Introduction

Let $M$ be a connected complete Riemannian manifold of dimension $m$. We call $M$ hyperbolic if it has a nonconstant positive superharmonic function. When $M$ is not hyperbolic, then we call $M$ parabolic. In the study of function theoretic properties of $M$, it is important to know whether $M$ is hyperbolic or parabolic. We know that the behavior of the sectional curvature tells us whether $M$ is hyperbolic or parabolic. For example, when $m=2$, a theorem of Blac-Fiala-Huber [16] states that $M$ is parabolic if the Gaussian curvature is nonnegative, and a theorem of Ahlfors says that $M$ is hyperbolic if it is simply connected and the Gaussian curvature is bounded from above by some negative constant\(^\dagger\). Moreover when $m\geq 3$, a theorem of Aomoto [1] tells us that $M$ is hyperbolic if it is simply connected and the sectional curvature is nonpositive. Recently, Ichihara [17] has given more general geometric criteria for $M$ to be hyperbolic or parabolic. Basically he used the Laplacian and Hessian comparison theorems due to Greene and Wu [14]. In the first half (§§ 2–3) of this paper, we shall generalize the comparison theorems of Greene and Wu. In the second half (§§ 4–7), we shall give their applications to obtain several geometric criteria for $M$ to be hyperbolic or parabolic in terms of the curvature behavior of $M$. Moreover we shall show that our criteria are optimum.

We shall now summarize our main results briefly. Let $N$ be a (proper) closed subset of $M$ and $\rho_n(x)$ be the distance between a point $x$ and $N$. Suppose there exists a continuous function $R$ on $[0, \infty)$ such that, for any $x \in M$, the Ricci curvature at $x$ is bounded from below by $(m-1)R(\rho_n(x))$. Then we shall show in Theorem (2.28) (Laplacian comparison theorem) that there is a continuous function $\lambda: (0, +\infty) \to [-\infty, +\infty)$, depending only on $R$, such that

\(^\dagger\) This is explained in the paper of R. Osserman: On the inequality $\Delta u \geq f(u)$, Pacific Journal of Math., 7 (1957), p. 1646.
for any $x \in M \setminus N$. Now suppose there exists a continuous function $K$ on $[0, \infty)$ such that, for any $x \in M$, the sectional curvature at $x$ is bounded from below by $K(p_N(x))$. Then by the same method as in the proof of Theorem (2.28), we shall obtain Theorem (2.31) showing that there is a continuous function $\mu: (0, +\infty) \to [-\infty, +\infty)$, depending only on $K$, such that

$$l^2 \rho_N(x) \leq \mu(p_N(x))$$

(0.2)

for any $x \in M \setminus N$ and every unit tangent vector $X$ orthogonal to the tangent vector $\sigma'(t)$ of a geodesic $\sigma: [0, \ell] \to M$ such that $p_N(\sigma(t)) = t$ for $t \in [0, \ell]$ and $\sigma(\ell) = x$. When $N$ is a point and $x$ is within the cut locus of $N$ (so that $p_N$ is a $C^\infty$ function near $x$), above inequalities (0.1) and (0.2) are exactly the Laplacian and Hessian comparison theorems of Greene and Wu. More generally, when $N$ is a closed (imbedded) submanifold of $M$ and $x$ is within the cut locus of $N$, (0.1) and (0.2) are implicit in a comparison theorem of Heintze and Karcher (cf. [15: Section 3, 2] and Remark (2.26)). However, in general, $\rho_N$ is merely a Lipschitz continuous function (even if $N$ is a closed submanifold). Therefore the global inequalities (0.1) and (0.2) are considered in an appropriate weak sense (cf. Section 1) and do not follow from the known comparison theorems such as the ones mentioned above. In fact, our proofs of (0.1) and (0.2) require much more delicate arguments (cf. Wu [33]). When $N$ is a closed hypersurface such that the length of its mean curvature is bounded, inequality (0.1) can become much stronger. Namely, we can replace $\lambda$ in (0.1) with a continuous function $\nu: (0, +\infty) \to [-\infty, +\infty)$ such that $\nu < \lambda$, where $\nu$ depends only on $R$ and the mean curvature of $N$. We emphasize here that inequality (0.1) with $\nu$ enjoys much more applicability than the one with $\lambda$ (cf. Lemma (2.27) and Remark (4.10)). Similarly when $N$ is a closed submanifold, we shall obtain a better estimate for $l^2 \rho_N$ than (0.2).

Now suppose we have an inequality of the form:

$$\Delta \rho_N \leq \eta(p_N)$$

(0.3)

on $M \setminus N$ for some continuous function $\eta$ on $(0, \infty)$. Put

$$\mathcal{F}_r(t) = \int_t^\infty \exp \left( -\int_1^t \eta(u)du \right) ds \quad (r > 0).$$

Then by (0.3), we see that $\Delta \mathcal{F}_r(\rho_N) \geq 0$ on $M \setminus N$, that is, $\mathcal{F}_r(\rho_N)$ is subharmonic on $M \setminus N$, and moreover it is positive on $\{ x \in M: 0 < \rho_N(x) < r \}$ and vanishes on $\{ x \in M: \rho_N(x) = r \}$. Now suppose $M$ is noncompact and $N$ is a hypersurface bounding a bounded domain. Then inequality (0.1) with $\nu$ implies an inequality of the form (0.3), and then by the above procedure constructing an
appropriate subharmonic function, we shall obtain in Theorem (4.9) a geometric criterion for $M$ to be parabolic. Our theorem contains as a special case a result of Ichihara (cf. [17: Theorem 2.1] and Remark (4.10)).

On the other hand, as for the lower estimate of $\mathcal{Z}_{N}$, we shall show in Theorem (2.49) (Hessian comparison theorem II) that if $N$ is a totally convex closed subset of $M$ and, for any $x \in M$, the sectional curvature of $M$ at $x$ is bounded from above by $K(\rho_{N}(x))$ for some nonpositive continuous function $K$ on $[0, \infty)$, then there is a nonnegative continuous function $\xi$ on $(0, \infty)$, depending only on $K$, such that

\begin{equation}
(F_{ \rho_{N}})'(X, X) \geq \xi(\rho_{N}(x))
\end{equation}

for any $x \in M \setminus N$ and every unit tangent vector $X$ orthogonal to $\sigma(\ell)$, where $\sigma: [0, \ell] \rightarrow M$ is the geodesic such that $\rho_{N}(\sigma(t)) = t$ for $t \in [0, \ell]$ and $\sigma(\ell) = x$. For the proof of (0.4), we shall use some of the results in [3] concerning Riemannian manifolds of nonpositive curvature.

Now suppose an inequality of the form:

\begin{equation}
A_{\rho_{N}} \geq \theta(\rho_{N})
\end{equation}

holds on $M \setminus N$ for some continuous function $\theta$ on $(0, \infty)$. Set

$\Phi_{t}(\rho_{N}) = \int_{1}^{\rho_{N}} \exp \left( - \int_{1}^{s} \theta(u)du \right) ds \ (r > 0)$.

Then (0.5) implies that $A_{\Phi_{t}(\rho_{N})} \leq 0$ on $M \setminus N$, that is, $\Phi_{t}(\rho_{N})$ is superharmonic on $M \setminus N$ and further it is positive on $\{ x \in M: 0 < \rho_{N}(x) < r \}$ and vanishes on $\{ x \in M: \rho_{N}(x) = r \}$. Then (0.4) implies an inequality of the form (0.5), and then by the argument just mentioned, we shall obtain in Theorem (5.5) and Theorem (5.7) geometric criteria for $M$ to be hyperbolic. Moreover we shall apply them to obtain certain geometric conditions for $M$ to possess a nonconstant bounded harmonic function or a nonconstant harmonic function with finite Dirichlet norm (cf. Corollary (5.8)).

Bishop and O'Neill [3] used (Riemannian) warped products to construct a large class of complete Riemannian manifolds of negative curvature. In Section 6, we shall also use warped products to give several examples and to show that our criteria in Sections 4 and 5 are optimum. In the last section, we consider as another application of Theorem (5.5) and Theorem (5.7) the Dirichlet problems on the geometric compactification of a Hadamard manifold of strictly negative curvature, which was introduced by Eberlein and O'Neill [10].

After the statements of Theorems (2.28) and (2.31), we shall add some corollaries (2.35)–(2.44) to them, which will yield some new results on the geometrical structure of a Riemannian manifold with boundary (cf. [22]) and
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isoperimetric inequalities (cf. [23]). However these corollaries will not be
used in the latter sections §§ 3-7 and included here for the matter of com-
pleteness and for the sake of the forthcoming papers [22] and [23], and there-
fore the reader may skip them. Moreover we remark that we can obtain an
upper estimate for the “Levi form” of a distance function on a Kähler mani-
fold, applying the technique used in the proofs of Theorems (2.28) and (2.31).
This topic will be taken up in [24].

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§ 1. Preliminaries

N-Jacobi fields and the basic inequality of the index form

We recall several fundamental facts in Riemannian geometry. (In this
paper, we assume a Riemannian manifold has no boundary, unless otherwise
stated.) Let $M$ be a connected Riemannian manifold of dimension $m$. We
write $M_x$ for the tangent space of $M$ at $x$ and $\langle X, Y \rangle$ for the inner product
of tangent vectors $X$ and $Y$. The length or norm of a tangent vector $X$ is
denoted by $||X||$. Let $N$ be an $n$-dimensional (imbedded) submanifold of $M$
passing through a point $x$ of $M$ with $0 \leq n \leq m - 1$. Let $\sigma: [0, l] \to M$ be a unit
speed geodesic in $M$ with initial point $x$ and initial tangent $\sigma(0)$ orthogonal
to the tangent space $N_x$ of $N$ at $x$. The second fundamental form assigns to
the normal vector $\sigma(0)$ a symmetric linear transformation $S_{\sigma(0)}$ on $N_x$ (i.e.
$\langle S_{\sigma(0)}X, Y \rangle = \langle \partial_x \xi, Y \rangle$). Let $L(\sigma, l, N)$ denote the linear space of all smooth
vector fields along $\sigma$ whose values are everywhere orthogonal to the tangent
vector $\sigma$ of $\sigma$ and whose initial value is an element of $N_x$.

A Jacobi field $Y \in L(\sigma, l, N)$ is called an $N$-Jacobi field if it satisfies the
boundary condition:

$$S_{\sigma(0)}Y(0) - \nu, Y(0) \in (N_x)^{\perp},$$
where \( \perp \) means orthogonal complement in \( M_\sigma \). A focal point on \( \sigma \) is a point \( \sigma(t) \, (t \neq 0) \) at which a non-trivial \( N \)-Jacobi field along \( \sigma \) vanishes. The set of \( N \)-Jacobi fields along \( \sigma \) forms an \((m-1)\)-dimensional linear space and spans the orthogonal complement to \( \sigma(t) \) in \( M_\sigma(t) \) for \( 0 < t \) prior to the first focal point on \( \sigma \). In the case when \( N = \{ x \} \), the set of \( N \)-Jacobi fields along \( \sigma \) is the set of Jacobi fields which vanish at \( x \), and focal points are also called conjugate points.

The index form \( I(1, N) \) is a symmetric bilinear form on \( L(\sigma, l, N) \) defined as follows:

\[
I(1, N)(X, Y) = \langle S_{\perp x} X(0), Y(0) \rangle + \int_0^1 \langle \nabla_x X, \nabla_y Y \rangle - \langle R(X, \sigma) \sigma, Y \rangle \, dt
\]

One of the basic properties of Jacobi fields is that they minimize the index form prior to focal points in the following sense.

(1.1) FACT (cf. Theorem 4 in chap. 11 of [2]). Suppose there is no focal point on \( \sigma \). For \( V \in L(\sigma, l, N) \), there is a unique \( N \)-Jacobi field \( Y \) such that \( Y(l) = V(l) \). Then \( I(1, N)(V, V) \geq I(1, N)(Y, Y) \) and equality occurs if and only if \( V = Y \).

Convex functions

Let \( f \) be a continuous function defined on an open set \( U \) of \( M \). \( f \) is said to be convex if its restriction to any geodesic is a convex function of one variable and strictly convex if in some neighborhood of each point of \( U \), \( f \) is the sum of a convex function and a \( C^2 \) strictly convex function. Here we call a \( C^2 \)-function \( h \) strictly convex if the Hessian \( \nabla^2 h \) is positive definite. Following Wu [33], we introduce an extended real number \( Cf(x: X) \) for a point \( x \in U \) and a tangent vector \( X \in M_x \). Let \( \gamma: [-\varepsilon, \varepsilon] \rightarrow M \) be a geodesic such that \( \gamma(0) = x \) and \( \gamma(0) = X \). Then the number \( Cf(x: X) \) is defined by

\[
Cf(x: X) = \lim_{t \to 0} \frac{1}{t} \left[ f \circ \gamma(t) + f \circ \gamma(-t) - 2f \circ \gamma(0) \right].
\]

Then we have the following

(1.2) FACT (cf. [33: Lemmas 1–3]). (1) If \( f \) is a \( C^2 \)-function near \( x \), \( Cf(x: X) = \nabla^2 f(X, X) = (f \circ \gamma')''(0) \). (2) \( f \) is convex on \( U \) if and only if \( Cf(y: Y) \geq 0 \) for any \( y \in U \) and every \( Y \in M_y \). (3) \( f \) is strictly convex on \( U \) if and only if for some positive continuous function \( k \) on \( U \), \( Cf(y: Y) \geq k(y) \) for any \( y \in U \) and every unit tangent vector \( Y \in M_y \).

