Unfoldings of complex analytic foliations with singularities

By Tatsuo SUWA*

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In order to study the singularities of complex analytic foliations, we develop a theory of unfoldings in this paper. Section 1 contains terminologies and definitions on foliations with singularities. In section 2, we introduce unfoldings of foliations and define the infinitesimal unfolding map using some homological algebra. In section 3, we give explicit descriptions of the objects that appear in section 2 for later use.

In section 4, we consider first order unfoldings of a local foliation F and give an embedding of the set $U(F)$ of equivalence classes of first order unfoldings of F into the group $\text{Ext}_\theta^1(\Omega_F, \mathcal{O})$ (cf. (4.5)). In general, $U(F)$ is smaller than $\text{Ext}_\theta^1(\Omega_F, \mathcal{O})$ and we may say that the difference of $U(F)$ from $\text{Ext}_\theta^1(\Omega_F, \mathcal{O})$ is the primary obstruction to finding integrating factors for a system of generators of F (cf. section 6).

We associate, in section 5, a local cohomology class to a 2-form satisfying a certain condition (cf. (5.2)). This is used to describe the set $U(F)$ in section 6 ((6.1) Theorem). Then we give an algorithm to find the first order unfoldings of a given local foliation (also a simpler one if it is of codimension 1). This together with a versality theorem in [10] enables us to construct universal unfoldings of some foliation singularities. Finally in section 7, we determine the set of equivalence classes of first order unfoldings of a global foliation.

A part of the contents of this paper is summarized in [13].

§ 1. Complex analytic foliations with singularities

In this section we give terminologies and definitions on complex analytic foliations with singularities (cf. Baum [1], Baum-Bott [2], see also Thom [16], [17]).

Let M be a connected complex manifold of dimension n and let $\mathcal{O}_M$, Ω_M^p and Θ_M denote, respectively, the sheaves of germs of holomorphic functions, holomorphic p -forms and holomorphic vector fields on M . The sheaf Ω_M^1

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will be denoted simply by Ω_M . For each point z in M , let $\Omega_{M,z}^p$ denote the stalk of Ω_M^p at z . The exterior product and the exterior differentiation make the direct sum $\Lambda_z = \bigoplus_{p=0}^n \Omega_{M,z}^p$ ($\Omega_M^0 = \mathcal{O}_M$) a differential graded $\mathcal{O}_{M,z}$ -algebra. If F is a subsheaf of Ω_M , we denote by $(\Omega_M \wedge F)_z$ the degree 2 component of the ideal in Λ_z generated by F_z .

For a coherent sub- $\mathcal{O}_M$ -module F of Ω_M , the quotient sheaf Ω_M/F is denoted by Ω_F and the singular set $S(F)$ of F is defined to be the singular set of the coherent sheaf Ω_F ;

$$S(F) = \{z \in M \mid \Omega_{F,z} \text{ is not a free } \mathcal{O}_{M,z}\text{-module}\}.$$

If $\omega_i = \sum_{j=1}^n f_{ij}(z) dz_j$, $1 \leq i \leq r$, denote generators of F on a coordinate neighborhood U in M with coordinates $(z_1, z_2, \dots, z_n)$, then $S(F) \cap U$ is the set of points z in U where the rank of the matrix $(f_{ij}(z))_{1 \leq i \leq r, 1 \leq j \leq n}$ fails to be maximal. Thus $S(F)$ is an analytic set of M . Also, the annihilator F^a of F is the sub- $\mathcal{O}_M$ -module of Ω_M defined by the presheaf

$$U \longmapsto \{\theta \in \Gamma(U, \Theta_M) \mid \omega(\theta) = 0 \quad \text{for all } \omega \in \Gamma(U, F)\},$$

where $\Gamma(U, \quad)$ denotes the space of sections over U . There is a canonical isomorphism between F^a and the sheaf of germs of $\mathcal{O}_M$ -homomorphisms of Ω_F into $\mathcal{O}_M$;

$$(1.1) \quad F^a \simeq \text{Hom}_{\mathcal{O}_M}(\Omega_F, \mathcal{O}_M).$$

The annihilator $(F^a)^a$ of F^a is defined dually and is a sub- $\mathcal{O}_M$ -module of Ω_M . Note that F is included in $(F^a)^a$. The subsheaf F is said to be full in Ω_M if for any open set U of M and for any ω in $\Gamma(U, \Omega_M)$, $\omega \in \Gamma(U - S(F), F)$ implies $\omega \in \Gamma(U, F)$. The sheaves F^a and $(F^a)^a$ are full, respectively, in Θ_M and Ω_M . It is easily shown that F is full if and only if $F = (F^a)^a$. In general, $S(F)$ contains the singular set $S(F^a)$,

$$S(F^a) = \{z \in M \mid (\Theta_M/F^a)_z \text{ is not a free } \mathcal{O}_{M,z}\text{-module}\},$$

of F^a . Hence if F is full, then we have $S(F) = S(F^a)$.

(1.2) DEFINITION. A foliation (with singularities) on M is a coherent sub- $\mathcal{O}_M$ -module F of Ω_M which is integrable in the sense that

$$dF_z \subset (\Omega_M \wedge F)_z \quad \text{for all } z \text{ in } M - S(F).$$

A foliation F is reduced if it is a full subsheaf of Ω_M .

Note that the above integrability condition is weaker than [1] Definition 1.2 (ii).

If (M, F) is a foliation, then by the Frobenius theorem, F defines an ordinary complex analytic foliation (without singularity) on $M - S(F)$. Its codimension is called the codimension of F . We call the quotient sheaf $\Omega_F = \Omega_M/F$ the sheaf of germs of holomorphic 1-forms along the leaves of F . We record the exact sequence for later use:

$$(1.3) \quad 0 \longrightarrow F \xrightarrow{\iota} \Omega_M \xrightarrow{\eta} \Omega_F \longrightarrow 0.$$

Let E be a coherent sub- $\mathcal{O}_M$ -module of Θ_M . We say that E is integrable if

$$(1.4) \quad [E_z, E_z] \subset E_z$$

holds for all z in $M - S(E)$. Note that if moreover E is full in Θ_M , then (1.4) holds for all z in M .

The proof of the following is straightforward.

(1.5) PROPOSITION. (i) *If (M, F) is a foliation, then F^a is a coherent full sub- $\mathcal{O}_M$ -module of Θ_M which is integrable.*

(ii) *If (M, F) is a foliation, then $(M, (F^a)^a)$ is a reduced foliation.*

(iii) *There is a canonical one-to-one correspondence between the set of reduced foliations on M and the set of coherent sub- $\mathcal{O}_M$ -modules of Θ_M which are full and integrable.*

For a foliation F on M , we call $(F^a)^a$ the reduction of F .

(1.6) EXAMPLE. Let M be the 2-dimensional complex affine space $\mathbb{C}^2$ with coordinates (z_1, z_2) and let F be generated by the 1-form $\omega = z_1 z_2 dz_1 - z_1^2 dz_2$. Then F^a is generated by $\theta = z_1(\partial/\partial z_1) + z_2(\partial/\partial z_2)$ and the reduction of F is generated by $\omega_0 = z_2 dz_1 - z_1 dz_2$. We have $S(F) = \{z_1 = 0\}$, whereas $S(F^a) = S((F^a)^a) = \{(0, 0)\}$.

Let $f: M' \rightarrow M$ be a holomorphic map of a complex manifold M' into another M and let F be a sub- $\mathcal{O}_M$ -module of Ω_M . For each open set U' of M' , we denote by $f^{**}F(U')$ the sub- $\Gamma(U', \mathcal{O}_{M'})$ -module of $\Gamma(U', \Omega_{M'})$ generated by the set

$$\{f^*\omega \mid \omega \in \Gamma(f(U'), F)\},$$

where $f^*\omega$ denotes the pull-back of the 1-form ω by f . Note that each section ω in $\Gamma(f(U'), F)$ can be extended to a 1-form on an open set containing $f(U')$.

(1.7) DEFINITION. The pull-back $f^{**}F$ of a sub- $\mathcal{O}_M$ -module F of Ω_M by a holomorphic map $f: M' \rightarrow M$ is the sub- $\mathcal{O}_{M'}$ -module of $\Omega_{M'}$ determined by the presheaf $U' \mapsto f^{**}F(U')$. Especially, if M' is a closed submanifold of M and

if f is the inclusion, then the pull-back $f^{**}F$ is also denoted by $F|_{M'}$ and is called the restriction of F to M' as forms.

It is readily checked that $f^{**}F$ is coherent.

(1.8) **REMARK.** If $f: M' \rightarrow M$ is a holomorphic map, then there is an exact sequence

$$f^*\Omega_M \longrightarrow \Omega_{M'} \longrightarrow \Omega_{M'/M} \longrightarrow 0,$$

where $f^*\Omega_M$ is the analytic inverse image of Ω_M by f and $\Omega_{M'/M}$ is the sheaf of germs of relative holomorphic 1-forms of f . The pull-back $f^{**}\Omega_M$ coincides with the image of the homomorphism $f^*\Omega_M \rightarrow \Omega_{M'}$. When the map f is generically submersive, the homomorphism is injective. Thus in this case the analytic inverse image and the pull-back are identical.

(1.9) **DEFINITION.** A foliation (M, F) is a Haefliger foliation if each point of M has a neighborhood on which F is generated by exact forms.

(1.10) **DEFINITIONS.** A foliation F is of complete intersection type if F is a locally free $\mathcal{O}_M$ -module.

Thus if F is a codim q foliation of complete intersection type, then each point of M has a neighborhood U on which F is generated by q 1-forms $\omega_1, \omega_2, \dots, \omega_q$ such that $\omega_1 \wedge \omega_2 \wedge \dots \wedge \omega_q(z) \neq 0$ for a general point z in U . The singular set $S(F)$ is given by

$$(1.11) \quad S(F) \cap U = \{z \in U \mid \omega_1 \wedge \omega_2 \wedge \dots \wedge \omega_q(z) = 0\}.$$

Also the integrability condition in (1.2) is given by

$$(1.12) \quad d\omega_i \wedge \omega_1 \wedge \omega_2 \wedge \dots \wedge \omega_q(z) = 0, \quad 1 \leq i \leq q,$$

for all z in U .

If F is a foliation of complete intersection type, then there is an associated vector bundle. However, in general, the bundle cannot be embedded into the holomorphic cotangent bundle of M in such a way that the embedding induces the inclusion $F \hookrightarrow \Omega_M$.

In what follows, we only deal with foliations of complete intersection type unless stated otherwise.

