{id}_{B(K)})(xy) = s- \sum f_\alpha \phi(\phi^{-1} x_\alpha y_\beta) \otimes f_\beta$. It follows from Lemma 1.1 that

(3.4) \[ x \circ y = s- \sum x_\alpha y_\beta (O- \sum x_\alpha y_\beta) \otimes f_\alpha, \]

where the sum in the parentheses order-converges in $I(V)$, and that $(1 \otimes f_\alpha)_{\alpha \in I}$ is a family of matrix units in $I(V) \otimes B(K)$. Moreover if $x$ or $y$ is in $I(V) \otimes B(K)$, then $x \circ y = (\phi \otimes \text{id}_{B(K)})(xy) = xy$ since $\phi \otimes \text{id}_{B(K)}$ is an $(I(V) \otimes B(K))$-module homomorphism.

Since $V \otimes B(K) \subseteq V \otimes B(K) \subseteq I(V) \otimes B(K) = I(V \otimes B(K))$, $V \otimes B(K)$ is a C*-algebra if and only if it is a C*-subalgebra of $I(V) \otimes B(K)$. Noting $V \otimes f_\alpha = (1 \otimes f_\alpha)(V \otimes B(K))(1 \otimes f_\alpha) = (1 \otimes f_\alpha) \cdot (V \otimes B(K)) \cdot (1 \otimes f_\alpha)$, we see the necessity of (ii) and that if $V \otimes B(K)$ is a C*-algebra, then $V$ is a C*-algebra.

Necesity of (i): We need only show that $V$ is $\mathfrak{N}$-additively complete. If $\{x_\alpha\}_{\alpha \in I}$ is a $\mathfrak{N}$-family in $V$ and $\beta \in I$ is fixed, then $s- \sum x_\alpha \otimes f_\beta$ exists in $V \otimes B(K)$ since the family $(\sum_{\alpha \in I'} x_\alpha \otimes f_\beta)^* (\sum_{\alpha \in I'} x_\alpha \otimes f_\beta) = \sum_{\alpha \in I'} x_\alpha^* x_\alpha \otimes f_\beta$ with $I' \subseteq I$ finite, is bounded. Hence by (3.4), $(s- \sum x_\alpha^* x_\alpha) \otimes f_\beta = (s- \sum x_\alpha \otimes f_\beta)^* \circ (s- \sum x_\alpha \otimes f_\beta) \in V \otimes B(K)$ and $s- \sum x_\alpha^* x_\alpha \in V$.

Sufficiency of (i): If $V$ is an $\mathfrak{N}$-additively complete C*-algebra, then it is $\mathfrak{N}$-additively closed C*-subalgebra of $I(V)$. For $x, y \in V \otimes B(K)$ we have $s- \sum x_\alpha y_\beta \in V$ by Lemma 1.5 and $x \circ.y = s- \sum (s- \sum x_\alpha y_\beta) \otimes f_\alpha \in V \otimes B(K)$. Hence $V \otimes B(K)$ is a C*-algebra.

We show that if $V \otimes B(K)$ is a C*-algebra, then it is $\mathfrak{N}$-additively complete. If $\mathfrak{N} = \text{dim} K$ is finite, we have nothing to prove. Hence suppose that $\mathfrak{N}$ is infinite. Then $(V \otimes B(K)) \otimes B(K) = V \otimes B(K \otimes K) \cong V \otimes B(K)$ is a C*-algebra and from the foregoing $V \otimes B(K)$ is additively complete.

Sufficiency of (ii): If $V$ is an additively complete C*-algebra, then as above $(V \otimes B(K)) \otimes B(K') \cong V \otimes B(K \otimes K')$ is a C*-algebra for each Hilbert space $K'$ and so $V \otimes B(K)$ is additively complete.

If $V$ is a monotone complete C*-algebra, then $V \otimes B(K)$ is monotone closed in $I(V) \otimes B(K)$, hence monotone complete. In fact, for a fixed $x_\alpha \in I$, $V \otimes f_{\alpha \alpha}$ is monotone closed in $I(V) \otimes B(K)$. If $\{x_\alpha\}$ is a bounded increasing net in $(V \otimes B(K))_{\alpha, \beta}$, then it has an order limit $O- \lim x$ in $(V \otimes B(K))$. Since $(1 \otimes f_{\alpha \alpha}) x \in V \otimes f_{\alpha \alpha}$ for all $\alpha, \beta$, Lemma 1.2 implies that $(1 \otimes f_{\alpha \alpha}) \cdot (1 \otimes f_{\beta \beta}) = O- \lim (1 \otimes f_{\alpha \alpha}) x \in V \otimes f_{\alpha \alpha} \otimes f_{\alpha \alpha}$, hence that $O- \lim x \in V \otimes B(K)$.

If $V$ is a W*-algebra, then it is unitally completely order isomorphic to a W*-subalgebra $V_i$ of $B(H_i)$ and $V \otimes B(K)$ is unitally completely order iso-
morphic to the $W^*$-tensor product $V \otimes B(K)$.

(iii) Since $\mathcal{F}(V, M) = V \otimes B(K) \cap B(H) \otimes M = V \otimes B(K) \cap (1 \otimes M')'$, where primes denote commutants, $\mathcal{F}(V, M)$ is the intersection of $V \otimes B(K)$ and the fixed point algebra under the inner *-automorphisms $x \mapsto (1 \otimes u) \star (1 \otimes u^*) = (1 \otimes u) x (1 \otimes u^*)$ of $I(V) \otimes B(K)$ with $u$ unitaries in $M'$. Hence the conclusion follows from (ii).

From the theorem and Lemma 2.3 we deduce the following:

**Corollary 3.13** If $\dim K = \mathcal{K}$ and $V \subseteq B(H)$ is an operator system which is an $\mathcal{K}$-additively complete $C^*$-algebra, then $V \otimes B(K)$ is a unique $\mathcal{K}$-additively complete $C^*$-algebra $B$ which has a family $\{e_a\}_{a \in \mathcal{F}}$ of matrix units such that $e_a^* B e_a$ is *-isomorphic to $V$ and $I = \mathcal{K}$.