Laplacian, Green's functions and superharmonic functions

The Laplacian \( \Delta \) is the elliptic differential operator defined by
$\Delta f$ = the trace of the Hessian $\nabla^2 f$,

where $f$ is a $C^2$-function. A $C^2$-function $h: M \to R$ is called harmonic if $\Delta h = 0$, an upper semi-continuous function $f: M \to [-\infty, +\infty)$ is subharmonic if for any relatively compact domain $D$ and every harmonic function $h$ on $D$ such that $f \leq h$ on $\partial D$, $f$ satisfies $f \leq h$ on $D$, and a lower semi-continuous function $g: M \to (-\infty, +\infty]$ is superharmonic if $-g$ is subharmonic. Let $x$ be a point of $M$, $i(x)$ the injectivity radius at $x$, and $B_r(x)$ the open ball centered at $x$ with radius $r$. The Green's function $G_r(y, z)$ of $B_r(x)$ $(r < i(x))$ with a pole $y \in B_r(x)$ is the unique function on $B_r(x)$ with the following properties:

(i) $G_r(y, z)$ is harmonic on $B_r(x) \setminus \{y\}$.
(ii) $G_r(y, z)$ is continuous on $\overline{B_r(x)} \setminus \{y\}$ and vanishes on $\partial B_r(x)$.
(iii) $G_r(y, z)$ has the singularity at $y$ such that

$$G_r(y, z) \sim \begin{cases} \frac{1}{(m-2)\omega_{m-1}} \text{dis}(y, z)^{-m+2} & (m > 2) \\ \frac{1}{2\pi} \log \text{dis}(y, z)^{-1} & (m = 2) \end{cases}$$

where $\omega_{m-1}$ is the volume of the unit sphere in $m$-dimensional Euclidean space. Then for any upper semi-continuous function $f$ defined near $x$, an extended real number $S_f(x)$ is defined by

$$S_f(x) = \liminf_{t \to 0} -\frac{2m}{r^2} \left\{ \int_{\partial B_r(x)} f \ast G_r(x, y) + f(x) \right\},$$

where $\ast$ is the usual star operator in Hodge theory. Then we have the following

(1.4) FACT (cf. [33: Lemmas 1 and 2]).

(1) For a $C^2$-function $f$ near $x$, $S_f(x) = \Delta f(x)$.
(2) Given an upper semicontinuous function $f: M \to [-\infty, \infty)$, $f$ is subharmonic if and only if $S_f \geq 0$.

(1.5) LEMMA. Let $f$ be a continuous function on an open subset $U$ of $M$. Suppose there is a continuous function $g$ on $U$ such that $S_f \geq g$ on $U$. Then $f$ satisfies $\Delta f \geq g$ as a distribution on $U$.

PROOF. Fix a point $x$ of $U$. For any positive constant $\varepsilon$, let $g_\varepsilon$ be a $C^\infty$-function on a small metric ball $B_r(x)$ such that $g \geq g_\varepsilon \geq g - \varepsilon$ on $B_r(x)$. Set

$$f_\varepsilon(y) = -\int_{\partial B_r(x)} G_r(y, z)g_\varepsilon(z) \quad (y \in B_r(x)).$$

Then $f_\varepsilon$ is a $C^\infty$-function on $B_r(x)$ satisfying $\Delta f_\varepsilon = g_\varepsilon$. Since $S(f - f_\varepsilon) \geq S_f - S_f \geq g - g_\varepsilon \geq 0$, $f - f_\varepsilon$ is subharmonic on $B_r(x)$. Therefore for any nonnegative
C^-function \psi on \( B_r(x) \) whose support is contained in \( B_r(x) \),
\[
\int f \Delta \psi \geq \int f, \Delta \psi = \int \Delta f, \psi = \int g, \psi \geq \int (g-\varepsilon)\psi,
\]
that is, \( f \Delta \psi \geq \int (g-\varepsilon)\psi \). Since \( \varepsilon \) is any positive constant, we see that
\[
\int f \Delta \psi \geq \int g\psi.
\]
This implies that \( \Delta f \geq g \) as a distribution on a neighborhood of any \( x \in U \), and hence on \( U \). This completes the proof of Lemma (1.5).

We call a continuous function \( G_M(x, \ast) : M \to (-\infty, \infty] \) the Green's function of \( M \) with a pole at \( x \) if it possesses the following properties:

(i) \( G_M(x, \ast) \) is a positive harmonic function on \( M \setminus \{x\} \),
(ii) \( G_M(x, \ast) \) has the singularity of type (1.3) at \( x \),
(iii) \( G_M(x, \ast) \) is minimal among all the functions satisfying (i) and (ii).

Then it is well known that \( M \) is hyperbolic if and only if for some \( x \in M \), the Green's function \( G_M(x, \ast) \) of \( M \) with a pole at \( x \) exists (cf. [19]).

Before we conclude this section, we recall an elementary but important technique in classical analysis, which played an essential role in Wu [33]. Let \( f \) and \( g \) be two continuous functions on a neighborhood of a point \( x \in M \). We say that \( g \) supports \( f \) at \( x \) if \( g \leq f \) near \( x \) and \( g(x) = f(x) \).

(1.6) FACT (cf. [33: Lemma 4]). If \( g \) supports \( f \) at \( x \), then \( Sg(x) \leq Sf(x) \) and \( Cg(x; X) \leq Cf(x; X) \) for any \( X \in M_x \).

§ 2. Statements of Laplacian and Hessian comparison theorems

In this section, we shall generalize the Laplacian and Hessian comparison theorems of Greene and Wu [14: Theorem A and Proposition 2.15]) (cf. also [30: p. 227]).

Let \( M \) be a connected Riemannian manifold of dimension \( m \). We assume \( M \) has no boundary, unless otherwise stated. Let \( U \) be a connected open subset of \( M \) and \( A \) be a closed subset of \( U \). We write \( \text{dis}_U(x, y) \) for the distance between two points \( x \) and \( y \) in \( U \) induced by the Riemannian metric of \( M \) restricted to \( U \), and we denote by \( \text{dis}_A(A, x) \) for the distance between \( A \) and \( x \) defined by \( \inf_{y \in A} \text{dis}_U(y, x) \). Let \( N \) be a closed (imbedded) submanifold of \( M \) and \( x \) be a point of \( M \setminus N \). Suppose there is a unit speed geodesic \( \sigma : [0, \ell] \to M \) joining \( N \) to \( x \), and suppose that, in the normal bundle \( N^\perp \) of \( N \) (we understand \( N^\perp = M_p \) if \( N \) is a point \( p \)), there is an open neighborhood \( \hat{C}_{N, \varepsilon} \) of a segment \( \{t\sigma(0) : 0 \leq t \leq \ell\} \) which has the following properties:

\[
\begin{align*}
(i) & \quad \text{For any } t \in [0, 1], \quad tv \in \hat{C}_{N, \varepsilon} \text{ if } v \in \hat{C}_{N, \varepsilon}, \\
(ii) & \quad \text{The exponential map } \exp_N \text{ of } N \text{ restricted to } \hat{C}_{N, \varepsilon} \text{ induces a}
\end{align*}
\]
diffeomorphism between \( \hat{C}_{N,x} \) and its image. We write \( C_{N,x} \) for the image \( \exp_N(\hat{C}_{N,x}) \) of such a \( \hat{C}_{N,x} \). Then for any \( y = \exp_N v \) (\( v \in \hat{C}_{N,x} \)), \( \text{dis}_{C_{N,x}} (C_{N,x} \cap N, y) = \|v\| \), so that the distance function \( \rho = \text{dis}_{C_{N,x}} (C_{N,x} \cap N, *) \) is a smooth function on \( C_{N,x} \setminus N \). Now in the following four lemmas, we fix a point \( x \) in \( M \setminus N \) as above and the geodesic \( \sigma : [0, l] \to M \). Let \( R, K, \) and \( \bar{K} \) be continuous functions on \([0, l]\) such that

\begin{align}
(2.2) & \quad R(t) \leq \frac{1}{(m-1)} \text{ the Ricci curvature in direction } \dot{\sigma}(t), \\
(2.3) & \quad K(t) \leq \text{ the sectional curvature of any plane containing } \dot{\sigma}(t) \\
(2.4) & \quad \bar{K}(t) \geq \text{ the sectional curvature of every plane tangent to } \dot{\sigma}(t).
\end{align}

Then we have the following

(2.5) **Lemma.** Let \( f \in C^2([0, l]) \) be the solution of the equation:

\begin{equation}
(2.6) \quad f''(t) + R(t)f(t) = 0 \quad \text{with } f(0) = 0 \quad \text{and } f'(0) = 1.
\end{equation}

Then the distance function \( \rho = \text{dis}_{C_{N,x}} (C_{N,x} \cap N, *) \) satisfies

\begin{equation}
(2.7) \quad S \rho(x) \leq (m-1)(f'/f)(\rho(x)).
\end{equation}

Moreover when \( N \) is a point of \( M \), the equality in (2.7) holds if and only if the sectional curvature of any plane tangent to \( \dot{\sigma}(t) \) is equal to \( R(t) \) (\( 0 \leq t \leq l \)).

**Proof.** We first note that \( f \) is positive on \((0, l]\). In fact, suppose \( f \) is positive on \((0, t_0)\) and \( f(t_0) = 0 \) for some \( t_0 \in (0, l] \). Let \( \{E_1, \ldots, E_{m-1}\} \) be the parallel vector fields along \( \sigma \) such that \( \{E_i(t), \ldots, E_{m-1}(t), \dot{\sigma}(t)\} \) is an orthonormal base of \( M(\sigma(t)) \) for each \( t \in [0, l] \). Since \( N \) has no focal point along \( \sigma \), the index \( I_{(t, N)}(fE_i, fE_i) \) is positive for each \( i = 1, \ldots, m-1 \). On the other hand, we have by (2.2) and (2.6)

\begin{equation}
\sum_{i=1}^{m-1} I_{(t, N)}(fE_i, fE_i) = \sum_{i=1}^{m-1} \langle S_{(t)}(fE_i)(0), (fE_i)(0) \rangle + \int_0^t \|F_\sigma(fE_i)\|^2 - \langle R(fE_i, \dot{\sigma})\dot{\sigma}, fE_i \rangle dt \\
= \int_0^t (m-1)(f')^2 - \text{Ric}(\dot{\sigma}, \dot{\sigma})f^2 dt \leq (m-1) \int_0^t (f')^2 - R \cdot f^2 dt \\
= (m-1)f'(t_0)f(t_0) = 0.
\end{equation}

This is a contradiction. Therefore \( f \) is positive on \((0, l]\). Let \( \{J_i\}_{i=1, \ldots, m-1} \) be the unique \( N \)-Jacobi fields along \( \sigma \) such that \( J_i(l) = E_i(l) \). Then by the second fundamental formula of arc length, we have

\begin{equation}
\Delta \rho(x) = \sum_{i=1}^{m-1} F_\sigma^2 \rho(E_i(l), E_i(l)) \\
= \sum_{i=1}^{m-1} \langle S_{(t)}(J_i(0), J_i(0)) \rangle + \int_0^t \|F_\sigma J_i\|^2 - \langle R(J_i, \dot{\sigma})\dot{\sigma}, J_i \rangle dt
\end{equation}
By the basic inequality (1.1) of index,

\[ \Delta \rho(x) = \sum_{i=1}^{m-1} I_{(i,N)}(J_i, J_i) \leq (m-1) f'(l) f(l). \]

This shows inequality (2.7). The discussion concerning the equality in the lemma follows from Fact (1.1). This completes the proof of Lemma (2.5).

In the same way, we see the following two lemmas.

(2.8) **Lemma.** Suppose \( N \) is a closed (imbedded) hypersurface of \( M \). Let \( h \in C^1([0, l]) \) be the solution of the equation:

\[ h''(t) + R(t) h(t) = 0 \quad \text{with} \quad h(0) = 1 \quad \text{and} \quad h'(0) \geq \frac{1}{m-1} \quad \text{the trace of} \ S_{\sigma(0)}. \]

Then the distance function \( \rho = \text{dist}_{C_{\sigma,x}}(C_{N_2 \cap N, *}) \) satisfies

\[ \rho(x) \leq (m-1) (h'/h)(\rho(x)). \]

Moreover the equality in (2.10) holds if and only if the sectional curvature of any plane tangent to \( \sigma(t) \) is equal to \( R(t) \) \( (0 \leq t \leq l) \) and \( N \) is umbilic at \( \sigma(0) \) (i.e., \( \langle S_{\sigma(0)} X, Y \rangle = h(0) \langle X, Y \rangle \)).