§ 2. Unfoldings of foliations with singularities

Let F be a codim q foliation (of complete intersection type) on M . Consider an analytic family of deformations (simply, a deformation)

$$\pi: \mathcal{M} \longrightarrow T$$

of M , i.e., π is a holomorphic submersion of a complex manifold $\mathcal{M}$ onto another T such that $\pi^{-1}(o)=M$ for some point o of T . We assume that π is proper when M is compact. Denoting by $\Omega_{\mathcal{M}/T}$ the sheaf of germs of relative holomorphic 1-forms of π , we have the exact sequence

$$(2.1) \quad 0 \longrightarrow \pi^* \Omega_T \longrightarrow \Omega_{\mathcal{M}} \longrightarrow \Omega_{\mathcal{M}/T} \longrightarrow 0.$$

If we denote by $\mathcal{I}_t$ the ideal sheaf of $M_t = \pi^{-1}(t)$ in $\mathcal{M}$, then we have $\mathcal{O}_{M_t} = \mathcal{O}_{\mathcal{M}} / \mathcal{I}_t$. For an $\mathcal{O}_{\mathcal{M}}$ -module $\mathcal{S}$, we set $\mathcal{S}|_{M_t} = \mathcal{S} \otimes_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{M_t}$. Since π is a holomorphic submersion, the natural homomorphism

$$\pi^* \Omega_T|_{M_t} \longrightarrow \Omega_{\mathcal{M}}|_{M_t}$$

is injective. Also there is a canonical isomorphism

$$\Omega_{\mathcal{M}/T}|_{M_t} \simeq \Omega_{M_t}.$$

Thus, from (2.1), we get the exact sequence

$$(2.1)_t \quad 0 \longrightarrow \pi^* \Omega_T|_{M_t} \longrightarrow \Omega_{\mathcal{M}}|_{M_t} \xrightarrow{\alpha_t} \Omega_{M_t} \longrightarrow 0.$$

It is easy to show the following

(2.2) LEMMA. *If $\mathcal{F}$ is a sub- $\mathcal{O}_{\mathcal{M}}$ -module of $\Omega_{\mathcal{M}}$, then the image in Ω_{M_t} of $\mathcal{F}|_{M_t}$ by the composition of the canonical homomorphism $\mathcal{F}|_{M_t} \rightarrow \Omega_{\mathcal{M}}|_{M_t}$ and the homomorphism α_t in (2.1)_t coincides with the restriction $\mathcal{F}|_{M_t}$ of $\mathcal{F}$ to M_t as forms ((1.7)).*

(2.3) DEFINITION. An unfolding of a codim q foliation (M, F) consists of

- (I) a deformation $\pi: \mathcal{M} \rightarrow T$ of M (we let $\pi^{-1}(o)=M$ for $o \in T$) and
- (II) a codim q foliation $\mathcal{F}$ (of complete intersection type) on $\mathcal{M}$ such that $\mathcal{F}|_M = F$.

(2.4) REMARK. If F is either a local foliation or a foliation on a compact M , then there exists a neighborhood V of o in T such that $F_t = \mathcal{F}|_{M_t}$ is a codim q foliation (of complete intersection type) on M_t for each t in V . Thus $\{F_t\}_{t \in V}$ is an analytic family of deformations F_t of F .

(2.5) DEFINITION. An unfolding $(\mathcal{M}' \xrightarrow{\pi'} T', \mathcal{F}')$ of (M, F) is induced from another $(\mathcal{M} \rightarrow T, \mathcal{F})$ if there exist holomorphic maps $\Phi: \mathcal{M}' \rightarrow \mathcal{M}$ and $\varphi: T' \rightarrow T$ such that

- (i) φ maps the specific point o' in T' to the specific point o in T ,
- (ii) $\varphi \circ \pi' = \pi \circ \Phi$,
- (iii) for each t' in T' , Φ map $\pi'^{-1}(t')$ biholomorphically onto $\pi^{-1}(\varphi(t'))$,

(iv) $\mathcal{F}' = \Phi^{**}\mathcal{F}$ ((1.7)).

An unfolding $(\mathcal{M} \xrightarrow{\pi} T, \mathcal{F})$ of (M, F) is versal if every unfolding of F is induced from $\mathcal{F}$.

Let $(\mathcal{M} \xrightarrow{\pi} T, \mathcal{F})$ be an unfolding of (M, F) . Then we have the diagram

$$\begin{array}{ccccc} \Omega_{\mathcal{M}}|_M & \xrightarrow{\alpha} & \Omega_M & \longrightarrow & 0 \\ \uparrow \iota|_M & & \uparrow & & \\ \mathcal{F}|_M & \xrightarrow{\beta} & F & \longrightarrow & 0 \\ & & \uparrow & & \\ & & 0 & & \end{array},$$

where ι denotes the inclusion $\mathcal{F} \subset \Omega_M$, $\alpha = \alpha_o$ ((2.1)_o) and $\beta = \alpha \circ \iota|_M$. By (2.3) (II), β is surjective. Moreover, since both $\mathcal{F}|_M$ and F are locally free $\mathcal{O}_M$ -modules of rank q , we see that β is an isomorphism. Hence the homomorphism $\iota|_M$ is injective. Setting $\Omega_{\mathcal{F}} = \Omega_{\mathcal{M}}/\mathcal{F}$, we obtain the commutative diagram with exact rows and columns:

$$(2.6) \quad \begin{array}{ccccccc} & & & 0 & & 0 & \\ & & & \uparrow & & \uparrow & \\ 0 & \longrightarrow & \pi^*\Omega_T|_M & \longrightarrow & \Omega_{\mathcal{F}}|_M & \longrightarrow & \Omega_F \longrightarrow 0 \\ & & \parallel & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \pi^*\Omega_T|_M & \longrightarrow & \Omega_{\mathcal{M}}|_M & \longrightarrow & \Omega_M \longrightarrow 0 \\ & & & & \uparrow & & \uparrow \\ & & & & \mathcal{F}|_M & \xrightarrow{\sim} & F \\ & & & & \uparrow & & \uparrow \\ & & & & 0 & & 0 \end{array}.$$

Letting $\mathcal{I} = \mathcal{I}_0$ be the ideal sheaf of M in $\mathcal{M}$, there is a canonical isomorphism $\mathcal{F}|_{\mathcal{I}} \simeq \mathcal{F} \otimes_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_M = \mathcal{F}|_M$. Hence there is a canonical surjection $\mathcal{F} \rightarrow F$. Let ω be a germ in F_z for some z in M . We call a germ $\tilde{\omega}$ in $\mathcal{F}_z$ an extension of ω if it is mapped to ω by the canonical surjection $\mathcal{F} \rightarrow F$. Now considering the exact sequence of the groups $\text{Hom}_{\mathcal{O}_M}(\cdot, \mathcal{O}_M)$ of $\mathcal{O}_M$ -module homomorphisms for the first row of (2.6), we have the connecting homomorphism

$$(2.7) \quad \text{Hom}_{\mathcal{O}_M}(\pi^*\Omega_T|_M, \mathcal{O}_M) \longrightarrow \text{Ext}_{\mathcal{O}_M}^1(\Omega_F, \mathcal{O}_M).$$

Denoting by $\pi^*\Theta_T$ the analytic inverse image of the sheaf Θ_T by π , we have the canonical isomorphisms

$$\mathrm{Hom}_{\mathcal{O}_M}(\pi^*\Omega_T|_M, \mathcal{O}_M) = \Gamma(M, \mathcal{H}om_{\mathcal{O}_M}(\pi^*\Omega_T|_M, \mathcal{O}_M)) \simeq \Gamma(M, \pi^*\Theta_T|_M).$$

Let $T_{T,o}$ denote the holomorphic tangent space of T at the specific point o and let T_M denote the constant sheaf $M \times T_{T,o}$ on M . Then the sheaf T_M can be thought of as a subsheaf of $\pi^*\Theta_T|_M$ in a canonical manner. We define the (first order) infinitesimal unfolding map

$$(2.8) \quad \rho: T_{T,o} \longrightarrow \mathrm{Ext}_{\mathcal{O}_M}^1(\Omega_F, \mathcal{O}_M)$$

of the unfolding $(\mathcal{M} \xrightarrow{\pi} T, \mathcal{F})$ at o to be the composition of the inclusion $T_{T,o} = \Gamma(M, T_M) \hookrightarrow \mathrm{Hom}_{\mathcal{O}_M}(\pi^*\Omega_T|_M, \mathcal{O}_M)$ and the homomorphism (2.7). We give an explicit description of the map (2.8) in the next section.

Now we consider the spectral sequence

$$(2.9) \quad E_2^{p,q} = H^p(M, \mathcal{E}xt_{\mathcal{O}_M}^q(\Omega_F, \mathcal{O}_M)) \implies \mathrm{Ext}_{\mathcal{O}_M}^n(\Omega_F, \mathcal{O}_M).$$

Also, replacing Ω_F by Ω_M , we have a similar spectral sequence, which degenerates since Ω_M is locally free. Noting that $\mathcal{E}xt_{\mathcal{O}_M}^0(\Omega_F, \mathcal{O}_M) = \mathcal{H}om_{\mathcal{O}_M}(\Omega_F, \mathcal{O}_M) = F^a$, we get the commutative diagram with exact rows

$$(2.10) \quad \begin{array}{ccccccc} 0 & \rightarrow & H^1(M, F^a) & \rightarrow & \mathrm{Ext}_{\mathcal{O}_M}^1(\Omega_F, \mathcal{O}_M) & \xrightarrow{\varepsilon} & H^0(M, \mathcal{E}xt_{\mathcal{O}_M}^1(\Omega_F, \mathcal{O}_M)) \rightarrow H^2(M, F^a) \\ & & \downarrow & & \downarrow & & \\ & & H^1(M, \Theta_M) & \xrightarrow{\sim} & \mathrm{Ext}_{\mathcal{O}_M}^1(\Omega_M, \mathcal{O}_M). & & \end{array}$$

It is easily checked that the composition of (2.8) and the second vertical map is identical with the usual infinitesimal deformation map ([5] (6.2)) of the deformation $\pi: \mathcal{M} \rightarrow T$ at o (cf. (3.7)). The sheaf $\mathcal{E}xt_{\mathcal{O}_M}^1(\Omega_F, \mathcal{O}_M)$ is supported on the singular set $S(F)$ of F and contains some information about the singularity (see [12]).

§ 3. The infinitesimal unfolding map

In this section we give explicit descriptions of the spectral sequence (2.9) and the infinitesimal unfolding map (2.8).