We follow the line of argument similar to that of Theorem 3.14 to obtain:

**Proposition 3.14.** Let $V \subseteq W \subseteq B(H)$ be operator systems with $W$ an $\mathcal{K}$-additively complete $C^*$-algebra and $\dim K = \mathcal{K}$. Then $V \otimes B(K)$, canonically embedded in $W \otimes B(K)$, is a $C^*$-subalgebra of $W \otimes B(K)$ if and only if $V$ is an $\mathcal{K}$-additively closed $C^*$-subalgebra of $W$. In this case $V \otimes B(K)$ is also $\mathcal{K}$-additively closed in $W \otimes B(K)$, and if in addition $V$ is monotone closed in $W$, then $V \otimes B(K)$ is monotone closed in $W \otimes B(K)$.

**Corollary 3.15.** Let $A$ be a $C^*$-subalgebra of $B(H)$. Then $A \otimes B(K)$ is a $C^*$-subalgebra of $B(H \otimes K)$ for each Hilbert space $K$ if and only if $A$ is a $W^*$-subalgebra of $B(H)$.

**Proof.** By the proposition $A \otimes B(K)$ is a $C^*$-subalgebra of $B(H \otimes K)$ for each Hilbert space $K$ if and only if $A$ is additively closed in $B(H)$. But the latter condition implies that $A$ is an $AW^*$-subalgebra of $B(H)$, hence that it is a $W^*$-subalgebra of $B(H)$ by Pedersen [18]. q.e.d.

**Corollary 3.16.** Let $V \subseteq W \subseteq B(H)$ be operator systems such that $W$ is a monotone complete $C^*$-algebra and $V$ is its monotone closed $C^*$-subalgebra and let $M \subseteq N \subseteq B(K)$ be $W^*$-subalgebras of $B(K)$. Then $\mathcal{F}(M, V)$ is a monotone closed $C^*$-subalgebra of $\mathcal{F}(N, W)$.

**Proof.** Combine the proposition with Theorem 3.12 (iii). q.e.d.

Theorem 3.12 is generalized as follows:

**Proposition 3.17.** Let $V \subseteq B(H)$ be an operator system and $M \subseteq B(K)$ a $W^*$-subalgebra. Then $\mathcal{F}(V, M)$ is a monotone complete $C^*$-algebra (resp. $W^*$-algebra) if and only if $V$ is a monotone complete $C^*$-algebra (resp. $W^*$-algebra).
PROOF. We need only show the necessity. If $g$ is a normal state of $M$, then the map $x \mapsto L_g(x) \otimes 1$ is a completely positive projection of $\mathcal{F}(V, M)$ onto $V \otimes 1$. Hence by Lemma 3.2 if $\mathcal{F}(V, M)$ is a monotone complete C*-algebra, then $V$ is also a monotone complete C*-algebra. Thus by Theorem 3.12 (iii), $V \otimes 1$ is a monotone closed C*-subalgebra of $\mathcal{F}(V, M)$, so that if $\mathcal{F}(V, M)$ is a W*-algebra, then $V$ is a W*-algebra. q.e.d.

We close this section with a result on the associativity of the Fubini products, whose straightforward proof is left to the reader.

**Proposition 3.18.** Let $V_j \subseteq B(H_j)$, $j = 1, 2, 3$, be operator systems. Then $\mathcal{F}(V_1, \mathcal{F}(V_2, V_3)) = \mathcal{F}(\mathcal{F}(V_1, V_2), V_3)$.

§ 4. **Monotone complete tensor products**

Henceforth we are concerned with only monotone complete C*-algebras. For simplicity we regard them as C*-subalgebras, containing the unit, of some $B(H)$. In this situation they are called **monotone complete C*-subalgebras** of $B(H)$.

**Definition 4.1.** Given a monotone complete C*-subalgebra $A$ of $B(H)$ (resp. W*-subalgebra $M$ of $B(K)$), the **monotone complete tensor product** of $A$ and $M$, written $A \otimes M$, is the monotone closure of $A \odot M$ in the Fubini product $\mathcal{F}(A, M)$ (i.e., the smallest monotone closed C*-subalgebra, containing $A \odot M$, of $\mathcal{F}(A, M)$).

The results in the preceding sections are summarized in the following:

**Theorem 4.2.** Let $A$ (resp. $M$) be a monotone complete C*-subalgebra of $B(H)$ (resp. W*-subalgebra of $B(K)$).

(i) The monotone complete tensor product $A \otimes M$ does not depend on the underlying Hilbert spaces $H$ and $K$. Namely if $\phi$ (resp. $\psi$) is a *-isomorphism of $A$ (resp. $M$) onto a monotone complete C*-subalgebra $A_i$ of $B(H_i)$ (resp. W*-subalgebra $M_i$ of $B(K_i)$), then there is a unique *-isomorphism $\phi \otimes \psi$ of $A \otimes M$ onto $A_i \otimes M_i$ which extends $\phi \odot \psi$.

(ii) The operation $\otimes$ defined for each pair of a monotone complete C*-algebra and a W*-algebra satisfies the following properties:

(a) $A \otimes M$ is a monotone complete C*-algebra which contains $A \otimes 1$ and $1 \otimes M$ as monotone closed C*-subalgebras and is the monotone closure of $A \odot M$.

(b) If $A$ (resp. $M$) is a monotone closed C*-subalgebra of a monotone complete C*-algebra $B$ (resp. W*-subalgebra of a W*-algebra $N$), then $A \otimes M$ is the monotone closure of $A \odot M$ in $B \otimes N$. 

(c) $A \bar{\otimes} M$ is a $W^*$-algebra if and only if $A$ is a $W^*$-algebra.
(iii) The operation $\bar{\otimes}$ is characterized uniquely by the properties (a) and (b).

**Proof.** (i) and (ii) are immediate from the construction and Theorem 3.12.

(iii) Let an operation $\bar{\otimes}$ be defined for each pair of a monotone complete $C^*$-algebra $A$ and a $W^*$-subalgebra $M$ of $B(K)$ and satisfy (a) and (b) with $\otimes$ replaced by $\bar{\otimes}$. Then $A \bar{\otimes} B(K)$ satisfies (a) with $\otimes$ and $M$ replaced by $\bar{\otimes}$ and $B(K)$, so that Theorem 2.5 implies the existence of a *-isomorphism $\pi$ of $A \bar{\otimes} B(K)$ onto $A \otimes B(K)$ which fixes $A \otimes B(K)$ elementwise. Moreover (b) shows that $\pi(A \otimes M) = \pi(m \text{-} \text{cl}_{\bar{\otimes}B(K)} A \otimes M) = m \text{-} \text{cl}_{\bar{\otimes}B(K)} \pi(A \otimes M) = A \bar{\otimes} M$.

q.e.d.