(2.11) **Lemma.** Suppose \( N \) is a closed (imbedded) submanifold of \( M \) with \( \text{dim} \ N = n \) \( (0 < n \leq m-1) \). Let \( F \in C^1([0, l]) \) and \( H \in C^1([0, l]) \) be, respectively, the solutions of the equations:

\[ F'''(t) + K(t) F(t) = 0 \quad \text{with} \quad F(0) = 0 \quad \text{and} \quad F'(0) = 1, \]

\[ H''(t) + K(t) H(t) = 0 \quad \text{with} \quad H(0) = 1 \quad \text{and} \quad H'(0) \geq \max \lambda_i, \]

where \( \lambda_i \) \( (1 \leq i \leq n) \) are the eigenvalues of \( S_{\sigma(0)} \). Then the distance function \( \rho = \text{dist}_{C_{\sigma,x}}(C_{N \cap N, *}) \) satisfies

\[ C \rho(x; X) \leq (F'/F)(\rho(x)) (\|X\|^2 - \langle \sigma(l), X \rangle^2), \]

for any tangent vector \( X \in M_x \) and, if \( X \in P_{\sigma(t), N_{\sigma(0)} + \varnothing(l)} R, \) where \( P_{\sigma(t)} : M_{\sigma(0)} \to M_x \) is the parallel displacement along \( \sigma \),

\[ C \rho(x; X) \leq (H'/H)(\rho(x)) (\|X\|^2 - \langle \sigma(l), X \rangle^2). \]

Moreover let \( H_* \in C^1([0, l]) \) be the solution of the equation:

\[ H_*'(t) + K(t) H_* (t) = 0 \quad \text{with} \quad H_* (0) = 1 \quad \text{and} \quad H_* (0) \geq \frac{1}{n} \quad \text{the trace of} \ S_{\sigma(0)}. \]

Then we have
The equality in (2.17) holds if and only if the sectional curvature of any plane tangent to \( \sigma(t) \) is equal to \( K(t) \) \((0 \leq t \leq l)\) and \( N \) is umbilic at \( \sigma(0) \) with respect to \( \sigma(0) \) (i.e., \( \langle S_{\sigma(0)}X, Y \rangle = H'(0)\langle X, Y \rangle \)).

We shall now give lower estimates of \( C_\rho \) at \( x \).

\[(2.18) \text{ LEMMA.} \text{ Suppose } N \text{ is a closed (imbedded) submanifold of } M \text{ with dim. } N = n \ (0 \leq n \leq m-1). \text{ Let } F \in C^t([0, l]) \text{ and } H \in C^t([0, l]) \text{ be, respectively,}\]
\[(2.19) F''(t) - K(t)F(t) = 0 \text{ with } F(0) = 1 \text{ and } F'(0) = 1,\]
and
\[(2.20) H''(t) + K(t)H(t) = 0 \text{ with } H(0) = 1 \text{ and } H'(0) \leq \min \lambda_i.\]

Suppose \( H \) is positive on \([0, l]\). Then the distance function \( \rho = \text{dist}_{C_{N,x}}(C_{N,x} \cap N, \ast) \) satisfies
\[(2.21) C_\rho(x, X) \geq \langle H'/H \rangle(\rho(x)) \{ \| X \|^2 - \langle \sigma(l), X \rangle \} \]
for any \( X \in M_x \) and, in addition, if \( X \in \exp_{N, \ast}(V_x) \),
\[(2.22) C_\rho(x, X) \geq \langle F'/F \rangle(\rho(x)) \{ \| X \|^2 - \langle \sigma(l), X \rangle \},\]
where \( \exp_{N} : N^\perp \rightarrow M \) is the exponential mapping from the normal bundle \( N^\perp \) of \( N \) to \( M \), \( \xi \) is the normal vector in \( N_{\sigma(0)} \) such that \( \exp_{N}\xi = x \), and \( V_\xi \) is the subspace of \( (N^\perp)_\xi \) tangent to the fibre \( N_{\sigma(0)} \).

\textbf{PROOF.} Obviously it suffices to prove (2.21) and (2.22) in the case \( X \in M_x \) is perpendicular to \( \sigma(l) \) and \( \| X \| = 1 \). Let \( Y \) be the \( N \)-Jacobi field along \( \sigma \) such that \( Y(l) = X \). First we assume \( Y(0) \neq 0 \). Set \( V(t) = \| Y(t) \| \ (0 \leq t \leq l) \). Suppose \( V(t) \) is positive on \((0, t_0)\) and \( V(t_0) = 0 \) for some \( t_0 \in (0, l] \). Then by Schwartz's inequality, we have
\[(2.23) | V'(t)| = \langle F, Y(t), Y(t) \rangle V(t) \leq \| F \| Y(t) \]
for any \( t \in (0, t_0) \). Set \( W(t) = V(t)/H(t) \). Since \( Y \) is an \( N \)-Jacobi field along \( \sigma \), we have for any \( t \in (0, t_0] \)
\[(2.24) I_{(t, N)}(Y, Y) = \frac{1}{2} \langle Y, Y \rangle'(t) \]
\[= \langle S_{\sigma(0)}Y(0), Y(0) \rangle + \int_0^t \| F \| Y \|^2 - \langle R(Y, \sigma) \sigma, Y \rangle ds \]
\[= \langle S_{\sigma(0)}Y(0), Y(0) \rangle + \int_0^t \| F \| Y \|^2 - \langle K(Y \wedge \sigma) \| Y \|^2 ds \]
where \( K(Y \wedge \dot{v}) \) denotes the sectional curvature of the plane spanned by \( Y \) and \( \dot{v} \). By (2.23) and (2.4), we get
\[
I_{(t, N)}(Y, Y) \geq \langle S_{v(0)}Y(0), Y(0) \rangle + \int_0^t (V)' - KV^2 ds.
\]
By simple computations, we have
\[
(V)' - KV^2 = (HW)' + (H'WV)' \geq (H'WV)'^2
\]
on \([0, t_0]\). Therefore we see that
\[
I_{(t, N)}(Y, Y) \geq \langle S_{v(0)}Y(0), Y(0) \rangle + \int_0^t (H'WV)' ds
\]
\[
= \langle S_{v(0)}Y(0), Y(0) \rangle - H'(0) \parallel Y(0) \parallel^2 + H'(t) \parallel Y(t) \parallel^2
\]
on \([0, t_0]\). Since we assume \( H'(0) \leq \min \lambda \), \( \langle S_{v(0)}Y(0), Y(0) \rangle - H'(0) \parallel Y(0) \parallel^2 \geq 0 \), and hence we have
\[
\langle Y, Y \rangle' - \langle Y, Y \rangle(t) \geq 2H(t) - H(t)
\]
on \((0, t_0)\). Integrating the both sides from \( \varepsilon \) to \( t \) \((0 < \varepsilon < t < t_0)\) we see that
\[
\parallel Y(t) \parallel \geq \parallel Y(\varepsilon) \parallel / H(\varepsilon) \cdot H(t)
\]
on \((\varepsilon, t_0)\). Since we assume \( H \) is positive on \([0, l]\), we see that \( Y \) does not vanish on \([0, l]\) and (2.24) and (2.25) hold on \([0, l]\) \((\varepsilon = 0)\). From the second variational formula of arc length and (2.24), it turns out that (2.21) holds. As for (2.22), we note that \( Y(0) = 0 \), if \( X \in \exp_{\varepsilon}(V) \). Then by taking \( F \) in place of \( H \) in the above arguments, we see that (2.22) holds. This completes the proof of Lemma (2.18).

(2.26) REMARK. After the preparation of this paper, we are informed of the paper of Heintze and Karcher [15]. Lemmas (2.5)–(2.18) are, roughly speaking, different formulations of their theorem (cf. ibid.: Section 3.2). However the full statements of Lemmas (2.5)–(2.18) do not follow directly from their statements. Therefore we have given here our version of proofs for Lemmas (2.5)–(2.18).

(2.27) LEMMA. Let \( k \) be a continuous function on \([0, l]\); let \( f \) and \( h \) be, respectively, the solutions of the equations:
\[
f'' + kf = 0 \text{ with } f(0) = 0 \text{ and } f'(0) = 1
\]
and
\[
h'' + kh = 0 \text{ with } h(0) = 1 \text{ and } h'(0) = \alpha \ (\alpha \in \mathbb{R}).
\]
Suppose \( h \) is positive on \([0, l]\). Then we have

\[
h'|h < f'|f
\]
on \([0, l]\).

**Proof.** Since \( hf' - h'f = 1 \) on \([0, l]\), we see that \( f'/f - h'/h = 1/(fh) > 0 \).

Let \( M \) be a connected, complete Riemannian manifold of dimension \( m \) and \( N \) be a (proper) closed subset of \( M \). Let \( \rho = \text{dis}_M(N, \ast) \), be the distance function to \( N \) on \( M \). Upper or lower estimates of \( S_\rho \) and \( C_\rho \) everywhere on \( M \setminus N \) as in Lemmas (2.5)~(2.18) cannot be expected to hold (even if \( N \) is a closed (imbedded) submanifold of \( M \)), since \( \rho \) is not necessarily differentiable on \( M \setminus N \). We can, however, overcome this difficulty in certain cases, making use of the method developed in Wu [33] (cf. Fact (1.6)).

(2.28) **Theorem (Laplacian comparison theorem).** Let \( M \) be a connected, complete Riemannian manifold of dimension \( m \) and \( N \) be a (proper) closed subset of \( M \). Let \( x \) be a point in \( M \setminus N \) and \( \sigma: [0, l] \to M \) be a unit speed geodesic joining \( N \) and \( x \) such that \( \text{dis}_M(N, \sigma(t)) = t \) for \( t \in [0, l] \) and \( \sigma(l) = x \). Then for any non-increasing \( C^0 \)-function \( \psi \) on \([0, l] \), we have

\[
S_\psi(\rho)(x) \geq (\psi'' + (m - 1)\psi'h'/h)(\rho(x)),
\]

where \( \rho = \text{dis}_M(N, \ast) \). Moreover in case \( N \) is a closed (imbedded) hypersurface of \( M \), we have

\[
S_\psi(\rho)(x) \geq (\psi'' + (m - 1)\psi'h'/h)(\rho(x)),
\]

where \( f \) and \( h \) are, respectively, the solutions of equations (2.6) and (2.9) defined by a continuous function \( R \) on \([0, l] \) satisfying \( (2.2) \) with respect to \( \sigma \).

(2.31) **Theorem (Hessian comparison theorem I).** Let \( M, N, x, \sigma, \rho \) and \( \psi \) be as in Theorem (2.28). Then for any \( X \in M_x \), we have

\[
C_\psi(\rho)(x; X) \geq \{\psi''(\sigma(l), X) + \psi'F|F|X^\perp - \langle \sigma(l), X^\perp \rangle\}(\rho(x)),
\]

and further in case \( N \) is a closed (imbedded) submanifold of \( M \), we have for any \( X \in P_{(x)}N_{(x)} \),

\[
C_\psi(\rho)(x; X) \geq \{\psi''(\sigma(l), X) + \psi'H'|H|X^\perp - \langle \sigma(l), X^\perp \rangle\}(\rho(x)),
\]

where \( F \) and \( H \) are, respectively, the solutions of equations (2.12) and (2.13) defined by a continuous function \( K \) on \([0, l] \) satisfying \( (2.3) \) with respect to \( \sigma \). Moreover in case \( N \) is a closed (imbedded) submanifold of \( M \), we have

\[
S_\psi(\rho)(x) \geq \{\psi'' + \psi'(m-n-1)F'/F + nH^\ast/H^\ast\}(\rho(x)),
\]
A Laplacian comparison theorem

where \( n = \dim N \) and \( H_\ast \) is the solution of equation (2.16) defined by a continuous function \( K \) on \([0, l]\) as above.

The proofs of Theorem (2.28) and Theorem (2.31) will be given in the next section. We remark here that in the above theorems, the completeness of \( M \) can be replaced by another weaker assumption. In fact, we shall carry out the proofs for them under the assumption that there is a geodesic \( \sigma : [0, l] \to M \) satisfying \( \operatorname{dis}_M(N, \sigma(t)) = t \) for \( t \in [0, l] \) and \( \sigma(l) = x \).

Now we shall give some corollaries to the above theorems. Let \( : \bar{N} \to M \) be an isometric immersion from a Riemannian manifold \( \bar{N} \) of dimension \( n \) to a connected, complete Riemannian manifold \( M \) of dimension \( m \) such that the image \( N = \iota(\bar{N}) \) is closed. Set \( l_N = \sup_{x \in N} \operatorname{dis}_M(N, x) \) \((0 < l_N \leq +\infty)\). Let \( \psi \) be a nonincreasing \( C^2 \)-function on \((0, l_N)\) \((\psi \in C^2((0, \infty)) \) if \( l_N = +\infty)\). Then we have the following

(2.35) **Corollary.** Let \( M, \bar{N}, N \) and \( \psi \) be as above. Suppose \( n = m - 1 \) and there are a continuous function \( R \) on \([0, \infty)\) and a nonnegative constant \( \alpha \) such that for any \( x \in M \), the Ricci curvature of \( M \) at \( x \geq (m - 1)R(\rho(x)) \) \((\rho = \operatorname{dis}_M(N, *))\) and the trace of the second fundamental form \( S_i \) of \( N \) with respect to any unit normal vector \( \xi \leq (m - 1)\alpha \). Then we have

\[
A\psi(\rho) \geq (\psi'' + (m - 1)\psi' h'/h)(\rho)
\]

as a distribution on \( \{x \in M \setminus N: (\psi'' + (m - 1)\psi' h'/h)(\rho(x)) < +\infty\} \), where \( h \) is the solution of the equation: \( h'' + Rh = 0 \) with \( h(0) = 1 \) and \( h'(0) = \alpha \).

