On the zeta function of an abelian scheme over
the Shimura curve

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Introduction

Let $F$ be a totally real number field of finite degree $g$ over $\mathbb{Q}$, and $B$ a division quaternion algebra over $F$ such that $B \otimes_\mathbb{Q} \mathbb{R} \cong M_2(\mathbb{R}) \times H^{g-1}$, where $H$ denotes the Hamilton quaternion algebra over $\mathbb{R}$. In a previous work [14], we constructed $l$-adic representations of the Galois group $\text{Gal}(\bar{F}/F)$ whose characteristic polynomials of a Frobenius element at $p$ coincide with the “$p$-th Hecke polynomials” attached to automorphic forms on $B^\times$, for almost all finite primes $p$ of $F$. Such representations were obtained as $l$-adic analogues of the Eichler-Shimura cohomology groups, which are étale cohomology groups on Shimura curves with coefficients in certain non-constant $\mathbb{Q}_l$-sheaves. The aim of this paper is to relate these cohomology groups with the usual $l$-adic cohomology groups with constant coefficients of some varieties.

When $F$ is the rational number field $\mathbb{Q}$, this had been done by Kuga and Shimura [13] (for division algebras $B$), and by Deligne [5] (for $M_2(\mathbb{Q})$). In this case, a modular curve obtained from $B$ has the meaning as a moduli variety of certain abelian varieties, and the canonical family of such abelian varieties played the crucial role in both [13] and [5]. In the general case where $g > 1$, however, there is no such canonical family; or rather, we do not use the families that were used to construct canonical models, but use the one that is obtained from an embedding of the canonical system into another one. (The possibility of using this latter family was suggested to me by Professor G. Shimura several years ago, to whom I am quite thankful.)

More precisely, let $\{V_s, \varphi_s, J_s(x), (s, T \in J; x \in G_2)\}$ be the canonical system for the group $G = B^\times$ in the sense of Shimura [18]. In the rest of this introduction, we assume that $g = [F : \mathbb{Q}]$ is odd. Then in [18] § 8, Shimura constructed an embedding of the above canonical system into a canonical system for another group $G^\times$. The canonical models for $G^\times$ naturally have the meaning as moduli varieties of certain ($2^g$-dimensional) abelian varieties. Using the above embedding, we obtain, by base change, an abelian scheme
Let $A_s \rightarrow V_s$ of relative dimension $2^s$ for each $S \in \mathcal{Z}$ which is sufficiently small (see § 2 for details). Let $A_S^k$ be the $k$-fold fibre product of $A_s$ over $V_s$. The purpose of this paper is to show that the abelian schemes $A_S^k$ thus obtained play the role of Kuga-Shimura varieties for general totally real fields of odd degree. Namely we will obtain:

(I) the determination of the Hasse-Weil zeta function of $A_S^k$ ((4.5.3)).

As in [13], the zeta function of $A_S^k$ is expressed as a product of Dedekind zeta functions and the Dirichlet series attached to automorphic forms on $B^\infty$. But our method of the proof is closer to [5], rather than that of [13]. As an application of the above result, combined with results of Deligne [6] and Jacquet and Langlands [10], we will prove:

(II) the Ramanujan-Petersson conjecture for certain automorphic forms on general quaternion algebras over $F$ (with odd $g$) ((1.4.1), (5.4.1)).

For example, when $B = M_2(F)$, our result says that the Ramanujan-Petersson conjecture is valid for Hilbert cusp forms of weight $(k_1 + 2, \ldots, k_n + 2)$ when $k_n$ are non-negative integers having the same parity, for almost all finite primes of $F$.

The content of each section is explained briefly at the beginning of the respective section.

**Notation and terminology**

For a field $k$, we denote by $\overline{k}$ a fixed algebraic closure of $k$. $k_{sep}$ (resp. $k_{ab}$) denotes the separable closure of $k$ (resp. the maximal abelian extension of $k$) in $\overline{k}$. For a Galois extension $K/k$ of fields, we denote by Gal($K/k$) its Galois group. The action of Gal($K/k$) on $K$ is on the right: $x^\sigma = (x^\tau)^\sigma$ for $\sigma, \tau \in \text{Gal}(K/k)$ and $x \in K$.

For a ring $R$ with unity, we denote by $M_n(R)$ the ring of matrices of size $n$ with entries in $R$. The unit matrix in $M_n(R)$ is denoted by $1_n$. For an element $A$ of $M_n(R)$, we denote $A^t$ the transpose of $A$. $GL_n(R)$ denotes the group of invertible matrices in $M_n(R)$. We denote by $R^n$ the free left $R$-module of rank $n$. $R^n$ is hence isomorphic to the $R$-module consisting of column vectors (resp. row vectors) of size $n$ with entries in $R$, on which the group $GL_n(R)$ naturally acts on the left (resp. on the right). We will often consider $R^n$ together with such an action of $GL_n(R)$. When we want to emphasize this, we will write it $R^n$ (column vectors) (resp. $R^n$ (row vectors)). When there is no explicit mention, we mean the former.

Let $k$ be a number field. For a finite prime $p$ of $k$, we denote by $k_p$ the $p$-adic completion of $k$. $\mathfrak{p}$ denotes the ring of integers in $k$ (resp. $k_p$). For an algebraic group $G$ over $k$, we denote by $G_k$ the adelization of $G$ over $k$. Its finite (resp. infinite) component is denoted by $G_f$ (resp. $G_{\infty}$).
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$G_{\ast+}$ denotes the connected component of the identity in $G_{\ast}$, and we put $G_{\ast+}=G_{\ast+}\times G_{0}$. For a $k$-algebra $R$, we denote by $G(R)$ or by $G_{R}$ the $R$-valued points of $G$. For a $k$-morphism $f: G\rightarrow G'$ of algebraic groups over $k$, we denote by $f_{R}$ the group homomorphism: $G_{R}\rightarrow G'_{R}$ induced by $f$.

The cohomology groups of schemes are always the étale cohomology groups, in this paper.

§ 1. The space $\Xi(S; \{k\}_{n}; w)$

In this section, we define and study the space of cusp forms $\Xi(S; \{k\}_{n}; w)$ and the Hecke operators acting on it. The space of this type was first studied by Shimura [16], and we also gave an exposition on such a space under a restriction on the quaternion algebras ([14] § 1). The formalism given in [14] works as well for general quaternion algebras, and we do this here.

1.1. Let $F$ be a totally real field of finite degree $g$ over $Q$, and $B$ a quaternion algebra over $F$. (The case where $B=M_{2}(F)$ is allowed). As an $R$-algebra, $B\otimes_{Q}R$ is isomorphic to $M_{2}(R)^{r} \times H^{g-r}$ with an integer $r$, where $H$ denotes the Hamilton quaternion algebra over $R$. We denote by $v_{1}, \ldots, v_{g}$ the different archimedean primes of $F$, and assume that the completions $B_{v_{n}}$ of $B$ at $v_{n}$ are isomorphic to $M_{2}(R)$ (resp. $H$) for $n \leqslant r$ (resp. $n > r$). Throughout this section, we assume that $r > 0$.

Suppose that non-negative integers $\{k\}_{n}$ ($1 \leqslant n \leqslant g$) and a real number $w$ are given. By our assumption, $B_{\ast}= (B \otimes_{Q}R)^{\times}$ is isomorphic to $GL_{2}(R)^{r} \times (H^{\times})^{g-r}$. We fix such an isomorphism once for all, and identify them. We denote by $B_{\ast+}$ the connected component of the identity element of $B_{\ast}$, i.e., $B_{\ast+}= \rho_{k_{n}}^{-1}GL_{2}(R)^{r} \times (H^{\times})^{g-r}$, the set of elements of $B_{\ast}$ whose reduced norms are "totally positive". Put $\lambda= \prod_{n=r+1}^{g} (k_{n}+1)$ and define a representation $\Psi(\{k\}_{n}; w)$ $=\Psi$ of $B_{\ast+}$ into $GL_{2}(C)$ by

\[
\Psi(x) = \prod_{n=r+1}^{g} \det(x_{n})^{w/2} \otimes \prod_{n=r+1}^{g} (\det(x_{n})^{-x_{n}/2} \rho_{k_{n}}(x_{n}))
\]

for $x=(x_{1}, \ldots, x_{g}) \in B_{\ast+}$, where we consider $H^{\times}$ as a subgroup of $GL_{2}(C)$ via the natural inclusion, and $\rho_{k_{n}}$ denotes the symmetric tensor representation of degree $k_{n}$ of $GL_{2}(C)$. Let $H$ be the complex upper half plane. For an element $x=(x_{1}, \ldots, x_{g}) \in B_{\ast+}$ and $z=(z_{1}, \ldots, z_{r}) \in H^{r}$, we put

\[
x(z) = (x_{1}(z_{1}), \ldots, x_{r}(z_{r})) \in H^{r}
\]

where $x_{n} \in GL_{2}(R)$ acts on $H$ by a linear fractional transformation in the usual manner. We also put
The relation
\[ j(x, z) = \prod_{n=1}^{\infty} j(x_n, z_n)^{k_n+1} \]
where \( j \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix}, z \right) = cz + d \). The relation
\[ j(xx', z) = j(x, x'(z))j(x', z) \]
is obvious. We define \( d(x) \) for \( x \in B_{\infty}^+ \) as above by
\[ d(x) = \prod_{n=1}^{\infty} | \det(x_n) |^{(k_n+2)/2} . \]

Let \( B_{\infty}^\times \) be the adelization of \( B_{\infty}^\times \), and \( B_{\infty}^+ \) the subgroup of \( B_{\infty}^\times \) consisting of \( x \in B_{\infty}^\times \) whose infinite component \( x_\infty \) is contained in \( B_{\infty}^\times \). For \( x \in B_{\infty}^\times \), we put \( x(z) = x_\infty(z) \), \( \varphi(x) = \varphi(x_\infty) \), \( j(x, z) = j(x_\infty, z) \), and \( d(x) = d(x_\infty) \) with the above notation.

1.2. For a function \( f \) on \( H^\times \) with values in \( C^i \), and \( x \in B_{\infty}^\times \), we put
\[ (f|\chi)(z) = d(x)j(x, z)^{-\chi}(z)j(x^{-1})f(x(z)) \]
with the notation of 1.1. Now let \( \varpi \) be an order of \( B \), and \( \Gamma \) a congruence subgroup of \( \varpi B_{\infty}^+ \).

**DEFINITION (1.2.2).** The notation being as above, we define the space of cusp forms \( \mathcal{S}(\Gamma; \{k_n\}; w) \) as the space of functions \( f \) on \( H^\times \) with values in \( C^i \) which satisfy the following three conditions:

(i) \( f \) is holomorphic,
(ii) \( f|\gamma = f \) for all \( \gamma \in \Gamma \),
(iii) When \( B = M_2(F) \), \( f \) vanishes at every cusp of \( \Gamma \).