Let (M, F) be a codim q foliation of complete intersection type and let

$$\cdots \xrightarrow{d''} \mathcal{L}_q \xrightarrow{d''} \cdots \xrightarrow{d''} \mathcal{L}_1 \xrightarrow{d''} \mathcal{L}_0 \longrightarrow \Omega_F \longrightarrow 0$$

be a resolution of Ω_F by locally free $\mathcal{O}_M$ -modules $\mathcal{L}_q$, $q \geq 0$. In our case, we may actually set ((1.3))

$$\mathcal{L}_0 = \Omega_M, \mathcal{L}_1 = F \quad \text{and} \quad \mathcal{L}_q = 0 \text{ for } q \geq 2.$$

We use this resolution hereafter. For an $\mathcal{O}_M$ -module $\mathcal{S}$, we set $\mathcal{S}^* =$

$\mathcal{H}om_{\mathcal{O}_M}(\mathcal{S}, \mathcal{O}_M)$. If $\mathcal{U} = \{U_\lambda\}_{\lambda \in A}$ is an open covering of M , we denote by $C^p(\mathcal{U}, \mathcal{S})$ the group of p -cochains on $\mathcal{U}$ with coefficients in the sheaf $\mathcal{S}$. We have the coboundary map $d': C^p(\mathcal{U}, \mathcal{S}) \rightarrow C^{p+1}(\mathcal{U}, \mathcal{S})$. The homomorphism $\mathcal{L}_q^* \rightarrow \mathcal{L}_{q+1}^*$ induced from d'' is also denoted by d'' . In the double complex

$$C^{**}(\mathcal{U}) = \{C^p(\mathcal{U}, \mathcal{L}_q^*); d', d''\}$$

we have

$$'H^{p''}H^q(C^{**}(\mathcal{U})) = H^p(\mathcal{U}, \mathcal{E}xt_{\mathcal{O}_M}^q(\Omega_F, \mathcal{O}_M)).$$

Hence we see that the spectral sequence (2.9) is obtained by taking the inductive limit (as $\mathcal{U}$ runs through open coverings of M) of each term of the first spectral sequence

$$(3.1) \quad 'E_2^{p,q} = 'H^{p''}H^q(C^{**}(\mathcal{U})) \implies H^n(C^{**}(\mathcal{U}))$$

associated with the double complex $C^{**}(\mathcal{U})$. Since the $\mathcal{O}_M$ -modules $\mathcal{E}xt_{\mathcal{O}_M}^q(\Omega_F, \mathcal{O}_M)$ are coherent for all $q \geq 0$, if $\mathcal{U}$ is a Stein covering of M , then the spectral sequence (3.1) itself can be identified with (2.9). Using (1.1), from (1.3), we get the exact sequence

$$(3.2) \quad 0 \longrightarrow F^a \longrightarrow \Theta_M \xrightarrow{\iota^*} F^* \longrightarrow \mathcal{E}xt_{\mathcal{O}_M}^1(\Omega_F, \mathcal{O}_M) \longrightarrow 0.$$

Let $\mathcal{U} = \{U_\lambda\}_{\lambda \in A}$ be a covering of M by small polydisks. We denote the group $C^{p,q}(\mathcal{U})$ simply by $C^{p,q}$ and we set $d = d' + (-1)^p d'' : C^{p,q} \rightarrow C^{p+1,q} + C^{p,q+1}$. Now we examine the groups $H^i(C^{**}) = \text{Ext}_{\mathcal{O}_M}^i(\Omega_F, \mathcal{O}_M)$, $'E_2^{1,0} = H^1(M, F^a)$ and $'E_2^{0,1} = H(M, \mathcal{E}xt_{\mathcal{O}_M}^1(\Omega_F, \mathcal{O}_M))$. First we have

$$\text{Ext}_{\mathcal{O}_M}^1(\Omega_F, \mathcal{O}_M) = H^1(C^{**}) = \frac{\{c \in C^{1,0} + C^{0,1} \mid dc = 0\}}{dC^{0,0}}.$$

Take an element c in $C^{1,0} + C^{0,1}$ and write $c = c_1 + c_2$, where c_1 is in $C^{1,0} = C^1(\mathcal{U}, \mathcal{L}_0^*) = C^1(\mathcal{U}, \Omega_M^*) = C^1(\mathcal{U}, \Theta_M)$ and c_2 is in $C^{0,1} = C^0(\mathcal{U}, \mathcal{L}_1^*) = C^0(\mathcal{U}, F^*)$. We have $dc = d'c_1 - d''c_1 + d'c_2 + d''c_2$. Since the elements $d'c_1$, $-d''c_1 + d'c_2$ and $d''c_2$ are, respectively, in $C^{2,0}$, $C^{1,1}$ and $C^{0,2}$, the identity $dc = 0$ is equivalent to

$$\begin{aligned} d'c_1 &= 0 & \text{in } C^{2,0} &= C^2(\mathcal{U}, \Theta_M), \\ d''c_1 &= d'c_2 & \text{in } C^{1,1} &= C^1(\mathcal{U}, F^*) \text{ and} \\ d''c_2 &= 0 & \text{in } C^{0,2} &= C^0(\mathcal{U}, \mathcal{L}_2^*) = 0. \end{aligned}$$

We may write $d''c_1 = \iota^*c_1$ ((3.2)). Setting $\theta = c_1$, $\varphi = c_2$ and $\delta = d'$, we obtain the

expression

$$(3.3) \quad \text{Ext}_{\mathcal{O}_M}^1(\Omega_F, \mathcal{O}_M) = \frac{\{(\theta, \varphi) \in Z^1(\mathcal{U}, \Theta_M) \times C^0(\mathcal{U}, F^*) \mid \iota^* \theta = \delta \varphi\}}{\{(\delta c, \iota^* c) \mid c \in C^0(\mathcal{U}, \Theta_M)\}},$$

where $Z^1(\mathcal{U}, \Theta_M)$ denotes the group of 1-cocycles on $\mathcal{U}$ with coefficients in Θ_M . Similarly we have

$$(3.4) \quad H^1(M, F^a) = \frac{Z^1(\mathcal{U}, F^a)}{\delta C^0(\mathcal{U}, F^a)}.$$

The edge homomorphism $H^1(M, F^a) \rightarrow \text{Ext}_{\mathcal{O}_M}^1(\Omega_F, \mathcal{O}_M)$ in (2.10) sends the class of a cocycle θ in $Z^1(\mathcal{U}, F^a)$ to the class of $(\theta, 0)$. Also we have

$$(3.5) \quad H^0(M, \mathcal{E}xt_{\mathcal{O}_M}^1(\Omega_F, \mathcal{O}_M)) = \frac{\{(\theta, \varphi) \in C^1(\mathcal{U}, \Theta_M) \times C^0(\mathcal{U}, F^*) \mid \iota^* \theta = \delta \varphi\}}{\{(\delta c + c', \iota^* c) \mid c' \in C^1(\mathcal{U}, F^a), c \in C^0(\mathcal{U}, \Theta_M)\}}.$$

The edge homomorphism ε in (2.10) sends the class of (θ, φ) to the class of (θ, φ) .

Now we examine the infinitesimal unfolding map (2.8). Let $(\mathcal{M} \xrightarrow{\pi} T, \mathcal{F})$ be an unfolding of (M, F) and let

$$\mathcal{L}_\bullet \longrightarrow \Omega_F \longrightarrow 0, \quad {}'\mathcal{L}_\bullet \longrightarrow \Omega_{\mathcal{F}}|_M \longrightarrow 0 \quad \text{and} \quad {}''\mathcal{L}_\bullet \longrightarrow \pi^* \Omega_T|_M \longrightarrow 0$$

be locally free resolutions such that we have the commutative diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \pi^* \Omega_T|_M & \longrightarrow & \Omega_{\mathcal{F}}|_M & \longrightarrow & \Omega_F \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & {}''\mathcal{L}_\bullet & \longrightarrow & {}'\mathcal{L}_\bullet & \longrightarrow & \mathcal{L}_\bullet \longrightarrow 0 \end{array}$$

with exact rows (and columns). In our case we may actually set ((2.6))

$$\begin{aligned} \mathcal{L}_0 &= \Omega_M, \quad \mathcal{L}_1 = F, \quad \mathcal{L}_q = 0 \quad \text{for } q \geq 2, \\ {}'\mathcal{L}_0 &= \Omega_{\mathcal{M}}|_M, \quad {}'\mathcal{L}_1 = \mathcal{F}|_M, \quad {}'\mathcal{L}_q = 0 \quad \text{for } q \geq 2, \\ {}''\mathcal{L}_0 &= \pi^* \Omega_T|_M, \quad {}''\mathcal{L}_q = 0 \quad \text{for } q \geq 1. \end{aligned}$$

We use these resolutions hereafter. Take a coordinate neighborhood V on T about the specific point o with coordinates $t = (t_1, \dots, t_m)$. Since π is a submersion, $\pi^{-1}(V)$ has a covering $\{\mathcal{U}_\lambda\}_{\lambda \in \Lambda}$ by small polydisks $\mathcal{U}_\lambda$ such that each $\mathcal{U}_\lambda$ is a coordinate neighborhood on $\mathcal{M}$ with coordinates of the form $(z^\lambda, t) = (z_1^\lambda, \dots, z_n^\lambda, t_1, \dots, t_m)$, $\pi(z^\lambda, t) = t$. In the intersection $\mathcal{U}_\lambda \cap \mathcal{U}_\mu$, z^λ is a vector-valued holomorphic function of (z^μ, t) ;

$$z^\lambda = f^{\lambda\mu}(z^\mu, t).$$

Let $\mathcal{U} = \{U_\lambda\}_{\lambda \in A}$ be the covering of M defined by $U_\lambda = \mathcal{U}_\lambda \cap M$ and consider the double complexes

$$C^{\bullet\bullet} = \{C^p(\mathcal{U}, \mathcal{L}_q^*)\}, {}'C^{\bullet\bullet} = \{C^p(\mathcal{U}, {}'\mathcal{L}_q^*)\} \quad \text{and} \quad ''C^{\bullet\bullet} = \{C^p(\mathcal{U}, ''\mathcal{L}_q^*)\}.$$

Denoting by $C^\bullet$, $'C^\bullet$ and $''C^\bullet$, respectively, the total complexes of these double complexes, we get the exact sequence

$$(3.6) \quad 0 \longrightarrow C^\bullet \longrightarrow {}'C^\bullet \xrightarrow{\psi} ''C^\bullet \longrightarrow 0.$$

The exact sequence of $\text{Hom}_{\mathcal{O}_M}(\cdot, \mathcal{O}_M)$ for the first row of (2.6) is identical with the cohomology exact sequence obtained from (3.6). Thus $\text{Hom}_{\mathcal{O}_M}(\pi^*\Omega_T|_M, \mathcal{O}_M) = H^0({}'C^\bullet) = \{a \in {}''C^0 \mid da = 0\}$ and $\text{Ext}_{\mathcal{O}_M}^1(\Omega_F, \mathcal{O}_M) = H^1(C^\bullet) = \{c \in C^1 \mid dc = 0\} / dC^0$. Take a tangent vector $v = \sum_{k=1}^m c_k(\partial/\partial t_k)$ in $T_{T,0} \hookrightarrow \text{Hom}_{\mathcal{O}_M}(\pi^*\Omega_T|_M, \mathcal{O}_M)$ and choose an element b in $'C^0 = C^0(\mathcal{U}, \mathcal{H}om_{\mathcal{O}_M}(\Omega_{\mathcal{A}}|_M, \mathcal{O}_M)) = C^0(\mathcal{U}, \Theta_{\mathcal{A}}|_M)$ such that $\psi(b) = v$. We may write $b = \{b^\lambda\}$,

$$b_\lambda = \sum_{k=1}^m c_k \frac{\partial}{\partial t_k} \quad \text{on } U_\lambda.$$