**Proof.** Let \( x \) be a point of \( M \setminus N \) and \( \sigma : [0, l] \to M \) be a geodesic such that \( \rho(\sigma(t)) = t \) for \( t \in [0, l] \) and \( \sigma(l) = x \). Let \( q \) be a point of \( \bar{N} \) such that \( \iota(q) = p \) and \( \bar{U} \) be a (sufficiently small) relatively compact neighborhood of \( q \) in \( \bar{N} \) such that \( \iota_\partial : \bar{U} \to M \) is one-to-one. Let \( \Omega \) be a connected (sufficiently small) open neighborhood of \( \iota([0, l]) \) such that \( \partial \Omega \cap \iota(\bar{U}) \subset \iota(\bar{U}) \). Put \( \beta = \operatorname{dis}_N((\bar{U} \cap \Omega), *) \). Then \( \beta \geq \rho \) and \( \rho(x) = \rho(x) \), and further \( \rho(\sigma(t)) = t \) for \( t \in [0, l] \). Hence we see by Fact (1.6) that

\[
S\psi(\rho)(x) \geq S\psi(\beta)(x),
\]

and by (2.30) in Theorem (2.28) (cf. the remark after the statement of Theorem (2.31)) that

\[
S\psi(\rho)(x) \geq (\psi'' + (m - 1)\psi' h'/h)(\rho(x)).
\]

Then (2.37) and (2.38) imply that for any \( x \in M \setminus N \),

\[
S\psi(\rho)(x) \geq (\psi'' + (m - 1)\psi' h'/h)(\rho(x)).
\]
Since the right-hand side of (2.39) is continuous on
\[ A = \{ x \in M \setminus N : (\psi'' + (m-1)\psi' h'/h)(\rho(x)) < +\infty \}, \]
we see by Lemma (1.5) that \( \psi(\rho) \) satisfies
\[ \Delta \psi(\rho) \geq (\psi'' + (m-1)\psi' h'/h)(\rho) \]
as a distribution on \( A \). This completes the proof of Corollary (2.35).

(2.40) Corollary. Let \( M, N, \rho, \psi \) be as above. Suppose there are a continuous function \( K \) on \([0, \infty)\) and a nonnegative constant \( a \) such that for any \( x \in M \), the sectional curvature of \( M \) at \( x \geq (m-1)K(\rho(x)) \) and the trace of the second fundamental form \( S \) of \( N \) with respect to any unit normal vector \( \xi \leq a \). Then we have
\[ (2.41) \]
\[ \Delta \psi(\rho) \geq (\psi'' + (m-n-1)F'/F + nH'_s/H_s)(\rho) \]
as a distribution on \( \{ x \in M \setminus N : (\psi'' + (m-n-1)F'/F + nH'_s/H_s)(\rho) < +\infty \} \), where \( F \) and \( H_s \) are, respectively, the solutions of the equations: \( F'' + KF = 0 \) with \( F(0) = 0 \) and \( F'(0) = 1 \) and \( H''_s + KH_s = 0 \) with \( H_s(0) = 1 \) and \( H'_s(0) = a \).

Proof. Combining the same argument as in the proof of Corollary (2.35) with inequality (2.24) in Theorem (2.31), we can prove Corollary (2.40).

(2.42) Corollary. Let \( M \) be a connected complete Riemannian manifold of dimension \( m \); let \( N \) be a closed subset of \( M \). Suppose there is a continuous function \( R \) on \([0, \infty)\) such that for any \( x \in M \), the Ricci curvature of \( M \) at \( x \geq (m-1)R(\rho(x)) \) (\( \rho = \text{dist}_M(N, 0) \)). Then for any non-increasing \( C^2 \)-function \( \psi \) on \([0, l_N]\) (\( \psi \in C^\infty((0, \infty)) \) if \( l_N = +\infty \)), we have
\[ (2.43) \]
\[ \Delta \psi(\rho) \geq (\psi'' + (m-1)\psi' f'/f)(\rho) \]
as a distribution on \( \{ x \in M : (\psi'' + (m-1)\psi' f'/f)(\rho(x)) < +\infty \} \), where \( f \) is the solution of the equation: \( f'' + Rf = 0 \) with \( f(0) = 0 \) and \( f'(0) = 1 \).

Proof. Inequality (2.43) is a direct consequence of (2.29) in Theorem (2.28) and Lemma (1.5). In the following corollary, we consider the case when the boundary \( \partial M \) of \( M \) is nonempty and smooth. We call \( M \) complete if it is complete as a metric space with the distance induced by its Riemannian metric. Set \( i(M) = \sup_{x \in M} \text{dist}_M(\partial M, x) \) (\( \leq +\infty \)).

(2.44) Corollary. Let \( M \) be a connected, complete Riemannian manifold of dimension \( m \) with smooth boundary \( \partial M \). Suppose the Ricci curvature
of $M \geq (m-1)R$ for some $R \in \mathbb{R}$. Let $p$ be a point of the interior $M_0$ of $M$. Then for any non-increasing $C^2$-function $\psi$ on $(0, l_p)$ ($l_p = \text{dis}_M(\partial M, p)$), the distance function $\rho_1 = \text{dis}_M(p, \ast)$ satisfies

\begin{equation}
D\psi(\rho_1) \geq (\psi'' + (m-1)\psi'f'/f)(\rho_1)
\end{equation}

as a distribution on $\{x \in M: 0 < \rho_1(x) < l_p\}$, where $f$ is the solution of the equation: $f'' + Rf = 0$ with $f(0) = 0$ and $f'(0) = 1$. Moreover suppose the trace of the second fundamental form $S_\xi$ of $\partial M$ with respect to any inner unit normal vector $\xi \leq (m-1)\alpha$ for some $\alpha \in \mathbb{R}$. Then for a non-increasing $C^2$-function $\psi$ on $(0, i(M))$ ($\psi \in C^1((0, \infty))$ if $i(M) = +\infty$), the distance function $\rho_2 = \text{dis}_M(\partial M, \ast)$ satisfies

\begin{equation}
D\psi(\rho_2) \geq (\psi'' + (m-1)\psi'h'/h)(\rho_2)
\end{equation}

as a distribution on $\{x \in M_0: \psi'' + (m-1)\psi'h'/h(\rho_2(x)) < +\infty\}$, where $h$ is the solution of the equation: $h'' + Rh = 0$ with $h(0) = 1$ and $h'(0) = \alpha$.

**Proof.** We shall prove (2.45). Put $U = \{x \in M: 0 < \rho_1(x) < l_p\}$. First we claim that for any $x \in U$, there is a geodesic $\gamma: [0, l] \to M_0$ such that $\rho_1(\gamma(t)) = t$ for $t \in [0, l]$ and $\gamma(l) = x$. In fact, suppose there is no geodesic as above for some $x \in U$. Then it is known that there is a half-line $r: [0, \infty) \to M_0$ such that $r(0) = x$ and $\text{dis}_M(x, r(t)) + \text{dis}_M(r(t), p) = \text{dis}_M(x, p)$ for $t \in [0, \infty)$ (cf. [31: Theorem 7.1]). Here we call a geodesic $\gamma: [0, \delta) \to M_0$ a half-line if the image $\gamma([0, \delta))$ is closed and $\text{dis}_M(\gamma(t), \gamma(s)) = |t - s|$ $(0 \leq t, s < \delta)$. Since $M$ is complete, there is a point $y \in \partial M$ such that $\gamma(t)$ converges to $y$ as $t \to \delta$. Therefore $\text{dis}_M(p, y) = \lim_{t \to \delta} \text{dis}_M(p, \gamma(t)) = \lim_{t \to \delta} (\text{dis}_M(x, p) - \text{dis}_M(x, \gamma(t))) = \text{dis}_M(x, p) - \delta$. This contradicts the assumption that $\text{dis}_M(x, p) < l_p \leq \text{dis}_M(p, y)$. Thus we see that for any $x \in U$, the distance between $x$ and $p$ can be realized by a geodesic $\sigma: [0, l] \to M_0$. Therefore we can apply Theorem (2.28) to the distance function $\rho_1$, restricted to $U$ (cf. the remark after the statement of Theorem (2.31)) and hence we see by (2.29) in the theorem that $\psi(\rho_1)$ satisfies

\begin{equation}
S\psi(\rho_1) \geq (\psi'' + (m-1)\psi'f'/f)(\rho_1)
\end{equation}

on $U$. Since the right-hand side of (2.47) is continuous on $U$, we see by Lemma (1.5) that $\psi(\rho_1)$ satisfies inequality (2.45) as a distribution on $U$. Next we shall show inequality (2.46). Let $x$ be any point of $M_0$. Set $V = \{y \in M: 0 < \text{dis}_M(x, y) < \text{dis}_M(x, \partial M)\}$. Then as we have seen in the preceding proof, for any $y \in V$, the distance between $x$ and $y$ can be realized by a geodesic in $M_0$, so that the same conclusion holds for the closure $\overline{V}$ of $V$. Therefore $\overline{V}$ is compact, and hence we see that $\overline{V} \cap \partial M$ is not empty, since $\text{dis}_M(V, \partial M) = 0$. Thus we see that for any $x \in M_0$, the distance between $x$ and $\partial M$ can be realized by a geodesic. Therefore we can apply Theorem (2.28) to the dis-
tance function \( \rho \), and we see by (2.30) in the theorem that \( \psi(\rho) \) satisfies
\[
S\psi(\rho) \geq (\psi'' + (m-1)\psi'\theta'/\eta)(\rho)
\]
on \( M \). Since the right-hand side of (2.48) is continuous on \( A = \{ x \in M; \{(\psi'' + (m-1)\psi'\theta'/\eta)(\rho(x)) \} < +\infty \} \), it follows from Lemma (1.5) that \( \psi(\rho) \) satisfies (2.46) as a distribution on \( A \). This completes the proof of Corollary (2.35).

In the rest of this section, we assume that a connected, complete Riemannian manifold \( M \) has nonpositive curvature (\( \partial M = \phi \)). Let \( N \) be a closed subset of \( M \) such that for any \( x, y \in N \) and every geodesic \( \sigma: [0, l] \to M \) from \( x \) to \( y \), we have \( \sigma([0, l]) \subset N \). We call such a closed subset \( N \) totally convex. We remark the following facts due to Bishop and O'Neill (cf. [3: Proposition 3.4 and Proposition 4.7]): Let \( N \) be a totally convex closed subset of \( M \). Then

(i) For each point \( x \in M \), there is a unique perpendicular from \( x \) to \( N \), which is the unique shortest geodesic segment for \( x \) to \( N \).

(ii) The distance function \( \rho = \text{dis}_x(N, \cdot) \) is a convex function on \( M \).
In addition, it is easily checked that \( \rho \) has continuous first derivatives on \( M \setminus N \) (cf. [27: Lemma 5]). Let \( x \) be a point of \( M \setminus N \) and \( \sigma: [0, l] \to M \) be the unique geodesic with unit speed such that \( \sigma(l) = x \) and \( \rho(\sigma(t)) = t \) for any \( t \in [0, l] \). Let \( \bar{K}: [0, l] \to \mathbb{R} \) be a nonpositive continuous function satisfying (2.4) relative to \( \sigma \); let \( \bar{H} \) be the solution of the equation: \( \bar{H}'' + \bar{K}\bar{H} = 0 \), subject to the initial condition \( \bar{H}(0) = 1 \) and \( \bar{H}'(0) = 0 \). Now we shall state our theorem, which will be proved in the next section.

(2.49) THEOREM (Hessian comparison theorem II). Let \( N \) be a totally convex closed subset in a connected, complete Riemannian manifold \( M \) of nonpositive curvature. Let \( x \) be a point of \( M \setminus N \) and \( \phi: [0, l] \to \mathbb{R} \) be the \( C^2 \)-function defined as above. Then for any non-decreasing \( C^2 \)-function \( \phi: (0, \infty) \to \mathbb{R} \), the distance function \( \rho = \text{dis}_x(N, \cdot) \) satisfies
\[
C\phi(\rho)(x; X) \geq (\phi'' d\rho(X, X) + \phi' \bar{H}'H(|X|^2 - d\rho(X, X)))(\rho(x)),
\]
where \( X \) is a tangent vector at \( x \).

As an immediate consequence of this theorem, we have the following corollary, which will be used in the proof of Theorem (5.6).

(2.51) COROLLARY. Let \( M, N \) and \( \rho \) be as in Theorem (2.49). Then for any \( C^2 \)-function \( \phi: (0, \infty) \to \mathbb{R} \) such that \( \phi' > 0 \) and \( \phi'' > 0 \), \( \phi(\rho) \) restricted to \( M \setminus N \) is strictly convex wherever the sectional curvature is negative, and everywhere strictly convex if the sectional curvature is negative on \( U \setminus N \) for some neighborhood \( U \) of \( N \).
REMARK. (1) Let $M$ be a closed convex subset in a complete Riemannian manifold and $N$ be the boundary of $M$. Then by the proof of Theorem (2.31) and some results on the structure of convex subsets due to Cheeger and Cromoll (cf. [7: Theorem 1.16 and Lemma 1.7]), we shall see that inequality (2.33) with $H'(0)=0$ holds also in this case. (2) The results in this section will yield interesting applications to Riemannian geometry. For example, we can obtain several known facts on complete Riemannian manifolds of nonnegative sectional (or Ricci) curvature (cf. e.g. [6], [7] and [33]) and some new results, which will be discussed elsewhere (cf. [22] and [23]).