Next consider a subgroup \( S = B_{\infty}^\times \times S_0 \) of \( B_{\infty}^\times \) with \( S_0 \) an open and compact subgroup of \( B_{\infty}^\times \), the finite component of \( B_{\infty}^\times \). Then \( B_{\infty}^+ S \), with \( B_{\infty}^+ = B_{\infty}^\times \cap B_{\infty}^+ \), is a normal subgroup of \( B_{\infty}^\times \), and the reduced norm \( \nu \) induces an isomorphism of \( B_{\infty}^+ / B_{\infty}^+ S \) onto \( F_{\infty}^\times / F_{\infty}^\times \nu(S) \) (cf. Shimura [18] 3.5). We also obtain a finite disjoint double coset decomposition

\[ B_{\infty}^+ = \bigcup_{i=1}^{h} S x_i B_{\infty}^+ = \bigcup_{i=1}^{h} B_{\infty}^+ y_i S \]
with \( h = [F_{\infty}^\times : F_{\infty}^\times \nu(S)] \). Note that this decomposition implies the disjoint decomposition

\[ B_{\infty}^+ = \bigcup_{i=1}^{h} S x_i B_{\infty}^+ = \bigcup_{i=1}^{h} B_{\infty}^+ y_i S . \]

We henceforth assume that \( y_i = x_i^{-1} \) (\( 1 \leq i \leq h \)). Fixing the representatives
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\{x_i\}, we put

(1.2.5) \quad S_i = x_i^{-1} S x_i = y_i S y_i^{-1},

(1.2.6) \quad \Gamma_{S_i} = B^\times \cap S_i.

**Definition (1.2.7).** The notation being as above, we put \( \mathcal{S}(S; \{k_i\}; w) = \bigoplus_{i=1}^\infty \mathcal{S}(\Gamma_{S_i}; \{k_i\}; w) \).

Here in the definition, we dropped the reference to \( \{x_i\} \), because such spaces are canonically isomorphic if one changes the representatives \( \{x_i\} \) (cf. [14] (1.4.4)). We also note that \( \mathcal{S}(S; \{k_i\}; w) = \{0\} \) if there exists an element \( c \in S \cap F^\times \) such that \( \prod_{n=1}^\infty \text{sgn}(c_n)^{k_n} = -1 \), where \( c_n \) are the conjugates of \( c \).

1.3. For a subgroup \( S \) of \( B_{n, +}^\times \) as in 1.2, take and fix a maximal order \( \mathfrak{o} \) of \( B \) such that \( S_0 \subseteq \prod \mathfrak{o}_n^\times \), where the product ranges over all the finite primes \( p \) of \( F \). \( S_0 \) contains \( \mathfrak{o}_n^\times \) as a direct factor for all but a finite number of \( p \). We denote by \( L(S) \) the product of \( p \) such that \( \mathfrak{o}_n^\times \) is not a direct factor of \( S \). Hence \( S_0 \) is a direct product of \( \prod_{p \in L(S)} \mathfrak{o}_n^\times \) and an open subgroup \( S'_0 \) of \( \bigcap_{p \in L(S)} \mathfrak{o}_n^\times \).

We shall define the Hecke operators \( \mathcal{X}(SxS) \) for \( x \) in the following set

(1.3.1) \quad V(S) = \{x \in B_{n, +}^\times | x^{-1} S_0 x = S_0\}.

For \( x \in V(S) \) and \( i (1 \leq i \leq h) \), let \( xx_i = x s_i \alpha \) with \( s \in S \), \( \alpha \in B_{n, +}^\times \) and some \( j (1 \leq j \leq h) \). Then we “recall” that

(1.3.2) \quad \Gamma_{S_i} \alpha \Gamma_{S_i} \text{ depends only on } SxS ([14] (1.5.2)).

(1.3.3) \quad The disjoint decomposition \( \bigcup_i \alpha_i \Gamma_{S_i} = \bigcup (x_i \alpha_i x_i^{-1}) S \) ([14] (1.5.6)).

Indeed, the proofs given in [14] (under the assumption that \( r=1 \)) works as well in the general case.

**Definition (1.3.4).** For \( x \in V(S) \) and \( f = (f_i) \in \mathcal{S}(S; \{k_i\}; w) \) with \( f_i \in \mathcal{S}(\Gamma_{S_i}; \{k_i\}; w) \), we define \( \mathcal{X}(SxS) f = (g_i) \in \mathcal{S}(S; \{k_i\}; w) \) by \( g_i = \sum_j f_j |\alpha^{-1}_j \) for each \( i \), where \( j \) and \( \alpha_i \) are as in (1.3.3).

Let \( D(B/F) \) be the discriminant of \( B \) over \( F \). For a finite prime \( p \) of \( F \) which is prime to \( D(B/F)L(S) \), we define the Hecke operators \( \mathcal{X}(\mathfrak{p}) \) and \( \mathcal{X}(\mathfrak{p}; \mathfrak{p}) \) as follows. Take \( x_0 \in \mathfrak{o}, \) whose reduced norm is a prime element of \( F^\times \), and let \( x \) be an element of \( V(S) \) whose \( \mathfrak{p} \)-component is \( x_0 \), and whose other components are all equal to 1. We then put \( \mathcal{X}(\mathfrak{p}) = \mathcal{X}(SxS) \) and \( \mathcal{X}(\mathfrak{p}; \mathfrak{p}) = \mathcal{X}(Sx(x)S) \), both of which depend only on \( \mathfrak{p} \).

**Definition (1.3.5).** The notation being as above, let \( \mathfrak{p} \) be a finite prime
of $F$ which is prime to $D(B/F)L(S)$. We define the $\wp$-th Hecke polynomial of $\Xi(S; \{k_n\}; w)$ by

$$H_\wp(T; \Xi(S; \{k_n\}; w)) = \det(1 - \Xi(\wp)T + N_{F/Q}(\wp)\Xi(\wp, \wp)T^2|\Xi(S; \{k_n\}; w)),$$

where $N_{F/Q}$ denotes the absolute norm. We also put

$$D(s; S; \{k_n\}; w) = \prod H_\wp(N_{F/Q}(\wp)^{-1}; \Xi(S; \{k_n\}; w))^{-1},$$

where $s$ is a complex variable, and the product ranges over all the finite primes $\wp$ of $F$ which are prime to $D(B/F)L(S)$.

**Remark (1.3.6).** It is easy to see that

$$H_\wp(T; \Xi(S; \{k_n\}; w+a)) = H_\wp(N_{F/Q}(\wp)^{w/2}T; \Xi(S; \{k_n\}; w)),$$

and hence

$$D(s; S; \{k_n\}; w+a) = D(s-a/2; S; \{k_n\}; w)$$

for any real number $a$. The Euler product $D(s; S; \{k_n\}; w)$ converges for $\text{Re}(s)$ sufficiently large, and can be continued to an entire function on the whole complex plane (Shimura [16], Jacquet and Langlands [10]).

**1.4.** We can now state one of the main result of this paper, whose proof will be completed in §5.

**Theorem (1.4.1).** The notation being as above, suppose that $[F: Q] = g$ is odd, and that $k_1 \equiv \cdots \equiv k_x \pmod{2}$. Then for almost all finite primes $\wp$ of $F$ (which are prime to $D(B/F)L(S)$), the roots of the equation $H_\wp(T; \Xi(S; \{k_n\}; w)) = 0$ have absolute values $N_{F/Q}(\wp)^{-(w+1)/2}$.

As the proof will show, the set of possible exceptional primes in the above assertion depends only on $B$ and $S$, and not on $\{k_n\}$ and $w$. We note that, when $F = Q$ and $B$ is a division quaternion algebra, the above assertion had been obtained by Kuga and Shimura [13]. When $B = M_2(Q)$, it had been proved by Deligne [5], by virtue of the validity of the Weil conjecture (Deligne [6]).

§ 2. An abelian scheme over the Shimura curve

The purpose of this section is to recall the theory of Shimura [18]. In 2.3, we will define an abelian scheme $A_\wp$ over the Shimura curve $V_\omega$, which will play the role of Kuga-Shimura variety [13] for general totally real field of odd degree. From this section until §4, we will assume that $r = 1$ for the quaternion algebra $B$. 
2.1. Let $F$ and $B$ be as in the previous section. We first recall Shimura's canonical system under a restrictive assumption that $r=1$, to simplify the exposition. Fix a positive integer $n$, and put

$$(2.1.1) \quad G = \{ \alpha \in GL_n(B) | \alpha \cdot \iota \alpha = \iota(\alpha)1_n, \iota(\alpha) \in F^\times \},$$

where $\iota$ denotes the canonical involution of $B$ over $F$. $G$ can be naturally regarded as an algebraic group over $\mathbb{Q}$, and hence we can define the group $G_\mathbb{A}$, $G_\mathbb{A}^+$, $G_\mathbb{A}^-$, and $G_0$ from $G_0$ (cf. Notation and terminology). We define a homomorphism $\sigma$ of $G_\mathbb{A}$ to $\text{Gal}(F_{ab}/F)$ by

$$(2.1.2) \quad \sigma(x) = [\iota(x)^{-1}, F]$$

where $F_{ab}$ denotes the maximal abelian extension of $F$, and $[\ , F]$ is the Artin symbol for $F$ in class field theory. Let $\mathcal{S}$ be the family of subgroups $S$ of $G_\mathbb{A}^+$ which are of the form $S = G_\mathbb{A}^+ \cap S_0$ with $S_0$ open and compact in $G_0$. For $S \in \mathcal{S}$, we put

$$(2.1.3) \quad k_S = \{ \alpha \in F_{ab} | \alpha^{\iota(x)} = \alpha \text{ for all } x \in S \}$$

$$(2.1.4) \quad \Gamma_S = S \cap G_0.$$

$\Gamma_S$ naturally acts on the Siegel upper half space $\mathbb{H}_n$ of degree $n$. With the above notation, there exists the canonical system $\{ V_S, \varphi_S, J_\tau_S(x), (S, T \in \mathcal{S}, x \in G_\mathbb{A}^+) \}$ for $G$ ([18] 2.5). We only recall that $V_S$ is defined over $k_S$, and that $\varphi_S$ is a holomorphic map from $\mathbb{H}_n$ onto $V_S(\mathbb{C})$ which induces an isomorphism of $\varphi^* \mathbb{H}_n$ onto $V_S(\mathbb{C})$.

We next recall the homomorphism $\tau$ ([18] 2.7). Let $L_S$ be the field of rational functions on $V_S$ defined over $k_S$, and put $\mathcal{L}_S = \{ f \circ \varphi_S | f \in L_S \}$. We also put $\mathcal{L} = \bigcup_{S \in \mathcal{S}} \mathcal{L}_S$ and $L = \lim_{\longrightarrow} L_S$, where the lim is taken in the obvious sense. In this paper, we identify $\mathcal{L}_S$ and $L_S$ and hence $\mathcal{L}$ with $L$ by means of $\varphi_S$.