In connection with the tensor product of normal completely positive maps, we obtain:

**Theorem 4.3.** For $j=1, 2$ let $A_j$ (resp. $M_j$) be a monotone complete $C^*$-subalgebra of $B(H_j)$ (resp. $W^*$-subalgebra of $B(K_j)$). If $\phi: A_1 \to A_2$ and $\psi: M_1 \to M_2$ are normal completely positive maps, then $\mathcal{F}(\phi, \psi): \mathcal{F}(A_1, M_1) \to \mathcal{F}(A_2, M_2)$ is a normal completely positive map, and $\phi \otimes \psi = \mathcal{F}(\phi, \psi)|_{\bar{\otimes}A_1, M_1}$ is a unique normal completely positive extension of $\phi \otimes \psi$ such that $(\phi \otimes \psi)(A_1 \bar{\otimes} M_1) \subset A_2 \bar{\otimes} M_2$.

The proof follows from the following two lemmas. Noting that $\psi$ is $\sigma$-weakly continuous, Lemma 3.9 (iii) shows the existence of the completely positive map $\mathcal{F}(\phi, \psi) = (\phi \otimes \text{id}_{B(K_2)}) \circ (\text{id}_{B(H_1)} \otimes \psi)|_{\bar{\otimes}A_1, M_1}: \mathcal{F}(A_1, M_1) \to \mathcal{F}(A_2, M_2)$.

**Lemma 4.4.** The restriction $(\text{id}_{B(H_1)} \otimes \psi)|_{\bar{\otimes}A_1, M_1}: \mathcal{F}(A_1, M_1) \to \mathcal{F}(A_1, M_2)$ is normal.

**Proof.** Take the injective envelope $I(A_i)$ of $A_i$ so that $A_i \subset I(A_i) \subset B(H_i)$ and let $\pi$ be a completely positive projection of $B(H_i)$ onto $I(A_i)$. Then $\pi \otimes \text{id}_{B(K_2)}|_{\bar{\otimes}A_1, M_2}$ is a completely positive projection of $B(H_i) \bar{\otimes} M_2$, and $\mathcal{F}(I(A_i), M_2)$ and $\mathcal{F}(A_1, M_2)$ is monotone closed in $\mathcal{F}(I(A_i), M_2)$ (Corollary 3.16). Hence the order limit of a net in $\mathcal{F}(A_1, M_2)$ coincides with that calculated in $\mathcal{F}(I(A_i), M_2)$, so that Lemma 3.2 implies that for each bounded increasing net $\{x_\alpha\}$ in $\mathcal{F}(A_1, M_2)$, $\text{O-lim}_{s, a} \mathcal{F}(A_1, M_2) x_\alpha = \text{O-lim}_{s, a} \mathcal{F}(I(A_i), M_2) x_\alpha = (\pi \otimes \text{id}_{B(K_2)})(\text{s-lim}_{B(H_1) \bar{\otimes} M_2} x_\alpha)$, where $\text{O-lim}_{s, a} x_\alpha$ (resp. $\text{s-lim}_{s, a} x_\alpha$) means the order (resp. strong) limit in $B$. Similarly

$\text{O-lim}_{\bar{\otimes}A_1, M_2}(\text{id}_{B(H_1)} \otimes \psi)(x_\alpha) = (\pi \otimes \text{id}_{B(K_2)})(\text{s-lim}_{B(H_1) \bar{\otimes} M_2}(\text{id}_{B(H_1)} \otimes \psi)(x_\alpha))$.

Hence by Lemma 3.9 (iii) and the $\sigma$-weak continuity of $\text{id}_{B(H_1)} \otimes \psi$,
(id_{B(H)} \otimes \psi)(O\lim_{\mathcal{F}(A, M)} x_a) = (id_{B(H)} \otimes \psi)\circ (\pi \otimes id_{B(K)}) \circ (s\lim_{\mathcal{F}(A, M)} x_a)

= (\pi \otimes id_{B(K)}) \circ (id_{B(H)} \otimes \psi) \circ (s\lim_{\mathcal{F}(A, M)} x_a)

= (\pi \otimes id_{B(K)}) \circ (s\lim_{\mathcal{F}(A, M)} (id_{B(H)} \otimes \psi)(x_a))

= O\lim_{\mathcal{F}(A, M)} (id_{B(H)} \otimes \psi)(x_a).

q.e.d.

If we write as in Section 3, \( x = \sum a_{\alpha} \otimes f_{\alpha} \in A_1 \otimes B(K_2) \), where \( x_{\alpha} \in A_1 \) and \( \{f_{\alpha}\} \) is the family of matrix units in \( B(K_2) \), then it is readily shown that for an increasing net \( \{x_i\} \), \( x_i \in A_1 \otimes B(K_2) \) if and only if \( (x_i)_{\alpha} \rightarrow x_{\alpha} \) (in \( A_1 \) for all \( \alpha, \beta \). From this fact, (3.1) and the normality of \( \phi \) we conclude the following:

**Lemma 4.5.** The map \( \phi \otimes id_{B(K_2)} : A_1 \otimes B(K_2) \rightarrow A_1 \otimes B(K_2) \) is normal.

If we denote the restriction to \( A \otimes M \) of the right (resp. left) slice map \( R_f \) (resp. \( L_g \)) on \( \mathcal{F}(A, M) \), \( f \in A_* \) (resp. \( g \in M_* \)), again by \( R_f \) (resp. \( L_g \)) and we write \( f \otimes g = \mathcal{F}(f, g) \mid_{A \otimes M} \), then we can show the next result, whose almost obvious proof is omitted.

**Proposition 4.6.** With notation as above \( \{R_f : f \in A_*\} \) (resp. \( \{L_g : g \in M_*\} \)) is a separating family of continuous linear maps from \( A \otimes M \) into \( M \) (resp. \( A \)) which satisfies the following properties:

(i) \( R_f \mid_{A \otimes M} = f \otimes id_M \) and \( L_g \mid_{A \otimes M} = id_A \otimes g \), where \( (f \otimes id_M)(\sum a_i \otimes b_i) = \sum f(a_i)b_i \) and \( (id_A \otimes g)(\sum a_i \otimes b_i) = \sum g(b_i)a_i \).