§ 3. Proofs of the comparison theorems in Section 2

PROOF OF THEOREM (2.28). Let $M$ be a connected Riemannian manifold of dimension $m$ and $N$ be a closed subset of $M$. Let $x$ be a point in $M \setminus N$. As we mentioned after the statement of Theorem (2.31), we shall prove Theorem (2.28) under the assumption that there is a geodesic $\sigma: [0, l] \to M$ such that $\text{dis}_M(N, \sigma(t))=t$ for $t \in [0, l]$ and $\sigma(l)=x$. Put $y=\sigma(0)$ and $\rho_y=\text{dis}_M(y, \ast)$. Let $\psi$, $R$, $f$ and $h$ be the functions as in the theorem. Since $\psi$ is a monotone non-increasing function, we see that $\psi(\rho_y)$ supports $\psi(\rho)$ at $x$, where $\rho=\text{dis}_M(N, \ast)$, and hence by (1.6), $S\psi(\rho_y)(x) \leq S\psi(\rho)(x)$. Therefore it suffices to prove inequality (2.29) in the case when $N$ is a point $y$. Suppose there is no conjugate point along $\sigma$. Then we see that, in the tangent space $M_y$ at $y$, there is an open neighborhood $C_y, x$ of a segment $\{\omega(t): 0 \leq t \leq l\}$ satisfying (2.1). Let $C_{y, z}$ be the image $\exp_y(C_{y, z})$. Set $\rho=\text{dis}_{C_{y, z}}(y, \ast)$. Clearly $\rho \geq \rho_y$ and $\rho(x)=\rho_y(x)$, so that $\psi(\rho)$ supports $\psi(\rho_y)$ at $x$. Therefore by inequality (2.7) in Lemma (2.5), we see that $\psi(\rho)(x) \geq \psi(\rho)(x) \geq (m-1)\psi'(f'/f)(\rho(x))$. Now suppose $x$ is the first conjugate point of $y$ along $\sigma$. Let $\varepsilon$ be a small positive constant. Put $\rho_\varepsilon=\text{dis}_M(x, \ast)$. Let $f_\varepsilon: [0, \rho_\varepsilon] \to \mathbb{R}$ be the solution of the equation: $f''_\varepsilon(t)+R(t+\varepsilon)f_\varepsilon(t)=0$ with $f_\varepsilon(0)=0$ and $f'_\varepsilon(0)=1$. Then, noting that $\sigma$ has no conjugate point on $[\varepsilon, l]$ (cf. e.g. [31: Theorem 6.3]), we see by the same reason as above that there is a smooth function $\hat{\rho}$ near $x$ which satisfies $\psi(\rho_\varepsilon+\varepsilon)(x) \geq \psi(\rho_\varepsilon+\varepsilon)(x)$ and $\psi(\rho_\varepsilon+\varepsilon)(x) \geq \psi'(l)+(m-1)\psi'(f'/f)(l-\varepsilon)$. Since $\psi(\rho_\varepsilon)$ supports $\psi(\rho_\varepsilon+\varepsilon)$ at $x$, we see that

$$\psi(\rho_\varepsilon)(x) \geq \psi'(l)+(m-1)\psi'(l)(f'/f)(l-\varepsilon).$$

Then the continuity of solutions on the initial conditions implies that the right-hand side of (3.1) tends to that of (2.29) as $\varepsilon \to 0$. This proves that (2.29) is valid for any closed subset $N$. Now we consider the case when $N$ is a closed (imbedded) hypersurface of $M$. In this case, we shall prove that $\psi(\rho)(x)$ ($\rho=\text{dis}_M(N, \ast)$) satisfies inequality (2.30). Let $P_{\rho_{y_1}}: M \to M_y$ ($y=\sigma(0) \in N$) be the parallel displacement along $\sigma$ from $M_y$ to $M_z$. Let $B$, be the ball of radius
r in \( M_r \) such that the exponential map \( \exp_r \) restricted to \( \hat{B}_r \) induces a diffeomorphism between \( \hat{B}_r \) and its image \( B_r(x) \). For any tangent vector \( X \in \hat{B}_r \), put \( X_1 = \langle X, \phi(l) \rangle \phi(l) \) and \( X_2 = X - X_1 \) (\( \in P_{s(t)}(N_r) \)), and we write \( \tilde{X}(t) \) for the parallel vector field along \( \sigma \) such that \( \tilde{X}(l) = X \). We remark here that \( f > 0 \) on \((0, l)\) and \( h > 0 \) on \((0, l)\) (cf. Lemmas (2.5) and (2.8)). Now suppose \( h(l) > 0 \).

Let \( k : [0, l] \times \hat{B}_r \to M \) be an \( m \)-parameter \( N \)-variation such that (i) \( k(t, 0) = \sigma(t) \), (ii) \( k(t, X) = \exp_r X \), (iii) \( \partial k/\partial s(t, sX) \bigg|_{s=0} = f(t)/f(l) \tilde{X}_1(t) + h(t)/h(l) \cdot \tilde{X}_2(t) \) for \( t \in [0, l] \), (iv) \( k(0, X) \subset N \) and (v) \( \partial k(t, s\phi(l)) = \sigma(t + (f(t)/f(l))s) \). (We see by e.g., [26: p. 25] that such an \( N \)-variation \( k \) exists, taking a sufficiently small \( r \) if necessary.) Now define a smooth map \( g : B_r(x) \to R \) by \( g(\exp_r X) = \) the length of the curve: \( t \to h(t, X) \). Let \( (x_1, \ldots, x_m) \) be a normal coordinates centered at \( x \) on \( B_r(x) \) such that \( (\partial/\partial x_1)(0) = \phi(l) \) and \( \{(\partial/\partial x_2)(0), \ldots, (\partial/\partial x_m)(0)\} \) is an orthonormal base of \( P_{s(t)}(N_r) \). Then we have

\[
\Delta g(x) = \sum_{i=1}^m \frac{\partial^2 g}{\partial x_i^2}(0).
\]

Since \( g(s, 0, \ldots, 0) = \int_0^t \| \partial k/\partial t(t, s\phi(l)) \| \, dt = \int_0^t (1 + (f'(t)/f(l))s) \, dt = 1 + s \) for small \( s \), we see that \( \partial^2 g/\partial x_i^2(0) = 0 \). Consider the case \( i > 1 \). The restriction of \( k \) to \([0, l] \times \{ \phi(\partial/\partial x_1)(0) : -r < s < r \} \) is an \( N \)-variation of \( \sigma \) which induces a variational vector field \( Z(t) = h(t)/h(l) \cdot (\partial/\partial x_1)(0) \) such that \( Z \in L(\sigma, l, N) \). Therefore \( \partial^2 g/\partial x_i^2(0) \) is the second variation of arc length corresponding to \( Z \).

Hence we have

\[
\Delta g(x) = \sum_{i=2}^m \langle S_{x(i)} Z_i(0), Z_i(0) \rangle + \int_0^t \| \nabla Z_i \|^2 - \langle R(Z_i, \phi) \phi, Z_i \rangle \, dt
\]

\[
= \left( \frac{1}{h(l)} \right)^2 \left( \text{tr}(S_{x(0)}) + \int_0^t (m-1)h'' - \text{Ric}(\phi, \phi)h \, dt \right)
\]

\[
\leq \left( \frac{1}{h(l)} \right)^2 \left( \text{tr}(S_{x(0)}) - (m-1)h'(0) + (m-1)h'(l)h(l) \right)
\]

\[
\leq (m-1)h'(l)/h(l).
\]

Therefore, noting that \( \psi(g) \) supports \( \psi(\rho) \) at \( x \), we have

\[
S_{\psi(\rho)}(x) \geq S_{\psi(g)}(x) = \psi''(l) + (m-1)\psi'(l)\Delta g(x)
\]

\[
\geq (\psi'' + (m-1)\psi'h'/h)(\rho(x)).
\]

Thus we see that \( S_{\psi(\rho)}(x) \) satisfies (2.30) in the case when \( h(l) > 0 \). Now suppose \( h(l) = 0 \). Then we choose a family \( \{h_s\}_{s>0} \) of the solutions of the equations: \( h_s' + R h_s = 0 \) with \( h_s(0) = 1 \) and \( h'(0) > h'(0) \) such that \( \lim_{s \to 0} h_s(0) = h'(0) \). Then \( h_s > 0 \) on \([0, l]\), since \( h_s(l)h'(l) = h'(0) - h'(0) < 0 \). Therefore it follows from the above argument that
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Since the right-hand side tends to that of (2.30) as \( \delta \to 0 \), we see that \( S\psi(\rho)(x) \) satisfies (2.30). This completes the proof of Theorem (2.28).

By the same method, we can show Theorem (2.31). We shall omit the proof for it.

**Proof of Theorem (2.49).** Let \( \tilde{M} \) be the Riemannian universal covering of \( M \) and \( \pi: \tilde{M} \to M \) be the projection. Let \( \rho_S (\tilde{N}=\pi^{-1}(N)) \) be the distance to \( \tilde{N} \). Then we see that \( \tilde{N} \) is a totally convex subset of \( M \) and \( \rho_S \circ \pi = \rho_S \). Therefore it suffices to prove Theorem (2.49) in the case \( M \) is simply connected. Let \( x \) be a point of \( M \setminus N \) and \( \sigma: [0, l] \to M \) be the unique geodesic such that \( \sigma(t) = x \) and \( \rho(\sigma(t)) = t \) for \( t \in [0, l] \). We assume that \( \sigma \) is defined on \([0, \infty)\). Let \( K_x: [0, \infty) \to \mathbb{R} \) be the non-positive continuous function defined by \( K_x(t) = \max \{ \text{the sectional curvature of all planes containing } \sigma(t) \} \). Let \( H_x: [0, \infty) \to \mathbb{R} \) be the solution of the equation: \( H_x'(s) + \rho'(\rho(x))d\rho(X) = 0 \) subject to the initial condition \( H_x(0) = 1 \) and \( H_x(0) = 0 \). Then it follows from the definition of \( K_x \) that \( H_x' = H_x' + K_xH_x = 0 \) subject to the initial condition \( H_x(0) = 1 \) and \( H_x(0) = 0 \). It is easily seen that for any \( X \in M \), \( C_\rho(\rho(x); X) = \rho'(\rho(x))d\rho(X) + \rho'(\rho(x))C_\rho(x; X) \), since \( \rho \) is a \( C^1 \)-function on \( M \setminus N \). Therefore Theorem (2.49) follows from the following

**Lemma.** For any unit speed geodesic \( \gamma: [0, \delta] \to M (\delta > 0) \) such that \( \gamma(0) = x \), we have

\[
C_\rho(x; \gamma(0)) \geq (\overline{H_x} - \overline{H_x}) \rho(x) \{ 1 - d\rho'(\gamma(0)) \}.
\]

**Proof.** For any number \( u \in [-\delta, \delta] \), we denote by \( \Gamma_{\tilde{N}}(\gamma(u)) \) the projection to \( N \) of \( \gamma(u) \). Let \( \xi_u: [-u, u] \to M (0 \leq u \leq \delta) \) be the unique geodesic joining \( \Gamma_{\tilde{N}}(\gamma(-u)) \) to \( \Gamma_{\tilde{N}}(\gamma(u)) \). Let \( \rho_\alpha \) be the distance to the image \( \xi_u([-u, u]) \) of \( \xi_u \). Note that \( \xi_u([-u, u]) \) is a closed totally convex subset in \( M \) contained in \( N \), since we assume that \( M \) is simply connected. Let \( \{ \sigma_{(u, t)}: [0, \infty) \to M \} \) be the family of unit speed geodesics continuously parametrized by the set \( \{ (u, t): 0 \leq t \leq u \} \) such that \( \rho_\sigma(\sigma_{(u, t)}(s)) = s \) for any \( s \geq 0 \) and \( \sigma_{(u, t)} \) runs through \( \gamma(t) \). Obviously \( \sigma_{(0, 0)}(s) = \sigma(s) \). Define a continuous function \( K^*_\gamma: [-\delta, \delta] \times [0, \infty) \to \mathbb{R} \) by \( K^*_\gamma(u, s) = \max \{ \text{the sectional curvatures of all planes containing } \sigma_{(u, t)}(s) \} \). Then \( K^*_\gamma(0, s) = K^*_\gamma(s) \). For each \( u \in [0, \delta] \), let \( H_u: [0, \infty) \to \mathbb{R} \) be the solution of the equation: \( H_u'(s) + K^*_\gamma(u, s)H_u(s) = 0 \) with \( H_u(0) = 1 \) and \( H_u(0) = 0 \). We now claim that for any \( u \in (0, \delta] \),

\[
\frac{1}{u^2} \{ \rho_\sigma(\gamma(u)) + \rho_\sigma(\gamma(-u)) - 2\rho_\sigma(\gamma(0)) \} \geq \overline{H_u}(\rho_\sigma(\gamma(0)) | \overline{H_u}(\rho_\sigma(\gamma(0))) \{ 1 - d\rho'(\gamma(0)) \} + \alpha(u),
\]
where $\alpha: [0, \delta] \to \mathbb{R}$ is a continuous function such that $\alpha(0) = 0$. Put $\chi_u(t) = \rho_u(\gamma(t))(-u \leq t \leq u)$ and $\eta_u(t) = H_u(\rho_u(\gamma(t)))/H_u(\rho_u(\gamma(t))) \cdot [1 - d\rho_u(\gamma(t))]( -u \leq t \leq u)$. We fix some $u$ for the moment. Then by dividing $M$ into three parts, depending on whether the nearest point on $\xi_u([-u, u])$ is in $\xi_u(( -u, u))$ or one or the other endpoint of $\xi_u$, we see that $\chi_u(t)$ is continuous with respect to $(u, t)$ and it is piecewise smooth and everywhere $C^1$ with respect to $t$. Furthermore by (2.21), we can choose a finite collection of numbers $\{t_i\}_{i=1, 2, \ldots, 2n}$ such that $-u = t_0 < t_1 < \cdots < t_n = 0 < t_{n+1} < \cdots < t_{2n} = u$ and on each interval $[t_i, t_{i+1}]$, $\chi_u(t)$ is smooth and satisfies

$$\chi''_u(t) \geq \xi_u(t).$$

Since $\chi_u(t)$ is a continuous function of $t$, we see that for any $t \in [0, u]$,

$$\chi'_u(t) - \chi'_u(0) \geq \int_0^t \xi_u(v)dv,$$

and hence integrating the both sides from 0 to $u$, we have

$$(3.4) \quad \chi_u(u) - \chi_u(0) \geq u\chi'_u(0) + \int_0^u \int_0^v \eta_u(v)dvdt.$$