In [18] 2.7, Shimura defined the homomorphism

$$(2.1.5) \quad \tau: G_\mathbb{A}^+ \rightarrow \text{Aut}(\mathcal{L}/F) = \text{Aut}(L/F)$$

which is continuous if we topologize $\text{Aut}(\mathcal{L}/F)$ as in [18] 2.7. Let $F^\times G_\mathbb{A}^+$ be the closure of $F^\times G_\mathbb{A}^+$ in $G_\mathbb{A}^+$. Then $\tau$ induces a continuous injective homomorphism of $G_\mathbb{A}^+ / F^\times G_\mathbb{A}^+$ into $\text{Aut}(L/F)$. It is also surjective when $\mathbb{H}_n/\Gamma_S$ is compact, or when $F = \mathbb{Q}$ and $G_\mathbb{Q} = \text{GL}_n(\mathbb{Q})$ ([18] 2.8). For a member $S$ of $\mathcal{S}$, put $E_S = S \cap F^\times$. Then $\tau$ induces a continuous injective homomorphism, which we also denote by the same letter

$$(2.1.6) \quad \tau: S_0/E_{S_0} \rightarrow \text{Gal}(L/L_S)$$

where $E_{S_0}$ is the projection of $E_S$ to $G_0$, and $\overline{E_{S_0}}$ is the closure of $E_{S_0}$ in $G_0$. 

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This map is surjective if (2.1.5) is surjective.

2.2. In [18] § 8, Shimura constructed a homomorphism of algebraic groups over $Q$:

$\beta: G \longrightarrow GL_q(C)$

where $C$ is a quaternion algebra over $Q$, and $q = 2^{n-1}n^2$. When $n = 1$, it is the map which we will call $\psi(1)$ in the next section. $\beta$ is actually a homomorphism into a subgroup

$G(C, v) = \{ y \in GL_q(C) \mid yv = y = v(y)v, v(y) \in Q^* \}$

with $v \in GL_q(C)$. By [18] 8.6, we know that $v' = (-1)^{q-1}v$, and that $C$ is indefinite if and only if $g$ is odd. In the rest of this section, we assume that $g$ is odd. Then, $G(C, v)$ is isomorphic (by an inner automorphism in $GL_q(C)$) to

$G^* = \{ y \in GL_q(C) \mid yy^* = v(y)v, v(y) \in Q^* \}$

which is also a group of the type considered in 2.1. We henceforth consider $\beta$ as a homomorphism of $G$ into $G^*$. Let $(V_s, \varphi_s, J_{x, s}(x), (S, T \in \mathcal{Z}; x \in G_{q,s}))$ and $(V^*_s, \varphi^*_s, J^*_{x, s}(y), (M, N \in \mathcal{Z}^*; y \in G^*_{q,s}))$ be the canonical systems for $G$ and $G^*$, respectively. Then the following assertions are proved in [18] 8.9:

(2.2.4) $\beta$ induces a holomorphic map $e: \tilde{\mathcal{H}} \longrightarrow \tilde{\mathcal{H}}_q$, which is equivariant with respect to the actions of $G_{q,s}$ and $G^*_{q,s}$.

(2.2.5) For $S \in \mathcal{Z}$ and $M \in \mathcal{Z}^*$ such that $\beta(S) \subseteq M$, we have $k_s \supseteq k^*_s$, and there exists a $k_s$-morphism $E_{M^*}: V^*_s \rightarrow V^*_M$ which satisfies $\varphi^*_s \circ e = E_{M^*} \circ \varphi_s$.

2.3. Let the notation be as above and assume either that $q > 1$ or that $C$ is a division algebra, so that the condition (3.7.1) in Shimura [17] is satisfied. We recall that $V^*_s$ has the meaning as a moduli variety of abelian varieties (plus additional data; the PEL-structures). To be precise, let $\mathcal{O}^* = (C, \Phi, \rho, \zeta T; m; d_1, \ldots, d_s)$ be a PEL-type for the group $G^*$ as in [18] 5.1, and put $M = M(\mathcal{O}^*) = \{ y \in G^*_{q,s} \mid m y = m, d_i y = d_i (1 \leq i \leq s) \}$. Then $V^*_M$ is obtained as the moduli variety for the PEL-structures of type $\mathcal{O}^*$ ([18] § 5). If moreover $\mathcal{I}_M = G^*_q \cap M$ is torsion free, then there exists the universal family of abelian varieties

$V^*_M \longrightarrow A^*_M$

with the PEL-type $\mathcal{O}^*$, by [17] 5.3. $A^*_M$ is a projective abelian scheme of relative dimension $2q = (2n)^2$ over $V^*_M$. We note that $m$ is a lattice in $C^*$ (row vectors), and that $\sigma = \{ x \in C^* \mid \alpha m \subseteq m \}$ acts on $A^*_M$ as $V^*_M$-endomorphisms on the left. For each point $P$ of $V^*_M$, the group of torsion points $(A^*_M)_P(C)_{\text{tors}}$ of the fibre $(A^*_M)_P$ is isomorphic to $C^*/m$ as left $\sigma$-modules.
Now consider the following condition on \( S \in \mathcal{Z} \):

(2.3.2) **There exists a PEL-type \( \Omega^* \) for \( G^* \) such that \( \beta(S) \) is contained in \( M = M(\Omega^*) \), and that \( \Gamma_{\overline{\mathbb{Q}}}^S \) is torsion free.**

**DEFINITION (2.3.3).** The notation and the assumption being as above, suppose that \( S \in \mathcal{Z} \) satisfies the above condition (2.3.2). Then we define the projective abelian scheme

\[
\tilde{f}_S : A_S \to V_S
\]
as the base change of \( \tilde{f}_S^* \) by \( E_{M^S} \), where \( M = M(\Omega^*) \).

It is easy to see that, when \( \Phi, \rho, kT \) and \( m \) are fixed, \( \tilde{f}_S \) and \( A_S \) depend only on \( S \), and not on the choice of the level structure. When \( F=\mathbb{Q} \), \( n=1 \) and \( B \) is a division algebra, we see that \( G=G^*, \beta=id \), and \( E_{M^S}=id \), and hence the above \( A_S \) is the abelian scheme considered by Kuga and Shimura [13].

**DEFINITION (2.3.4).** The notation being as above, for a positive integer \( k \), we denote by \( A^k_S \) the \( k \)-fold fibre product of \( A_S \) over \( V_S \), and by \( \tilde{f}^k_S \) the structure morphism: \( A^k_S \to V_S \).

When there is no fear of confusion, we will drop the subscript \( S \) for \( A^k_S \) and \( \tilde{f}^k_S \), and write them simply \( A^k \) and \( f^k \), respectively.

### § 3. The fundamental isomorphism

In this section, we assume that \( B \) is a division quaternion algebra over \( F \) with \( r=1 \), and that \( n=1 \) for \( G \), i.e. \( G_{\mathbb{Q}} = B^\times \). The aim of this section is to prove the isomorphism (3.3.3), which links the \( l \)-adic cohomology groups of \( A_S^k \) with the \( l \)-adic representation spaces constructed in [14].

**3.1.** We first recall some facts about \( \mathbb{Q}_l \)-sheaves ([9] VI). Let us fix a noetherian scheme \( X \), and denote by \( X_{\text{ét}} \) the étale site of \( X \). We also fix a prime number \( l \) in the following. For the definitions of (constructible) \( \mathbb{Z}_l \)-sheaves and (constructible) \( \mathbb{Q}_l \)-sheaves on \( X_{\text{ét}} \) (we will drop the adjective “constructible” following SGA 4\( \frac{1}{2} \)), we refer to [9] VI. Categories of such objects will be denoted by \( \mathbb{Z}_l \mathcal{C}(X) \) and \( \mathbb{Q}_l \mathcal{C}(X) \), respectively. For \( F \in \text{Ob}(\mathbb{Z}_l \mathcal{C}(X)) \), we write \( F \otimes Q_l \) for the corresponding \( \mathbb{Q}_l \)-sheaf. A \( \mathbb{Z}_l \)-sheaf \( F=(F_n) \) is called twisted constant (“constant tordu” in French) if all the \( F_n \) are locally constant, or equivalently all the \( F_n \) are represented by finite étale group schemes over \( X \). A \( \mathbb{Q}_l \)-sheaf is called twisted constant if it is isomorphic to \( F \otimes Q_l \), with a twisted constant \( \mathbb{Z}_l \)-sheaf \( F \) on \( X_{\text{ét}} \).

Suppose now that \( X \) is integral and normal. Let \( \gamma \) be the generic point
of $X$, and identify $k(\gamma)$, the residue field of $\gamma$, with the field of rational functions $K$ of $X$. We denote by $\gamma$ the geometric point of $X$ over $\gamma$ which corresponds to the separable closure $K_{\text{sep}}$ of $K$. Let $\mathcal{F} = (F_n) \otimes \mathbb{Q}_l$ be a twisted constant $\mathbb{Q}_l$-sheaf on $X_{\text{ét}}$ with locally constant $F_n$. Then we can consider its associated $\text{Gal}(K_{\text{sep}}/K)$-module

$$(3.1.1) \quad \mathcal{F}_{\gamma} = (\text{lim} F_n(K_{\text{sep}})) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$$

which is a $\mathbb{Q}_l$-vector space of finite dimension on which $\text{Gal}(K_{\text{sep}}/K)$ acts continuously and $\mathbb{Q}_l$-linearly. We will frequently use the following easy

**Lemma (3.1.2).** Suppose that $X$ is integral and normal. Then the functor $\mathcal{F} \mapsto \mathcal{F}_{\gamma}$ from the category of twisted constant $\mathbb{Q}_l$-sheaves on $X_{\text{ét}}$, to the category of finite dimensional $\mathbb{Q}_l$-vector spaces on which $\text{Gal}(K_{\text{sep}}/K)$ acts continuously and $\mathbb{Q}_l$-linearly, is fully faithful.

**Proof.** This follows from the $l$-adic version of Grothendieck’s Galois theory ([9] VI 1.4.2) and the fact that there is a canonical surjective homomorphism of $\text{Gal}(K_{\text{sep}}/K)$ onto the fundamental group $\pi_1(X; \gamma)$ of $X$ at $\gamma$ ([8] V 8.2). Q.E.D.

For twisted constant $\mathbb{Q}_l$-sheaves $\mathcal{F}$ and $\mathcal{G}$ on $X_{\text{ét}}$, we can define their tensor product $\mathcal{F} \otimes \mathcal{G}$ and the exterior products $\wedge^n \mathcal{F}$, which are also twisted constant. The following isomorphisms of $\text{Gal}(K_{\text{sep}}/K)$-modules are obvious:

$$(3.1.3) \quad (\mathcal{F} \otimes \mathcal{G})_{\gamma} \cong \mathcal{F}_{\gamma} \otimes \mathcal{G}_{\gamma}$$

$$(3.1.4) \quad (\wedge^n \mathcal{F})_{\gamma} \cong \wedge^n \mathcal{F}_{\gamma}$$

where $\otimes$ and $\wedge$ in the right hand sides are the usual ones as vector spaces over $\mathbb{Q}_l$.