We have $\psi d(b) = d\psi(b) = 0$. Hence $d(b)$ is in (the image of) $C^1 = C^{1,0} + C^{0,1}$. Write $d(b) = d'(b) + d''(b)$ with $d'(b)$ in $C^{1,0} = C^1(\mathcal{U}, \Theta_M)$ and $d''(b)$ in $C^{0,1} = C^0(\mathcal{U}, \mathcal{H}om_{\mathcal{O}_M}(F, \mathcal{O}_M))$. If we set $\theta = d'(b)$, θ is a collection $\{\theta^{\lambda\mu}\}$, where

$$(3.7) \quad \begin{aligned} \theta^{\lambda\mu} &= b^\mu - b^\lambda = \sum_{i=1}^n \sum_{k=1}^m c_k \frac{\partial z_i^\lambda}{\partial t_k} \frac{\partial}{\partial z_i^\lambda} + \sum_{j,k=1}^m c_k \frac{\partial t_j}{\partial t_k} \frac{\partial}{\partial t_j} - \sum_{k=1}^m c_k \frac{\partial}{\partial t_k} \\ &= \sum_{i=1}^n \sum_{k=1}^m c_k \frac{\partial f_i^{\lambda\mu}}{\partial t_k}(z^\mu, 0) \frac{\partial}{\partial z_i^\lambda}. \end{aligned}$$

Next we find $d''(b)$. Note that $d''(b)$ is a priori in $'C^{0,1} = C^0(\mathcal{U}, \mathcal{H}om_{\mathcal{O}_M}(\mathcal{F}|_M, \mathcal{O}_M))$ and that d'' is induced from the inclusion $\mathcal{F} \hookrightarrow \Omega_M$. If we set $\varphi = d''(b)$, φ is a collection $\{\varphi^\lambda\}$, where, for each λ , φ^λ is in $\Gamma(U_\lambda, \mathcal{H}om_{\mathcal{O}_M}(F, \mathcal{O}_M)) \simeq \text{Hom}_{\mathcal{O}_M|U_\lambda}(F|U_\lambda, \mathcal{O}_M|U_\lambda)$. If $\omega = \sum_{i=1}^n f_i(z^\lambda) dz_i^\lambda$ is a 1-form in $\Gamma(U_\lambda, F)$, then there is a 1-form $\tilde{\omega}$ in $\Gamma(\mathcal{U}_\lambda, \mathcal{F})$ which extends ω . We may express

$$\tilde{\omega} = \sum_{i=1}^n f_i(z^\lambda, t) dz_i^\lambda + \sum_{k=1}^m h_k(z^\lambda, t) dt_k,$$

where f_i and h_k are holomorphic functions on $\mathcal{U}_\lambda$ with $f_i(z^\lambda, 0) = f_i(z^\lambda)$. Although $\tilde{\omega}$ is not uniquely determined by ω , the functions $h_k(z^\lambda, 0)$ on U_λ are determined because of the isomorphism $\mathcal{F}|_M \xrightarrow{\sim} F$. Then we have

$$(3.8) \quad \varphi^i(\omega) = \sum_{k=1}^m c_k h_k(z^i, 0).$$

In summary, the infinitesimal unfolding map (2.8) assigns to the vector $v = \sum_{k=1}^m c_k (\partial/\partial t_k)$ the class of $(\theta, \varphi) = (\{\theta^{\lambda\mu}\}, \{\varphi^i\})$ ((3.3)) given by (3.7) and (3.8).

(3.9) REMARK. Let $f: X \rightarrow Y$ be a holomorphic map of a compact complex manifold X onto another Y . In this case the analytic inverse image $f^*\Omega_Y$ of Ω_Y by f is identical with the pull-back $f^{**}\Omega_Y$ ((1.8)). If we set $F = f^*\Omega_Y$, comparing (3.3) and [4] Definition 4.1, we have

$$\text{Ext}_{\mathcal{O}_X}^1(\Omega_F, \mathcal{O}_X) = D_{X/Y}.$$

We also have $\Omega_F = \Omega_{X/Y}$ and $F^a = \Theta_{X/Y}$ and the first sequence of (2.10) is identical with [4] (4.2).

§ 4. First order unfoldings of local foliations

We denote by ${}_n\mathcal{O}$ or simply by $\mathcal{O}$ the local ring $\mathcal{O}_{C^n,0}$ of germs of holomorphic functions at the origin 0 in $C^n = \{(z_1, \dots, z_n)\}$, which is isomorphic to the ring $C\{z_1, \dots, z_n\}$ of convergent power series in $(z_1, \dots, z_n)$. We also denote the ${}_n\mathcal{O}$ -module $\Omega_{C^n,0}$ by ${}_n\Omega$. A local foliation F at 0 in C^n is defined to be the stalk F_0 at 0 of a foliation F on a neighborhood U of 0 in C^n ; $F = F_0$. If we denote the quotient $\mathcal{O}$ -module ${}_n\Omega/F$ by Ω_F , then this is identified with the stalk $\Omega_{F,0}$ and we have the exact sequence

$$(4.1) \quad 0 \longrightarrow F \longrightarrow {}_n\Omega \longrightarrow \Omega_F \longrightarrow 0$$

of $\mathcal{O}$ -modules. If F is a local foliation represented by (U, F) , then the singular set $S(F)$ of F is defined to be the germ at 0 of the singular set $S(F)$ of F . A local foliation F is of complete intersection type if it is represented by a foliation of complete intersection type, in other words $F \simeq \mathcal{O}^q$ for some q . We only deal with local foliations of complete intersection type unless stated otherwise. Thus a codim q local foliation at 0 in C^n is a rank q free sub- $\mathcal{O}$ -module $F = (\omega_1, \dots, \omega_q)$ of ${}_n\Omega$ such that $d\omega_i \wedge \omega_1 \wedge \dots \wedge \omega_q = 0$, $1 \leq i \leq q$, where $(\omega_1, \dots, \omega_q)$ is a system of generators of F .

(4.2) DEFINITION. Let F be a local foliation at 0 in C^n . An unfolding $\mathcal{F}$ of F is a local foliation at 0 in C^{n+m} represented by an unfolding $(U \times V \xrightarrow{\pi} V, \mathcal{F})$ of (U, F) , where (U, F) is a representative of F , V is a neighborhood of 0 in C^m and π is the projection.

If $\mathcal{F}$ is an unfolding of a codim q local foliation F , then the infinitesimal unfolding map

$$(4.3) \quad \rho: T \longrightarrow \text{Ext}_\theta^1(\Omega_F, \mathcal{O}),$$

where $T = T_{C^m, 0}$, of $\mathcal{F}$ is defined in a similar manner as in section 2. From (4.1), we have a presentation of $\text{Ext}_\theta^1(\Omega_F, \mathcal{O})$;

$$(4.4) \quad \cdots \longrightarrow \text{Hom}_\theta(\Omega, \mathcal{O}) \longrightarrow \text{Hom}_\theta(F, \mathcal{O}) \longrightarrow \text{Ext}_\theta^1(\Omega_F, \mathcal{O}) \longrightarrow 0.$$

Let $\omega_1, \omega_2, \dots, \omega_q$ be a system of generators of F . We write

$$\omega_i = \sum_{j=1}^n f_{ij} dz_j, \quad f_{ij} \in \mathcal{O}, \quad 1 \leq i \leq q.$$

If we denote by f_j the element $(f_{1j}, \dots, f_{qj})$ in $\mathcal{O}^q$ and by $(f_1, \dots, f_n)$ the sub- $\mathcal{O}$ -module of $\mathcal{O}^q$ generated by $f_1, \dots, f_n$, then we have, from (4.4),

$$(4.5) \quad \text{Ext}_\theta^1(\Omega_F, \mathcal{O}) = \mathcal{O}^q / (f_1, \dots, f_n).$$

If $h = (h_1, \dots, h_q)$ is an element in $\mathcal{O}^q$, we denote by $[h] = [h_1, \dots, h_q]$ its image in $\text{Ext}_\theta^1(\Omega_F, \mathcal{O})$ by the canonical surjection. For each i , $1 \leq i \leq q$, let $\tilde{\omega}_i$ be an extension of ω_i in $\mathcal{F}$ and write $\tilde{\omega}_i = \sum_{j=1}^n f_{ij}(z, t) dz_j + \sum_{k=1}^m h_{ik}(z, t) dt_k$ in a neighborhood of 0 in $C^n \times C^m = \{(z_1, \dots, z_n; t_1, \dots, t_m)\}$. Then the map (4.3) assigns to a tangent vector $\sum_{k=1}^m c_k (\partial/\partial t_k)$ in T the element $\left[\sum_{k=1}^m c_k h_{1k}, \dots, \sum_{k=1}^m c_k h_{qk} \right]$ in $\text{Ext}_\theta^1(\Omega_F, \mathcal{O})$, where h_{ik} is the germ at $z=0$ of the function $h_{ik}(z, 0)$ (see (3.8)).

Now let U and V be neighborhoods of the origins in $C^n = \{(z_1, \dots, z_n)\}$ and $C = \{t\}$, respectively, and set $\mathcal{M} = U \times V$. The exterior product makes the direct sum $\Lambda = \bigoplus_{p=0}^{n+1} H^p(\mathcal{M}, \Omega_{\mathcal{M}}^p)$ into a graded ring. Let (t^2, tdt) denote the ideal in Λ generated by t^2 and tdt . Two forms σ and τ in Λ are said to be identical up to the first order if $\sigma - \tau$ is in (t^2, tdt) . If this is the case, we write

$$\sigma \equiv \tau \quad \text{mod}(t^2, tdt).$$

(4.6) DEFINITION. A first order unfolding of a codim q foliation (U, F) is a rank q free sub- $\mathcal{O}_{U \times V}$ -module $\mathcal{F}^{(1)}$ of $\Omega_{U \times V}$, for some neighborhood V of 0 in C , such that

- (i) $\mathcal{F}^{(1)}|_{U \times 0} = F$ and
- (ii) $\mathcal{F}^{(1)}$ satisfies the first order integrability condition, i.e., if $(\tilde{\omega}_1, \dots, \tilde{\omega}_q)$ is a system of generators of $\mathcal{F}^{(1)}$ on $U \times V$, then

$$d\tilde{\omega}_i \wedge \tilde{\omega}_1 \wedge \dots \wedge \tilde{\omega}_q \equiv 0 \quad \text{mod}(t^2, tdt).$$

A direct computation shows the following

(4.7) LEMMA. Let $(\tilde{\omega}_1, \dots, \tilde{\omega}_q)$ be a system of holomorphic 1-forms on $U \times V$. If

$$\omega_i \equiv \omega_i + \omega_i^{(1)}t + h_i dt \pmod{(t^2, tdt)}, \quad 1 \leq i \leq q,$$

where ω_i and $\omega_i^{(1)}$ are 1-forms involving only z and dz and h_i are holomorphic functions of z , then the first order integrability condition (4.6) (ii) is given by

$$\begin{aligned} d\omega_i^{(1)} \wedge \Omega - d\omega_i \wedge \left(\sum_{j=1}^q (-1)^j \omega_j^{(1)} \wedge \Omega_j \right) &= 0 \quad \text{and} \\ (dh_i - \omega_i^{(1)}) \wedge \Omega + d\omega_i \wedge \left(\sum_{j=1}^q (-1)^j h_j \Omega_j \right) &= 0, \quad 1 \leq i \leq q, \end{aligned}$$

where $\Omega = \omega_1 \wedge \cdots \wedge \omega_q$ and $\Omega_j = \omega_1 \wedge \cdots \wedge \hat{\omega}_j \wedge \cdots \wedge \omega_q$ (omit ω_j).