Similarly, we have

$$(3.5) \quad \chi_u(-u) - \chi_u(0) \geq -u\chi'_u(0) + \int_{-u}^0 \int_0^v \eta_u(v)dvdt.$$

From (3.4) and (3.5) it follows that

$$(3.6) \quad \chi_u(u) + \chi_u(-u) - 2\chi_u(0) \geq \int_0^u \int_0^v \eta_u(v)dvdt + \int_{-u}^0 \int_0^v \eta_u(v)dvdt \geq (\eta_u(0) - \max_{-u \leq v \leq u} |\eta_u(v) - \eta_u(0)|)\left(\int_0^u \int_0^t dvdt + \int_{-u}^0 \int_0^t dvdt\right)$$

$$= (\eta_u(0) + \alpha(u)) \cdot u^2,$$

where $\alpha(u) = -\max_{-u \leq v \leq u} |\eta_u(v) - \eta_u(0)|$. Clearly $\alpha: [0, \delta] \to \mathbb{R}$ is continuous and $\alpha(0) = 0$. Therefore (3.6) implies (3.3). Now we shall complete the proof of Lemma (3.2). By the definition of $\xi_u$, we see that $\rho(\gamma(u)) = \rho_u(\gamma(u))$, $\rho(\gamma(-u)) = \rho_u(\gamma(-u))$ and $\rho(\gamma(0)) \leq \rho_u(\gamma(0))$. Therefore we have by (3.3)

$$C \rho(x: X) = \lim_{u \to 0} \inf \frac{1}{u^2} \{\rho(\gamma(u)) + \rho(\gamma(-u)) - 2\rho(\gamma(0))\} \geq \lim_{u \to 0} \inf \frac{1}{u^2} \{\rho_u(\gamma(u)) + \rho_u(\gamma(-u)) - 2\rho_u(\gamma(0))\} \geq \lim_{u \to 0} \inf \{\eta_u(0) + \alpha(u)\} = \eta_0(0).$$
This completes the proof of Lemma (3.2).

§ 4. A class of parabolic Riemannian manifolds

Let $M$ be a connected, complete and noncompact Riemannian manifold of dimension $m$. Let $x$ be a point of $M$ and $B_r(x)$ be the open metric ball around $x$ with radius $r$. Let $G_r(x, y)$ be the Green function of $B_r(x)$ with a pole at $x$. We choose a continuous function $R_x: [0, \infty) \to \mathbb{R}$ such that for any unit speed minimizing geodesic $\alpha: [0, l] \to M$ issuing from $x$,

\begin{equation}
R_x(t) \leq \frac{1}{m-1} \text{ the Ricci curvature in the direction } \dot{\alpha}(t).
\end{equation}

Let $f_x: [0, \infty) \to \mathbb{R}$ be the unique solution of the equation:

\begin{equation}
f''_x + R_x f_x = 0 \quad \text{with } f_x(0) = 0 \text{ and } f'_x(0) = 1.
\end{equation}

Then we have the following

\begin{enumerate}
  \item \textbf{Theorem.} Let $M$, $x$, $B_r(x)$ and $G_r(x, y)$ be as above. Let $R_x: [0, \infty) \to \mathbb{R}$ be a continuous function satisfying (4.1) and $f_x: [0, \infty) \to \mathbb{R}$ be the solution of (4.2). Then, for any point $y \in B_r(x)$,

\begin{equation}
G_r(x, y) \geq \frac{1}{\omega_{m-1}} \int_0^r (1/f_x)^{m-1} dt,
\end{equation}

where $\omega_{m-1}$ is the volume of $(m-1)$-dimensional unit sphere in Euclidean space $\mathbb{R}^m$ and $\rho_x = \text{dis}_x(x, *)$. In particular, if there exists the Green's function $G_n(x, y)$ on $M$, it satisfies

\begin{equation}
G_n(x, y) \geq \frac{1}{\omega_{m-1}} \int_{\rho_x(y)}^\infty (1/f_x)^{m-1} dt.
\end{equation}

\item \textbf{Remark.} Obviously the latter part in the theorem (4.3) implies that if $\int_0^\infty (1/f_x)^{m-1} = +\infty$, then $M$ is parabolic. As mentioned in Introduction, this fact was first proved by Ichihara (cf. [17: Theorem 2.1]) from a different point of view.

\item \textbf{Proof of Theorem (4.3).} Put \( \psi_r(t) = 1/\omega_{m-1} \cdot \int_t^r (1/f_x)^{m-1} ds \). Then $\psi_r: (0, \infty) \to \mathbb{R}$ is a monotone non-increasing $C^\infty$-function which vanishes at $r$ and satisfies $\lim_{t \to 0} \psi_r(t) t^{m-2} = 1/\omega_{m-1}$. Applying Theorem (2.29) to $\psi_r$, we see that $\psi_r(\rho_x)$ is subharmonic on $M \setminus \{x\}$ and we have by (1.3)

\begin{equation}
\lim_{y \to x} \rho_x^{m-1}(y) (\psi_r(\rho_x(y)) - G_r(x, y)) = 0.
\end{equation}

\end{enumerate}
Put $V_{e} = \psi_{e} - (1 + e)G_{e}(x, *),\ \text{for a positive constant } e$. Then $V_{e}$ is subharmonic on $B_{r}(x)\backslash \{x\}$ which satisfies $\limsup_{y \to x \in \partial B_{r}(x)} V_{e}(y) \leq \limsup_{y \to x \in \partial B_{r}(x)} \psi_{e}(\rho_{e}(y)) = 0$ and $\limsup_{y \to x \in \partial B_{r}(x)} V_{e}(y) < 0$ because of (4.5). Therefore the maximum principle of subharmonic functions implies that $V_{e} \leq 0$ i.e. $\psi_{e}(\rho_{e}) \leq (1 + e)G_{e}(x, *)$ on $B_{r}(x)\backslash \{x\}$. Since $\varepsilon$ is any positive constant, we get the required estimate. This completes the proof of the theorem.

As we have mentioned it in the above remark, the latter part of Theorem (4.3) gives a criterion for $M$ to be parabolic. We shall now improve this criterion in the next theorem. Let $\Omega$ be an open subset of $M$ whose complement $M\backslash \Omega$ is compact and whose boundary is a $C^{1}$ hypersurface $N$. We choose a continuous function $R_{N} : [0, \infty) \to \mathbb{R}$ and a constant $a_{N}$ such that for any geodesic $\sigma : [0, l] \to \Omega$ with $\rho_{N}(\sigma(t)) = t$ for $t \in [0, l]$ ($\rho_{N} = \text{dis}_{M}(N, *)$),

\begin{equation}
R_{N}(t) \leq \frac{1}{m-1}\ \text{the Ricci curvature in the direction } \sigma(t),
\end{equation}

and

\begin{equation}
a_{N} \geq \frac{1}{m-1}\ \text{the trace of } S_{N(0)},
\end{equation}

where $S_{N(0)}$ is the second fundamental form of $N$ with respect to $\sigma(0)$. Let $h_{N} : [0, \infty) \to \mathbb{R}$ be the solution of the equation:

\begin{equation}
h''_{N} + R_{N}h_{N} = 0 \ \text{with } h_{N}(0) = 1 \ \text{and } h'_{N}(0) = a_{N}.
\end{equation}

Note that $h_{N} > 0$ on $[0, \infty)$, as we have seen in Section 2. Then we have the following

\begin{equation}
\text{(4.9) Theorem. Under the above notations, if the integral } \int_{0}^{\infty}(1/h_{N})^{n-1}ds = +\infty, \text{ then } M \text{ is parabolic.}
\end{equation}

**Proof.** For any positive constant $r$, put $\psi_{r}(t) = 1/C_{r}$, $\int_{0}^{r}(1/h_{N})^{n-1}ds$, where $C_{r} = \int_{0}^{r}(1/h_{N})^{n-1}ds$. Then it follows from (2.30) that $\psi_{r}(\rho_{N})$ is subharmonic on $\Omega$ and $\psi_{r}(\rho_{N}) = 1$ on $N$. Suppose there exists a nonconstant positive superharmonic function $A$ on $M$. Without loss of generality, we may assume that $\inf_{M}A = 0$ and $A \geq 1$ on $M\backslash \Omega$. Then the maximum principle for subharmonic functions implies that $\psi_{r}(\rho_{N}) \leq A$ on $\Omega$. Since $\psi_{r}(\rho_{N})$ tends to 1 as $r \to \infty$ on $\Omega$, we see that $A \geq 1$ on $\Omega$ and hence $A \geq 1$ on $M$. This is a contradiction. This completes the proof of Theorem (4.9).

\begin{equation}
\text{(4.10) Remark. As a direct consequence of Theorem (4.9) but not Theorem (4.3), we see that if } M \text{ has nonnegative Ricci curvature and it con-}
\end{equation}
contains a compact (imbedded) minimal hypersurface, then $M$ is parabolic. Thus Theorem (4.9) is much more useful than Theorem (4.3) as a criterion for $M$ to be parabolic. See also Remark (6.4).

§ 5. A class of hyperbolic Riemannian manifolds

Let $M$ be a connected, complete Riemannian manifold of dimension $m$. Let $x$ be a point of $M$. We write $i(x)$ for the injectivity radius at $x$. We choose a continuous function $K_x: [0, \infty) \to \mathbb{R}$ such that for any unit speed geodesic $\sigma: [0, \infty) \to M$ issuing from $x$,

$$K_x(t) \leq \text{the sectional curvature of any plane containing } \sigma(t).$$

Let $F_x: [0, \infty) \to \mathbb{R}$ be the unique solution of the equation:

$$F''_x + K_x F_x = 0 \text{ with } F_x(0) = 0 \text{ and } F'(0) = 1.$$  \hspace{1cm} (5.2)

Then by inequality (2.19) in Lemma (2.18) and the arguments in the proof of Theorem (4.3), we see the following

**Theorem.** Under the same notations as above, if $F_x$ is positive on $(0, i(x))$, then for any $r \in (0, i(x))$ and $y \in B_r(x) \backslash \{x\}$,

$$G_r(x, y) \leq \frac{1}{\omega_{m-1}} \int_{\rho_t(y)}^{r} (1/F_x)^{m-1} dt,$$

where $\rho_t(x) = \text{dis}_{\rho_t}(x, y)$ and $\omega_{m-1}$ is the volume of $(m-1)$-dimensional unit sphere in Euclidean space. If, in addition, $i(x) = \infty$ and $\int_0^{\infty} (1/F_x)^{m-1} dt < +\infty$, then the Green's function $G_x(x, \cdot)$ of $M$ with a pole at $x$ exists and satisfies

$$G_x(x, y) \leq \frac{1}{\omega_{m-1}} \int_{\rho_t(y)}^{\infty} (1/F_x)^{m-1} dt,$$

for any $y \in M \backslash \{x\}$.

**Remark.** 1. Theorem (4.3) and Theorem (5.3) generalize a theorem of Debiard, Gaveau and Mazet (cf. [9: Theoreme 3.1]). 2. Suppose that the above $F_x$ is positive on $(0, \infty)$. Then $x$ has no conjugate point along any geodesic issuing from $x$ (cf. the proof of Lemma (2.18)) and hence if, in addition, $M$ is simply connected, then $i(x) = \infty$ (cf. [18: Proposition 1]). 3. Suppose the above $K_x$ satisfies: $K_x(t) \leq (1+\varepsilon)(m-2-\varepsilon)/(m-1)^2t^2$ for some $\varepsilon > 0$. Then Winter's theorem (cf. [32: p. 369]) and Strum's comparison theorem imply that $F_x$ is positive on $(0, \infty)$ and $F_x(t) \geq a \cdot t^{1/((m-1)2)}$ on $[b, \infty)$ for some $a > 0$ and $b > 0$, and hence $\int_{b}^{\infty} (1/F_x)^{m-1} dt < +\infty$. Therefore if, in
addition, $M$ is simply connected, we see that the Green's function $G_M(x, \ast)$ of $M$ with a pole at $x$ exists and satisfies: $G_M(x, \ast) \leq c/\rho^2$ for some $c > 0$. (For other related topics, see [20] or [21].)

We shall now prove the hyperbolicity for a certain Riemannian manifold of nonpositive curvature. In the sequel, we assume that $M$ has nonpositive curvature. Let $N$ be a closed totally convex subset of $M$. Let $K_N: [0, \infty) \to \mathbb{R}$ be a nonpositive continuous function which satisfies that, for any geodesic $\sigma: [0, l] \to M$ such that $\rho_N(\sigma(t)) = t$ for $t \in [0, l] \ (\rho_N = \text{dis}_N(N, \ast))$,

$$K_N(t) \geq \text{the sectional curvature of any plane containing } \sigma(t). \quad (5.4)$$

Then as an application of Theorem (2.49), we have the following

(5.5) **Theorem.** Let $M$ be an $m$-dimensional ($m \geq 3$), connected complete Riemannian manifold with nonpositive curvature. Suppose $M$ contains a closed totally convex subset $N$ and suppose there is a nonpositive continuous function $K_N: [0, \infty) \to \mathbb{R}$ which is not identically zero and satisfies (5.4). Then $M$ possesses a super-harmonic function $\Psi_N$ such that $0 < \Psi_N \leq 1$ on $M$, $\Psi_N = 1$ on $N$ and $\Psi_N(x)$ tends to 0 as $\rho_N(x) \to \infty$.