(ii) \( R_f ((1 \otimes b) x (1 \otimes b)) = b R_f (x) b \) and \( L_g ((a_1 \otimes 1) x (a_2 \otimes 1)) = \sum a_i L_g (x) a_i \), for all \( x \in A \otimes M \), \( a_i, a_2 \in A \) and \( b_1, b_2 \in M \).

(iii) If \( g \in M_* \) is a state, then \( L_g \) is a unique normal completely positive extension of \( id_A \otimes g \).

(iv) \( (f \otimes g)(x) = g(R_f(x)) = f(L_g(x)) \) for all \( x \in A \otimes M \).

**Proposition 4.7.** Let \( A \) (resp. \( M \)) be a monotone complete C*-subalgebra of \( B(H) \) (resp. W*-subalgebra of \( B(K) \)) and \( G_1 \) (resp. \( G_2 \)) a group of *-automorphisms of \( A \) (resp. \( M \)). Then the fixed point algebra \( \mathcal{F}(A, M) \mid_{G_1 \times G_2} \) of \( \mathcal{F}(A, M) \) under the *-automorphisms \( \mathcal{F}(\pi, \rho) \) with \( \pi \in G_1 \) and \( \rho \in G_2 \) coincides with the Fubini product \( \mathcal{F}(A^{G_1}, M^{G_2}) \) of the fixed point algebras \( A^{G_1} \) and \( M^{G_2} \) under \( G_1 \) and \( G_2 \) respectively.

**Proof.** For an \( x \in \mathcal{F}(A, M) \) we have \( (\pi \otimes id_{B(K)})(x) = x \) for all \( \pi \in G_1 \) if and only if \( (id_{B(H)} \otimes g)(x) = x \) for all \( g \in B(K)^* \) since by (3.3), \( \pi \circ (id_{B(H)} \otimes g)(x) = (id_{B(H)} \otimes g)(x) \). Similarly \( (id_{B(H)} \otimes \rho)(x) = x \) for all \( \rho \in G_2 \) if and only if \( (f \otimes id_{B(K)})(x) = x \) for all \( f \in B(H)^* \) since by the \( \sigma \)-weak continuity of \( \rho \) and Lemma 3.9 (iii), \( \rho \circ (f \otimes id_{B(K)})(x) = (f \otimes id_{B(K)})(\rho)(x) \). Hence \( \mathcal{F}(\pi, \rho)(x) = (\pi \otimes id_{B(K)})(x) = x \) for all \( x \in \mathcal{F}(A, M) \) and \( \rho \in G_2 \) if and only if \( (id_{B(H)} \otimes g)(x) = x \) for all \( f \in B(H)^* \) and \( g \in B(K)^* \).
if and only if \( x \in \mathcal{F}(A, M) \) (Lemma 3.6 (ii)). q.e.d.

**Corollary 4.8.** Let \( A \) (resp. \( M \)) be a monotone complete C*-algebra (resp. W*-algebra) and \( A_i \) (resp. \( M_i \)) a C*-subalgebra of \( A \) (resp. \( M \)).

(\( i \)) \( (A_i \otimes M_i) \cap \mathcal{F}(A, M) = \mathcal{F}(A_i \cap A, M_i \cap M) \), where \( B' \cap C \) denotes the relative commutant of \( B \) in \( C \).

(\( ii \)) If \( A \) and \( M \) are factors, then \( A \otimes M \) and \( \mathcal{F}(A, M) \) are factors.

**Proof.** (\( i \)) Take as the above \( G_i \) (resp. \( G_i \)) the group of all inner *-automorphisms of \( A \) (resp. \( M \)) implemented by unitaries in \( A_i \) (resp. \( M_i \)).

(\( ii \)) Putting \( A_1 = A \) and \( M_1 = M \) in (i) we get \( Z_{A\otimes M} \subset (A \otimes M)' \cap \mathcal{F}(A, M) \) = \( \mathcal{F}(Z_A, Z_M) = C1 \otimes 1 \) and similarly for \( Z_{A\otimes M} \). q.e.d.

**Theorem 4.9.** Let \( A \) be the injective envelope of a separable infinite dimensional simple C*-algebra and \( M \) the type II_1 W*-factor generated by the left regular representation of the free group on two generators. Then \( A \) and \( A \otimes M \) are non *-isomorphic σ-finite monotone complete non W*, AW*-factors of type III.

**Proof.** By [11, Corollary 3.8], \( A \) is injective (hence monotone complete) σ-finite non W*, AW*-factor of type III with a faithful state, say, \( f \). Then it follows from Theorem 4.2 and Corollary 4.8 that \( A \otimes M \) is a non W*, monotone complete AW*-factor. Moreover \( A \otimes M \) is noninjective since the map \( x \mapsto 1 \otimes R_x(x) \) is a completely positive projection of \( A \otimes M \) onto \( 1 \otimes M \) and \( M \) is noninjective, so that it is not *-isomorphic to \( A \). If \( g \) is a faithful normal state of \( M \), the product functional \( f \otimes g \) on \( A \otimes M \) is also a faithful state (Lemma 3.9 (iii)). Hence \( A \otimes M \) is σ-finite and is of type III by Wright [24, Corollary]. q.e.d.

The associativity for monotone complete tensor products is valid.

**Proposition 4.10.** If \( A \) is a monotone complete C*-subalgebra of \( B(H) \) and \( M_1, M_2 \) are W*-subalgebras of \( B(K_1), B(K_2) \) respectively, then \( (A \otimes M_1) \otimes M_2 = A \otimes (M_1 \otimes M_2) \).

**Proof.** Since \( (A \otimes B(K_1)) \otimes B(K_2) = A \otimes (B(K_1) \otimes B(K_2)) \), it follows from the normality of the maps \( B \ni x \mapsto x \otimes 1 \in B \otimes N, N \ni y \mapsto 1 \otimes y \in B \otimes N \) with \( B \) a monotone complete C*-algebra and \( N \) a W*-algebra that the both sides in the above equality are the monotone closure of \( A \otimes M_1 \otimes M_2 \) in \( A \otimes B(K_1) \otimes B(K_2) \). q.e.d.

**Corollary 4.11.** If \( A \) (resp. \( M \)) is a type I AW*-algebra (resp. type I W*-algebra), then \( A \otimes M \) is a type I AW*-algebra.