(4.8) REMARK. As is shown in the proof of (6.1) Theorem, the second equality above implies the first.

(4.9) LEMMA. Let $(U \times V, \mathcal{F}^{(1)})$ be a first order unfolding of a codim q foliation (U, F) and let $(\omega_1, \dots, \omega_q)$ be a system of generators of F on U . Suppose $(\tilde{\omega}_1, \dots, \tilde{\omega}_q)$ and $(\tilde{\omega}'_1, \dots, \tilde{\omega}'_q)$ are two systems of generators of $\mathcal{F}^{(1)}$ extending the given system and write

$$\begin{aligned} \tilde{\omega}_i &\equiv \omega_i + \omega_i^{(1)}t + h_i dt, \\ \tilde{\omega}'_i &\equiv \omega_i + \omega_i^{(1)}t + h'_i dt \pmod{(t^2, tdt)} \end{aligned}$$

as in the previous lemma. Then $h_i = h'_i$ for $i = 1, \dots, q$.

PROOF. There exist holomorphic functions f_{ij} , $1 \leq i, j \leq q$, on $U \times V$ such that $\tilde{\omega}'_1 = \sum_{j=1}^q f_{1j}(z, t)\omega_j$. If we write

$$f_{ij}(z, t) \equiv f_{ij}(z) + f_{ij}^{(1)}(z)t \pmod{t^2},$$

then we see that $f_{ij}(z) = \delta_{ij}$. Hence we have $h'_i = h_i$ for $i = 1, \dots, q$, Q.E.D.

(4.10) DEFINITION. Two first order unfoldings $(U \times V, \mathcal{F}^{(1)})$ and $(U \times V', \mathcal{F}'^{(1)})$ of (U, F) are equivalent if there is a biholomorphic map Φ of a neighborhood N of $U \times 0$ in $U \times (V \cap V')$ into $U \times (V \cap V')$ such that

- (i) $\pi \circ \Phi = \pi|_N$, where π is the projection $U \times (V \cap V') \rightarrow V \cap V'$,
- (ii) $\Phi|_{U \times 0} = 1_U$ (the identity map of U) and
- (iii) $\mathcal{F}'^{(1)}|_N \equiv \Phi^*(\mathcal{F}^{(1)}|_{\Phi(N)}) \pmod{(t^2, tdt)}$.

(4.11) DEFINITION. Let F be a local foliation at 0. A first order unfolding $\mathcal{F}^{(1)}$ of F is the stalk at 0 of a first order unfolding $\mathcal{F}^{(1)}$ of a representative F of F . Two first order unfoldings $\mathcal{F}^{(1)}$ and $\mathcal{F}'^{(1)}$ of F are equivalent if they are represented by equivalent first order unfoldings of a representative of F .

Now let F be a codim q local foliation and fix a system $(\omega_1, \dots, \omega_q)$ of

generators of F . Then, by (4.9), each first order unfolding $\mathcal{F}^{(1)}$ of F determines a system of germs of functions $(h_1, \dots, h_q)$.

(4.12) LEMMA. *Let $\mathcal{F}^{(1)}$ and $\mathcal{F}'^{(1)}$ be two first order unfoldings of F and let $(h_1, \dots, h_q)$ and $(h'_1, \dots, h'_q)$ be the systems of germs determined by $\mathcal{F}^{(1)}$ and $\mathcal{F}'^{(1)}$, respectively. Then $\mathcal{F}^{(1)}$ and $\mathcal{F}'^{(1)}$ are equivalent if and only if*

$$[h_1, \dots, h_q] = [h'_1, \dots, h'_q] \quad \text{in } \text{Ext}_\theta^1(\Omega_F, \mathcal{O}).$$

PROOF. Let $(\tilde{\omega}_1, \dots, \tilde{\omega}_q)$ and $(\tilde{\omega}'_1, \dots, \tilde{\omega}'_q)$ be systems of generators of $\mathcal{F}^{(1)}$ and $\mathcal{F}'^{(1)}$, respectively, extending the given system $(\omega_1, \dots, \omega_q)$. We write

$$\begin{aligned} \tilde{\omega}_i &\equiv \omega_i + \omega_i^{(1)}t + h_i dt, \\ \tilde{\omega}'_i &\equiv \omega_i + \omega_i'^{(1)}t + h'_i dt \quad \text{mod}(t^2, tdt). \end{aligned}$$

Suppose $\mathcal{F}^{(1)}$ and $\mathcal{F}'^{(1)}$ are equivalent and let Φ be a biholomorphic map as in (4.10). If we write $\Phi(z, t) = (\varphi(z, t), t)$ and $\varphi(z, t) \equiv z + \varphi^{(1)}(z)t \text{ mod}(t^2)$, then we have

$$\Phi^* \tilde{\omega}_i \equiv \omega_i + \omega_i''^{(1)}t + \left(h_i + \sum_{j=1}^n f_{ij} \varphi_j^{(1)} \right) dt \quad \text{mod}(t^2, tdt),$$

where $\varphi_j^{(1)}$ is the j -th component of $\varphi^{(1)}$ and f_{ij} are coefficients of ω_i ; $\omega_i = \sum_{j=1}^n f_{ij}(z)dz_j$, $1 \leq i \leq q$. Hence, by (4.9) Lemma, we have

$$(4.13) \quad h'_i = h_i + \sum_{j=1}^n f_{ij} \varphi_j^{(1)}, \quad 1 \leq i \leq q.$$

This shows that $[h'_1, \dots, h'_q]$ and $[h_1, \dots, h_q]$ are identical in $\text{Ext}_\theta^1(\Omega_F, \mathcal{O})$. Conversely, suppose $[h'_1, \dots, h'_q] = [h_1, \dots, h_q]$. Then there are holomorphic functions $\varphi_j^{(1)}(z)$, $1 \leq j \leq n$, such that (4.13) holds. Let Φ be defined by $\Phi(z, t) = (z + \varphi^{(1)}(z)t, t)$. Then it is easily shown that Φ satisfies the conditions of (4.10), Q.E.D.

We denote by $U(F)$ the set of equivalence classes of first order unfoldings of F . Then by the above lemma, there is an embedding

$$U(F) \hookrightarrow \text{Ext}_\theta^1(\Omega_F, \mathcal{O}),$$

which assigns $[h_1, \dots, h_q]$ to the equivalence class of $\mathcal{F}^{(1)}$.

§ 5. Certain local cohomology class associated with a 2-form τ satisfying $\tau \wedge \omega_1 \wedge \dots \wedge \omega_q = 0$

Let F be a codim q local foliation at 0 in C^n and let $S(F)$ be the singular

set of F . We say that F is reduced if it is represented by a reduced foliation. First we prove the following

(5.1) LEMMA. *The codimension of the germ $S(F)$ of analytic set at 0 in C^n is greater than one if and only if F is reduced.*

PROOF. We let F be represented by a foliation F on a small neighborhood U of 0 in C^n and take a system $(\omega_1, \dots, \omega_q)$ of generators of F on U . Suppose $\text{codim } S(F) \geq 2$. Then the Hartogs' continuation theorem shows that F is reduced. We now suppose that $\text{codim } S(F) = 1$. Then there is a germ f in the ring $\mathcal{O}$ such that every germ in the ideal of $S(F)$ is divisible by f in $\mathcal{O}$. We write $\omega_i = \sum_{j=1}^n f_{ij}(z) dz_j$, $1 \leq i \leq q$, where f_{ij} are holomorphic functions on U . Let $f(i_1, \dots, i_r; j_1, \dots, j_r)$ denote the $r \times r$ minor

$$\begin{vmatrix} f_{i_1 j_1} & \cdots & f_{i_1 j_r} \\ \vdots & & \vdots \\ f_{i_r j_1} & \cdots & f_{i_r j_r} \end{vmatrix}$$

of the $q \times n$ matrix (f_{ij}) . If all the f_{ij} are divisible by f , then, for example, the form ω_1/f is holomorphic on U and is in $\Gamma(U - S(F), F)$ but not in $\Gamma(U, F)$. Hence F is not reduced. From now on we assume that there is an f_{ij} which is not divisible by f . By (1.12), $S(F)$ is the set of common zeros of the functions

$$f(1, 2, \dots, q; j_1, j_2, \dots, j_q), \quad 1 \leq j_1 < j_2 < \dots < j_q \leq n.$$

Hence all the $q \times q$ minors are divisible by f . Therefore, there exists a (unique) integer p ($2 \leq p \leq q$), such that all the $p \times p$ minors of (f_{ij}) are divisible by f but there is a $(p-1) \times (p-1)$ minor which is not divisible by f . Without loss of generality, we may assume that the minor $f(1, 2, \dots, p-1; 1, 2, \dots, p-1)$ is not divisible by f . Using the identities $\omega_i = \sum_{j=1}^n f_{ij} dz_j$, $1 \leq i \leq p$, we have

$$\begin{aligned} & \sum_{i=1}^p (-1)^{p+i} f(1, \dots, \hat{i}, \dots, p; 1, \dots, p-1) \omega_i \\ &= \sum_{j=p}^n f(1, \dots, p; 1, \dots, p-1, j) dz_j. \end{aligned}$$

If we denote this form by ω , then the 1-form ω/f is holomorphic on U , since the functions $f(1, \dots, p; 1, \dots, p-1, j)$ are divisible by f . The function (representing) f does not vanish outside of $S(F)$. Hence ω/f is in $\Gamma(U - S(F), F)$. However, it is not in $\Gamma(U, F)$, since the function $f(1, \dots, p-1; 1, \dots, p-1)$ is not divisible by f . This shows that F is not reduced.

Q.E.D.