**Proof.** We shall first prove the theorem in the case when the boundary $\partial N$ of $N$ is a smooth hypersurface. Then we see that the distance $\rho_N$ to $N$ is smooth on $M \setminus N$ (cf. Section 2). Let $H: [0, \infty) \to \mathbb{R}$ be the unique solution of the equation: $H'' + K_N H = 0$ with $H(0) = 1$ and $H'(0) = 0$. Then by Theorem (2.49), we have

$$\Psi_N(x; X) = C_{\rho_N}(x; X) \geq (H' | H)(\rho_N(x)) \| X \| - d\rho_N^2(X)$$

for any $x \in M \setminus N$ and every $X \in M_x$. Therefore we get

$$\Delta \rho_N(x) \geq (m - 1)(H' | H)(\rho_N(x)) \quad (5.6)$$

for any $x \in M \setminus N$. On the other hand, since $K_N$ is nonpositive and not identically zero, we see that $H(t) \geq bt + c$ for some $b > 0$ and $c$. Therefore the integral $\int_0^\infty (1/H)^{m-1} dt (m \geq 3)$ is finite. Define a continuous function $\Psi_N: M \to \mathbb{R}$ by $\Psi_N = 1/C_N \cdot \int_0^\infty (1/H)^{m-1} dt$ on $M \setminus N$ and $\Psi_N = 1$ on $N$, where $C_N = \int_0^a (1/H)^{m-1} dt$. Then it turns out from (5.6) that the function $\Psi_N$ is the required one.

Next we shall consider a closed totally convex subset $N$ in general. Let $a$ be a number such that $K_N(a) < 0$. Set $N_a = \{x \in M: \rho_N(x) \leq a\}$. Then $N_a$ is a closed totally convex subset in $M$, since $\rho_N$ is a convex function on $M$. Furthermore Corollary (2.51) implies that the square of the distance $\rho_N$ to $N_a$ is strictly convex on $M \setminus N_a$. Therefore by the smooth approximation theorem.
of Greene and Wu [13], we see that there exists a family \( \{N_\varepsilon \} \) of totally convex subsets of \( M \) such that (1) the boundary \( \partial N_\varepsilon \) of \( N_\varepsilon \) is smooth for each \( \varepsilon > 0 \), (2) \( \lim_{\varepsilon \to 0} \rho_\varepsilon = \rho_0 \) uniformly on each compact set of \( M \setminus N_\varepsilon \) (\( \rho_\varepsilon = \text{dis}_M(N_\varepsilon, *) \)), and (3) for every compact set \( U \) of \( M \setminus N_\varepsilon \), there is a family \( \{\delta_\varepsilon \} \) of real numbers so that \( \lim_{\varepsilon \to 0} \delta_\varepsilon = 0 \) and \( \Delta \Psi_\varepsilon (\rho_\varepsilon) \leq \delta_\varepsilon \) on \( U \). Here

\[
\Psi_\varepsilon(t) = 1/C_\varepsilon \cdot \int_t^\infty (1/\overline{H}_\varepsilon)_{n-1} dt, \quad C_\varepsilon = \int_0^\infty (1/\overline{H}_\varepsilon)_{n-1} dt,
\]

and \( \overline{H}_\varepsilon(t) \) is the solution of the equation: \( \overline{H}_\varepsilon''(t) + K_\varepsilon(a + t) \overline{H}_\varepsilon(t) = 0 \), with \( \overline{H}_\varepsilon(0) = 1 \) and \( \overline{H}_\varepsilon'(0) = 0 \). Therefore we see that \( S(-\Psi_\varepsilon(\rho)) \geq 0 \) on \( M \setminus N_\varepsilon \), that is, \( \Psi_\varepsilon(\rho_\varepsilon) \) is superharmonic on \( M \setminus N_\varepsilon \) (cf. [33: Lemma 5]). Putting \( \Psi_N = \Psi_\varepsilon(\rho_\varepsilon) \) on \( M \setminus N_\varepsilon \) and \( \Psi_N = 1 \) on \( N_\varepsilon \), we see that \( \Psi_N \) is the required function. This completes the proof of Theorem (5.5).

In the case \( m = 2 \), we have the following

(5.7) THEOREM. Let \( M \) be a 2-dimensional, connected, complete Riemannian manifold with nonpositive Gaussian curvature. Suppose there exists a closed totally convex subset \( N \) in \( M \) and suppose that for some positive constants \( a \geq 1 \) and \( \varepsilon > 0 \), the Gaussian curvature at a point \( x \) where \( \rho_N(x) \geq a \) is bounded from above by \( -(1+\varepsilon)/(\rho_N(x) \log \rho_N(x)) \) (\( \rho_N = \text{dis}_M(N, *) \)). Then \( M \) possesses a superharmonic function \( \Psi_N \) on \( M \) such that \( 0 < \Psi_N \leq 1 \) on \( M \), \( \Psi_N = 1 \) on \( N \) and \( \lim_{\rho_N \to \infty} \Psi_N(x) = 0 \).

PROOF. We choose a nonpositive continuous function \( K_\varepsilon : [0, \infty) \to \mathbb{R} \) such that the Gaussian curvature at \( x \in M \setminus N \leq K_\varepsilon(\rho_N(x)) \) and \( K_\varepsilon(t) \leq -(1+\varepsilon)/(t \log t) \) on \([a, \infty)\). Let \( \overline{H} \) be the solution of the equation: \( \overline{H}'' + K_\varepsilon \overline{H} = 0 \) with \( \overline{H}(0) = 1 \) and \( \overline{H}'(0) = 0 \). Then by Milnor's computations (cf. [25]), we see that the integral \( \int 1/\overline{H} dt \) is finite. Therefore the same arguments as in the proof of the above theorem prove Theorem (5.7).

(5.8) COROLLARY. Let \( M \) and \( N \) be as in Theorem (5.5) or Theorem (5.7).

(1) Suppose \( N \) separates \( M \). Then \( M \) possesses a nonconstant bounded harmonic function. (2) In addition, if \( M \setminus N \) has a connected component with compact boundary, then there is a nonconstant harmonic function with finite Dirichlet norm on \( M \).

PROOF. Let \( C_1 \) and \( C_2 \) be two of the connected components of \( M \setminus N \). Let \( \Psi_N \) be a superharmonic function as in Theorem (5.5) or Theorem (5.7). Define a continuous subharmonic function \( \Phi_1 \) on \( M \) by \( \Phi_1 = 1 - \Psi_N \) on \( C_1 \) and \( \Phi_1 = 0 \) on \( M \setminus C_1 \), and define a continuous superharmonic function \( \Phi_2 \) on \( M \) by \( \Phi_2 = \Psi_N \) on \( C_2 \) and \( \Phi_2 = 1 \) on \( M \setminus C_2 \). Then we have
on $M$. Let $\{M_i\}_{i=1,2,\ldots}$ be a family of increasing relatively compact domains of $M$ with smooth boundaries such that $M = \bigcup_{i=1}^{\infty} M_i$. Let $\omega_1$ be the unique harmonic function on $M_i$ which equals $\Phi_i$ on $\partial M_i$. Since $\Phi_i$ is a subharmonic function and $\Phi_2$ is a superharmonic function such that $\Phi_1 \leq \Phi_2$ on $M$, we see that for any $i$,

$$0 \leq \Phi_i \leq \Phi_2$$

on $M_i$, and by the choice of $M_i$,

$$\omega_i \leq \omega_{i+1}$$

on $M_i$. Therefore $\omega = \lim_{i \to \infty} \omega_i$ exists and is harmonic on $M$ such that $0 < \Phi_i \leq \omega \leq \Phi_2 \leq 1$ on $M$. This proves (1). As for (2), we may assume that $M$ is oriented (taking the double covering of $M$ if necessary) and the boundary $\partial C_1$ of $C_1$ is a smooth compact hypersurface (cf. the proof of Theorem (5.5)). Then we see that there exists a harmonic function $A$ on $C_1$ such that $A = 0$ on $\partial C_1$ and the Dirichlet norm $\int_{C_1} \|FA\|^2$ is finite. Setting $A = 0$ on $M \setminus C_1$, we extend $A$ to a subharmonic function on $M$ and use it in place of $\omega_1$ in the preceding. Then Dirichlet's principle implies that the resulting harmonic function $\omega$ is the required one. This completes the proof of Corollary (5.9).

REMARK. (1) The assertion (1) contains as a special case a result of Greene and Wu (cf. [14: Proposition 7.1]). (2) In the assertion (2), we can not delete the condition that $M \setminus N$ has a connected component with compact boundary. In fact, it is known that the simply connected hyperbolic space form with dimension $\geq 3$ has no nonconstant harmonic function with finite Dirichlet norm (cf. [29: pp. 48-49]).

§ 6. Examples and remarks

Let $B$ and $F$ be Riemannian manifolds with dim $B = m$ and dim $F = n$ and $f$ be a positive smooth function on $B$. Consider the product differentiable manifold $B \times F$ with its projections $\pi: B \times F \to B$ and $\gamma: B \times F \to F$. The warped product $M = B \times_f F$ is the manifold $B \times F$ furnished with the Riemannian structure such that

$$\|X\|^2 = \|\pi_\# X\|^2 + f^2(\pi(m)) \|\gamma_\# X\|^2,$$

for every tangent vector $X \in M_\ast$. We denote the Riemannian metric on $M$ (resp. $F$) by $\langle \cdot, \cdot \rangle$ (resp. $(\cdot, \cdot)$). The sectional curvature of $F$ is denoted by $L$: that of $M$ by $K$. The covariant differential operator of $M$ is denoted by $\nabla$. 

Let $II$ be a plane tangent to $M$ at $m=(b, p)$, and let vectors $x+v, y+w$ form an orthonormal basis for $II$, where $x$ and $y$ are the components tangent to the horizontal leaf $\gamma^{-1}(p)$ and $v$ and $w$ are the components tangent to the vertical fibres $\pi^{-1}(b)$. Then by the computations in Section 7 of [3], we have the sectional curvature formula:

\[
K(II) = K(x, y) \left\| x \wedge y \right\|_g^2
- f(b)(w, w)F^2f(x, x) - 2(v, w)F^2f(x, y) + (v, v)F^2f(y, y)
+ f^2(b)[L(v, w) - \| G(b) \|_g^2](v \wedge w, v \wedge w),
\]

where $G$ denotes the gradient of $f$.

We shall now consider a class of warped products whose base space $B$ is the usual real line $\mathbb{R}$ or $\mathbb{R}^m$ $(m \geq 2)$ as a differential manifold and with the metric $g$ which can be written in the form: $g=dr^2+q^2(r)d\theta^2$ in the polar coordinates, where $q$ is a smooth function on $[0, \infty)$ which satisfies: $q(0)=0$, $q'(0)=1$ and $q>0$ on $(0, \infty)$. We call such a Riemannian manifold $B=(\mathbb{R}^m, g)$ a Riemannian model (cf. [14]). Let $f>0$ be a $C^\infty$-function on $(-\infty, \infty)$, which is an even function if $m \geq 2$. Let $F$ be any Riemannian manifold and $M$ the warped product $B \times f(r)F$ if $m \geq 2$ or $\mathbb{R} \times fF$ if $m=1$. Then for a $C^2$-function $\phi$ on an open set $U$ in $B$, we see that if $m \geq 2$ and $\phi$ depends only on $r$,

\[
\Delta(\phi \circ \pi) = |\partial^2 \phi/\partial r^2 + ((m+1)q'(r)/q(r) + nf'(r)/f(r) \cdot \partial \phi/\partial r| \circ \pi
\]

$(n=\dim N)$ and if $m=1$, 

\[
\Delta(\phi \circ \pi) = (\phi'' + nf'/f \cdot \phi') \circ \pi
\]

on $\pi^{-1}(U) \subset M$. Then the following assertions hold:

(i) $(m \geq 2)$ $M=\mathbb{R} \times f(r)F$ is hyperbolic if $\int_1^\infty 1/f^a q^{a-1} < +\infty$, and $M$ is parabolic if $\int_1^\infty 1/f^a q^{a-1} = +\infty$ and $F$ is compact.

(ii) $(m=1)$ $M=\mathbb{R} \times fF$ is hyperbolic if $\int_0^\infty 1/f^a < +\infty$ or $\int_0^1 1/f^a < +\infty$, $M$ has a nonconstant bounded harmonic function if $\int_0^\infty 1/f^a < +\infty$, $M$ has a nonconstant harmonic function with finite Dirichlet norm if $\int_0^\infty 1/f^a < +\infty$ and $F$ is compact, and $M$ is parabolic if $\int_0^\infty 1/f^a = +\infty$, $\int_0^\infty 1/f^a = +\infty$, and $F$ is compact.

PROOF. Suppose $m \geq 2$. Define a continuous function $\phi_\epsilon$ $(\epsilon \geq 1)$ on $B$ as follows: $\phi_\epsilon = \left( \int_1^\infty 1/f^a q^{a-1} \right)^{-1} \cdot \int_1^\infty 1/f^a q^{a-1}$ on $\{ b \in B : r(b) \geq 1 \}$ and $\phi_\epsilon = 1$ on $\{ b \in B : r(b) \leq 1 \}$. Then by (6.2), $\Delta(\phi_\epsilon \circ \pi) = 0$ on $\pi^{-1}(\{ b \in B : r(b) \geq 1 \})$. Therefore if $\int_1^\infty 1/f^a q^{a-1} < +\infty$, $\phi \circ \pi = \lim_{\epsilon \to 0} \phi_\epsilon \circ \pi$ is a nonconstant superharmonic function on $M$. On the other hand, by the same arguments as in the proof of Theorem
we see that $M$ has no nonconstant positive superharmonic function if \( \int_{1}^{\infty} 1/f^a q^{n-1} = \infty \) and $F$ is compact. In the same way, the assertion (ii) can be proved.

