$C^\ast$-algebras and Boolean valued analysis

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Let $\mathcal{D}$ be an abelian von Neumann algebra and $\mathcal{B}$ the complete Boolean algebra of all the projections in $\mathcal{D}$. Then Boolean valued analysis, developed in [8] and [9], provides us with a transfer principle on operator theory. Such a transfer principle about the theory of von Neumann algebras was studied in [10]. In this paper we investigate a transfer principle obtained for the theory of $C^\ast$-algebras. This transfer principle can be considered as a uniform method to tell what happens to the theory of $C^\ast$-algebras if we use $\mathcal{D}$ in the place of $\mathcal{C}$. $C^\ast$-algebra, *-homomorphism, separability, countable approximate identities, nuclearity etc. should be replaced by the notions $\mathcal{D}$-$C^\ast$-algebra, $\mathcal{D}$-$*$-homomorphism, $\mathcal{D}$-separability, $\mathcal{D}$-countable approximate identities, $\mathcal{D}$-nuclearity etc., respectively. First we develop a general theory. Then we apply our theory to Brown-Douglas-Filmore's $C^\ast$-algebra extension theory, more precisely, we apply our theory to a special case of Kasparov's operator K-functor theory where $G$ is trivial and $\mathcal{C}$ is $\mathcal{C}$. Our main objective is to investigate the nature of our transfer principle. However since $\mathcal{D}$-separability is weaker than separability, and $\mathcal{D}$-nuclearity is weaker than nuclearity etc., there are $C^\ast$-algebras to which the standard $C^\ast$-extension theory cannot be applied but to which our $\mathcal{D}$-$C^\ast$-extension theory can be applied.

Since operator theory is outside of the author's fields, he has asked many elementary questions to operator theorists and they have answered very kindly. He is very grateful to all these people especially to Jun Tomiyama who has been most helpful.

§1. $\mathcal{D}$-$C^\ast$-algebras

Let $\mathcal{D}$ be an abelian von Neumann algebra, and $\mathcal{B}$ the complete Boolean algebra of all projections in $\mathcal{D}$. $\mathcal{B}$ is a subset of $\mathcal{D}$. We identify an element $z \in \mathcal{D}$ and a complex number in $V^{(\mathcal{B})}$ which corresponds to $z$.

**Definition.** Let $\mathcal{A} = \langle \mathcal{A}, +, \cdot, \| \cdot \| \rangle$ be a normed space in $V^{(\mathcal{B})}$. The bounded part $\mathcal{A}$ of $\mathcal{A}$ is defined as follows.
\[ \mathcal{A} = \langle A, +, \cdot, \| \| \rangle \] where

1) \[ A = \{ x \in A | \exists M \in \mathbb{R} \| x \| \leq M \} \] and for every pair \( x, y \) in \( A \), \( x = y \) iff \[ [x = y] = 1. \]

2) For every pair \( x, y \) in \( A \), the sum \( u \) of \( x \) and \( y \) is defined by \[ [u = x + y] = 1. \] \( u \) is uniquely defined as a member of \( A \). The sum of \( x \) and \( y \) in \( A \) is also denoted by \( x + y \).

3) For every \( x \in A \) and for every \( z \in \mathcal{Z} \), the product of \( z \) and \( x \) is defined by \[ [u = z \cdot x] = 1. \] \( u \) is uniquely defined as a member of \( A \). The product of \( z \) and \( x \) in \( A \) is also denoted by \( z \cdot x \).

4) For every \( x \in A \), \( \| x \| \) is defined by

\[ \| x \| = \inf \{ M \in \mathbb{R} | \| x \| \leq M \} \text{ holds in } V^{(\mathfrak{C})}. \]

From now on, we use symbols \( \hat{A} \) and \( A \) in the place of \( \mathcal{A} \) and \( \mathcal{A} \) respectively.

**Definition.** Let \( \mathcal{A} = \langle A, +, \cdot, \| \| \rangle \) be a normed space where \( z \cdot x \) is defined for every pair \( x \in A \) and \( z \in \mathcal{Z} \). \( \mathcal{A} \) is said to be a normed \( \mathcal{Z} \)-space if \( \mathcal{A} \) is a \( \mathcal{Z} \)-module satisfying the following condition

\[ x \in A, z \in \mathcal{Z} \implies \| z \cdot x \| \leq \| z \| \cdot \| x \|. \]

**Proposition 1.** If \( \hat{A} \) is a normed space in \( V^{(\mathfrak{C})} \), then the bounded part \( A \) of \( \hat{A} \) is a normed \( \mathcal{Z} \)-space.

**Proof.** Let \( x \) and \( y \) be members of \( A \), \( \alpha \) a complex number, and \( z \) a member of \( \mathcal{Z} \).

1) \[ \| x + y \| \leq \| x \| + \| y \|. \]
   This follows from \( \| x + y \| \leq \| x \| + \| y \| \).

2) \[ \| x \| \geq 0 \text{ and } \| x \| = 0 \text{ iff } x = 0. \]
   \[ \| x \| = 0 \text{ iff } \| x \| = 0 \text{ iff } x = 0. \]

3) \[ \| \alpha x \| = |\alpha| \cdot \| x \|. \]
   \[ \| \alpha x \| = |\alpha| \cdot \| x \|. \]

4) \[ \| z x \| = \| z \| \cdot \| x \|. \]
   \[ \| z x \| = \| z \| \cdot \| x \|. \]

where \( |z|' \) denotes the absolute value of \( z \) as a complex number in \( V^{(\mathfrak{C})} \).

**Proposition 2.** Let \( \hat{A} \) be a normal space in \( V^{(\mathfrak{C})} \) and \( A \) the bounded part of \( \hat{A} \). \( \hat{A} \) is complete in \( V^{(\mathfrak{C})} \) iff \( A \) is complete.

**Proof.** Let \( \hat{A} \) be complete in \( V^{(\mathfrak{C})} \) and \( \{ x_i \} \) a Cauchy sequence in \( A \) i.e.

\[ \forall \varepsilon > 0 \exists N \forall i, j \geq N \| x_i - x_j \| < \varepsilon. \]

Then \( \{ x_i \} \) is a Cauchy sequence in \( \hat{A} \). Therefore there exists a unique \( x \) in \( \hat{A} \) such that \( x_i \to x \) in \( \hat{A} \). Since \( \{ x_i \} \) is a Cauchy sequence, there exists a real
number $M$ such that $\forall i(\|x_i\| \leq M)$. Then $\forall i(\|x_i\|^* \leq M^*)$. Therefore $\|x\|^* \leq M^*$ and $x \in A$. Now for every $\varepsilon > 0$ there exist a partition of unity $\{b_i\}$ in $\mathcal{B}$ and $\{N_i\} \subseteq N$ such that every $m \geq N_i$,

$$\left[\|x - x_m\|^* < \varepsilon\right] \geq b_i.$$

Now there exists $N(\varepsilon)$ such that,

$$i, j \geq N(\varepsilon) \implies \|x_i - x_j\| < \varepsilon.$$

Let $i \geq N(\varepsilon)$. For every $b_i$, there exists $m$ such that $m \geq N(\varepsilon)$ and

$$\left[\|x - x_m\|^* \leq \varepsilon\right] \geq b_i.$$

Therefore we have

$$\left[\|x - x_m\|^* \leq \|x - x_n\|^* + \|x_m - x_n\|^* \leq 2\varepsilon\right] \geq b_i.$$

Therefore $\|x - x_n\|^* \leq 2\varepsilon$ holds in $V(\varepsilon)$ and $x_n \to x$ in $A$.