Next, let $L$ be a finite extension of $\mathbb{Q}_l$, and $A$ the ring of integers in $L$. Then one can naturally define the categories of $A$-sheaves and $L$-sheaves on $X_{\text{ét}}$ ([9] VI), which we denote by $A_{\text{fc}}(X)$ and $L_{\text{fc}}(X)$, respectively. The above formalism, especially (3.1.2) also holds for $A$-sheaves and $L$-sheaves, with obvious modifications. For $F = (F_n) \in \text{Ob}(\mathbb{Z}_l_{\text{fc}}(X))$, put

$$(3.1.5) \quad F \otimes A = (F_n \otimes \mathbb{Z}_l/l^n+1 \mathbb{Z}_l, A/l^{n+1} A).$$

Then $F \otimes A$, together with natural action of $A$, is an object of $A_{\text{fc}}(X)$. It is twisted constant if $F$ is. Also, we have a functor $\otimes_A L : A_{\text{fc}}(X) \rightarrow L_{\text{fc}}(X)$, which is by definition $\otimes \mathbb{Q}_l$ for the underlying $\mathbb{Z}_l$-sheaves. The functor: $\mathbb{Z}_l_{\text{fc}}(X) \rightarrow L_{\text{fc}}(X)$, which associates $F$ to $(F \otimes A) \otimes_A L$, factors through $\mathbb{Q}_l_{\text{fc}}(X)$, and hence we obtain a functor
It is obvious that all these functors are exact, and that $H^i(X, \mathcal{F} \otimes_{\mathbb{Q}_L} L) \cong H^i(X, \mathcal{F}) \otimes_{\mathbb{Q}_L} L$ for any $\mathcal{F} \in \text{Ob}(\mathcal{Q}_{\text{fc}}(X))$. Suppose again that $X$ is integral and normal. Then for a twisted constant $\mathbb{Q}_L$-sheaf $\mathcal{F}$ on $X_{\text{ét}}$, the following isomorphism of $\text{Gal}(K_{\text{sep}}/K)$-modules is also obvious:

\[(\mathcal{F} \otimes_{\mathbb{Q}_L} L)^g \cong \mathcal{F} \otimes_{\mathbb{Q}_L} L.\]

3.2. We next recall the results of [14] § 5. Let us take and fix a finite Galois extension $K$ of $\mathbb{Q}$ in $C$ which contains $F$ and splits $B$, i.e. $B \otimes_K \mathbb{Q}_L \cong M_2(\mathbb{Q}_L)^g$. We denote by $\mathfrak{g}$ the Galois group of $K$ over $\mathbb{Q}$, and by $\mathfrak{g}$ the subgroup of $\mathfrak{g}$ corresponding to $F$. Thus each coset in $\mathfrak{g}/\mathfrak{g}$ may be identified with an embedding of $F$ into $C$, or equivalently, with an archimedean prime of $F$. Identifying the set $\{v_1, \ldots, v_g\}$ of archimedean primes of $F$ with the index set $\{1, \ldots, g\}$ (in that order), we define a homomorphism of $\mathfrak{g}$ into $S_g \cong \mathbb{Z}_L$, the symmetric group of degree $g$, by right regular representation on $\mathfrak{g}$. Fix non-negative integers $k_1, \ldots, k_g$ and an integer $w$ satisfying

\[(k_1 \equiv \cdots \equiv k_g \equiv w \pmod{2}), \quad \text{and } k_n < w \quad (1 \leq n \leq g).\]

We put

\[\mathfrak{h} = h_\mathfrak{h}(\{k_n\}, w) = \{\sigma \in \mathfrak{g} | k_{\sigma - 1} = k_n \quad (1 \leq n \leq g)\}.\]

Then in [14] (5.4.3), we constructed a homomorphism of $\mathbb{Q}$-algebraic groups

\[\rho = \rho(\{k_n\}, w) : G \longrightarrow GL_{\text{red}}(\mathbb{Q})\]

with $e = e(\{k_n\}, w) = \prod_{n=1}^g (k_n + 1)$, $e = e(\{k_n\}, w) = 1$ or 2, and $d = d(\{k_n\}, w) = [\mathfrak{g} : \mathfrak{h}(\{k_n\}, w)]$. We note that $e(\{k_n\}, w) = 1$ if $w$ is even ([14] 5.3). We recall that the representation $\rho_K$ of $(B \otimes_K \mathbb{Q}_L)^e \cong GL_{\text{red}}(K)$ induced by $\rho$ is equivalent to the one which maps an element $(A_1, \ldots, A_g)$ of $GL_{\text{red}}(K)^g$ to $\bigoplus_{\mathfrak{g} \in \mathfrak{h}(\{k_n\}, w)} \left( \prod_{n=1}^g \det(A_n)^{w - k_{\sigma - 1} - 1/n} \right) \otimes \rho_{k_{\sigma - 1} - 1}(A_n)).$

Now fix a rational prime $l$, and a member $S$ of $\mathcal{X}$ with respect to $G$ satisfying

\[(3.2.4) \quad \text{For any } x \in G_{\mathfrak{h}}, \quad x^{-1} \mathcal{X} \cap B^x \text{ has no non-trivial elliptic elements (considered as a Fuchsian group of the first kind acting on } \mathcal{X} = H).\]

Then we can define from $\rho(\{k_n\}, w)$ a projective system $\{F_n(\rho(\{k_n\}, w))\} = \{F_n(\{k_n\}, w)\}$ of finite étale group schemes of $l$-power orders over $V_S$ ([14] (2.4.3)). This system, considered as a projective system of sheaves on $V_{S, \text{ét}}$, is a twisted constant $\mathbb{Z}_L$-sheaf. Its associated $\mathbb{Q}_L$-sheaf will be denoted by $\mathcal{F}_S(\{k_n\}, w; l)$. When $S$ and $l$ are fixed, and there is no fear of confusion, we
simply write it $\mathcal{F}(\{k_s\}; w)$.

3.3. We consider here the special case where $k_1 = \cdots = k_g = w = 1$. In this case, $g = g$, and we obtain a $\mathbb{Q}$-rational homomorphism

\begin{equation}
\psi(1) : G \longrightarrow GL_{g-1}(C)
\end{equation}

by [14] (5.2.10), where $C$ is a quaternion algebra over $\mathbb{Q}$ (which may be $M_1(\mathbb{Q})$); the algebra $Y$ in [14] is $M_{g-1}(C)$ in this case. Note that this is nothing but Shimura's homomorphism $\beta$ (cf. (2.2.1)). Fixing a $\mathbb{Q}$-linear isomorphism of $C$ onto $\mathbb{Q}$, we obtain from $\psi(1)$ a $\mathbb{Q}$-rational homomorphism

\begin{equation}
\rho(1)^g : G \longrightarrow GL_g(Q)
\end{equation}

via the natural inclusion of $M_{g-1}(C)$ into $M_{g+1}(Q)$. We note that the representation $\rho(1)^g$ of $GL_1(Q)^g$ is equivalent to the one which maps $(A_1, \cdots, A_g) \in GL_1(Q)^g$ to $\bigoplus_{n=1}^g \rho(A_n)$. This representation $\rho(1)^g$ also satisfies the properties $(p_1) - (p_4)$ of [14] Introduction, and hence we can define a twisted constant $\mathbb{Q}$-sheaf $\mathcal{F}_\delta(\rho(1)^g; l) = \mathcal{F}(\rho(1)^g)$ on $V_{S,\text{et}}$, for $S$ satisfying (3.2.4), by the same procedure as above.

**Theorem (3.3.3).** The notation being as above, suppose that $g$ is odd, and that $S \in \mathcal{I}$ satisfies (2.3.2) and (3.2.4). Let $f_S : A_\delta \rightarrow V_\delta$ be the abelian scheme defined in (2.3.3). Then for each prime number $l$, the $\mathbb{Q}_l$-sheaf $R^1f_\delta^*(Q_l)$ on $V_{S,\text{et}}$ is isomorphic to $\mathcal{F}(\rho(1)^g; l)$.

The proof will be given in 3.4-3.5.

3.4. Fix $S \in \mathcal{I}$ as above, and write $f$ and $A$ for $f_S$ and $A_\delta$, respectively. We also fix a prime number $l$. First note that $R^1f_\delta^*(Q_l)$ is a twisted constant $\mathbb{Q}_l$-sheaf on $V_{S,\text{et}}$; this follows from a general result ([2] XVI 2.1 and XV 2.1) or from the Kummer theory ([2] IX 3). Let $\eta$ be the generic point of $V_\delta$, and identify $k(\eta)$ with $L_\delta$. By (3.1.2), the above theorem is equivalent to saying that $\text{Gal}(L_\delta/L_\delta)$-modules $R^1f_\delta^*(Q_l)_\eta$ and $\mathcal{F}(\rho(1)^g)_\eta$ are isomorphic. We will describe these two $\text{Gal}(L_\delta/L_\delta)$-modules in the following. For this purpose, let us prepare some notation. Suppose that a continuous representation $\varphi : \text{Gal}(L_\delta/L_\delta) \rightarrow GL_m(Q_l)$ is given. If $\varphi$ factors through $\text{Gal}(L_\delta/L_\delta)$, then combining $\varphi$ with the homomorphism (2.1.6), we obtain a representation of $S_\delta$ into $GL_m(Q_l)$, which we denote by $\varphi$;

\begin{equation}
\varphi : S_\delta \longrightarrow GL_m(Q_l).
\end{equation}

Let $S_i$ be the projection of $S_\delta$ to the $l$-component $G(Q_l) = (B \otimes Q_l)^c \subset G_\delta$. If moreover the above $\varphi$ factors through $S_i$, we denote by $\bar{\varphi}_i$ the associated representation of $S_i$;
Thus if $p_i : G_0 \to G(Q_l)$ denotes the projection map, then $\tilde{\phi} = \tilde{\phi}_1 \circ p_i$.

To see the representation of $\text{Gal}(\overline{L}/L)$ on $\mathcal{F}(\rho(1)\ell)_\ell$, we need the following elementary

**Lemma (3.4.3).** Let $X$ be an integral scheme with its field of rational functions $K$. Let $G$ be a finite étale group scheme over $X$. Then there is an isomorphism $i$ of $G \times X \text{Spec}(K_{\text{sep}})$ to the constant group scheme $G(K_{\text{sep}}) \times \text{Spec}(K_{\text{sep}})$ over $\text{Spec}(K_{\text{sep}})$, and the following diagram commutes for all $\sigma \in \text{Gal}(K_{\text{sep}}/K)$:

$$
\begin{array}{ccc}
G \times X \text{Spec}(K_{\text{sep}}) & \xrightarrow{i} & G(K_{\text{sep}}) \times \text{Spec}(K_{\text{sep}}) \\
\downarrow & & \downarrow \\
G \times X \text{Spec}(K_{\text{sep}}) & \xrightarrow{i} & G(K_{\text{sep}}) \times \text{Spec}(K_{\text{sep}})
\end{array}
$$

where the left vertical map is $\text{id}_G \times X \text{Spec}(\sigma)$, and the right vertical map is $\sigma^{-1} \times \text{Spec}(\sigma)$.