**Example 1.** Let $F$ be a compact Riemannian manifold of dimension $n$. Let $f: \mathbb{R} \to \mathbb{R}$ be a positive, smooth and even function such that $\int_{1}^{\infty} 1/f^a = \infty$. Then the warped product $M = \mathbb{R} \times fF$ satisfies all the conditions in Theorem (4.9).

**Remark.** Let $M$ be a complete Riemannian manifold. Let $V(r)$ be the volume of the geodesic ball centered at any fixed point $p$ with radius $r$. Cheng and Yau proved that if $\lim \inf \frac{V(r)}{r^a} < +\infty$ for some $a \leq 2$, then $M$ is parabolic (cf. [8: Corollary 1]). Their result does not necessarily imply Theorem (4.9). In fact, in Example 1, we choose a function $f(t)$ so that $f(t) = (t \log t)^{1/n}$ for large $t$. Then $\int_{1}^{\infty} 1/f^a = \infty$ and $\lim \inf \frac{V(r)}{r^a} = \infty$ for any $a \geq 2$.

**Remark.** Cheeger and Gromoll proved the following splitting theorem in [6]: if $M$ is a complete Riemannian manifold of non-negative Ricci curvature, then $M$ is the isometric product $\overline{M} \times \mathbb{R}^k$ where $\overline{M}$ contains no line and $\mathbb{R}^k$ has its standard flat metric. By the above assertions (i) and (ii), we see that, in this decomposition, if $k \geq 3$, then $M$ is hyperbolic and if $k \leq 2$ and $\overline{M}$ is compact, then $M$ is parabolic.

**Example 2.** Let $F$ be a complete Riemannian manifold of non-positive curvature. Let $B$ be a Riemannian model with the metric $g$ which can be written in the form: $g = dr^2 + q(r)^2 d\theta^2$ in the polar coordinates $(r, \theta)$ and let $f$ be a positive, smooth and even function on $\mathbb{R}$. Suppose $q'' \geq 0$ everywhere on $[0, \infty)$ and $q''(a) > 0$ for some $a > 0$, and suppose $f'' \geq 0$ on $[0, \infty)$ and $f''(a) > 0$. Then it follows from (6.1) that the warped product $M = B \times fF$ satisfies all the conditions in Theorem (5.5).

**Remark.** In Theorem (5.5), we can not delete the condition: the non-positive function $K$ is not identically zero. In fact, the Riemannian product $M$ of 2-dimensional Euclidean space $\mathbb{R}^2$ and a compact negatively curved manifold has nonpositive curvature and totally convex sets, but by the assertion (i) as above, $M$ is parabolic.

**Example 3.** Let $F$ be a complete Riemannian manifold of nonpositive curvature. Let $f: \mathbb{R} \to \mathbb{R}$ be a positive, smooth and convex function such that $f''(a) > 0$ and $f''(-a) > 0$ for some $a > 0$. Then the warped product $M = \mathbb{R} \times fF$ satisfies all the conditions in the first assertion (1) of Corollary (5.8). In addition, if $F$ is compact, $M$ satisfies all the conditions in the second assertion of the corollary.
§ 7. Dirichlet problems on visibility manifolds

Let $H$ be a complete simply connected Riemannian manifold of nonpositive curvature. We call such a manifold $H$ a Hadamard manifold. Throughout this section, $H$ will always denote a Hadamard manifold whose sectional curvature is bounded from above by some negative constant, and all geodesics have unit speed. Two geodesic rays $\gamma_1, \gamma_2$ are called equivalent if $\text{dis}_p(\gamma_1(t), \gamma_2(t))$ is bounded for $t \geq 0$. The set of all equivalence classes of geodesic rays is denoted by $H(\infty)$. We assume that $\overline{H} = H \cup H(\infty)$ is equipped with the "cone topology" (i.e., a subbase for the topology is the set of open cones of geodesic rays), which makes $\overline{H}$ homeomorphic to a cell (cf. [10: Theorem 2.10]).

We shall now consider the Dirichlet problems on $\overline{H}$, using the Perron-Wiener-Brelot method (cf. e.g., [4: Chap. V] or [5]). The following lemma is obvious, since $\overline{H}$ is compact.

(7.1) Lemma. For any superharmonic function $V$, the condition:

$$\lim \inf_{H \ni p \to x} V(p) \geq 0,$$

implies $V \geq 0$.

Let $f$ be an extended real valued function on $H(\infty)$, and $\sum f$ be a family of lower bounded superharmonic functions $\phi$ such that $\lim \inf_{H \ni p \to x} \phi(p) \geq f(x)$ for any $x \in H(\infty)$. Then the lower envelope $D_f$ of $\sum f \cup \{+\infty\}$ is $+\infty$, $-\infty$, or harmonic, and $D_f \leq \overline{D}_f$, where $\overline{D}_f$ is by definition $-D_f$ (cf. [4: Theorem 16]).

We introduce a certain geometrical condition on $\overline{H}$ to solve the Dirichlet problems on it. We say that the condition (R) holds at a point $x \in H(\infty)$ if for any neighborhood $U$ of $x$, there exists a neighborhood $V$ such that $V \subseteq U$ and $H \cap V$ is totally convex. The condition (R) holds at a point $x \in H(\infty)$, for example, if $x$ has a neighborhood $U$ such that the sectional curvature is constant on $U \cap H$, or if the dimension of $H$ is two. The condition (R) at a point $x \in H(\infty)$ implies that $x$ is a regular boundary point (cf. [5: sec. 18]). That is, we have the following

(7.2) Lemma. Suppose that the condition (R) holds at a point $x \in H(\infty)$. Then for any function $f$ bounded above,

$$\lim \sup_{H \ni p \to x} \overline{D}_f(p) \leq \lim \sup_{H(\infty) \ni y \to x} f(y).$$

Proof. Set $\alpha = \lim \sup_{H(\infty) \ni y \to x} f(y)$. Since $\overline{D}_{f-\alpha} \leq \overline{D}_{f-\alpha-\alpha}$, it suffices to prove that $\lim \sup_{H \ni p \to x} \overline{D}_f(p) \leq 0$ if $\lim \sup_{H(\infty) \ni y \to x} f(y) = 0$. Fix any positive constant $\varepsilon$. Let
V₁ and V₂ be open neighborhoods of x in H(∞) such that V₁ ⊂ V₁ ⊂ V₂ and supₓ₁₂ f < ε. By the assumption, there exists an open neighborhood U of x in H such that U ∩ H(∞) ⊂ V₁, and H − U is totally convex. Then by Theorem (5.5) or Theorem (5.7), there is a continuous superharmonic function Ψ on H such that 0 < Ψ ≤ 1 on H, Ψ = 1 on H − U, and Ψ(p) → 0 as disₓ(p, H − V) → ∞. Therefore, for any z ∈ H(∞) − V₂, lim inf Ḡ(p) = η · lim inf Ḡ(p) + ε = η + ε ≥ f(z), where η = sup f, and for any z ∈ V₂, lim inf Ḡ(p) ≥ ε ≥ f(z). This implies 0 < Ψ + ε ∈ Σₓ and hence

\[ \limsup_{\psi \to \infty} \bar{D}_f(p) \leq \limsup_{\psi \to \infty} (\Psi + \epsilon)(p) = \epsilon. \]

Since ε is an arbitrary positive constant, we obtain lim sup \( D_f(p) \) ≤ 0. This completes the proof of Lemma (7.2).

Note that, in the lemma, if we assume that f is bounded on H(∞), then we have lim inf f(y) ≤ lim inf \( D_f(p) \) ≤ lim sup \( D_f(p) \) ≤ lim sup f(y). Therefore we have the following

(7.3) THEOREM. Let H be a Hadamard manifold whose sectional curvature is bounded from above by some negative constant. Suppose that the condition (R) holds at every point x ∈ H(∞). Then any continuous function f on H(∞) is resolutive, that is, \( D_f = \bar{D}_f \) and lim \( D_f(p) = f(x) \) for every x ∈ H(∞), where we set \( D_f = \bar{D}_f \).

Before we state the corollary of Theorem (7.3), we recall some definitions in [10]. We refer the reader to it for details. Let D be a freely acting, properly discontinuous group of isometries of H and M be the quotient manifold H/D of H by D. A unit speed geodesic \( \gamma(t) (t \geq 0) \) in M is called an almost minimizing geodesic if there is a number \( c > 0 \) such that disₓ(\( \gamma(0), \gamma(t) \)) ≥ t − c for \( t \geq 0 \). Two unit speed geodesics \( \gamma_1, \gamma_2 \) in M are called equivalent if disₓ(\( \gamma(0), \gamma(t) \)) is bounded for \( t \geq 0 \). The set of all equivalence classes of almost minimizing geodesics is denoted by M(∞). Let \( \gamma \) be an almost minimizing geodesic in M and \( \tilde{\gamma} \) be some lift of \( \gamma \) in H. If \( \tilde{\gamma} \) represents an equivalence class in H(∞) − L(D), where L(D) is the cone limit set of D, \( \gamma \) represents, by definition, a class of F(M). We assume that \( M = M \cup M(\infty) \) is equipped with the topology induced from the cone topology and the "horocycle topology" (i.e. a subbase of the neighborhoods of a point x ∈ H(∞) with respect to the topology is the set of all limit balls at x) on H. Then the covering map \( \pi: H \to M \) extends naturally to the covering map, also denoted by \( \pi \), from \( H \cup O(D) \) onto \( M \cup F(M) \) and the restriction map \( \pi: O(D) \to F(M) \) is also the covering map, where \( O(D) = H(\infty) − L(D) \). Then by the same arguments as in the proof of Theorem (7.3) we have the following
COROLLARY. Let $H$ be a Hadamard manifold as in Theorem (7.3). Let $D$ be a freely acting, properly discontinuous group of isometries of $H$ and $M$ be the quotient manifold $H/D$. Suppose $\bar{M}$ is compact. Then for any continuous function $f$ on $M(\infty)$, there is a harmonic function $U$ on $M$ such that $\lim_{H \ni p \to x} U(p) = f(x)$ for any $x \in F(M)$.

REMARK. (1) $\bar{M}$ is compact, for example, when $M$ is core-compact, that is, $M$ contains a compact totally convex set, or when the dimension of $M$ is 2 and $D$ is finitely generated (cf. [11]). (2) Let $H$ be a Hadamard manifold as in Theorem (7.3) and $D$ be a freely acting, properly discontinuous group of isometries of $H$. Then by Theorem (5.5) and Theorem (5.7), we see that the quotient manifold $M=H/D$ possesses nonconstant bounded harmonic functions, if $L(D)$ is not equal to $H(\infty)$. In particular, if $\dim H=2$ and $D$ is finitely generated, the following three statements are mutually equivalent: (1) $L(D)=H(\infty)$, (2) $M$ does not possess a nonconstant bounded harmonic function, and (3) $M$ is parabolic. In fact, (1) implies that the volume of $M$ is finite (cf. [11]) and hence $M$ is parabolic (cf. Remark (6.4)).

Before we finish this section, we add some remarks and questions concerning the function theoretic properties of the geometric compactification of a Hadamard manifold described as above. Let $H$ be a Hadamard manifold whose sectional curvature is bounded from above by some negative constant, say $-1$. Let $\gamma(t) (t \geq 0)$ be a geodesic ray in $H$. The Busemann function $F_r: H \to R$ with respect to $\gamma$ is defined by

$$F_r(p) = \lim_{t \to -\infty} \{-\langle m-1 \rangle \gamma(t) - t\}$$

for $p \in H$ (cf. [10: Section 3]). Then from (2.50) it turns out that $SF_r \geq m-1$ ($m=\dim H$) and hence $S(\exp\{-\langle m-1 \rangle F_r\}) \leq 0$, that is, $\exp\{-\langle m-1 \rangle F_r\}$ is superharmonic on $H$. In addition by the definition of $F$, we see that cone-lim $\exp\{-\langle m-1 \rangle F_r(p)\} = 0$ for any $y \in H(\infty) - \{\gamma\}$ and horocycle-lim $\exp\{-\langle m-1 \rangle F_r(p)\} = +\infty$. (In particular, if the sectional curvature of $H \equiv -1$, $\exp\{-\langle m-1 \rangle F_r\}$ is a positive minimal harmonic function on $H$.) We now assume that every point of $H(\infty)$ satisfies the condition $(R)$. Then Theorem (7.3) implies that any continuous function on $H(\infty)$ is resolutive and every point of $H(\infty)$ is regular. Then the measure $\theta$ on $H(\infty)$ is defined as follows: $\theta(f) = D, (p)$ for any continuous function $f$ on $H(\infty)$, where $p \in M$ is any fixed point. On the other hand, let $\hat{H}$ be the Martin’s compactification of $H$ and $\Gamma$ be the Martin’s boundary (cf. e.g. [4] or [5]). Let $\mu$ be the canonical measure on $\Gamma$ with respect to constant function 1. Then by theorems of Gowrisankaran (cf. [12: Theorem 4 and Theorem 6]), we see that there exists
a Borel subset $I_1$ of $I'$ such that $\mu(I' - I_1) = 0$ and a Borel mapping $\Phi: I_1 \to H(\infty)$ such that the measure $\theta$ on $H(\infty)$ is the image of the canonical one on $I'$ and $\theta(\Phi(I_1)) = 1$. We conjecture that in our situations $I_1 = I'$ and the above mapping $\Phi: I_1 \to H(\infty)$ is homeomorphic. We refer the reader to [29] for this question.

References

A Laplacian comparison theorem

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