Conversely let $\{x_i\}$ be a Cauchy sequence in $\tilde{A}$. Since $\{x_i\}$ is a Cauchy sequence, there exists $u \in \mathbb{R}^+$ such that $u$ is positive and $\forall i(\|x_i\|^* \leq u)$ holds in $V(u)$. Since the convergence of $\{x_i\}$ is equivalent to the convergence of $\{x_i/u\}$, we may assume that $x_i$ satisfies $\forall i(\|x_i\|^* \leq 1)$ and $\{x_i\} \subseteq A$. For every natural number $k$ there exist a partition of unity $\{b_k^i\}$ in $\mathcal{B}$ and $\{N_k^i\} \subseteq N$ such that for every $n, m \geq N_k^i$, the following holds in $V(\varepsilon^k)$

$$\|b_k^i x_n - b_k^i x_m\|^* \leq 1/2^k.$$

Without loss of generality, we assume that for every pair $k_1, k_2$ with $k_1 < k_2$, $\{b_k^i\}$ is a refinement of $\{b_k^i\}$ and if $b_k^i \wedge b_k^j \neq 0$, then $N_k^i \geq N_k^j$. We define $y_k$ by the following equation

$$y_k = \sum_i b_k^i x_n^i.$$

Let $n, m \geq k$. Then for every $i$, we have

$$\left[\|y_n - y_m\|^* \leq 1/2^k\right] \geq b_k^i.$$

Therefore $\|y_n - y_m\| \leq 1/2^k$ and there exists $y$ in $A$ such that $y_n \to y$ in $A$. From the construction of $y$, we have

$$n \geq N_k^i \implies \|b_k^i x_n - b_k^i y\| \leq 1/2^{k-1}.$$

Therefore $x_n \to y$ in $\tilde{A}$.

**Definition.** Let $A$ be a $C^*$-algebra and a $\mathcal{Z}$-module. $A$ is said to be a normal $\mathcal{Z}$-module if the following conditions are satisfied.
1) For every pair \( a \in A \) and \( z \in \mathcal{Z} \),
\[
za = az, \quad (az)^* = z^*a^*, \quad ||az|| \leq ||a|| \cdot ||z||.
\]

2) Let \( b \in \mathcal{B} \) and \( b \neq 0 \). Then there exists \( a \in A \) such that \( ba \neq 0 \).

3) Let \( \{b_a\} \) be a partition of unity in \( \mathcal{B} \) and \( \{a_a\} \subseteq A \) and \( \{||a_a||\} \) be bounded. Then there exists a unique \( a \in A \) such that for every \( b_a, \ ab_a = a_a b_a \).

Such an \( a \) is also denoted by \( \sum_a b_a a_a \).

Let \( A \) be a normal \( \mathcal{Z} \)-module and \( z \in \mathcal{Z} \). Let \( a \in A \) satisfy
\[
x \in A \implies ax = xa = zx.
\]
Such an \( a \) is obviously unique if it exists. We identify such an \( a \) and \( z \).

Let \( D = \{ b \in \mathcal{B} | b \in A \} \). Let \( \{b_a\}_{a \leq a_0} \) be a well-ordering of \( D \). We define \( b_a' \) and \( b_a \) by the following conditions
\[
b_a' = b_a - \bigvee_{b \leq a} b \quad \quad b_a = \bigvee_{b \leq a} b_a.
\]
Then \( \{b_a'\}_{a \leq a_0} \cup \{1 - b_a\} \) is a partition of unity and \( \{b_a'\}_{a \leq a_0} \subseteq A \). We define \( \{a_a\}_{a \leq a_0} \) by the following equations:
\[
\alpha < a_0 \implies a_a = b_a' \quad \text{and} \quad a_{a_0} = 0.
\]

Let \( a = \sum_{a \leq a_0} b_a a_a + (1 - b_{a_0}) \cdot 0 \). Then \( a = b_a \) and \( b_{a_0} \in A \). Namely we have \( D = \{ b \in \mathcal{B} | b \leq b_{a_0} \} \). This \( b_{a_0} \) is called the characteristic of \( \mathcal{B} \) in \( A \). Obviously \( \mathcal{B} \subseteq A \) and \( 1 \in A \) are equivalent.

If \( A \) is not unital, then we can embed \( A \) into a normal unital \( \mathcal{Z} \)-module \( A(\mathcal{Z}) \) by the following method. Let \( b_0 \) be the characteristic of \( \mathcal{B} \) in \( A \). Then \( \mathcal{Z} = \mathcal{Z}_0 \oplus \mathcal{Z}_1 \), where \( \mathcal{Z}_0 = b_0 \mathcal{Z} \) and \( \mathcal{Z}_1 = (1 - b_0) \mathcal{Z} \). Then \( \mathcal{Z}_0 \subseteq A \). Now we claim \( A \cap \mathcal{Z}_1 = \{0\} \). Suppose that \( z \in A \cap \mathcal{Z}_1 \) and \( z \neq 0 \). Then there exists \( \varepsilon > 0 \) such that \( ||z|| > \varepsilon > 0 \) where \( ||z|| \in R \) is the absolute value of \( z \) as a complex number in \( V(\mathcal{Z}) \). Let \( ||z|| \geq \varepsilon \). Then \( b_0 \leq 1 - b_0 \). Let \( z \leq b_0 \). Then \( z_1, z_0 \neq 0 \) and \( z_1 \in A \cap \mathcal{Z}_1 \). Define \( z_i = (1 - b_0) + z_1 \). Then \( z_i \) is invertible in \( \mathcal{Z} \). Therefore \( b_i = z_i, z_0 \in A \cap \mathcal{Z}_1 \) which is a contradiction. We take the direct sum \( A \oplus \mathcal{Z}_1 \) as a linear space \( A(\mathcal{Z}) \) in which we define multiplication and the involution by the following equations:
\[
(x, z_1)(y, z_2) = (xy + z_1 x + z_2 y, z_1 z_2)
\]
\[
(x, z)^* = (x^*, z^*).
\]
We identify \( x \in A \) and \( (x, 0) \in A(\mathcal{Z}) \) and we write \( (x, z) = x + z \in A(\mathcal{Z}) \). Then \( A \) is an ideal of \( A(\mathcal{Z}) \). We put \( L_x y = xy \) for \( x \in A(\mathcal{Z}) \) and \( y \in A \). For each \( x \in A \), we have \( ||x|| = ||L_x|| \). Therefore we define \( ||x|| = ||L_x|| \) for \( x \in A(\mathcal{Z}) \) and the new norm is an extension of the old norm for \( A \). Suppose \( ||L_x|| = 0 \) for
some $x=x'+z$, $x' \in A$, $z \in \mathcal{Z}$, $z \neq 0$. Then $z = -x'$ which is a contradiction.

In order to prove that $A(\mathcal{Z})$ is complete, it suffices to show that for every $z \in \mathcal{Z}$, $\|L_z\| = \|z\|$, where $\|z\|$ is the norm of $z$ in $\mathcal{Z}$. Let $0 \neq b' \leq 1 - b_0$. Then there exists $x \in A$ such that $b'x \neq 0$. Put $y = b'x$. Then $y \neq 0$ and $b'y = y$. Therefore $\|L_y\| = 1$. In order to prove that $\|L_z\| = \|z\|$ for every $z \in \mathcal{Z}$, it suffices to show it for $z \in \mathcal{Z}_+$. Let $z > 0$, $\|z\| = M$, and $\varepsilon > 0$. Then $[(M-\varepsilon)^\vee \leq z] > 0$. Put $b_2 = [(M-\varepsilon)^\vee \leq z]$. Then $z \geq (M-\varepsilon)b_2$. Since $x \geq 0$ and $z \geq 0$ imply $zx \geq 0$, $\|L_z\| \geq M-\varepsilon$. Therefore we have $\|L_z\| = \|z\|$.