Now we fix a system $(\omega_1, \dots, \omega_q)$ of generators of the germ F . Let τ be a germ of holomorphic 2-form at 0 in $\mathbb{C}^n$ satisfying

$$(5.2) \quad \tau \wedge \omega_1 \wedge \dots \wedge \omega_q = 0.$$

We choose a small neighborhood U_0 of 0 so that the germs $\tau, \omega_1, \dots, \omega_q$ have representatives on U_0 . Let F be the foliation on U_0 generated by $\omega_1, \dots, \omega_q$ and set $S = S(F)$ and $S = S(F)$. Take a polydisk $U (\subset U_0)$ with center 0 and let $\{U_\lambda\}_{\lambda \in A}$ be an open covering of $U - S$ small polydisks U_λ . Since $\omega_1, \dots, \omega_q$ are linearly independent at each point of $U - S$, from (5.2), τ may be written as

$$(5.3) \quad \tau = \sum_{i=1}^q \eta_i^\lambda \wedge \omega_i \quad \text{on } U_\lambda,$$

where $\eta_i^\lambda, 1 \leq i \leq q$, are holomorphic 1-forms on U_λ . In the intersection $U_\lambda \cap U_\mu$, we have

$$\sum_{i=1}^q (\eta_i^\lambda - \eta_i^\mu) \wedge \omega_i = 0.$$

Hence we may write

$$(5.4) \quad \eta_i^\lambda - \eta_i^\mu = \sum_{j=1}^q \psi_{ij}^{\lambda\mu} \omega_j \quad \text{on } U_\lambda \cap U_\mu,$$

where $\psi_{ij}^{\lambda\mu}$ are holomorphic functions on $U_\lambda \cap U_\mu$ satisfying

$$(5.5) \quad \psi_{ij}^{\lambda\mu} = \psi_{ji}^{\mu\lambda}, \quad 1 \leq i, j \leq q,$$

and for each (i, j) , the cocycle condition

$$(5.6) \quad \psi_{ij}^{\lambda\mu} + \psi_{ij}^{\mu\nu} + \psi_{ij}^{\nu\lambda} = 0 \quad \text{on } U_\lambda \cap U_\mu \cap U_\nu.$$

If we set

$$\Psi = \{\Psi^{\lambda\mu}\}, \quad \Psi^{\lambda\mu} = (\psi_{ij}^{\lambda\mu})_{1 \leq i, j \leq q},$$

then, by (5.6), Ψ defines a cohomology class in $H^1(U - S, \mathcal{O}_U^{q^2})$. It is not difficult to check that the cohomology class does not depend on the choice of η_i^λ or $\psi_{ij}^{\lambda\mu}$ and is uniquely determined by the form τ . We denote this cohomology class by $c_U(\tau)$. For our purpose, it is more convenient to think of $c_U(\tau)$ as a class in the cohomology $H_{S \cap U}^2(U, \mathcal{O}_U^{q^2})$ with supports in $S \cap U$. In the local cohomology exact sequence

$$\begin{aligned} \dots &\longrightarrow H^1(U, \mathcal{O}_U^{q^2}) \longrightarrow H^1(U - S, \mathcal{O}_U^{q^2}) \longrightarrow H_{S \cap U}^2(U, \mathcal{O}_U^{q^2}) \\ &\longrightarrow H^2(U, \mathcal{O}_U^{q^2}) \longrightarrow \dots, \end{aligned}$$

we have $H^p(U, \mathcal{O}_U^{q^2}) = 0$ for $p \geq 1$. Hence we have a canonical isomorphism

$$H^1(U - S, \mathcal{O}_U^{q^2}) \simeq H_{S \cap U}^2(U, \mathcal{O}_U^{q^2}).$$

Thus we may think of $c_U(\tau)$ as a class in $H_{S \cap U}^2(U, \mathcal{O}_U^{q^2})$. Let $\mathcal{H}_S^p(\mathcal{O}_{U_0}^{q^2})$ denote the p -th cohomology sheaf (over U_0) with coefficients in $\mathcal{O}_{U_0}^{q^2}$ and supports in S . We denote by $\mathcal{H}_S^p(\mathcal{O}^{q^2})$ the stalk $\mathcal{H}_S^p(\mathcal{O}_{U_0}^{q^2})_0$. Then we have ([3] Proposition 1.2)

$$\mathcal{H}_S^p(\mathcal{O}^{q^2}) = \lim_{\substack{\longrightarrow \\ U}} H_{S \cap U}^p(U, \mathcal{O}_U^{q^2}),$$

where U runs through open neighborhoods of 0 in U_0 . Clearly the image of $c_U(\tau)$ by the canonical homomorphism

$$(5.7) \quad H_{S \cap U}^2(U, \mathcal{O}_U^{q^2}) \longrightarrow \mathcal{H}_S^2(\mathcal{O}^{q^2})$$

does not depend on the choice of U and is uniquely determined by τ . We denote the image by $c(\tau)$. We note that $\mathcal{H}_S^2(\mathcal{O}^{q^2}) = 0$ if $\text{codim } S \geq 3$.

Next we examine the relation between the cohomology groups $\mathcal{H}_S^2(\mathcal{O}^{q^2})$ and H^2 , the latter being defined as

$$\begin{aligned} Z^2 &= \{\tau \in \Omega_{C^n, 0}^2 \mid \tau \wedge \omega_1 \wedge \cdots \wedge \omega_q = 0\}, \\ B^2 &= \left\{ \tau \in \Omega_{C^n, 0}^2 \mid \tau = \sum_{i=1}^q \eta_i \wedge \omega_i \text{ for some } \eta_i \text{ in } \Omega_{C^n, 0}^1 \right\}, \\ H^2 &= Z^2 / B^2. \end{aligned}$$

(5.8) LEMMA. *If $\text{codim } S \geq 2$, then there is a canonical embedding*

$$H^2 \subset \mathcal{H}_S^2(\mathcal{O}^{q^2}).$$

PROOF. We have a homomorphism

$$(5.9) \quad Z^2 \longrightarrow \mathcal{H}_S^2(\mathcal{O}^{q^2})$$

which assigns to τ the cohomology class $c(\tau)$ defined as above. Suppose $c(\tau) = 0$. Then there is a small polydisk U with center 0 such that τ is represented by a 2-form on U and that $c_U(\tau) = 0$ in $H_{S \cap U}^2(U, \mathcal{O}_U^{q^2})$. Choose η_i^λ and $\psi_{ij}^{\lambda\mu}$ so that (5.3) and (5.4) hold. Since $c_U(\tau) = 0$, we have, for each λ , a matrix-valued holomorphic function $(\varphi_{ij}^\lambda)_{1 \leq i, j \leq q}$ on U_λ such that

$$\psi_{ij}^{\lambda\mu} = \varphi_{ij}^\mu - \varphi_{ij}^\lambda \quad \text{on } U_\lambda \cap U_\mu.$$

From (5.4), we have

$$\eta_i^\lambda + \sum_{j=1}^q \varphi_{ij}^\lambda \omega_j = \eta_i^\mu + \sum_{j=1}^q \varphi_{ij}^\mu \omega_j \quad \text{on } U_\lambda \cap U_\mu.$$

This shows that we get, for each i , a holomorphic 1-form ξ_i on $U-S$ by setting $\xi_i = \eta_i^\lambda + \sum_{j=1}^q \varphi_{ij}^\lambda \omega_j$ on U_λ . By assumption, $\text{codim } S \geq 2$. Hence ξ_i can be extended to a holomorphic 1-form on U , which we denote also by ξ_i . From (5.5), we have

$$\varphi_{ji}^\lambda - \varphi_{ij}^\lambda = \varphi_{ji}^\mu - \varphi_{ij}^\mu \quad \text{on } U_\lambda \cap U_\mu.$$

Hence we get a holomorphic function f_{ij} on $U-S$ by setting $f_{ij} = \varphi_{ji}^\lambda - \varphi_{ij}^\lambda$ on U_λ . Again, each f_{ij} can be extended to a holomorphic function on U . We then compute

$$\begin{aligned} \tau &= \sum_{i=1}^q \eta_i^\lambda \wedge \omega_i = \sum_{i=1}^q \xi_i \wedge \omega_i + \sum_{i,j=1}^q \varphi_{ij}^\lambda \omega_j \wedge \omega_i \\ &= \sum_{i=1}^q \xi_i \wedge \omega_i + \sum_{1 \leq i < j \leq q} f_{ij} \omega_i \wedge \omega_j. \end{aligned}$$

This shows that τ is in B^2 . Hence the homomorphism (5.9) induces a monomorphism $H^2 \rightarrow \mathcal{H}_S^2(\mathcal{O}^{q^2})$. Q.E.D.

(5.10) COROLLARY. *If $\text{codim } S \geq 3$, then $H^2 = 0$.*

(5.11) REMARK. The above is a special case of the generalized de Rham lemma ([6] Proposition (2.1), [9] Theorem).

Now we take a system $\tau = (\tau_1, \dots, \tau_q)$ of germs of holomorphic 2-forms with each τ_i , $1 \leq i \leq q$, satisfying (5.2). Let U be a small polydisk with center 0 on which $\omega_1, \dots, \omega_q, \tau_1, \dots, \tau_q$ have representatives. We may write (see (5.3))

$$(5.12) \quad \tau_i = \sum_{j=1}^q \eta_{ij}^\lambda \wedge \omega_j, \quad 1 \leq i \leq q, \quad \text{on } U_\lambda.$$

Then we have ((5.4) and (5.5))

$$(5.13) \quad \eta_{ij}^\lambda - \eta_{ij}^\mu = \sum_{k=1}^q \psi_{ijk}^{\lambda\mu} \omega_k, \quad 1 \leq i, j \leq q, \quad \text{on } U_\lambda \cap U_\mu,$$

$$(5.14) \quad \psi_{ijk}^{\lambda\mu} = \psi_{ikj}^{\lambda\mu}, \quad 1 \leq i, j, k \leq q.$$

Let $h = (h_1, \dots, h_q)$ be an element in $\mathcal{O}^q$. We may suppose, by taking a smaller U if necessary, that each h_i has a representative on U . We define $hc_U(\tau)$ to be the cohomology class in $H^1(U-S, \mathcal{O}_U^{q^2}) \simeq H_S^2(U, \mathcal{O}_U^{q^2})$ represented by the cocycle

$$(5.15) \quad \left\{ \left(\sum_{j=1}^q h_j \psi_{ijk}^{\lambda\mu} \right)_{1 \leq i, k \leq q} \right\}.$$

We denote by $hc(\tau)$ the image in $\mathcal{H}_S^2(\mathcal{O}^{q^2})$ of $hc_\tau(\tau)$ by the homomorphism (5.7). It is easily checked that $hc(\tau)$ depends only on h and τ and not on the representatives. Now we write

$$\omega_i = \sum_{j=1}^n f_{ij} dz_j,$$

$1 \leq i \leq q$, and consider the systems $f_k = (f_{1k}, \dots, f_{qk})$, $1 \leq k \leq n$. If we write $\eta_{ij}^\lambda = \sum_{k=1}^q g_{ijk}^\lambda dz_k$ on U_λ , then from (5.13) and (5.14) we have

$$\sum_{l=1}^q f_{lk} \psi_{ilj}^{\lambda\mu} = \sum_{l=1}^q f_{lk} \psi_{ijl}^{\lambda\mu} = g_{ijk}^\lambda - g_{ijl}^\mu \quad \text{on } U_\lambda \cap U_\mu.$$

This shows that $f_k c(\tau) = 0$ in $\mathcal{H}_S^2(\mathcal{O}^{q^2})$, $1 \leq k \leq n$. Therefore, for each element $[h]$ in $\text{Ext}_\phi^1(\Omega_F, \mathcal{O})$, $[h]c(\tau)$ is a well defined element in $\mathcal{H}_S^2(\mathcal{O}^{q^2})$.

(5.16) REMARK. In this section we did not use the integrability of F . We only need that F is a rank q free sub- $\mathcal{O}$ -module of $\Omega_{C^n,0}$.

§ 6. $U(F)$

Let F be a codim q local foliation at 0 in C^n . We fix a system $(\omega_1, \dots, \omega_q)$ of generators of F and set $d\omega = (d\omega_1, \dots, d\omega_q)$. By the integrability

$$d\omega_i \wedge \omega_1 \wedge \dots \wedge \omega_q = 0, \quad 1 \leq i \leq q,$$

of F , for each element $[h]$ in $\text{Ext}_\phi^1(\Omega_F, \mathcal{O})$, we have the class $[h]c(d\omega)$ in $\mathcal{H}_S^2(\mathcal{O}^{q^2})$ as defined in the previous section.