**Proof.** Easy and left to the reader. Q.E.D.

**Corollary (3.4.4).** The representation $\phi : \text{Gal}(\overline{L}/L) \to \text{Aut}_{Q_l}(\mathcal{F}(\rho(1)\ell; l)_\ell)$ (resp. $\text{Aut}_{Q_l}(\mathcal{F}(\rho(1)\ell; (k_n); w; l)_\ell)$) factors through $\text{Gal}(L/L)$, and the associated representation $\tilde{\phi}$ of $S_i$ factors through $S_i$. The representation $\tilde{\phi}_1$ of $S_i$ is equivalent to the representation which maps an element $s$ of $S_i$ to $\rho(1)\ell_q(s^{-1})$ (resp. $\rho((k_n); w)\ell_q(s^{-1})$).

**Proof.** This follows from the very construction of $\mathcal{F}(\rho(1)\ell; l)$ and $\mathcal{F}(\rho((k_n); w; l))$ ([14] 2.4) and (3.4.3). Q.E.D.

**3.5.** We next consider the $\text{Gal}(\overline{L}/L)$-module $R^1f_*\mathcal{L}(Q)$. Let $f : A \to \text{Spec}(L)$ be the generic fibre of $f : A \to V_\eta$. Then by the Kummer theory, we have canonical isomorphisms of $\text{Gal}(\overline{L}/L)$-modules:

$$
R^1f_*\mathcal{L}(Q) \cong V_i(\text{Pic}^0(A))(-1),
$$

where $\text{Pic}^0(A)$ is the dual abelian variety of $A$, $V_i(\text{Pic}^0(A)) = T_i(\text{Pic}^0(A)) \otimes_{\mathbb{Z}} Q_l$ with $T_i$ the usual l-adic Tate module, and $(-1)$ is the Tate twisting. On the other hand, Weil's $e_{\mathcal{L}}$-pairings induce a canonical non-degenerate pairing

$$
V_i(A) \times V_i(\text{Pic}^0(A))(-1) \to Q_l.
$$

Therefore the representation of $\text{Gal}(\overline{L}/L)$ on $\text{Aut}_{Q_l}(R^1f_*\mathcal{L}(Q))$, attached to $A$, is equivalent to the one which is contragredient to the natural l-adic representation $\psi'$: $\text{Gal}(\overline{L}/L) \to \text{Aut}_{Q_l}(V_i(A))$ attached to $A$. But the quaternion algebra $C$
acts on $V_i(A_i)$ on the left, and $V_i(A_s)$ is isomorphic to $(C \otimes Q_1)^{q_s-1}$ (row vectors) as left $C$-modules (cf. 2.3). Put $C_i = C \otimes Q_1$ and $q = 2^{q_s-1}$ for simplicity. Since the actions of $C$ and $\text{Gal}(\bar{L}/L)$ on $V_i(A_s)$ commute, $\psi'$ in fact factors through $\text{Aut}_C(V_i(A_s))$, the group of automorphisms of $V_i(A_s)$ which commute with the action of $C$. We have thus obtained a representation

$$\psi: \text{Gal}(\bar{L}/L) \rightarrow \text{Aut}_C(V_i(A_s)) \cong GL_q(C)$$

from $\psi'$. We recall that $\text{Gal}(\bar{L}/L)$ acts on $L$ on the right, and that $\psi(a) \in GL_q(C)$ acts on $C^\dagger$ (row vectors) on the right.

To conclude the proof of (3.3.3), we need the following result which is essentially due to Shimura:

**Theorem (3.5.4) (Shimura [18] § 8).** The notation and the assumption being as above, $\psi$ factors through $\text{Gal}(L/L)$, and the associated representation $\tilde{\psi}$ of $S_i$ factors through $S_i$. The representation $\tilde{\psi}$ of $S_i$ is equivalent to the representation $\psi(1)_{q_s}$, where $\psi(1)$ is as in (3.3.1).

**Proof.** Take a point $z \in H = \mathfrak{H}$ so that $c_{ps}(z) \in V_s(C)$ is generic, i.e. $\dim_{k_s}k_s(c_{ps}(z)) = 1$. Then for any $T \in \mathfrak{H}$ which is contained in $S$, $c_{ps}(z)$ is also generic. Hence we obtain a natural isomorphism: $L_T \rightarrow k_T(c_{ps}(z))$ which sends $f \in L_T$ to $f(c_{ps}(z))$ for each $T$ as above. When $T$ varies, the above isomorphisms are compatible in the obvious sense, and induce an isomorphism of $L$ onto the union of all $k_T(c_{ps}(z))$ with $T \in \mathfrak{H}$ and $T \subseteq S$. The latter field is the one which was denoted by $\mathfrak{H}_u$ with $u = c_{ps}(z)$ in [18] § 7. Now fix $z \in H$ as above, and identify $L_s$ (resp. $L$) with $k_s(c_{ps}(z)) = k_s(u)$ (resp. $k_s$) by means of the above isomorphisms, and hence $\text{Gal}(L/L)$ with $\text{Gal}(k_s/k_s(u))$. Call $i$ the isomorphism of $\text{Gal}(L/L)$ onto $\text{Gal}(k_s/k_s(u))$ thus obtained.

By [18] 7.20 and (8.13.1), we know that the map $\psi$ factors through $\text{Gal}(L/L)$, and moreover that the map $\psi \circ i^{-1}: \text{Gal}(k_s/k_s(u)) \rightarrow GL_q(C)$ is the composite of the three maps: (i) $h: \text{Gal}(k_s/k_s(u)) \rightarrow S_0/E_0$ which is defined in [18] 7.8, (ii) $\psi(1)_0: S_0/E_0 \rightarrow GL_q(C)$ ($= \text{the finite part of } GL_q(C)$), and (iii) the projection: $GL_q(C) \rightarrow GL_q(C)$). (In [18] §§ 7–8, it is assumed that $c_{ps}(z) = u$ is not generic. But the same proof works for generic $u$.) Our conclusion will then follow from an easy observation that the map $h \circ i$ and the map (2.1.6) are inverse to each other, which follows from the definitions. Q.E.D.

Finally, consider a non-degenerate $Q_1$-bilinear pairing of $C_i$ (row vectors) and $C_i^\dagger$ (column vectors) defined by $((x_1, \ldots, x_q), (y_1, \ldots, y_q)) = tr(\sum_{i=1}^{q_s} x_i y_i^*)$, where $tr$ is the reduced trace of $C_i$ over $Q_1$. Then it is obvious that $(x \psi(1)_{q_s}(s), \psi(1)_{q_s}(s^{-1})y) = (x, y)$ for all $x \in C_i$ (row vectors), $y \in C_i^\dagger$ (column vectors), and $s \in S_i$. This shows that the representations of $\text{Gal}(\bar{L}/L)$ on
The zeta function of $A^k$

In this section, we will prove our main result (4.5.3) of this paper. Throughout this section, we again assume that $B$ is a division quaternion algebra over $F$ with $r=1$, and that $g$ is odd. $G$ is the algebraic group over $Q$ such that $G_0 = B^\times$.

4.1. We first recall general facts about the cohomology groups of abelian schemes. Let $V$ be a noetherian scheme, and $f : A \to V$ an abelian scheme over $V$. We fix a rational prime $l$ which is invertible in $V$.

**Lemma (4.1.1) (Deligne [5]).** With the above notation, we have a canonical isomorphism:

$$H^i(A, Q_\ell) \cong \bigoplus_{p \neq \ell} H^i(V, R^{q-p}f_* Q_\ell)$$

for each $i$.

**Proof.** This is contained in the proof of [5] Lemma 5.3. Q.E.D.

The following fact may be well known.

**Lemma (4.1.2).** With the above notation, we have a canonical isomorphism:

$$R^qf_* Q_\ell \cong \bigotimes_{q \geq 0} R^qf_* Q_\ell$$

of $Q_\ell$-sheaves on $V_{\text{et}}$ for each $q \geq 0$.

**Proof.** The cup product defines a morphism: $\bigotimes_{q \geq 0} R^qf_* Q_\ell \to R^qf_* Q_\ell$. By the anticommutativity of the cup product, this factors through $\bigotimes_{q \geq 0} R^qf_* Q_\ell$, and hence induces a morphism $I: \bigotimes_{q \geq 0} R^qf_* Q_\ell \to R^qf_* Q_\ell$. To prove that $I$ is an isomorphism, it is enough to show that each fibre of $I$ at a geometric point of $V$ is an isomorphism, because $V$ is noetherian (cf. [9] VI 1.2.6). Hence it is enough to prove the assertion for an abelian variety over an algebraically closed field. But this is known (Kleiman [11] Th. 2A8).

Q.E.D.

Let $A^k$ be the $k$-fold fibre product of $A$ over $V(k \geq 1)$, and $f^* : A^k \to V$ the structure morphism. Then the Künneth formula gives a canonical isomorphism
(4.1.3) \[ R^p f^*_k (Q_i) \cong \bigoplus_{[p_i]} \bigotimes_{i=1}^k R^p f_{[p_i]} (Q_i) \]

where the sum in the right hand side ranges over all the \( k \)-tuples \( [p_i] \) of non-negative integers satisfying \( p_1 + \cdots + p_k = p \). Noting that \( f_{[p]} (Q_i) \cong Q_i \), we especially obtain

(4.1.4) \[ R^p f^*_k (Q_i) \cong \bigoplus \bigotimes^k R^p f_{[p_i]} (Q_i) \]

4.2. Let \( K \) be a field of characteristic zero. It is known that any \( K \)-rational representation of \( GL_2 (K) \) is completely reducible. It is also known that any irreducible \( K \)-rational representation of \( GL_2 (K) \) is isomorphic to the one which maps each \( A \in GL_2 (K) \) to \( \det (A)^a \rho_k (A) \) with an integer \( a \) and a non-negative integer \( k \). The above representation is a polynomial representation if and only if \( a \geq 0 \).

Now let \( H = GL_2 (K)^g \), and define a \( K \)-rational representation \( \varphi (1) \) of \( H \) by

(4.2.1) \[ \varphi (1)( (A_1, \cdots, A_g) ) = \bigotimes_{n=1}^g \rho_i (A_n) \]

for \( (A_1, \cdots, A_g) \in GL_2 (K)^g \). For non-negative integers \( k_1, \cdots, k_g \) and \( w \) satisfying (3.2.1), we define a \( K \)-rational representation \( \varphi ([k_n]; w) \) of \( H \) by

(4.2.2) \[ \varphi ([k_n]; w)( (A_1, \cdots, A_g) ) = \bigotimes_{n=1}^g (\det (A_n)^{w - k_n}/\rho_{k_n} (A_n)) \]

for \( (A_1, \cdots, A_g) \in H \).

**Lemma (4.2.3).** Suppose that a non-negative integer \( w \) and a positive integer \( k \) are given. Then there exists a non-negative integer \( a(k; [k_n]; w) \) for each \( g \)-tuple \( [k_n] \) of non-negative integers satisfying (3.2.1) for the above \( w \), and the representation \( \bigotimes_{n=1}^g (\bigoplus_{[k_n]} \varphi (1)) \) is \( K \)-equivalent to \( \bigoplus_{[k_n]} \varphi ([k_n]; w) \bigotimes_{[k_n]} \rho_{k_n} (A_n) \), where the sum in the right hand side ranges over all \( [k_n] \) satisfying (3.2.1).