For every $x \in A(\mathcal{Z})$ and $\varepsilon > 0$, there exists $y \in A$ such that $\|xy\| \geq (1-\varepsilon)\|x\|\|y\|$ and $\|y\| > 0$. Since $xy \in A$, we have

$$\|y\|\|x^*x\| \geq \|(xy)^*(xy)\| = \|xy\|^2 \geq (1-\varepsilon)^2\|x\|^2\|y\|^2.$$  

Therefore we have $\|x\| \leq \|x^*x\| \leq \|x^*\|\|x\|$ and $\|x^*x\| = \|x\|^2$.

$A(\mathcal{Z})$ obtained by this construction is called the $\mathcal{Z}$-extension of $A$. If $A$ is a unital normal $\mathcal{Z}$-module, then the $\mathcal{Z}$-extension of $A$ is $A$ itself.

Now we give a definition of a normal unital $\mathcal{Z}$-module $\tilde{A}$ which is obtained by adjunction of an identity to $A$. Let $b_0$ be the characteristic of $\mathcal{B}$ in $A$, $\mathcal{Z}_0 = b_0\mathcal{Z}$, and $\mathcal{Z}_1 = (1 - b_0)\mathcal{Z}$ as before. Let $A_1$, the $\mathcal{Z}$-extension of $A$. We take the direct sum $A_1 \oplus \mathcal{Z}_0$ as a linear space $\tilde{A}$ in which we define multiplication and involution by the following equations:

$$(x, z_1)(y, z_2) = (xy + z_2x + z_1y, z_1z_2)$$

$$(x, z)^* = (x^*, z^*)$$

where $z$, $z_1$, and $z_2$ are members of $\mathcal{Z}_0$. We identify $x \in A_1$ and $(x, 0) \in \tilde{A}$ and we write $(x, z) = x + z \in \tilde{A}$. We denote the unit of $A_1$ by $\varepsilon$ and $z$ in $A_1$ by $ze$. A member of the form $(0, z)$ in $\tilde{A}$ is denoted by $z \cdot 1$. Then $(1 - \varepsilon) \cdot \mathcal{Z}_0$ and $A_1$ are complementary self-adjoint two sided $\mathcal{Z}$-ideals in $\tilde{A}$. Then we define $\|x + z\| = \max(\|x + z \cdot \varepsilon\|, \|z\|)$, where $\|x + z \cdot \varepsilon\|$ is the norm of $x + ze$ in $A_1$. It is immediately seen that $\tilde{A}$ is a normal unital $\mathcal{Z}$-module and the unit of $\tilde{A}$ is $(1 - b_0) \cdot \varepsilon + b_0 \cdot 1$. From now on the unit of $\tilde{A}$ is denoted by $1$.

**Definition.** Let $A$ be a $C^*$-algebra and a normal $\mathcal{Z}$-module. $A$ is said to be a $\mathcal{Z}$-$C^*$-algebra if $A$ satisfies the following condition. For every $x \in A$ and $z \in \mathcal{Z}_+$,

$$x^*x \leq z \iff xx^* \leq z$$

holds in the $\mathcal{Z}$-extension of $A$.

**Definition.** Let $A$ be a $\mathcal{Z}$-$C^*$-algebra and $x \in A$. $\|x\|^\mathcal{Z}$ is defined by the equation

$$\|x\|^\mathcal{Z} = \inf\{z \in \mathcal{Z}_+, x^*x \leq z^2 \text{ in } A(\mathcal{Z})\},$$
where \( A(\mathcal{A}) \) is the \( \mathcal{A} \)-extension of \( A \). Obviously \( \| x \|^\varepsilon \in \mathcal{A} \).

**Proposition 3.** Let \( A \) be a \( \mathcal{A} \)-C*-algebra. Then \( \| \cdot \|^\varepsilon \) satisfies the following properties, where \( x \) and \( y \) stand for an arbitrary member of \( A \) and \( z \) stands for an arbitrary member of \( \mathcal{A} \).

1) \( \| x \|^\varepsilon = 0 \) iff \( x = 0 \).
2) \( \| zx \|^\varepsilon = \| z \| \| x \|^\varepsilon \).

where \( |z| \) is the absolute value of \( z \) as a complex number in \( V(\mathcal{A}) \).

3) \( \| xy \|^\varepsilon \leq \| x \|^\varepsilon \| y \|^\varepsilon \).
4) \( \| x \|^\varepsilon = \| x^* \|^\varepsilon \).
5) \( \| xx^* \|^\varepsilon = \| x \|^\varepsilon \cdot \| x^* \|^\varepsilon \).
6) \( \| x + y \|^\varepsilon \leq \| x \|^\varepsilon + \| y \|^\varepsilon \).

**Proof.** 1) \( \| x \|^\varepsilon = 0 \) iff \( x^* x \leq 0 \) iff \( x^* x = 0 \) iff \( \| x^* x \|^\varepsilon = 0 \) iff \( \| x \|^\varepsilon = 0 \) iff \( x = 0 \).

2) First we prove three lemmas.

2.1) \( \| z \|^\varepsilon = |z| \).

This is obvious since \( z^* z = |z|^2 \).

2.2) \( b \in \mathcal{B} \implies \| bx \|^\varepsilon = b \| x \|^\varepsilon \).

It suffices to show \( b \| x \|^\varepsilon \leq \| bx \|^\varepsilon \).

We have

\[
x^* x = x^* x (1 - b) + x^* xb \leq \| x \|^\varepsilon (1 - b) + (\| bx \|^\varepsilon)^2.
\]

Therefore we have \( \| x \|^\varepsilon \leq \| x \|^\varepsilon (1 - b) + \| bx \|^\varepsilon \) and \( b \| x \|^\varepsilon \leq \| bx \|^\varepsilon \).

2.3) \( If z \in \mathcal{A}^+ \) and \( z \) is invertible in \( \mathcal{A} \), then

\[
|z| \| x \|^\varepsilon = \| zx \|^\varepsilon.
\]

**Proof.** \( \| zx \|^\varepsilon \leq z \) iff \( |z|^x x \leq z \)

iff \( x^* x \leq (z_i / |z|)^2 \)

iff \( \| x \|^\varepsilon \leq z_i / |z| \)

where \( z_i \in \mathcal{A}^+ \).

**Proof of 2.** Let \( b_0 = [z = 0] \) and \( b_1 = [\| z \| \geq \varepsilon] \). Put \( z_0 = (1 - b_0) + b_1 z \). Then we have

\[
\| z \| \| x \|^\varepsilon = (1 - b_0) |z| \| x \|^\varepsilon
\leq (1 - b_0) (z_0^2 + \varepsilon) \| x \|^\varepsilon
\leq (1 - b_0) |z_0| \| x \|^\varepsilon + \varepsilon \| x \|
\leq (1 - b_0) \| z_0 x \|^\varepsilon + \varepsilon \| x \|
\leq \| z_0 x \|^\varepsilon + \varepsilon \| x \|.
\]
Therefore we have $|z|\|x\|^p \leq |zx|\|x\|^p$. The direction $\|zx\|^p \leq |z|\|x\|^p$ is obvious.

3) First we claim the following.