(6.1) THEOREM. If F is a codim q reduced local foliation, the set $U(F)$ of equivalence classes of first order unfoldings of F is given by

$$U(F) = \{[h] \in \text{Ext}_\phi^1(\Omega_F, \mathcal{O}) \mid [h]c(d\omega) = 0 \quad \text{in } \mathcal{H}_S^2(\mathcal{O}^{q^2})\}.$$

PROOF. Let U be a small polydisk with center 0 in C^n on which the germs ω_i have representatives and let F be the foliation on U generated by $\omega_1, \dots, \omega_q$. We set $S = S(F)$ and take a covering $\{U_\lambda\}_{\lambda \in A}$ of $U - S$ by small polydisks U_λ . If we write (see (5.12))

$$(6.2) \quad d\omega_i = \sum_{j=1}^q \eta_{ij}^\lambda \wedge \omega_j, \quad 1 \leq i \leq q, \quad \text{on } U_\lambda,$$

then we have the identities (5.13) and (5.14). Now we take an element $[h] = [h_1, \dots, h_q]$ in $\text{Ext}_\phi^1(\Omega_F, \mathcal{O})$. We may assume that the germs h_i have representatives on U . Recall that the cohomology class $hc_\tau(\tau)$ is represented by the cocycle (5.15).

Suppose that there exists a first order unfolding $\mathcal{F}^{(1)}$ of F which determines the system $h=(h_1, \dots, h_q)$. Let $\mathcal{F}^{(1)}$ be a first order unfolding of F on $U \times V$ which represents $\mathcal{F}^{(1)}$, where V is a small disk with center 0 in $C=\{t\}$. We choose a system $(\tilde{\omega}_1, \dots, \tilde{\omega}_q)$ of generators of $\mathcal{F}^{(1)}$ which extends the system $(\omega_1, \dots, \omega_q)$. We may write

$$(6.3) \quad \tilde{\omega}_i = \omega_i + \omega_i^{(1)}t + h_i dt \quad \text{mod}(t^2, tdt), \quad 1 \leq i \leq q,$$

where $\omega_i^{(1)}$ are 1-forms involving only z and dz . Substituting (6.2) in the second identity of (4.7), we obtain

$$\left(dh_i - \omega_i^{(1)} - \sum_{j=1}^q h_j \eta_{ij}^\lambda \right) \wedge \Omega = 0.$$

Hence we may write

$$dh_i - \omega_i^{(1)} - \sum_{j=1}^q h_j \eta_{ij}^\lambda = \sum_{k=1}^q \varphi_{ik}^\lambda \omega_k, \quad 1 \leq i \leq q, \quad \text{on } U_\lambda,$$

where φ_{ik}^λ are holomorphic functions on U_λ . In the intersection $U_\lambda \cap U_\mu$, we have

$$\sum_{j=1}^q h_j \psi_{ij}^{\lambda\mu} \omega_k = \sum_{j=1}^q h_j \eta_{ij}^\lambda - \sum_{j=1}^q h_j \eta_{ij}^\mu = \sum_{k=1}^q (\varphi_{ik}^\mu - \varphi_{ik}^\lambda) \omega_k.$$

Since $\omega_1, \dots, \omega_q$ are linearly independent at each point of $U_\lambda \cap U_\mu$, we get

$$(6.4) \quad \sum_{j=1}^q h_j \psi_{ij}^{\lambda\mu} = \varphi_{ik}^\mu - \varphi_{ik}^\lambda.$$

This shows that the cohomology class $hc_U(d\omega)$ is zero. Hence $[h]c(d\omega)=0$ in $\mathcal{H}_S^2(\mathcal{O}^{q^2})$.

Conversely, if $[h]c(d\omega)=0$, then, by taking smaller U and U_λ 's if necessary, there are functions φ_{ik}^λ satisfying (6.4). If we set $\omega_i^\lambda = \sum_{j=1}^q h_j \eta_{ij}^\lambda + \sum_{k=1}^q \varphi_{ik}^\lambda \omega_k$ on U_λ , then it is easily checked that $\omega_i^\lambda = \omega_i^\mu$ on $U_\lambda \cap U_\mu$. Hence, for each i , ω_i^λ defines a 1-form on $U-S$. By the assumption that F is reduced, we see that $\text{codim } S \geq 2$ ((5.1) Lemma). Thus ω_i^λ extends to a 1-form on U . We set

$$(6.5) \quad \omega_i^{(1)} = dh_i - \sum_{j=1}^q h_j \eta_{ij}^\lambda - \sum_{k=1}^q \varphi_{ik}^\lambda \omega_k \quad \text{and}$$

$$(6.6) \quad \tilde{\omega}_i = \omega_i + \omega_i^{(1)}t + h_i dt, \quad 1 \leq i \leq q.$$

Then the system $(\tilde{\omega}_1, \dots, \tilde{\omega}_q)$ satisfies the second identity of (4.7). We now show that the first is also satisfied. Using (6.5) we compute

$$d\omega_i^{(1)} \wedge \Omega - d\omega_i \wedge \left(\sum_{j=1}^q (-1)^j \omega_j^{(1)} \wedge \Omega_j \right) = \sum_{j=1}^q h_j \left(\sum_{k=1}^q \eta_{ik}^\lambda \wedge \eta_{kj}^\lambda - d\eta_{ij}^\lambda \right) \wedge \Omega.$$

On the other hand, computing $d^2\omega_i$, we get

$$\sum_{j=1}^q \left(d\eta_{ij}^{\lambda} - \sum_{k=1}^q \eta_{ik}^{\lambda} \wedge \eta_{kj}^{\lambda} \right) \wedge \omega_j = 0.$$

Hence we have

$$\left(d\eta_{ij}^{\lambda} - \sum_{k=1}^q \eta_{ik}^{\lambda} \wedge \eta_{kj}^{\lambda} \right) \wedge \Omega = (-1)^j \sum_{l=1}^q \left(d\eta_{il}^{\lambda} - \sum_{k=1}^q \eta_{ik}^{\lambda} \wedge \eta_{kl}^{\lambda} \right) \wedge \omega_l \wedge \Omega_j = 0.$$

Therefore, the system $(\tilde{\omega}_1, \dots, \tilde{\omega}_q)$ satisfies the first order integrability condition. If we let $\mathcal{F}^{(1)}$ be the germ of the first order unfolding generated by $(\tilde{\omega}_1, \dots, \tilde{\omega}_q)$, then $\mathcal{F}^{(1)}$ determines $h = (h_1, \dots, h_q)$.

(6.7) REMARK. By the above theorem, $U(F)$ is a sub- $\mathcal{O}$ -module, thus a sub- $\mathcal{C}$ -vector space, of $\text{Ext}_\mathcal{O}^1(\Omega_F, \mathcal{O})$.

(6.8) REMARK. To find the first order unfoldings of a local foliation F ,

1°. Find $\text{Ext}_\mathcal{O}^1(\Omega_F, \mathcal{O})$ by (4.5).

2°. Find η_{ij}^{λ} and $\psi_{ij}^{\lambda\mu}$ satisfying (6.2) and (5.13).

3°. Pick up the elements $[h] = [h_1, \dots, h_q]$ in $\text{Ext}_\mathcal{O}^1(\Omega_F, \mathcal{O})$ such that (6.4) holds for some φ_{ik}^{λ} .

4°. For each $[h]$ in 3°, determine $\tilde{\omega}$ by (6.5) and (6.6).

When the codimension of F is one, there is a simpler way. If we set $\omega = \omega_1$, then the first order integrability condition in (4.7) becomes

$$(6.9) \quad d\omega^{(1)} \wedge \omega + \omega^{(1)} \wedge d\omega = 0 \quad \text{and}$$

$$(6.10) \quad h d\omega + (\omega^{(1)} - dh) \wedge \omega = 0.$$

It is easy to show that (6.10) implies (6.9). If we use the cohomology group H^2 instead of $\mathcal{H}_S^2(\mathcal{O})$ ((5.8)), then the class $[h]c(d\omega)$ is represented by the form $hd\omega$. Hence we have ((6.1) Theorem)

$$(6.11) \quad U(F) = \{[h] \in \text{Ext}_\mathcal{O}^1(\Omega_F, \mathcal{O}) \mid h d\omega = \eta \wedge \omega \text{ for some } \eta \in \Omega\}.$$

Thus if $[h]$ is in $U(F)$, and if $h d\omega = \eta \wedge \omega$,

$$\tilde{\omega} = \omega + \omega^{(1)}t + h dt, \quad \omega^{(1)} = dh - \eta,$$

is a first order unfolding representing $[h]$.

Note that if $\mathcal{F}$ is an unfolding of F , then the image of the infinitesimal unfolding map of $\mathcal{F}$ is contained in $U(F)$.

(6.12) DEFINITION. An element $[h]$ in $U(F)$ is said to be unobstructed if it is in the image of the infinitesimal unfolding map of a certain unfolding of F , i.e., if the equivalence class $[h]$ is represented by an unfolding.

If $F=(df_1, \dots, df_q)$ is a codim q Haefliger foliation, then by (4.5),

$$\mathrm{Ext}_\mathcal{O}^1(\Omega_F, \mathcal{O}) = \mathcal{O}^q / \left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n} \right),$$

where $\partial f / \partial z_j$ denotes the vector $(\partial f_1 / \partial z_j, \dots, \partial f_q / \partial z_j)$. We have $U(F) = \mathrm{Ext}_\mathcal{O}^1(\Omega_F, \mathcal{O})$ and none of the elements in $U(F)$ is obstructed. In fact, for any element $[h] = [h_1, \dots, h_q]$ in $\mathrm{Ext}_\mathcal{O}^1(\Omega_F, \mathcal{O})$, $\mathcal{F} = (d(f_1 + h_1 t), \dots, d(f_q + h_q t))$ is an unfolding of F representing $[h]$.

Also if $\mathcal{H}_S^2(\mathcal{O}^{q^2}) = 0$, then we have $U(F) = \mathrm{Ext}_\mathcal{O}^1(\Omega_F, \mathcal{O})$. This is the case if $\mathrm{codim} S(F) \geq 3$. In fact we have

(6.13) THEOREM (Malgrange [6], [7], Moussu [8]). *Let F be a codim q local foliation. If $\mathrm{codim} S(F) \geq 3$, then F is a Haefliger foliation.*

The proof of this theorem can be interpreted as a construction of a certain non-singular unfolding of F . The most difficult part is the convergence of the formal construction. The result then follows from the Frobenius theorem.

It is not difficult to show the following

(6.14) PROPOSITION. *Let $F=(\omega)$ be a codim 1 local foliation and let $[1]$ denote the class in $\mathrm{Ext}_\mathcal{O}^1(\Omega_F, \mathcal{O})$ represented by the constant function 1. Then the following are equivalent:*

- (i) *F is a Haefliger foliation, i.e., $\omega = gdf$ for some f in $\mathcal{O}$ and a unit g in $\mathcal{O}$.*
- (ii) *$U(F) = \mathrm{Ext}_\mathcal{O}^1(\Omega_F, \mathcal{O})$ and none of the elements in $U(F)$ is obstructed.*
- (iii) *$[1]$ is in $U(F)$ and is not obstructed.*

(6.15) DEFINITION. An unfolding $\mathcal{F}$ of a local foliation F is infinitesimally versal if the image of the infinitesimal unfolding map of $\mathcal{F}$ is $U(F)$.