**Proof.** Obvious from the above considerations. Q.E.D.

**Remark (4.2.4).** When \( g = 1 \), \( a(k; [k_n]; w) \) are explicitly calculable (cf. Kuga [12] IV–2–1). When \( g \geq 2 \), we do not know the explicit formula of \( a(k; [k_n]; w) \), and take the above lemma for their definition.

4.3. We now return to the situation considered in § 3. As before, we fix a member \( S \) of \( \mathcal{Z} \) with respect to \( G \), which satisfies (2.3.2) and (3.2.4). Let \( f : A \rightarrow V_g \) be the abelian scheme defined in (2.3.3). We also fix a rational prime \( l \) and a prime \( \mathfrak{l} \) above \( l \) in the field \( K (3.2) \).
An abelian scheme over the Shimura curve

PROPOSITION (4.3.1). For a non-negative integer \( w \) and a positive integer \( k \), the \( K \)-sheaf \( R^w f_*(Q_\eta) \otimes \eta K \) on \( V_s, \text{et} \) is isomorphic to \( \bigoplus_{\{k_i\}} (\mathcal{F}_s([k_i]; w; l) \otimes \eta \eta K)^{\oplus (3k; [k_i]; w)} \), where \( b(2k; [k_i]; w) = a(2k; [k_i]; w) / s([k_i]; w) \) (cf. 3.2), and the sum ranges over all the \( g \)-tuples \( \{k_1, \ldots, k_g\} = \{k_i\} \) of non-negative integers satisfying (3.2.1) for the above \( w \), taken modulo \( g \); i.e. modulo the equivalence relation defined by: \( \{k_i\} \sim \{k'_i\} \) if and only if \( k'_i = k_{i - 1} \) \((1 \leq n \leq g)\) for some \( \sigma \in \Gamma \).

PROOF. Since \( R^w f_*(Q_\eta) \otimes \eta K \) is twisted constant by [2] XVI 2.1, it is enough to show that the associated \( \text{Gal}(L_s/L_s) \)-modules are isomorphic ((3.1.2)). By (4.1.2), (4.1.4), (3.1.4) and (3.1.7), the \( \text{Gal}(L_s/L_s) \)-module \( (R^w f_*(Q_\eta) \otimes \eta K) \otimes \eta \eta K \) is isomorphic to \( \bigwedge^w (\bigoplus (\mathcal{F}_s(\rho(1); l) \otimes \eta \eta K)) \). This latter module is isomorphic to \( \bigwedge^w (\bigoplus (\mathcal{F}_s(\rho(1); l) \otimes \eta \eta K)) \) by (3.3.3). Call \( \varphi \) the natural representation: \( \text{Gal}(L_s/L_s) \to \text{Aut}_{K_1}(\bigwedge^w (\bigoplus (\mathcal{F}_s(\rho(1); l) \otimes \eta \eta K))) \). Then \( \varphi \) factors through \( \text{Gal}(L/L_s) \), and we obtain a representation \( \varphi \) of \( S_0 \). Also \( \varphi \) factors through the projection \( p_t : S_t \to S_s \), and we obtain a representation \( \varphi_t \) of \( S_t \) into \( \text{Aut}_{K_1}(\bigwedge^w (\bigoplus (\mathcal{F}_s(\rho(1); l) \otimes \eta \eta K)) \) (cf. 3.4). But \( S_t \) is a subgroup of \( (B \otimes \eta K)^n \cong GL_n(K)^n \), and, considering \( S_t \) as a subgroup of \( GL_n(K)^n \), \( \varphi_t \) is isomorphic to the restriction of the representation:

\[
GL_n(K)^n \ni s \mapsto \bigwedge^w (\bigoplus (\mathcal{F}_s(\rho(1); l) \otimes \eta \eta K))
\]

((3.4.4)). Noting that \( a(2k; [k_i]; w) \) is even if \( w \) is odd, we conclude, by (4.2.3) and (3.4.4), that the representations of \( \text{Gal}(L_s/L_s) \) on the \( K \)-vector spaces

\[
\bigwedge^w (\bigoplus (\mathcal{F}_s(\rho(1); l) \otimes \eta \eta K)) \quad \text{and} \quad \bigoplus_{\{k_i\}} (\mathcal{F}_s([k_i]; w; l) \otimes \eta \eta K)^{\oplus (3k; [k_i]; w)}
\]

are equivalent.

Q.E.D.

4.4. As in [14], we put \( V_s = V_\eta \otimes_F \eta \) and \( \bar{A}^k = A^k \otimes_F \eta \) \((k \geq 1)\), considering \( V_s \) and \( A^k \) as \( F \)-schemes via the inclusion of \( F \) into \( k_\eta \). Now by (4.1.1) and [2] X 4.3, we have a canonical isomorphism

\[
H^i(\bar{A}, Q_\eta) \cong H^i(\bar{V}_s, R^i f_*(Q_\eta)) \oplus H^i(V_s, R^{i-1} f_*(Q_\eta)) \oplus H^i(\bar{V}_s, R^{i-1} f_*(Q_\eta))
\]

for each \( i \); here we understand that \( R^j f_* = 0 \) if \( j < 0 \). In this subsection, we will compute the \( H^n \)- and \( H^2 \)-terms in the above sum.

PROPOSITION (4.4.2). Let \( k_1, \ldots, k_g \) and \( w \) be non-negative integers satisfying (3.2.1). Then \( H^i(V_s, \mathcal{F}_s([k_i]; w; l)) = 0 \) unless \( k_1 = \cdots = k_g = 0 \). If \( k_1 = \cdots = k_g = 0 \), then \( H^i(V_s, \mathcal{F}_s([0]; w; l)) \) is canonically isomorphic to
$Q[[\text{Gal}(k_s/F)](-w/2)$, where $Q[[\text{Gal}(k_s/F)]$ is the group ring of $\text{Gal}(k_s/F)$ over $Q$.

**Proof.** Take a $Z$-lattice $X$ in the representation space of $\rho(k_s); w$ satisfying the condition [14] (2.1.6). Then by the same argument as in [14] 2.5, we see that $H^i(\overline{V}_s, \mathcal{F}_s([k_s]; w; l)) \cong \bigoplus_{i=1}^n (X \otimes \mathbb{Q})^n$, with the notation of loc cit. Since $\Gamma_{s_0)|(S_i \cap F^*)$ is commensurable with an arithmetic subgroup of $B^*|\{\pm 1\}$, with $B^* = \{\alpha \in B^*|\nu(\alpha) = 1\}$ (cf. e.g. [14] (3.2.3)), we conclude that $H^i(\overline{V}_s, \mathcal{F}_s([k_s]; w; l))$ vanishes unless all the $k_s$ are zero, by a theorem of Borel ([3] Th. 1). If $k_1 = \cdots = k_s = 0$, we have $\rho([0]; w) = N_{F/F}(\nu(\alpha))^{w/2}$ for all $\alpha \in B^*$, and hence $\mathcal{F}_s([0]; w; l) \cong Q(-w/2)$ by [14] (4.1.8). Thus we obtain canonical isomorphisms:

$$H^i(\overline{V}_s, \mathcal{F}_s([0]; w; l)) \cong H^j(\overline{V}_s, Q)\cong (-w/2) \cong \bigoplus_{i=1}^n H^j(\overline{V}_s^{\times 2} \otimes \mathbb{Q}, Q)\cong (-w/2).$$

Q.E.D.

**Corollary (4.4.3).** If $w$ is odd, the cohomology group $H^i(\overline{V}_s, \mathcal{F}_s([Q]))$ vanishes. If $w$ is even, $H^i(\overline{V}_s, \mathcal{F}_s([Q])) \otimes Q$ is isomorphic to $K_i[\text{Gal}(k_s/F)](-w/2)$ as $\text{Gal}(F/F)$-modules.

**Proof.** This follows from (4.4.2) and (4.3.1). Q.E.D.

We next compute the 2nd cohomology groups.

**Proposition (4.4.4).** The notation and the assumption being as in (4.4.2), we have $H^i(\overline{V}_s, \mathcal{F}_s([k_s]; w; l)) = 0$ unless $k_1 = \cdots = k_s = 0$. If $k_1 = \cdots = k_s = 0$, then $H^i(\overline{V}_s, \mathcal{F}_s([0]; w; l))$ is isomorphic to $Q_i[\text{Gal}(k_s/F)](-1-w/2)$ as $\text{Gal}(F/F)$-modules.

**Proof.** We have a canonical non-degenerate $Q$-bilinear pairing

$$H^i(\overline{V}_s, \mathcal{F}_s([k_s]; w; l)) \times H^j(\overline{V}_s, \mathcal{F}_s([k_s]; w; l)) \longrightarrow Q(-w-1)$$

by the Poincaré duality theorem (cf. [14] 4.2). Our assertion follows immediately from this and (4.4.2). Q.E.D.

**Corollary (4.4.5).** If $w$ is odd, the cohomology group $H^i(\overline{V}_s, \mathcal{F}_s([Q]))$ vanishes. If $w$ is even, $H^i(\overline{V}_s, \mathcal{F}_s([Q])) \otimes Q$ is isomorphic to $K_i[\text{Gal}(K_s/F)](-1-w/2)$ as $\text{Gal}(F/F)$-modules.

**Proof.** This follows from (4.4.4) and (4.3.1). Q.E.D.

4.5. In general, let $k$ be an algebraic number field of finite degree, and $X$ a projective smooth scheme over $k$ of dimension $d$. For a finite prime $\mathfrak{p}$ of $k$, let $D_{\mathfrak{p}}$ (resp. $L_\mathfrak{p}$) be a decomposition subgroup of $\text{Gal}(\overline{k}/k)$ (resp. the
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Inertia subgroup of $D_\nu$ attached to $p$, and let $\sigma_\nu \in D_\nu$ be a Frobenius element. Following Serre [15], we put

\begin{equation}
Z_{\nu}(s; X/k) = \det(1 - N_{v'/v}(\psi)^{-1}\sigma_\nu^{-1})H'(X \otimes_{k_\nu} \bar{k}, Q_i)^{(s)}
\end{equation}

with a prime number $l$ which is prime to $p$, for $0 \leq i \leq 2d$. We also put

\begin{equation}
Z^{(i)}(s; X/k) = \prod Z_{\nu}^{(i)}(s; X/k)
\end{equation}

where the product ranges over all the finite primes of $k$. $Z_{\nu}^{(i)}(s; X/k)$ is known to be independent of the choice of $l$ (at least) when $X$ has good reduction at $\nu$ (Deligne [6]), or for $i = 0, 1, 2d - 1, 2d$. If $k'$ is a subfield of $k$, and if we consider $X$ as a $k'$-scheme, then we easily see that the Gal($\bar{k}/k'$)-module $H'(X \otimes_{k'} \bar{k}, Q_i)$ is isomorphic to the one which is induced from the Gal($\bar{k}/k$)-module $H'(X \otimes_{k} \bar{k}, Q_i)$. Hence (at least) up to finitely many Euler factors, $Z^{(i)}(s; X/k)$ is independent of $k$ (cf. Yoshida [20] Prop. 1).