**Proof.** We may assume that \( A \) and \( M \) are homogeneous, i.e., \( A = \ldots \)
$Z_A \boxtimes B(H)$ and $M = Z_M \boxtimes B(K)$ for some Hilbert spaces $H$ and $K$. Then $A \boxtimes M = (Z_A \boxtimes Z_M) \boxtimes B(H \otimes K)$ and $Z_A \boxtimes Z_M$, being the monotone closure of $Z_A \otimes Z_M$, is a commutative $AW^*$-algebra. q.e.d.

§ 5. Tensor products of $*$-automorphisms

A $*$-automorphism $\sigma$ of a monotone complete $C^*$-algebra $A$ acts freely on $A$ if $xy = \sigma(y)x$ for all $y \in A$ with $x \in A$ implies $x = 0$ (cf. Kallman [13]).

The following is an analogue of [13, Theorem 1.11] and Connes [6, Proposition 1.5.1].

**Proposition 5.1.** Let $A$ be a monotone complete $C^*$-algebra and $\sigma$ a $*$-automorphism of $A$.

(i) There is the largest projection, written $p(\sigma)$, in the set of all projections $e \in A$ such that $\sigma(e) = e$ and the restriction $\sigma|_{e \Delta e}$ is an inner $*$-automorphism of $e \Delta e$. This $p(\sigma)$ belongs to the center of $A$.

(ii) The restriction $\sigma|_{p(\sigma)A}$ (resp. $\sigma|_{(1-p(\sigma))A}$) is an inner (resp. freely acting) $*$-automorphism of $p(\sigma)A$ (resp. $(1-p(\sigma))A$) and this decomposition $\sigma = \sigma|_{p(\sigma)A} \oplus \sigma|_{(1-p(\sigma))A}$ to inner and freely acting parts is unique.

(ii) follows from (i), and as observed by Connes [6], (i) is a consequence of [13, Lemma 1.9] and the following lemma.

**Lemma 5.2.** With $A$ and $\sigma$ as above let $e$ be a projection of $A$, $C(e)$ the central cover of $e$ and let $u \in A$ be such that $u^*u = uu^* = e$ and $\sigma(x) = uxu^*$ for all $x \in e \Delta e$. Then there is a unique $v \in A$ such that $v^*v = vv^* = C(e)$, $\sigma(x) = vxv^*$ for all $x \in C(e)A$ and $ve = ev = u$.

**Proof.** We may assume that $C(e) = 1$. It follows readily from the comparability theorem [2, p. 80, Corollary] that the maximal orthogonal family $\{e_\alpha\}_{\alpha \in I}$ of nonzero projections in $A$ with $e_\alpha \sim f_\alpha \leq e$ (say, via a partial isometry $v_\alpha$) satisfies $O - \sum e_\alpha = 1$. We may also assume that $e_{\alpha_0} = v_{\alpha_0} = e$ for some $\alpha_0 \in I$. For each $x \in A$ we have

$\sigma(x) = \sigma(O - \sum_{e_\alpha \in I} e_\alpha x e_\alpha) = O - \sum_{e_\alpha \in I} \sigma(v_{\alpha_0}^* uv_{\alpha_0}) x v_{\alpha_0}^* u^* \sigma(v_{\alpha_0}^*)$.

Lemma 1.5 implies the existence of $O - \sum \sigma(v_{\alpha_0}^* uv_{\alpha_0}) = v$ in $A$; hence $\sigma(x) = vxv^*$ for all $x \in A$ and $ev = \sigma(v_{\alpha_0}^* u) = u = vu_{\alpha_0}$ for all $x \in A$ and $ev' = \sigma(v_{\alpha_0}^* u v' e) = u v' e = e$, so that $v^* v' e = v^* v' v^* e v_{\alpha_0} = v_{\alpha_0}^* v^* v' ev_{\alpha_0} = e_{\alpha_0}$ for all $\alpha$. Hence $v^* v' = 1$ and $v' = v$. q.e.d.

**Proposition 5.3.** Let $A$ (resp. $M$) be a monotone complete $C^*$-algebra (resp. $W^*$-algebra), $\pi$ (resp. $\rho$) a $*$-automorphism of $A$ (resp. $M$) and $\pi \otimes \rho$ the $*$-automorphism of $A \otimes M$ defined above.
(i) $\pi \otimes \rho$ is freely acting if and only if either $\pi$ or $\rho$ is freely acting.
(ii) $\pi \otimes \rho$ is inner if and only if both $\pi$ and $\rho$ are inner.

**Proof.** Let $A_i=p(\pi)A, A_2=(1-p(\pi))A, M_i=p(\sigma)M$ and $M_2=(1-p(\sigma))M$; then $A \otimes M=\sum_{r,s} A_r \otimes M_s$. Suppose that the sufficiency of (i) was proved. Then $\pi \otimes \rho|_{A_i \otimes M_i \otimes A_2 \otimes M_2}$ is freely acting and $\pi \otimes \rho|_{A_1 \otimes M_1}$ is inner. Hence $\pi \otimes \rho$ is freely acting (resp. inner) if and only if $p(\pi)\otimes p(\rho)=0$ (resp. $=1 \otimes 1$) if and only if $p(\pi)=0$ or $p(\rho)=0$ (resp. $p(\pi)=p(\rho)=1$). So it suffices to prove the sufficiency of (i). Suppose that $\pi$ is freely acting and $xy=(r_0p)(y)x$ for all $y \in A_0M$. Then for each $g \in M_*$ and $a \in A$ we have $L_g(x)a=L_g(x(a \otimes 1))=L_g((\pi \otimes \rho)(a \otimes 1)x)=L_g((\pi(a \otimes 1)x)=\pi(a)L_g(x)$, so that $L_g(x)=0$ for all $g \in M_*$ and $x=0$. Thus $\pi \otimes \rho$ is freely acting, and if $\rho$ is freely acting, then a similar reasoning applies with $L_g$ replaced by $R_f$, $f \in A^*$. q.e.d.

§ 6. Applications

First we state a simple fact about the matrix representation of a monotone complete C*-algebra and determine the injective envelope (resp. regular monotone completion) of a hereditary C*-subalgebra of a C*-algebra.

For a monotone complete C*-algebra $B$ and a Hilbert space $K$ we may and shall identify $B \times B(K)$ with the set of all matrices $[x_{\alpha \beta}]_{\alpha, \beta} \in B$ such that $\tilde{I}=\dim K$ and $\sup \{||x_{\alpha \beta}||: \alpha \in I, \beta \in \tilde{I} \}$ is finite ($x_{\alpha \beta}$ being regarded as an element of $B \otimes M_\alpha$ with $n=\tilde{I}$), in which the involution and multiplication are given by (2.1).