In [10], it is proved that an infinitesimally versal unfolding of a codim 1 local (reduced) foliation is versal.

(6.16) DEFINITION. An unfolding $\mathcal{F}$ of F is universal if it is versal and if the infinitesimal unfolding map of $\mathcal{F}$ is injective.

(6.17) EXAMPLES. Let $n=2$, $q=1$ and let $F=(\omega)$ be given by

$$1^\circ. \quad \omega = z_2 dz_1 - z_1 dz_2.$$

We have $\mathrm{Ext}_\mathcal{O}^1(\Omega_F, \mathcal{O}) = C\{z_1, z_2\} / (z_1, z_2) = C$ and we may take $[1]$ as a basis element. Since $d\omega = -2dz_1 \wedge dz_2$, we see that $U(F) = 0$. Hence by the versality theorem in [10], F is a universal unfolding of F itself and every unfolding of F is trivial. We remark that if we try to construct an unfolding of F representing $[1]$, we obtain a "meromorphic" foliation. Namely, since

$$d\omega = \eta \wedge \omega, \quad \eta = \frac{2}{z_1} dz_1,$$

setting $\omega^{(1)} = dh - \eta = -(2/z_1) dz_1$, we have

$$\tilde{\omega} = \omega + \omega^{(1)}t + hdt = \left(z_2 - \frac{2}{z_1}t\right) dz_1 - z_1 dz_2 + dt.$$

It is easily shown that $\tilde{\omega}$ actually satisfies the integrability $d\tilde{\omega} \wedge \tilde{\omega} = 0$.

$$2^\circ. \quad \omega = z_2^2 dz_1 - z_1 dz_2.$$

We have $\text{Ext}_\theta^1(\Omega_F, \mathcal{O}) = C\{z_1, z_2\}/(z_1, z_2^2) = C^2$. However, it is easily checked that $U(F) = 0$.

$$3^\circ. \quad \omega = z_2^2 dz_1 - z_1^2 dz_2.$$

We have $\text{Ext}_\theta^1(\Omega_F, \mathcal{O}) = C^4$ and we may take $[1], [z_1], [z_2]$ and $[z_1 z_2]$ as basis elements. Also it is not difficult to see that $U(F) = C^2$ and we may take $[z_1 - z_2]$ and $[z_1 z_2]$ as basis elements. If we set $h_1 = z_1 z_2$, then we have

$$h_1 d\omega = \eta_1 \wedge \omega, \quad \eta_1 = 2d(z_1 z_2).$$

Thus $\mathcal{F}_1 = (\tilde{\omega}_1)$,

$$\tilde{\omega}_1 = \omega + \omega_1^{(1)}t_1 + h_1 dt_1, \quad \omega_1^{(1)} = dh_1 - \eta_1 = -d(z_1 z_2),$$

is a first order unfolding of F representing $h_1 = z_1 z_2$. Moreover, it is easily checked that $\tilde{\omega}_1$ is integrable; $d\tilde{\omega}_1 \wedge \tilde{\omega}_1 = 0$, and thus $\mathcal{F}_1$ is actually an unfolding of F . Also if we set $h_2 = z_1 - z_2$, we have

$$h_2 d\omega = \eta_2 \wedge \omega, \quad \eta_2 = 2d(z_1 - z_2).$$

Thus $\mathcal{F}_2 = (\tilde{\omega}_2)$,

$$\tilde{\omega}_2 = \omega + \omega_2^{(1)}t_2 + h_2 dt_2, \quad \omega_2^{(1)} = dh_2 - \eta_2 = -d(z_1 - z_2),$$

is a first order unfolding of F representing $h_2 = z_1 - z_2$. It is easily checked that $\tilde{\omega}_2$ is integrable and $\mathcal{F}_2$ is also an unfolding of F . We combine $\tilde{\omega}_1$ and $\tilde{\omega}_2$ to obtain an integrable form

$$\begin{aligned} \tilde{\omega} &= \omega + \omega_1^{(1)}t_1 + \omega_2^{(1)}t_2 + h_1 dt_1 + h_2 dt_2 - t_2 dt_1 + t_1 dt_2 \\ &= (z_2^2 - z_2 t_1 - t_2) dz_1 - (z_1^2 + z_1 t_1 - t_2) dz_2 + (z_1 z_2 - t_2) dt_1 + (z_1 - z_2 + t_1) dt_2. \end{aligned}$$

Then by the versality theorem in [10], $\mathcal{F} = (\tilde{\omega})$ is a universal unfolding of F .

$$4^\circ. \quad \omega = z_2 dz_1 + (z_1 + z_1^2 z_2) dz_2.$$

We have $\text{Ext}_\theta^1(\Omega_F, \mathcal{O}) = C\{z_1, z_2\}/(z_1, z_2) = C$. Since

$$d\omega = \eta \wedge \omega, \quad \eta = -2z_1 dz_2,$$

we have $U(F) = \text{Ext}_\theta^1(\Omega_F, \mathcal{O})$ and $\mathcal{F}^{(1)} = (\tilde{\omega})$,

$$\tilde{\omega} = z_2 dz_1 + (z_1 + z_1^2 z_2 + 2z_1 t) dz_2 + dt$$

is a first order unfolding representing [1]. However, since ω does not admit integrating factors, $\mathcal{F}^{(1)}$ is not equivalent to an unfolding. Thus the element [1] (and hence every non-zero element) in $U(F)$ is obstructed.

For more examples and applications, see [11], [14] and [15].

§ 7. First order unfoldings of global foliations

Let M be a compact complex manifold. If we denote by $D(M)$ the set of equivalence classes of first order deformations of M , then it is well-known that there is a bijection

$$(7.1) \quad D(M) \simeq H^1(M, \Theta_M).$$

(7.2) REMARK. If $\mathcal{M}^{(1)}$ is a first order deformation of M , then it is represented by the following data:

(i) a finite open covering $\{U_\lambda\}_{\lambda \in A}$ of M by coordinate neighborhoods U_λ (If we denote by $z^\lambda = (z_1^\lambda, \dots, z_n^\lambda)$ a coordinate system on U_λ , then in the intersection $U_\lambda \cap U_\mu$, z^λ is a vector-valued holomorphic function of z^μ ; $z^\lambda = f^{\lambda\mu}(z^\mu)$),

(ii) a neighborhood V of 0 in $C = \{t\}$ and

(iii) a collection $\{\Psi_{\lambda\mu}\}$, where for each pair (λ, μ) , $\Psi_{\lambda\mu}$ is a biholomorphic map of $(U_\lambda \cap U_\mu) \times V$ onto itself of the form $\Psi_{\lambda\mu}(z^\mu, t) = (f^{\lambda\mu}(z^\mu, t), t)$, satisfying

$$f^{\lambda\mu}(z^\mu, 0) = f^{\lambda\mu}(z^\mu),$$

$$f^{\lambda\nu}(z^\nu, t) \equiv f^{\lambda\mu}(f^{\mu\nu}(z^\nu, t), t), \quad \text{for } (z^\lambda, t) \text{ in } (U_\lambda \cap U_\mu \cap U_\nu) \times V, \text{ mod } t^2.$$

Under the bijection (7.1), the equivalence class of $\mathcal{M}^{(1)}$ corresponds to the cohomology class represented by the cocycle $\left\{ \sum_{i=1}^n \frac{f_i^{\lambda\mu}}{\partial t} (z^\mu, 0) \frac{\partial}{\partial z_i^\lambda} \right\}$.

Now let F be a codim q foliation (of complete intersection type) on M .

(7.3) DEFINITION. A first order unfolding $\mathcal{F}^{(1)}$ of F consists of

(i) a first order deformation $\mathcal{M}^{(1)} = (\{U_\lambda\}, V, \{\Psi_{\lambda\mu}\})$ of M ,

(ii) a collection $\{\mathcal{F}_\lambda^{(1)}\}$, where, for each λ , $\mathcal{F}_\lambda^{(1)}$ is a first order unfolding of $F|_{U_\lambda}$ on $U_\lambda \times V$, satisfying

$$\mathcal{F}_\mu^{(1)}|_{(U_\lambda \cap U_\mu) \times V} \equiv (\Psi_{\lambda\mu})^{**}(\mathcal{F}_\lambda^{(1)}|_{(U_\lambda \cap U_\mu) \times V}) \quad \text{mod}(t^2, tdt).$$

The equivalence between two first order unfoldings of (M, F) is defined in an obvious way. Let $U(M, F)$ denote the set of equivalence classes of

first order unfoldings of (M, F) . Then we have an embedding

$$U(M, F) \hookrightarrow \text{Ext}_{\mathcal{O}_M}^1(\Omega_F, \mathcal{O}_M),$$

which assigns to the equivalence class of a first order unfolding $(\mathcal{M}^{(1)}, \mathcal{F}^{(1)}) = (\{U_\lambda\}, V, \{\Psi^{\lambda\mu}\}, \{\mathcal{F}_\lambda^{(1)}\})$, the class in $\text{Ext}_{\mathcal{O}_M}^1(\Omega_F, \mathcal{O}_M)$ (see (3.3)) represented by $(\theta, \varphi) = (\{\theta^{\lambda\mu}\}, \{\varphi^\lambda\})$ with

$$\theta^{\lambda\mu} = \sum_{i=1}^n \frac{\partial f_i}{\partial t}(z^\mu, 0) \frac{\partial}{\partial z_i^\mu} \quad \text{and} \quad \varphi^\lambda(\omega) = h_\lambda,$$

where for a 1-form ω in $\Gamma(U_\lambda, F)$, h_λ is the holomorphic function on U_λ given as the coefficient of dt of an extension $\tilde{\omega}$ of ω in $\Gamma(U_\lambda \times V, \mathcal{F}_\lambda^{(1)})$.

Now we determine the set $U(M, F)$. Recall that for each point z in M , the set $U(F_z)$ of equivalence classes of first order unfoldings of the germ F_z of the foliation F at z is canonically embedded in $\text{Ext}_{\mathcal{O}_{M,z}}^1(\Omega_{F,z}, \mathcal{O}_{M,z}) = \mathcal{E}xt_{\mathcal{O}_M}^1(\Omega_F, \mathcal{O}_M)_z$ (section 4) and the set is determined in section 6. We define $\mathcal{U}(M, F)$ to be the subsheaf of $\mathcal{E}xt_{\mathcal{O}_M}^1(\Omega_F, \mathcal{O}_M)$ given by $\mathcal{U}(M, F) = \bigcup_{z \in M} U(F_z)$ and call it the sheaf of germs of first order unfoldings of (M, F) .

Let ε be the edge homomorphism in (2.10). The proof of the following theorem is straightforward.

(7.4) **THEOREM.** *The set $U(M, F)$ of equivalence classes of first order unfoldings of (M, F) is given by*

$$U(M, F) = \varepsilon^{-1}(\Gamma(M, \mathcal{U}(M, F))).$$

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DEPARTMENT OF MATHEMATIC
HOKKAIDO UNIVERSITY
SAPPORO 060, JAPAN