We can now state and prove our main result:

**Theorem (4.5.3).** Suppose that $[F:Q] = g$ is odd, and let $S$ be a member of $\mathcal{X}$ which is stable under the canonical involution, and which satisfies the conditions (2.3.2) and (3.2.4). Let $f: A \to V_s$ be the abelian scheme defined in (2.3.3), and $A^k$ the $k$-fold fibre product of $A$ over $V_s(k > 1)$. Then for any $i$ ($0 \leq i \leq \dim A^k = 2(2g + 1)$) and any prime $p$ of $F$ which is prime to $D(B/F)L(S)$, $Z_{\nu}^{(i)}(s; A^k/F)$ is defined independently of the choice of $l$, and the following equalities hold.

(i) $Z^{(i)}(s; A^k/F) = Z^{(i)}(s; A^k/F) = Z(s; k) = \zeta(s; k)$, the Dedekind zeta function of $k$.

(ii) $Z^{(i)}(s; A^k/F) = \prod Z(s; S; \{k_n\}; i - 1)n(2k; \{k_n\}; i - 1)$ (cf. (1.3.5)) if $i$ is odd, where the product ranges over all the $g$-tuples $\{k_n\}$ satisfying $k_1 \equiv \cdots \equiv k_g \equiv i - 1 \pmod{2}$ and $0 < k_n < i - 1 (1 \leq n \leq g)$.

(iii) $Z^{(i)}(s; A^k/F) = \prod Z(s; S; \{k_n\}; i - 1)n(2k; \{k_n\}; i - 1, \zeta(s - i/2; k)\zeta(2s; \{k_n\}; i - 1))$ if $i$ is even and $\geq 2$, where the product has the same meaning as in (ii).

**Proof.** First note that the zeta function attached to the system of $l$-adic representations: Gal($\overline{F}/F$) $\to$ Aut$_Q(Q_t[Gal(k_s/F)])$ is $\zeta(s; k_s)$, since $Q_t[Gal(k_s/F)]$ is the Gal($\overline{F}/F$)-module which is induced from the trivial representation of Gal($\overline{F}/k_s$). For a finite prime $\nu$ of $F$ which is prime to $D(B/F)L(S)$, take a prime number $l$, which is prime to $\nu$, and a prime $\mathfrak{p}$ above $l$ in the field $K$. Then by (4.3.1), $H'(V_s, \mathcal{F}_s(k)[\mathfrak{p}]_{i - 1}; I) \otimes_{Q_t} K_t$ is isomorphic to $H'(V_s, \mathcal{F}_s(k)[\mathfrak{p}]_{i - 1}; I) \otimes_{Q_t} K_t$. By the main results (4.3.1) and (5.4.6) of [14], this Gal($\overline{F}/F$)-module is unramified at $\nu$, and the characteristic polynomial of $\sigma_\nu^{-1}$ on it is equal to $\prod H_i(T; \mathcal{E}(S; \rho(\{k_n\}; i - 1)))^{1/2|\{k_n\}; i - 1}$.
with the notation of loc. cit.. It is easy to see that this is equal to \[ \prod H_x(T; \mathbb{Q}(S; \{ k_s \}; i-1))^{o_{(S; \{ k_s \}; k^-1)}} \], where the product has the same meaning as in our assertions (ii) and (iii) above. Our assertion follows from what we have said above, (4.4.1), (4.4.3) and (4.4.5).

Q.E.D.

**Remark (4.5.4).** In the above discussions, we assumed that \( k \geq 1 \), to avoid confusions. If \( k = 0 \), i.e., \( A^s = V_s \), the zeta function of \( V_s \) is known, and easily determined from the above discussions. The result is that \( Z^{(v)}(s; V_s/F) = \zeta(s; k) \), \( Z^{(w)}(s; V_s/F) = Z(s; S; \{ 0 \}; 0) \) and \( Z^{(z)}(s; V_s/F) = \zeta(s-1; k) \), up to the Euler factors at \( p \) which divide \( D(B/F)L(S) \).

**4.6.** As an application of the above result, we prove here the assertion (1.4.1) for division quaternion algebras \( B \) over \( F \) with \( r = 1 \). In view of (1.3.6), the validity of (1.4.1) for one \( w \) implies (1.4.1) for general \( w \). We may therefore take \( w \) so that (3.2.1) is satisfied. For \( S \in \mathcal{Z} \), we can find a subgroup \( S' \subseteq \mathcal{Z} \) of \( S \) which is stable under the canonical involution, and which satisfies (2.3.2) and (3.2.4). For a prime \( p \) of \( F \) not dividing \( D(B/F)L(S') \), we know that the Hecke polynomial \( H_x(T; \mathbb{Q}(S; \{ k_s \}; w)) \) divides \( H_x(T; \mathbb{Q}(S'; \{ k_s \}; w)) \) (cf. [14] (1.6.2)), and hence we may and do assume that \( S \) itself satisfies the assumption of (4.5.3) to prove (1.4.1).

It is easy to see that there is a finite set \( P(S) \) of finite primes of \( F \) which contains the prime factors of \( D(B/F)L(S) \), and for which the following assertion holds for all \( \mathfrak{p} \in P(S) \):

\[(4.6.1) \quad \text{There exists a projective smooth curve } V_{s_x} \text{ over } \mathbb{F}_x, \text{ and a projective abelian scheme } f_x : A_x \to V_{s_x}, \text{ of relative dimension } 2g \text{ such that the base change of } f_x \text{ from } \mathbb{F}_x \text{ to } \mathbb{F}_x \text{ is isomorphic to the base change of } f : A \to V_s \text{ from } F \text{ to } \mathbb{F}_x. \]

Take a prime \( \mathfrak{p} \in P(S) \), and let \( k(\mathfrak{p}) \) be the residue field of \( \mathfrak{p} \). Then it is known that the characteristic polynomial of \( \sigma^{-1}_x \) on \( H^i(A_k, \mathbb{Q}_l) \) is equal to the characteristic polynomial of the geometric Frobenius relative to \( k(\mathfrak{p}) \) acting on \( H^i(A_k \otimes_{k(\mathfrak{p})} k(\mathfrak{p}), \mathbb{Q}_l) \), where \( l \) is a prime number which is prime to \( \mathfrak{p} \), and \( A_k \) is the \( k \)-fold fibre product of \( A_{\mathfrak{p}} \) over \( V_{s_x} \) (cf. [2] XVI 2.1). The assertion (1.4.1) is therefore a consequence of the Weil conjecture (Deligne's theorem [6] 1.6) combined with (4.5.3) and the following

**Lemma (4.6.2).** Let \( \{ k_s \} \) and \( w \) be non-negative integers satisfying (3.2.1) with \( w \geq 1 \). Then the integer \( a(2w; \{ k_s \}; w) \) is not zero.

**Proof.** We want to show that the representation \( \bigwedge^w \left( \bigoplus \varphi(1) \right) \) of \( GL_2(K)^w \) contains \( \varphi([k_s]; w) \) as a direct factor. It is easy to see that the former representation contains \( \bigotimes \varphi(1) \) as a direct factor, which is equivalent to the
representation that maps an element \((A_1, \ldots, A_g) \in GL_2(K)^g\) to \((\otimes \rho_i(A_i)) \otimes \ldots \otimes (\otimes \rho_i(A_j))\). But it is also easy to see that the representation \(\otimes \rho_i\) of \(GL_2(K)\) contains the representation: \(GL_2(K) \ni A \mapsto \det(A)^{(w-k)/2} \rho_k(A)\) as a direct factor for each integer \(k\) satisfying \(k \equiv w \pmod{2}\) and \(0 \leq k \leq w\).

Q.E.D.

§ 5. Reformulation in terms of automorphic representations

In this final section, we will reformulate the assertion (1.4.1) in terms of automorphic representations, and then extend the result in 4.6 for arbitrary quaternion algebras (over a totally real number field of odd degree), using the correspondence of Jacquet and Langlands [10].

5.1. We first explain the relation between classical automorphic forms and automorphic representations. This must be well-known to specialists, and is actually given in several literatures, at least in the elliptic modular case (cf. e.g. Casselman [4]). However, in the general case, the author could not find a better reference than master’s thesis of Takagi [19], which was not published. Thus in 5.1-5.3, we explicitly write down the correspondence for the sake of the convenience of the reader and the author, following the method of Takagi. For the proof, we refer to [19], or otherwise, the proof is direct.

Now let us return to the situation considered in § 1. Thus we fix a totally real field \(F\), a quaternion algebra \(B\) over \(F\) with \(r > 0\), and the data \(\{k_n\}\) and \(w\) as in 1.1. In this section, we take the representatives \(\{x_i\}\) and \(\{y_i\}\) in (1.2.3) and (1.2.4) from \(B^{r+1}\). Let \(W(\leq C^I\) with \(\lambda = \prod_{n=1}^{g} (k_n + 1)\)) be the representation space of \(\mathbb{F}((k_n); w)\). For an element \(f = (f_i) \in \mathbb{S}(S; \{k_n\}; w)\) for \(B\) (cf. (1.2.7)), we first define a function \(\varphi_f\) on \(B^{r+1}\) with values in \(W\) as follows.

**DEFINITION (5.1.1).** Let \(g = ay \in (a \in B^x, s \in S; 1 \leq i \leq h; \text{cf. (1.2.4)})\) be an element of \(B^{r+1}\). Then we put \(\varphi_f(g) = \varphi_f(ay)(i)\), where \(i = (\sqrt{-1}, \ldots, \sqrt{-1}) \in \mathbb{H}^{r}\) (cf. (1.2.1)).

It is easy to see that \(\varphi_f\) is well-defined. We moreover have the following

**PROPOSITION (5.1.2).** The map which sends \(f \in \mathbb{S}(S; \{k_n\}; w)\) to \(\varphi_f\) gives a bijective \(C\)-linear map from \(\mathbb{S}(S; \{k_n\}; w)\) to the space of continuous functions \(\varphi\) on \(B^{r+1}\) with values in \(W\) which satisfy the following six conditions:

(i) \(\varphi\) is left \(B^x\)-invariant.

(ii) \(\varphi\) is right \(S_i\)-invariant.

(iii) For any \(a \in F^{r+1}_x \cap S\) and \(g \in B_x^{r+1}\), \(\varphi(ga) = \prod_{n=1}^{g} (a_{x_n}^{-w} \text{sgn}(a_{x_n})^{k_n}) \varphi(g)\).
where $a_{vn}$ is the $v_n$-component of $a \in F^n$.

(iv) For any $c \in \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \in GL_n(\mathbb{R}) \mid a^2 + b^2 = 1 \right\} \times (H^*)^{r-r} \subset B^{\infty}_{vn}$ and $g \in B^{\infty}_{\mathbb{A}}$, $\varphi(gc) = j(c, i)^{-1} \Psi(c^{-1}) \varphi(g)$.