$x^*x \leq z_1^1 \implies y^*x^*xy \leq y^*z_1^1y$.

Embedding $A(\mathcal{F})$ in some $L(\mathcal{H})$, we have for every $\xi$ in $\mathcal{H}$

$\langle y^*x^*xy\xi, |\xi\rangle \leq \langle z_1^1y\xi, |\xi\rangle = \langle y^*z_1^1y\xi, |\xi\rangle$.

Therefore we have

$\langle (xy)^*xy, |\xi\rangle = \langle y^*x^*xy, y_1^1y \leq z_2^1z_1^2 \rangle$,

where $x^*x \leq z_1^1$ and $y^*y \leq z_2^1$.

4) This is obvious from the fact that

$x^*x \leq z$ iff $xx^* \leq z$.

5) Let $h = xx^*$ and $xx^*xx^* \leq z^2$ for some $z \in \mathcal{L}_+$. Then $h^2 \leq z^2$ iff $h \leq z$.

6) First we prove three lemmas.

6.1) If $\|x\|^p = z$ is invertible in $\mathcal{L}$, then

$\|x + y\|^p \leq \|x\|^p(1 + \|y/z\|^p)$.  

**Proof.**

$\|x + y\|^p \leq z(\|x/z + y/z\|^p)$

$\leq z(\|x/z\|^p + \|y/z\|)$

$\leq z(1 + \|y/z\|)$.

6.2) $\|x + y\|^p \leq \|x\|^p + \|y\|^p$.

**Proof.**

Let $\|x\|^p = z_0$ and $[z_0 \geq \delta] = b$. Put $x_i = bx + \delta(1 - b)$ and $z = \|x_i\|^p$. Then $z$ is invertible and we have

$\|x + y\|^p = \|x_i + (x - x_i) + y\|^p$

$\leq z(1 + \|x - x_i/z\| + \|y/z\|)$

$\leq \|x\|^p + \|y\|$.

Since $\delta$ is an arbitrary positive number, we have

$\|x + y\|^p \leq \|x\|^p + \|y\|^p$.

6.3) If $\|x\|^p = z$ is invertible, then

$\|x + y\|^p \leq \|x\|^p + \|y\|^p$.

**Proof.**

$\|x + y\|^p = z\|x/z + y/z\|^p \leq z(1 + \|y/z\|^p) = \|x\|^p + \|y\|^p$.  


PROOF of 6). There exists \( x_i \) such that \( \| x_i - x \| \leq \varepsilon \) and \( \| x_i \|^p = z \) is invertible. Then we have

\[
\begin{align*}
\| x + y \|_p^p &= \| x_i + (x - x_i) + y \|_p^p \\
&\leq \| x_i \|_p^p + \| (x - x_i) + y \|_p^p \\
&\leq \| x + (x_i - x) \|_p^p + \varepsilon + \| y \|_p^p \\
&\leq \| x \|_p^p + 2\varepsilon + \| y \|_p^p.
\end{align*}
\]

**Theorem 1.** Let \( \hat{A} \) be a \( C^* \)-algebra in \( V^{(q)} \) and \( A \) be the bounded part of \( \hat{A} \). Then \( A \) is a \( \hat{Z} \)-\( C^* \)-algebra. Conversely if \( A \) is a \( \hat{Z} \)-\( C^* \)-algebra, then there exists a \( C^* \)-algebra \( \hat{A} \) in \( V^{(q)} \) such that the bounded part of \( \hat{A} \) is isomorphic to \( A \).

**Proof.** First let \( \hat{A} \) be a \( C^* \)-algebra in \( V^{(q)} \) and \( A \) be the bounded part of \( \hat{A} \). By Proposition 1, \( A \) is a normal \( \hat{Z} \)-space. We prove that \( A \) is a \( C^* \)-algebra. In the following, \( x \) and \( y \) stand for a member of \( A \) and \( \alpha \) stands for a complex number.

1) \( \| x^* \| = \| x \| \).
This is obvious since \( \| x^* \|_p = \| x \|_p \).

2) \((x^*)^* = x, (x + y)^* = x^* + y^*, \ (\alpha x)^* = \alpha x^*, \) and \((xy)^* = y^* x^* \).
These are immediate since they hold in \( V^{(q)} \).

3) \( \| xy \| \leq \| x \| \| y \| \)
This is immediate since \( \| xy \|_p \leq \| x \|_p \| y \|_p \) hold in \( V^{(q)} \).

4) \( \| x^* x \| = \| x^* \| \| x \| \).
Since \( \| x^* x \|_p = \| x^* \|_p \| x \|_p \) holds in \( V^{(q)} \), it suffices to show \( \| x^* \| \| x \| \leq \| x^* x \| \).
This is proved as follows.

\[
(\| x \|_p) \leq \bar{M} \implies \| x \|_p \leq (\sqrt{M})^q \implies \| x \| \leq \sqrt{M} \\
\implies \| x^* \| \| x \| \leq M.
\]

By Proposition 2, \( A \) is complete and \( A \) is a \( C^* \)-algebra.

Next we show that \( A \) is a normal \( \hat{Z} \)-module. The first condition of a normal \( \hat{Z} \)-module was already proved in 2) and 3). The second condition of a normal \( \hat{Z} \)-module is obvious since \([\hat{A} \text{ is a } C^*-\text{algebra}] = 1\). In order to show the third condition of a normal \( \hat{Z} \)-module, let \( \{b_\alpha\} \) be a partition of unity in \( \mathcal{B} \), \( \{a_\alpha\} \subseteq A \), and \( \| [a_\alpha] \| \) be bounded i.e. \( \| a_\alpha \| \leq M \) for every \( \alpha \). Let \( a = \sum_\alpha b_\alpha a_\alpha \) in \( V^{(q)} \). Then \( a \in A \) and \( \| a \| \leq M \) hold in \( V^{(q)} \). Therefore \( a \in A \). Obviously \( b_\alpha a = b_\alpha a_\alpha \) for every \( \alpha \). Suppose there exist \( a_\alpha \in A \) such that \( b_\alpha a_\alpha = b_\alpha a_\alpha \) for every \( \alpha \). Then \( [a_\alpha \alpha] \geq b_\alpha \) for every \( \alpha \) and \( [a_\alpha \alpha] = 1 \). Then \( a_\alpha \) and \( a \) represent the same element in \( A \).