**Lemma 6.1.** Let $A$ be a monotone complete C*-algebra and $e$ a projection of $A$ with central cover $C(e)=1$. As in Lemma 5.2 take an orthogonal family $\{e_{\alpha}\}_{\alpha \in I}$ of nonzero projections in $A$ such that $0=\sum e_{\alpha}=1$ and $e_{\alpha}^{-1}f_{\alpha} \leq e$ (via a partial isometry $v_{\alpha}$) for all $\alpha \in I$. Then putting $\sigma(x)=[v_{\alpha}xv_{\beta}^{*}]_{\alpha, \beta} \in B(K)$ we obtain a *-isomorphism $\sigma$ of $A$ onto the reduced algebra $E(eAe \otimes B(K))$, where $K$ is the Hilbert space of $\dim K=\tilde{I}$ and $E=\text{diag}(f_{\alpha})$ is the projection of $eAe \otimes B(K)$ whose matrix representation has $f_{\alpha}$ in the $(\alpha, \alpha)$ position, $\alpha \in I$, and zeros elsewhere.

**Proof.** Clearly $\sigma$ is a *-homomorphism of $A$ into $eAe \otimes B(K)$. If $\sigma(x)=[v_{\alpha}xv_{\beta}^{*}]_{\alpha, \beta} \in B(K)$, then $e_{\alpha}xv_{\beta}^{*}=v_{\alpha}^{*}(v_{\alpha}xv_{\beta}^{*})v_{\alpha}=0$ for all $\alpha, \beta \in I$; hence $x=0$. If $[x_{\alpha \beta}]_{\alpha, \beta} \in E(eAe \otimes B(K))$, then the existence of $x=0=\sum v_{\alpha}^{*}x_{\alpha \beta}v_{\beta}$ in $A$ follows from Lemma 1.5. Moreover $\sigma(x)=[x_{\alpha \beta}]_{\alpha, \beta}$ since $v_{\alpha}xv_{\beta}^{*}=f_{\alpha}x_{\alpha \beta}f_{\beta}=x_{\alpha \beta}$. q.e.d.

**Corollary 6.2.** Let $A$ be a monotone complete C*-algebra, $e$ a projection of $A$ and $C(e)$ the central cover of $e$. If $eAe$ is injective, then $C(e)A$ is injective.

**Proof.** We have $C(e)A \cong E(eAe \otimes B(K))E$ and the right hand side is
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injective by Proposition 3.10. q.e.d.

**Proposition 6.3.** For a C*-algebra \( A \) and a projection \( e \) of \( A \) we have \( I(eAe) = eI(A)e \).

**Proof.** Replacing \( A \) by its regular monotone completion \( \tilde{A} \) we may assume that \( A \) is a monotone complete C*-algebra. In fact, \( (eAe) = e\tilde{A}e \) \([11,\ \text{Proposition 1.11}]\); hence \( I(eAe) = I((eAe)\tilde{=}) = I(e\tilde{A}e) \) and \( eI(A)e = eI(\tilde{A})e \). We may also assume that \( C(e) = 1 \) since \( I(C(e)A) = C(e)I(A) \) \([11,\ \text{Lemma 6.2}]\). (Note that, being \( Z_{A} = Z_{I(A)} \) \([11,\ \text{Theorem 6.3}]\), the central covers of \( e \) as calculated in \( A \) and \( I(A) \) coincide.) As in Lemma 6.1 take families \( \{e_{a}\}_{a \in I}, \{f_{a}\}_{a \in I}, \{v_{a}\}_{a \in I} \) in \( A \) and the \(*\)-isomorphism \( \sigma \) of \( I(A) \) onto \( E(eI(A)e \otimes B(K))E \) with \( \sigma(A) = E(eAe \otimes B(K))E \). Since \( eAe \subseteq eI(A)e \) and \( eI(A)e \) is injective, it suffices to show that a completely positive map \( \phi: eI(A)e \rightarrow eI(A)e \) with \( \phi|_{eAe} = \text{id}_{eAe} \) is necessarily \( \text{id}_{eI(A)e} \). Then the completely positive map \( \psi = \phi \otimes \text{id}_{B(K)} |_{E(eI(A)e \otimes B(\mathbb{K}))E} \) satisfies \( \psi(E(eI(A)e \otimes B(K))E) \subset E(eI(A)e \otimes B(K))E \) and \( \psi \circ \sigma|_{A} = \sigma|_{A} \). In fact, since \( E \in eAe \otimes B(K) \) and \( \psi|_{E(A)e \otimes B(K)} = \text{id}_{E(A)e \otimes B(K)} \), we have \( \psi(ExE) = E\psi(x)E \) for all \( x \in eI(A)e \otimes B(K) \). Therefore \( \psi = \text{id}_{E(eI(A)e \otimes B(K))E} \) and \( \phi = \text{id}_{eI(A)e} \) q.e.d.

In the remainder of this section C*-algebras are not necessarily unital. Let \( A \) be a not necessarily unital C*-algebra. We write \( A^{1} \) for the unital C*-algebra obtained by adjoining a unit to \( A \) if \( A \) is nonunital and for \( A \) if \( A \) is unital. We define the regular monotone completion (resp. injective envelope) of \( A \) as that of \( A^{1} \) and denote it again by \( \tilde{A} \) (resp. \( I(A) \)).

Let \( H \) be the universal Hilbert space of \( A \). Then the \( \sigma \)-weak closure \( \tilde{A}^{\omega} \) of \( A \) in \( B(H) \) is the enveloping von Neumann algebra \( A^{**} \) of \( A \) and \( A^{1} \) is identified with \( A + C1 \) (1 denoting the unit of \( A^{**} \)). If \( \phi \) is a minimal \( A^{1} \)-projection on \( B(H) \) \([9]\), then \( I(A) \) is identified with \( \text{Im} \phi \supset A \) equipped with the order, involution and norm induced by those of \( B(H) \) and the multiplication \( * \) given by \( x \circ y = \phi(xy) \). Moreover \( I(A) + I_{A} \) is a C*-subalgebra of \( B(H) \) and \( \phi \) restricted to \( I(A) + I_{A} \) is a \(*\)-homomorphism of \( I(A) + I_{A} \) onto \( I(A) \) with kernel \( I_{A} = \{ x \in B(H) : \phi(x^{*}x) = 0 \} \). Note that \( I_{A}^{+} = (\text{Ker} \phi)^{+} \) since \( I_{A}^{+} \) is a C*-subalgebra of \( B(H) \), hence that \( I_{A} \) is the linear span of \( (\text{Ker} \phi)^{+} \). We write \( (A_{a,s})^{m} \) for the set of all elements \( x \in A_{a,s}^{**} \) such that \( a_{s} \wedge x \) strongly in \( B(H) \) for some increasing net \( \{a_{s}\} \) in \( A_{a,s} \). We say that a projection \( e \in I(A) \) is open if \( a_{s} \wedge e \) (O) in \( I(A) \) for some increasing net \( \{a_{s}\} \) in \( A^{+} \).