(v) Let $V_n = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \sqrt{-1} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ be an element of the complexified Lie algebra of $SL_2(\mathbb{R})$, and let $V_{n-1}$ be an element of $Lie(SL_2(\mathbb{R})^r) \otimes C = \mathfrak{g}(C)^r$ whose $n$-th component is $V_n$ and whose other components are all equal to zero. Then $\varphi$ is $C^n$ with respect to the coordinates of $GL_n(F_{vn}) \subset B^{\infty}_{vn}$, and is annihilated by $\rho(V_{n-1}) (1 \leq n \leq r)$.

(vi) When $B = M_2(F)$, $\varphi$ is slowly increasing and cuspidal in the sense of [10] § 10.

5.2. Let $W^* = Hom(W, C)$ be the dual space of $W$, and $\varphi^*$ the representation of $B^*_\infty$ on $W^*$ contragredient to $\varphi$; $(\varphi^*(\alpha)x, \varphi^*(\alpha)y) = (x, y)$ for all $x \in W$, $y \in W^*$ and $\alpha \in B^*_\infty$, if we denote by $\langle, \rangle$ the natural pairing of $W$ and $W^*$.

Following Takagi [19], we fix a non-zero element $L$ of $W^*$, and consider the following space of functions on $B^*_\infty$:

$$(5.2.1) \quad \mathfrak{U}_L(S; \{k_n\}; w) = \{L \circ \varphi_f | f \in \mathfrak{S}(S; \{k_n\}; w)\}.$$ 

It is easy to see that the map $f \mapsto L \circ \varphi_f$ gives a $C$-linear isomorphism of $\mathfrak{S}(S; \{k_n\}; w)$ onto $\mathfrak{U}_L(S; \{k_n\}; w)$. We call this map $I_L$.

$$(5.2.2) \quad I_L : \mathfrak{S}(S; \{k_n\}; w) \rightarrow \mathfrak{U}_L(S; \{k_n\}; w).$$

**Proposition (5.2.3).** $\mathfrak{U}_L(S; \{k_n\}; w)$ is contained in the space $\mathcal{A}_G(B^*_\infty)$ of cusp form on $B^*_\infty$. (For the sake of simplicity, we use the same symbol $\mathcal{A}_G(B^*_\infty)$ to denote the space of automorphic forms on $B^*_\infty$ when $B$ is a division algebra.)

For an irreducible constituent $\pi = \otimes_v \pi_v$ of $\mathcal{A}_G(B^*_\infty)$, let $V(\pi) = \otimes_v V(\pi)_v$ be its representation space. Then $\mathfrak{U}_L(S; \{k_n\}; w) \cap V(\pi) \neq \{0\}$ if and only if $\pi$ satisfies the following four conditions:

(i) If we denote by $\omega$ the central quasi-character of $\pi$, then $\omega(a) = \prod_{n=1}^{r} (a_{vn})^{-w} \text{sgn}(a_{vn})^{k_n}$ for all $a \in F^n \cap S$.

(ii) For $1 \leq n \leq r$, $\pi_v$ is isomorphic to the discrete series representation $\sigma(\mu_v, \mu_v)$ with $\mu_v(t) = |t|^{-k_v - 1 - \frac{w}{2}} \text{sgn}(t)^{k_v}$ and $\mu_v(t) = |t|^{-k_v - 1 - \frac{w}{2}}$.

(iii) For $r + 1 \leq n \leq g$, $\pi_v$ is isomorphic to the finite dimensional representation which maps $x \in B^*_\infty$ (identifiable $GL_n(C)$) to $\det(x)^{-k_v - n + w/2} \rho_{vn}(x)$.

(iv) There exists a non-zero vector $x \in \bigotimes_{n=r+1}^{g} V(\pi)_v$ which is fixed under $S_v$.

Notice that when $\pi$ satisfies the condition (iii) above, $\bigotimes_{n=r+1}^{g} V(\pi)_v$ is isomorphic to $W^*$ as a representation space of $(H^*)^{r-r} \subset B^{\infty}_{vn}$. In the following, we fix such an isomorphism and identify these two spaces. On the other hand, if $\pi_v$ is a representation as in (ii) above, there is a vector $\varphi_{k_v+1}$ in...
V(\tau)_{\text{vs}} which is annihilated by V. (the holomorphic vector), which is unique up to scalar multiples. With these terminologies, \( \mathcal{U}(S; \{k_n\}; w) \cap V(\tau) \) is characterized as follows.

**Proposition (5.2.4).** The notation being as above, let \( \pi \) be an irreducible constituent of \( \mathcal{A}(B^\vee) \) satisfying the four conditions in (5.2.3). Then we have
\[
\mathcal{U}_L(S; \{k_n\}; w) \cap V(\tau) = (\bigotimes_{n=1}^{r} \varphi_{k_n+1}) \otimes L \otimes (\bigotimes_{\text{finite}} V(\tau))_{\text{vs}}.
\]

5.3. It remains to describe Hecke operators on \( \mathcal{U}_L(S; \{k_n\}; w) \). Let \( x \) be an element of \( V(S) \) (cf. (1.3.1)), and let \( SxS = \bigcup_{\tau} \xi_\tau S \) be a disjoint decomposition.

**Proposition (5.3.1).** The notation being as above, take the representatives \( \xi_\tau \) from \( B^\vee_\tau \). Let \( \mathcal{X}(SxS) \) be an endomorphism of \( \mathcal{U}_L(S; \{k_n\}; w) \) defined by:
\[
(\mathcal{X}(SxS)\varphi)(g) = \sum_\tau \varphi(g_{\xi_\tau}) \text{ for } \varphi \in \mathcal{U}_L(S; \{k_n\}; w).
\]
Then we have \( I_L \circ \mathcal{X}(SxS) = \mathcal{X}(SxS) \circ I_L \) on \( \mathcal{U}(S; \{k_n\}; w) \).

For a finite prime \( p \) of \( F \) which is prime to \( D(B/F)L(S) \), we can define endomorphisms \( \mathcal{X}(p) \) and \( \mathcal{X}(p, p) \) on \( \mathcal{U}_L(S; \{k_n\}; w) \) just as we defined \( \mathcal{X}(p) \) and \( \mathcal{X}(p, p) \) (cf. 1.3).

**Proposition (5.3.2).** Let \( \pi = \bigotimes_\tau \pi_\tau \) be an irreducible constituent of \( \mathcal{A}(B^\vee) \) satisfying the four conditions in (5.2.3). Then for each finite prime \( p \) of \( F \) which is prime to \( D(B/F)L(S) \), \( \pi_\tau \) is an unramified principal series representation. If \( \pi_\tau \cong \pi(\mu, \mu) \), then the operator \( \mathcal{X}(p) \) (resp. \( \mathcal{X}(p, p) \)) coincides with the multiplication by \( N_{F/Q}(\mu)^{(2)}(\mu(\sigma_\tau) + \mu(\sigma_\tau)) \) (resp. \( \mu(\sigma_\tau)\mu(\sigma_\tau) \)) on \( \mathcal{U}_L(S; \{k_n\}; w) \cap V(\tau) \), where \( \sigma_\tau \) is a prime element in \( F_\tau \).

5.4. The following theorem is a representation theoretic reformulation of (1.4.1):

**Theorem (5.4.1).** Let \( F \) be a totally real number field of odd degree \( g \) over \( Q \), and \( B \) a quaternion algebra over \( F \). For infinite primes \( v_1, \ldots, v_g \) of \( F \), we assume that \( B_{v_n} \) are isomorphic to \( M_2(R) \) (resp. \( H \)) for \( n < r \) (resp. \( n > r+1 \)). Let \( \{k_1, \ldots, k_g\} \) be a set of non-negative integers which are congruent mod 2, and let \( w \) be a real number. Let \( \pi = \bigotimes_\tau \pi_\tau \) be an infinite dimensional irreducible constituent of \( \mathcal{A}(B^\vee) \) satisfying:

(i) For \( n \leq r \), \( \pi_{v_n} \cong \sigma(\mu, \mu) \) with \( \mu(t) = |t|^{(k_n+1-w)/2} \text{sgn}(t)^{k_n} \) and \( \mu(t) = |t|^{(-k_n-1-w)/2} \).

(ii) For \( n > r+1 \), \( \pi_{v_n}(x) = \text{det}(x)^{-(k_n+w)/2} \rho_{v_n}(x) \) for all \( x \in B_{v_n}^\times \cong H_\times^\times \). Then for almost all finite primes \( p \) of \( F \) which are prime to \( D(B/F) \), \( \pi_\tau \) are isomorphic to \( \pi(\nu, \nu) \) with quasi-characters \( \nu_\tau \) of \( F_\tau^\times \) satisfying: \( \nu_\tau(t) = |t|^{\nu_\tau} \) with \( \text{Re}(\nu_\tau) = -w/2 \) (\( i = 1, 2 \)).
5.5. We now prove (1.4.1) and (5.4.1). First, we show that (1.4.1) and (5.4.1) are equivalent for a fixed quaternion algebra $B$ with $r>0$. Indeed, let $\pi$ be an automorphic representation of $B^+_\mathbb{A}$ as in (5.4.1). Then taking $S_0$ sufficiently small, we may suppose that $\pi$ satisfies the conditions in (5.2.3) for $S=B_\mathbb{A}^+ \times S_0$. By (5.2.3), $V(\pi)$ then contains some $I_\nu(f)$ with non-zero $f \in \mathfrak{Z}(S; \{k_s\}; w)$, which is an eigen function of all $\mathfrak{Z}(p)$ and $\mathfrak{Z}(p, p)$ for $p|D(B/F)\mathbb{L}(S)$ by (5.3.1) and (5.3.2). By (5.3.2), we conclude that the assertion (1.4.1) for $S$ implies the assertion (5.4.1) for this $\pi$. The converse is obvious from the above considerations.

We next show that if (5.4.1) holds for $B$ with $r=1$, then it also holds for arbitrary $B$. Since we assumed that $g=[F: \mathbb{Q}]$ is odd, there exists a quaternion algebra $B$ for which $r=1$ and $D(B/F)=(1)$. It is known that the representation of type (ii) in (5.4.1) corresponds to the representation of type (i) (with the same $k_\nu$) via the Weil representation. Thus by [10] Th. 16.1 (cf. Arthur [1]; cf. also Gelbart [7]), the assertion (5.4.1) for this $B$ implies (5.4.1) for $M_2(F)$. Again by the above mentioned result, the assertion (5.4.1) for general $B$ reduces to the case where $B=M_2(F)$.

Now to prove (1.4.1) and (5.4.1), it is enough to show them for one $w$; for (1.4.1), this follows from (1.3.6), and for (5.4.1), this follows by considering twists by powers of the idele norm quasi-character. Thus we may assume that $w$ is a positive integer satisfying (3.2.1) for $\{k_s\}$, and that $r=1$ for $B$, to prove (1.4.1) and (5.4.1). When $F=\mathbb{Q}$, and hence $B=M_2(\mathbb{Q})$, they reduce to a theorem of Deligne [5]. Otherwise, $B$ is a division quaternion algebra, for which we have proved (1.4.1) in 4.6.

References

An abelian scheme over the Shimura curve


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