Now let \( b_\alpha = [\hat{A} \text{ is unital}] \). Then \( b_\alpha \) is the characteristic of \( \mathcal{B} \) in \( A \). As before we define \( \mathcal{B} = b \hat{Z} \) and \( \mathcal{I} = (1 - b_\alpha) \hat{Z} \). Now let \( \hat{A} ' \) be obtained from \( \hat{A} \)
by adjunction of an identity to \( \hat{A} \) if \( \hat{A} \) is not unital and \( \hat{A}' \) be \( \hat{A} \) itself otherwise. Since \([\hat{A} \text{ is not unital}] = 1 - b_0\), \( \hat{A}' \) is obtained as \( \hat{A} \oplus (1 - b)C^* \). We claim that the bounded part of \( \hat{A}' \) is \( \hat{A} \oplus \mathbb{Z}' \) as a vector space and an algebra. It is obvious that \( \hat{A} \oplus \mathbb{Z}' \) is contained in the bounded part of \( \hat{A}' \). Now let \( x \) belong to the bounded part of \( \hat{A}' \). Then \( x = b_0x_0 + (1 - b_0)x_1 + u \), where \( x_0, x_1 \in \hat{A} \) and \( u \in (1 - b_0)C^* \) and \( u \neq 0 \). Since \( \|L_x\|^* \) is bounded, \( \|b_0x_0\|^* \) is also bounded, where \( L_x \) is the left multiplication of \( x \) to \( \hat{A} \) in \( V^* \). Therefore we may assume that \( x = (1 - b_0)x_1 + u \). Since \( (1 - b_0)x^*_1 + u^* \) is also bounded, we may also assume that \( x_1 \) and \( u \) are self-adjoint. If one of \( (1 - b_0)x_1 \) and \( u \) is bounded, then another is also bounded and \( x \) is in \( \hat{A} \oplus \mathbb{Z}' \). Therefore we always assume that both \( (1 - b_0)x_1 \) and \( u \) are unbounded. Now \( [u \geq 0] > 0 \) or \( [u \leq 0] > 0 \). Without loss of generality, we assume that \( [u \leq 0] = b_1 > 0 \) and \( u = -v \). Obviously \( b_1 \leq 1 - b_0 \) and we may assume that \( b_1x_1 \) is unbounded and \( b_1x_1 - v \) is bounded. Now \( b_1x_1 = x_2 + x_3 \), where \( x_2 = \max(b_1x_1, 0) \) and \( x_3 = \min(b_1x_1, 0) \). Then \( x_1 \) is bounded since \( b_1x_1 - v \) is bounded. Therefore \( x_3 \) is bounded. Let \( y = b_1b_2x_2 + b_3x_3 \). Since \( h^{1/2}, h^{1/4}, h^{1/8}, \ldots : b_1, b_2, b_3 \in A \) which is a contradiction because of \( 0 < b_1, b_2, b_3 < 1 - b_0 \).

Let \( x \in \hat{A} \oplus \mathbb{Z}' \). We define \( \|x\|' \) by the following equation.

\[
\|x\|' = \inf\{M | \|x\|^* \leq M \text{ holds in } V^* \}.
\]

We claim \( \|x\| = \|x\|' \). First we prove \( \|x\| \leq \|x\|' \). Suppose there exist \( \varepsilon > 0 \) such that \( \|x\| < \|x\|' + \varepsilon \). Then there exists \( y \in \hat{A} \) such that \( y \neq 0 \) and \( \|xy\| \geq \|x\|\|y\| - \varepsilon \|y\| \). Let \( x_1 \geq 0 \) satisfy \( 2x_1 < \varepsilon \|y\| \). Then \( \|xy\|^* \geq \|xy\|' - \xi \geq 0 \). Therefore \( \|xy\|^* \geq (\|x\|\|y\| - \varepsilon \|y\|)^* = b > 0 \). Now we have

\[
\|xy\|^* \geq b(\|x\|\|y\| - \varepsilon \|y\|)^* \\
> b(\|x\|' + \varepsilon \|y\| - \varepsilon \|y\|)^* \\
> b(\|x\|^* \|y\|^* + (2\varepsilon \|y\| - \varepsilon \|y\|)^*) \\
> b(\|x\|^* \|y\|^* + \varepsilon \|y\|),
\]

which is a contradiction because of \( b > 0 \). Next we show \( \|x\|' \leq \|x\| \). Suppose there exist \( \varepsilon > 0 \) such that \( \|x\| + \varepsilon \leq \|x\|' \). Then there exists \( y \in \hat{A} \) such that

\[
\|xy\|^* > (\|x\| + \varepsilon)^* \|y\|^* = b > 0.
\]

By replacing \( y \) by \( y/1 + \|y\| \) if necessary, we may assume that \( y \in A \). Then we have

\[
b\|x\|^* \|y\|^* \geq b\|x\|^* \|y\|^* \geq b\|xy\|^* \\
> b(\|x\| + \varepsilon)^* \|y\|^* \\
\geq b\|x\|^* \|y\|^* + b\varepsilon \|y\|^*.
\]
which is a contradiction.

In order to show that \( A \) is a \( \mathcal{A} \)-\( \mathcal{C}^* \)-algebra, it now suffices to show that for every \( x \in A \) and \( z \in \mathcal{A} \), \( x^*x \leq z \) iff \( xx^* \leq z \). However it is obvious since it holds in \( V(\omega) \).

Now we are going to show the second part of the theorem. Let \( A \) be a \( \mathcal{A} \)-\( \mathcal{C}^* \)-algebra. Then let \( \hat{A} = \{ \langle f, \tilde{x} \rangle | f \in C^* \text{ and } x \in A \} \) in \( V(\omega) \). We denote \( \langle f, \tilde{x} \rangle \in \hat{A} \) by \( f\tilde{x} \). For \( f\tilde{x} \in A \), we define \( \|f\tilde{x}\|^2 \) by the equation \( \|f\tilde{x}\|^2 = \|f\|^2 \|x\|^2 \). We define \( [f\tilde{x} = 0] \) by the following equation

\[
[f\tilde{x} = 0] = [\|f\tilde{x}\|^2 = 0] = [\|f\| = 0] \lor [\|x\|^2 = 0].
\]

We also define \( f\tilde{x} + g\tilde{y} \) by the equation

\[
f\tilde{x} + g\tilde{y} = (1 + |f| + |g|)(z_1x + z_2y),
\]

where \( z_1 = f/1 + |f| + |g| \) and \( z_2 = g/1 + |f| + |g| \). We are going to show many properties from which the second part of the theorem follows immediately.

5) \( [\tilde{x} = 0] \lor [\|x\|^2 = 0] \).

**Proof.** \( [\tilde{x} = 0] = [\|x\|^2 = 0] = [\|f\| = 0] \lor [\|x\|^2 = 0] \).

6) \( [\tilde{x} = \tilde{y}] \lor [\|x\|^2 = \|y\|^2] \).

This follows from 5).

7) \( [f\tilde{x} = g\tilde{y}] \lor [\|f\tilde{x}\|^2 = \|g\tilde{y}\|^2] \).

This follows from 5) and the definition of \( f\tilde{x} - g\tilde{y} \).

8) If \( \|f\tilde{x}\|^2 \leq M \) folds in \( V(\omega) \), then there exists \( y \in A \) such that \( f\tilde{x} = y \) holds in \( \hat{A} \).

**Proof.** Let \( b_i = [(i-1)<|f| \leq i] \). Then \( \{b_i\} \) is a partition of unity. Then \( b_i f \in \mathcal{A} \). Put \( x_i = b_i f\tilde{x} \) and \( y = \sum b_i x_i \). Then \( y \in A \) and \( b_i f\tilde{x} = b_i y \). Therefore \( f|1+|f| = y/1 + |f| \) and \( [f\tilde{x} = y] = 1 \).

9) \( [f\tilde{x} = 0] = [f = 0 \lor \tilde{x} = 0] \).

This is obvious from the definition of \( \| \| \) and the proof of 5).

10) \( \|f\tilde{x} + g\tilde{y}\|^2 \leq \|f\tilde{x}\|^2 + \|g\tilde{y}\|^2 \).

**Proof.** \( \|f\tilde{x} + g\tilde{y}\|^2 \leq (1+|f|+|g|)(\|f\tilde{x}\|^2 + \|g\tilde{y}\|^2) \)

\[
= \|f\tilde{x}\|^2 + \|g\tilde{y}\|^2.
\]
11) \[ \| f\hat{x} \| ^{\ast} \geq 0 \text{ and } \| f\hat{x} \| ^{\ast} = 0 \text{ iff } f\hat{x} = 0. \]

\[ \| f\hat{x} \| ^{\ast} \geq 0 \text{ is obvious. We also have} \]

\[ \| f\hat{x} \| ^{\ast} = \| f \| \| x \| ^{\ast} = 0 \]

\[ = [ f = 0 \lor \hat{x} = 0 ] \]

\[ = [ f\hat{x} = 0 ] . \]

We define \( f(g\hat{x}) \) by the equation \( f(g\hat{x}) = (fg)\hat{x} \).