**Lemma 6.4.** With notation as above we have \( (A_{a,s})^{m} \subset I(A) + I_{A} \) and \( \phi((A_{a,s})^{m}) \subset \tilde{A}_{a,s} \). Moreover \( \phi \) maps the set of all open projections of \( A^{**} \) onto that of \( I(A) \).

**Proof.** If \( a_{s} \wedge x \) strongly in \( B(H) \) with \( \{a_{s}\} \) an increasing net in \( A_{a,s} \), then \( a_{s} \wedge \phi(x) \) (O) in \( I(A) \) and \( \phi(x) \subseteq x \). Since \( \phi(x) - x \geq 0 \) and \( \phi(\phi(x) - x) = 0 \),
we have \( x = \phi(x) - (\phi(x) - x) \in I(A) + I_\cdot \) Hence \( (A_{s,x})^* \subset I(A) + I_\cdot \) and \( \phi((A_{s,x})^*) \subset A_{s,x} \). If \( p \) is an open projection of \( A^{**} \), i.e., \( a_{s,x}/p \) strongly for some increasing net \( \{a_{s,x}\} \) in \( A^+ \), then \( a_{s,x}/\phi(p)(O) \) in \( I(A) \) and \( \phi(p) \) is an open projection of \( I(A) \). Conversely if \( e \) is an open projection of \( I(A) \), then \( a_{s,x}/e(O) \) in \( I(A) \) with \( \{a_{s,x}\} \) an increasing net in \( A^+ \). Hence \( a_{s,x}/a \leq e \) strongly for some \( a \in (A_{s,x})^* \) and \( \phi(a) = e \). The support projection \( p \) of \( a \) in \( A^{**} \) is open [5, 2.2.9, 2.2.12] and from the foregoing \( \phi(p) \) is an open projection of \( I(A) \). Moreover \( e = \phi(p) \) since \( a \leq p \leq e \) and so \( e = \phi(a) \leq \phi(p) \leq \phi(e) = e \). q.e.d.

**Theorem 6.5.** Let \( A \) be a not necessarily unital \( C^* \)-algebra and \( B \) its hereditary \( C^* \)-subalgebra. Then there is a unique open projection \( p \) of \( A \) with \( a_{s,x}/p \) (O) in \( A \) for some increasing net \( \{a_{s,x}\} \) in \( B^+ \) such that \( B = pA_\cdot p \) and \( I(B) = pI(A)p \). In particular if \( B \) is a closed two-sided ideal of \( A \), then \( p \) is a central projection of \( A \).

**Proof.** Since \( B \) is also a hereditary \( C^* \)-subalgebra of \( A^+ \), we may assume that \( A \) is unital. Let \( H \) and \( \phi \) be as above. Then there is an open projection \( p' \) of \( A^{**} \) such that \( B^\cdot = p'A^{**}p' \) and \( a_{s,x}/p' \) strongly with \( \{a_{s,x}\} \) in \( B^+ \). We show that \( pI(A)p = I(B) \). Since \( B^\cdot \leq B + Cp \subset pI(A)p \), we need only show that a completely positive map \( \psi : pI(A)p \rightarrow pI(A)p \) with \( \psi|_{pI(A)p} = id_{pI(A)p} \) is \( id_{pI(A)p} \). For each \( x \in pAp \) we have \( a_{s,x}a_{s} \in B \) and \( a_{s,x}a_{s} = \psi(a_{s}a_{s}) = a_{s}\psi(x)a_{s} \) since \( \psi \) is a \( B^1 \)-module homomorphism. It follows from \( a_{s,x}/p \) (O) in \( I(A) \) that \( pxp = p\psi(x)p \) (Lemma 1.2) and \( x = \psi(x) \). Hence \( \psi|_{pAp} = id_{pAp} \). Since \( A \) is a regular extension of \( A \), for \( x \in A_{s,x} \), we have \( x = sup_{A_{s,x}}(-\infty, x] = sup_{I(A)}(-\infty, x], pxp = sup_{I(A)}p(-\infty, x]p \) by [11, Lemma 1.9]; hence \( pxp \leq \psi(px)p \) since \( p(-\infty, x]p = \psi(p(-\infty, x]p) \leq \psi(px)p \). Similarly the reverse inequality holds and \( \psi|_{pAp} = id_{pAp} \). It follows from Proposition 6.3 that \( I(pAp) = pI(A)p = I(A)p \), hence that \( \psi = id_{pAp} \). Therefore \( I(B) = pI(A)p \).

By definition \( \bar{B} = m-cl_{I(B)}(B + Cp) = m-cl_{I(A)p}(B + Cp) = m-cl_{A}(B + Cp) \subset p\bar{A}p \) and by Lemma 2.4, \( m-cl_{A}pAp = p\bar{A}p \). We have \( pAp \subset m-cl_{A}(B + Cp) \). In fact, for each \( x \in A^+ \),

\[
4a_{s,x}^2a_{s} = \sum_{j=0}^{3} i^j(a_{s,x}^2x + i^j)a_{s}(a_{s,x}^2x + i^j)^*.
\]