12) \[ \| f(g\hat{x}) \| ^{\ast} = \| f \| \| g\hat{x} \| ^{\ast} . \]

This is immediate from the definition. We define \((f\hat{x})^{\ast}\) by the equation \((f\hat{x})^{\ast} = f^{\ast}(x^{\ast})^{\lor}\).

13) \((f\hat{x})^{\ast \ast} = f\hat{x} \) and \((f\hat{x} + g\hat{y})^{\ast} = (f\hat{x})^{\ast} + (g\hat{y})^{\ast}\).

This is immediate from the definition. We define \((f\hat{x}) \cdot (g\hat{y})\) by the equation

\[ (f\hat{x}) \cdot (g\hat{y}) = (fg)(xy)^{\lor} . \]

14) \((f\hat{x} \cdot g\hat{y})^{\ast} = (g\hat{y})^{\ast}(f\hat{x})^{\ast}, \| f\hat{x} \cdot g\hat{y} \| ^{\ast} \leq \| f\hat{x} \| ^{\ast} \| g\hat{y} \| ^{\ast}, \text{ and } \| (f\hat{x})^{\ast}(f\hat{x}) \|^{\ast} = \| (f\hat{x})^{\ast} \| ^{\ast} \| f\hat{x} \| ^{\ast} . \]

These are immediate from the definition.

15) \( \hat{A} \) is complete with respect to \( \| \| ^{\ast} \).

\textbf{Proof.} It is obvious that the bounded part of \( A \) is isomorphic to \( A \). Therefore this follows from Proposition 2.

\textbf{Corollary 1.} Let \( \hat{A} \) be a C*-algebra in \( V^{(a)} \), \( A \) its bounded part, and \( \hat{A} \) the C*-algebra in \( V^{(a)} \) described in the proof of the theorem. Then the bounded part of \( \hat{A} \) is the \( \mathcal{E} \)-extension of \( A \).

\textbf{Corollary 2.} Let \( \hat{A} \) be a C*-algebra in \( V^{(a)} \), \( A \) its bounded part, and \( \hat{A} \) the C*-algebra in \( V^{(a)} \) obtained by adjunction of an identity to \( A \). Then the bounded part of \( \hat{A} \) is isomorphic to the \( \mathcal{E} \)-C*-algebra \( \hat{A} \) obtained from \( A \) by adjunction of an identity to \( A \).

\textbf{Definition.} Let \( A_{1} \) and \( A_{2} \) be \( \mathcal{E} \)-C*-algebras and \( \phi: A_{1} \rightarrow A_{2} \) be a \( \ast \)-homomorphism. \( \phi \) is said to be a \( \mathcal{E} \)-\( \ast \)-homomorphism if the following condition is satisfied.

\[ z \in Z, a \in A_{1} \implies \phi(za) = z\phi(a) \]
It is easily seen that if $\phi$ is a $\mathcal{L}^*$-homomorphism and $a = \sum b_i a_i$, then $\phi(a) = \sum b_i \phi(a_i)$.

**Theorem 2.** Let $\hat{A}_1$ and $\hat{A}_2$ be $C^*$-algebras in $V^{(\sigma)}$ and $\phi: \hat{A}_1 \to \hat{A}_2$ be a $*$-homomorphism in $V^{(\sigma)}$. Let $A_1$ and $A_2$ be the bounded part of $\hat{A}_1$ and $\hat{A}_2$ respectively. Let $\hat{\phi}: A_1 \to A_2$ be the restriction of $\phi$ to $A_1$. Then $\phi$ is a $\mathcal{L}^*$-homomorphism and the following equivalences hold.

1) $\phi$ is injective in $V^{(\sigma)}$ iff $\hat{\phi}$ is injective.

2) $\phi$ is surjective in $V^{(\sigma)}$ iff $\hat{\phi}$ is surjective.

**Proof.** Obviously $\phi$ is a $\mathcal{L}^*$-homomorphism. It is also obvious that $\phi$ is injective if $\hat{\phi}$ is injective in $V^{(\sigma)}$. Now let $\phi$ be injective and $[a_i \in \hat{A}_1] = 1$ $(i=1, 2)$, $[\hat{\phi}(a_i) = \hat{\phi}(a_2)] = 1$. Let $z = 1 + \|a_1\| + \|a_2\|$ and $a_i' = a_i / z$. Then $a_i'$ and $a_i''$ are in $A_i$ and $\phi(a_i') = \phi(a_i'')$. Hence $a_i' = a_i''$ since $\phi$ is injective. Therefore $[a_i = a_i] = 1$.

Now let $\phi$ be surjective and $a_1 \in A_1$. Let $\hat{\phi} = \phi^{-1}(0)$ in $V^{(\sigma)}$. Then we can identify $\hat{A}_1/A$ with $\hat{A}_2$ and $\phi$ with the canonical $*$-homomorphism from $\hat{A}_1$ onto $\hat{A}_2$. Let $\|a_1\| \leq M$ and $\varepsilon > 0$. Then there exists $a_i \in \hat{A}_1$ such that $\|a_i\|^2 \leq (M+\varepsilon)$ and $\hat{\phi}(a_i) = a_i$. Then $a_i \in A_i$ and $\phi(a_i) = a_i$. Now let $\phi$ be surjective and $a_i \in \hat{A}_i$. Let $z = 1 + \|a_i\|$ and $a_i' = a_i / z$. Then $a_i' \in A_i$ and there exists $a_i' \in A_i$ such that $\phi(a_i') = a_i$. Put $a_i = za_i'$. Then $\phi(a_i) = a_i$.

**Theorem 3.** Let $\hat{A}_1$ and $\hat{A}_2$ be $C^*$-algebras in $V^{(\sigma)}$ and $A_1$ and $A_2$ be the bounded part of $\hat{A}_1$ and $\hat{A}_2$ respectively. If $\phi: A_1 \to A_2$ is a $\mathcal{L}^*$-homomorphism, then there exists a $*$-homomorphism $\hat{\phi}: \hat{A}_1 \to \hat{A}_2$ in $V^{(\sigma)}$ such that the restriction of $\hat{\phi}$ to $A_1$ is $\phi$.

**Proof.** Let $a_1 \in \hat{A}_1$ and $z = \|a_1\|^2 + 1$. Put $a_i = a_i / z$. Then $a_i \in A_i$. We define $\hat{\phi}(a_i) = za_i$. In order to prove $\hat{\phi}: \hat{A}_1 \to \hat{A}_2$, it suffices to show that

$$ba_1 = bc_1 = b \hat{\phi}(a_1) = b \hat{\phi}(c_1)$$

for $a_i, c_i \in A$ and $b \in \mathcal{B}$. This is obvious since $ba_1 = bc_1$ implies $ba_i' = bc_i'$ and $b(1 + \|a_i\|) = b(1 + \|c_i\|)$ where $a_i' = a_i/1 + \|a_i\|^2$ and $c_i' = c_i/1 + \|c_i\|^2$.

**Corollary.** There exists a one-to-one correspondence between the $*$-isomorphism classes of $C^*$-algebras in $V^{(\sigma)}$ and the $\mathcal{L}^*$-isomorphism classes of $\mathcal{L}$-$C^*$-algebras.