With \( \alpha \) fixed \( B_{s,x} \), \( \exists (a_{s,x}^2x + i^j)a_{s}(a_{s,x}^2x + i^j)^*(a_{s,x}^2x + i^j)p(a_{s,x}^2x + i^j)^*(O) \in \bar{A} as b \uparrow \). Since \( m-cl_{A}(B + Cp) \) is a \( C^* \)-subalgebra of \( A \) [11, Lemma 1.4], we have \( 4a_{s,x}^2xp = \sum_{j=0}^{3} i^j(a_{s,x}^2x + i^j)p(a_{s,x}^2x + i^j)^* \in m-cl_{A}(B + Cp) \) and \( pxp = (a_{s,x}^2xp)^*(a_{s,x}^2xp) \in m-cl_{A}(B + Cp) \), so that \( m-cl_{A}(B + Cp) \supset pxp = pxp \in m-cl_{A}(B + Cp) \) and \( pxp = [pxp]^* \in m-cl_{A}(B + Cp) \). Hence \( pAp \subset m-cl_{A}(B + Cp) \). Therefore \( p\bar{A}p = m-cl_{A}pAp \subset m-cl_{A}(B + Cp) \subset p\bar{A}p \) and \( \bar{B} = p\bar{A}p \).
If $B$ is a closed two-sided ideal of $A$, then the above $p'$ is a central projection of $A^{**}$. Since $\phi|_{I(A)_{+,\mathbb{R}}}$ is a *-homomorphism which fixes $A$ elementwise, $p=\phi(p')$ commutes with $A$ elementwise in $I(A)$ and so belongs to $Z_{I(A)_{+,\mathbb{R}}}=Z_A$. q.e.d.

Now we concern ourselves with the regular monotone completion (resp. injective envelope) of a minimal C*-tensor product.

**Lemma 6.6.** Let $V \subset B(H)$ and $W \subset B(K)$ be operator systems and $I(V)$ the injective envelope of $V$ with $V \subset I(V) \subset B(H)$. Then the operator system $I(V) \otimes W$ is an essential extension of the operator system $V \otimes W$ in the sense of [10].

**Proof.** We may and shall assume that $I(V)$ is a C*-subalgebra of $B(H)$. Then $V \otimes W \subset I(V) \otimes W \subset I(V) \otimes B(K)$ and $I(V) \otimes B(K)$ is injective. Hence for the proof we need only show that if $\varphi: I(V) \otimes W \to I(V) \otimes B(K)$ is a completely positive map with $\varphi|_{I(V)}=\text{id}_{I(V)}$, then $\varphi$ is a complete order injection. For each normal state $g \in B(K)_{+,\mathbb{R}}$ the map $\psi_g: I(V) \otimes 1 \to I(V) \otimes 1$ given by $\psi_g(x)=L_x(\varphi(x)) \otimes 1$ is completely positive with $\psi_g|_{I(V)}=\text{id}_{I(V)}$, and so $\psi_g=\text{id}_{I(V) \otimes 1}$. Hence $\varphi|_{I(V) \otimes 1}=\text{id}_{I(V) \otimes 1}$. Since $I(V) \otimes B(K)$ is injective, $\varphi$ extends to a completely positive map $\varphi: B(H) \otimes B(K) \to I(V) \otimes B(K)$ and $\varphi$ fixes the C*-subalgebra $I(V) \otimes 1$ elementwise. Thus for each $a \in I(V)$ and $b \in W$ we have $\varphi(a \otimes b)=\varphi((a \otimes 1)(1 \otimes b))=(a \otimes 1)\psi(1 \otimes b)=a \otimes b$, so that $\varphi=\text{id}_{I(V) \otimes W}$ as desired. q.e.d.

**Theorem 6.7.** For not necessarily unital C*-algebras $A$ and $B$ we have $A \otimes B \subset (A \otimes B)^-= (A' \otimes B')$ and $I(A) \otimes I(B) \subset (A \otimes B)=I(A' \otimes B')$.

**Proof.** Applying Lemma 6.6 twice, we see that $I(A') \otimes I(B')$ is an essential extension of $A' \otimes B'$ and is contained in $I(A' \otimes B')$ as a C*-subalgebra. Since $A \otimes B$ is a closed two-sided ideal of $A' \otimes B'$, there is a central projection $h$ of $(A' \otimes B')^-$ such that $(A \otimes B)^-=h(A' \otimes B')$ and $I(A \otimes B)=hI(A' \otimes B')$ (Theorem 6.5). If $1_a$ and $1_b$ are units of $A'$ and $B'$ respectively, then $a_{+1_a(0)}$ in $I(A)$ and $b_{y1_b(0)}$ in $I(B)$ for some increasing nets $\{a_n\}$ in $A'$ and $\{b_n\}$ in $B'$ and $1_{a_{+1_a(0)}}$ is the unit of $A' \otimes B'$. It follows from the normality of the maps $I(A) \ni x \mapsto x \otimes 1_b \in I(A) \otimes I(B) = I(A') \otimes I(B') \subset I(A' \otimes B')$ and $I(B) \ni y \mapsto 1_a \otimes y \in I(A' \otimes B')$ that $h \geq a_{+1_a} \otimes b_{1_b(0)}$ in $I(A' \otimes B')$, $h=1_{a_{+1_a(0)}}$, hence that $(A \otimes B)^-=h(A' \otimes B')$ and $I(A \otimes B)=hI(A' \otimes B')$. Moreover $A \otimes 1_b = (m\text{-cl}_{I(A)_{+,\mathbb{R}}})A \otimes 1_b \subset m\text{-cl}_{I(A)_{+,\mathbb{R}}} A \otimes 1_b \subset m\text{-cl}_{I(A)_{+,\mathbb{R}}} A' \otimes 1_b \subset m\text{-cl}_{I(A)_{+,\mathbb{R}}} A \otimes B = (A' \otimes B')^-$. Hence $A \otimes 1_b \subset (A' \otimes B')^-$. q.e.d.

**Theorem 6.8.** Let $V \subset B(H)$ (resp. $W \subset B(K)$) be an operator system and $A$ (resp. $B$) its C*-envelope in the sense of [10]. Then the minimal C*-tensor product $A \otimes B$ is the C*-envelope of the operator system $V \otimes W$. 


Proof. By Lemma 6.6, \( I(V) \otimes I(W) \) is contained in \( I(V \odot W) \) as a C*-subalgebra. Since \( A \) (resp. \( B \)) is the C*-subalgebra of \( I(V) \) (resp. \( I(W) \)) generated by \( V \) (resp. \( W \)), the C*-envelope of \( V \odot W \), which is the C*-subalgebra of \( I(V \odot W) \) generated by \( V \odot W \), coincides with \( A \otimes B \subseteq I(V) \otimes I(W) \).

q.e.d.

References

Tensor products for monotone complete C*-algebras


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