§ 2. Properties of $\mathcal{L}$-$C^*$-algebras.

**Definition.** Let $\mathcal{H}$ be a Hilbert space and $\mathcal{L}$ its abelian von Neumann algebra. $\mathcal{H}$ is said to be $\mathcal{L}$-separable if $\mathcal{L}$ is generated as a von Neumann algebra by its countable members together with $\mathcal{L}$.
PROPOSITION 1. Let $\mathcal{H}$ be the Hilbert space in $V^{(\omega)}$ obtained from $\mathcal{H}$ and $\mathcal{I}$ (cf. [10]). Then $\mathcal{H}$ is separable in $V^{(\omega)}$ iff $\mathcal{H}$ is $\mathcal{I}$-separable.

PROOF. $\mathcal{H}$ is separable in $V^{(\omega)}$ if $\mathcal{I}'=\mathcal{I}(\mathcal{H})$ is generated as a von Neumann algebra by its countable members in $V^{(\omega)}$. Since von Neumann algebra is absolute as far as it is included in $\mathcal{I}'$ and countability is absolute, the separability in $V^{(\omega)}$ is equivalent to the $\mathcal{I}$-separability.

We are going to define the $\mathcal{I}$-projections of norm one, the $\mathcal{I}$-injective von Neumann algebras, and the $\mathcal{I}$-nuclear $\mathcal{I}$-$C^*$-algebras. We use the definitions of the projections of norm one, the injective von Neumann algebras, and the nuclear $C^*$-algebras based on Tomiyama [11] and [12]. First we recall them. Let $A$ be a $C^*$-algebra and $B$ be its $C^*$-subalgebra. A surjective linear map $\varepsilon: A\to B$ is said to be a projection of norm one if the following conditions are satisfied:

1) $x \in B \implies \varepsilon(x) = x$.
2) $\|\varepsilon(x)\| \leq \|x\|$.

If $\varepsilon$ is a projection of $A$ onto $B$ of norm one, then the following properties hold. (See [11], [12].)

1) $x \in A \implies \varepsilon(x^*x) \geq 0$.
2) $a, b \in B, x \in A \implies \varepsilon(axb) = a\varepsilon(x)b$.
3) $x \in A \implies \varepsilon(x^*\varepsilon(x)) \leq \varepsilon(x^*x)$.

Let $M$ be a von Neumann algebra acting on $\mathcal{H}$. $M$ is said to be injective if there exists a projection of norm one from $\mathcal{I}(\mathcal{H})$ onto $M$. Let $A$ be a $C^*$-algebra. $A$ is said to be nuclear if every representation of $A$ on Hilbert space generates an injective von Neumann algebra.

DEFINITION. Let $A$ be a $\mathcal{I}$-$C^*$-algebra and $B$ be its $\mathcal{I}$-$C^*$-subalgebra. Let $E: AB$ be a surjective $\mathcal{I}$-linear map. $E$ is said to be a $\mathcal{I}$-projection of norm one if the following conditions are satisfied:

1) $x \in B \implies \varepsilon(x) = x$.
2) $\|\varepsilon(x)\|^2 \leq \|x\|^2$.

DEFINITION. Let $M$ be a von Neumann algebra satisfying $\mathcal{I} \subseteq M \subseteq \mathcal{I}'$. $M$ is said to be $\mathcal{I}$-injective if there exists a $\mathcal{I}$-projection from $\mathcal{I}'$ onto $M$ of norm one.

DEFINITION. Let $\mathcal{I}$ act on a Hilbert space $\mathcal{H}$ and $\mathcal{I}'$ be its commutant with respect to $\mathcal{H}$. Let $A$ be a $\mathcal{I}$-$C^*$-algebra. A $\mathcal{I}$-$*$-homomorphism $\pi: A \to \mathcal{I}'$ is called a $\mathcal{I}$-representation of $A$ on $\mathcal{H}$.

DEFINITION. Let $A$ be a $\mathcal{I}$-$C^*$-algebra. $A$ is said to be $\mathcal{I}$-nuclear if every $\mathcal{I}$-representation of $A$ on Hilbert spaces and $\mathcal{I}$ generate a $\mathcal{I}$-injective von Neumann algebra.
PROPOSITION 2. Let \( \hat{A} \) be a \( C^* \)-algebra in \( V(\mathcal{H}) \) and \( \hat{B} \) be its \( C^* \)-sub-algebra in \( V(\mathcal{H}) \). Let \( A \) and \( B \) be the bounded part of \( \hat{A} \) and \( \hat{B} \) respectively. If \( \epsilon: \hat{A} \to \hat{B} \) is a projection of \( \hat{A} \) onto \( \hat{B} \) of norm one in \( V(\mathcal{H}) \), then \( \epsilon: A \to B \) is a \( \mathcal{Z} \)-projection of \( A \) onto \( B \) of norm one, where \( \epsilon \) is the restriction of \( \epsilon \) to \( A \).

Conversely if \( \epsilon: \hat{A} \to \hat{B} \) is a \( \mathcal{Z} \)-projection of \( A \) onto \( B \) of norm one, then there exists \( \epsilon: \hat{A} \to \hat{B} \) such that \( \epsilon \) is a projection of \( \hat{A} \) onto \( \hat{B} \) of norm one in \( V(\mathcal{H}) \) and \( \epsilon \) is the restriction of \( \epsilon \) to \( A \).

PROOF. First let \( \epsilon \) be a projection of \( \hat{A} \) onto \( \hat{B} \) of norm one in \( V(\mathcal{H}) \) and \( \epsilon \) be its restriction to \( A \). Then \( \epsilon \) is obviously a \( \mathcal{Z} \)-linear map from \( A \) onto \( B \). Let \( x \in A \). Then we have

\[
\| \epsilon(x) \| = \| \epsilon(x) \| = \| \epsilon(x) \| = \| x \| = \| x \|.
\]

Conversely let \( \epsilon \) be a \( \mathcal{Z} \)-projection of \( A \) onto \( B \) of norm one. Every member of \( \hat{A} \) is of the form \( fx \) where \( f \in C^* \) and \( x \in A \). We define \( \tilde{\epsilon} \) by the equation \( \tilde{\epsilon}(fx) = f \epsilon(x) \). Then \( \tilde{\epsilon} \) is obviously a projection of \( \hat{A} \) onto \( \hat{B} \) of norm one in \( V(\mathcal{H}) \) and \( \epsilon \) is the restriction of \( \tilde{\epsilon} \) to \( A \).

PROPOSITION 3. Let \( \mathcal{Z} \) act on a Hilbert space \( \mathcal{H} \) and \( \mathcal{Z}' \) its commutant with respect to \( \mathcal{H} \). Let \( \mathcal{H}' \) be the Hilbert space in \( V(\mathcal{H}) \) corresponding to \( \mathcal{H} \). Let \( M \) be a von Neumann algebra satisfying \( \mathcal{Z} \subseteq M \subseteq \mathcal{Z}' \) and \( \hat{M} \) be the von Neumann algebra in \( V(\mathcal{H}) \) corresponding to \( M \). Then \( M \) is \( \mathcal{Z} \)-injective iff \( \hat{M} \) is injective in \( V(\mathcal{H}) \).

PROOF. This is implied by Proposition 2 as follows: \( \hat{M} \) is injective in \( V(\mathcal{H}) \) iff \( \exists \epsilon: \mathcal{Z}(\mathcal{H}) \to \hat{M} \) a projection of norm one in \( V(\mathcal{H}) \) iff \( \exists \epsilon: \mathcal{Z}' \to M \) a \( \mathcal{Z} \)-projection of norm one iff \( M \) is \( \mathcal{Z} \)-injective.

PROPOSITION 4. Let \( \hat{A} \) be a \( C^* \)-algebra in \( V(\mathcal{H}) \) and \( A \) its bounded part. Then \( \hat{A} \) is nuclear in \( V(\mathcal{H}) \) iff \( A \) is \( \mathcal{Z} \)-nuclear.

PROOF. This immediately follows from Proposition 3, Theorem 2 in § 1, and the one-to-one correspondence between the von Neumann algebras on \( \mathcal{H} \) in \( V(\mathcal{H}) \) and the von Neumann algebras on \( \mathcal{H} \) with \( \mathcal{Z} \subseteq M \subseteq \mathcal{Z}' \).

PROPOSITION 5. Let \( A \) be a \( \mathcal{Z} \)-\( C^* \)-algebra. If \( A \) is nuclear, then \( A \) is \( \mathcal{Z} \)-nuclear.

PROOF. Let \( \pi: A \to \mathcal{Z}' \) be a \( \mathcal{Z} \)-\( * \)-homomorphism on a Hilbert space \( \mathcal{H} \). Then \( \pi \) is also a representation of \( A \) on \( \mathcal{H} \). Therefore the image \( M \) of \( \pi \) generates an injective von Neumann algebra. Since \( M \subseteq \mathcal{Z}' \), the von Neumann algebra \( \hat{M} \) generated by \( M \) and \( \mathcal{Z} \) is injective. Therefore \( M \) is \( \mathcal{Z} \)-injective. Since \( \hat{M} \) is generated by \( M \) and \( \mathcal{Z} \), the proposition is proved.
**Definition.** Let $A$ be a $\mathcal{Z}$-$C^*$-algebra. $A$ is said to be $\mathcal{Z}$-separable if there exists a countable subset $D$ of $A$ which generates $A$ as a $\mathcal{Z}$-$C^*$-algebra.

**Proposition 6.** Let $A$ be a $C^*$-algebra in $V(\omega)$ and $A$ its bounded part. $\hat{A}$ is separable in $V(\omega)$ iff $A$ is $\mathcal{Z}$-separable.

**Proof.** Let $\hat{D}$ be a countable subset of $\hat{A}$ in $V(\omega)$ which generates $\hat{A}$ in $V(\omega)$. We may assume that every element of $\hat{D}$ has a norm less than 1. Then $\hat{D}$ is also a countable subset of $A$ which generates $A$ as a $\mathcal{Z}$-$C^*$-algebra. The converse is obvious.

**Proposition 7.** Let $A$ be a $\mathcal{Z}$-$C^*$-algebra. If $A$ is separable, then $A$ is $\mathcal{Z}$-separable.

**Proof** is obvious.

**Definition.** Let $A$ be a $\mathcal{Z}$-$C^*$-algebra. A sequence $\{u_i\}_{i \in \omega}$ of elements of $A$ is said to be a $\omega$-countable approximate identity of $A$ if it satisfies the following properties.

1) $x \in A \implies \lim_{i \to \infty} \|u_i x - x\|^\omega = 0,$
2) $x \in A \implies \lim_{i \to \infty} \|x u_i - x\|^\omega = 0,$
3) $i \leq j \implies 0 \leq u_i \leq u_j,$
4) $\|u_i\|^\omega \leq 1,$
where $\lim_{i \to \infty}$ is the limit in the sense of $m$-convergence.

**Proposition 8.** Let $\hat{A}$ be a $C^*$-algebra in $V(\omega)$ and $A$ its bounded part. Let $\{u_i\}_{i \in \omega}$ be a countable sequence of elements of $\hat{A}$. Then $\{u_i\}_{i \in \omega}$ is a countable approximate identity of $A$ in $V(\omega)$ iff $\{u_i\}_{i \in \omega}$ is a $\omega$-countable approximate identity of $A$.

**Proof.** $\|u_i\|^\omega \leq 1$ in $V(\omega)$ is equivalent to $\|u_i\|^\omega \leq 1$ and also to $\|u_i\| \leq 1$. $0 \leq u_i \leq u_j$ is also absolute. If $x \in A$, then $\lim_{i \to \infty} \|u_i x - x\|^\omega = \lim_{i \to \infty} \|x u_i - x\|^\omega = 0$. If $x = \sum f y$, where $f \in C^*$, $\|f\| = 0$, and $y \in A$. Then $\lim_{i \to \infty} \|u_i x - x\|^\omega = \lim_{i \to \infty} \|x u_i - x\|^\omega = 0$ in $V(\omega)$ is equivalent to $\lim_{i \to \infty} \|u_i y - y\|^\omega = \lim_{i \to \infty} \|y u_i - y\|^\omega = 0$ in $V(\omega)$. Therefore the proposition is clear.

**Corollary.** Let $\hat{A}$ be a $\mathcal{Z}$-$C^*$-algebra. If $\{u_i\}_{i \in \omega}$ is a countable approximate identity of $A$, then $\{u_i\}_{i \in \omega}$ is a $\mathcal{Z}$-countable approximate identity of $A$.

**Definition.** Let $A$ be a $\mathcal{Z}$-$C^*$-algebra and $f: A \to \mathcal{Z}$. $f$ is said to be a $\mathcal{Z}$-linear function if it satisfies the condition

$$z_1, z_2 \in \mathcal{Z}, a_1, a_2 \in A \implies f(z_1 a_1 + z_2 a_2) = z_1 f(a_1) + z_2 f(a_2).$$
An element \( x \in A \) is said to be \( \mathcal{Z} \)-strictly positive if for every nonzero positive \( \mathcal{Z} \)-linear functional \( f \), \( f(x) \) is positive.

**Proposition 9.** Let \( \hat{A} \) be a \( C^* \)-algebra in \( V^{(s)} \), \( A \) its bounded part, and \( x \) an element of \( A \). \( x \) is strictly positive in \( \hat{A} \) in \( V^{(s)} \) iff \( x \) is \( \mathcal{Z} \)-strictly positive in \( A \).

**Proof.** Let \( \hat{f} \) be a positive linear functional of \( \hat{A} \) in \( V^{(s)} \). Then \( \hat{f} \) is continuous in \( V^{(s)} \). Let \( z \) be the norm of \( f \) in \( V^{(s)} \). Let \( f \) be the restriction of \( \hat{f}/1+z \) to \( A \). Then \( f \) is a positive \( \mathcal{Z} \)-linear functional on \( A \). Now the proposition immediately follows.

**Corollary.** Let \( \hat{A} \) be a \( C^* \)-algebra in \( V^{(s)} \) and \( A \) its bounded part. \( \hat{A} \) has a strictly positive element in \( V^{(s)} \) iff \( A \) has a \( \mathcal{Z} \)-strictly positive element.

**Proposition 10.** Let \( A \) be a \( \mathcal{Z} \)-\( C^* \)-algebra. \( A \) has a \( \mathcal{Z} \)-strictly positive element iff it has a \( \mathcal{Z} \)-countable approximate identity.

**Proof.** This follows from the fact that a \( C^* \)-algebra has a strictly positive element iff it has a countable approximate identity.

**Corollary.** Let \( A \) be a \( \mathcal{Z} \)-\( C^* \)-algebra. If it has a strictly positive element, then it has a \( \mathcal{Z} \)-strictly positive element.

**Definition.** Let \( A \) be a \( \mathcal{Z} \)-\( C^* \)-algebra, \( a, b \in \mathbb{R} \) and \( a \leq b \). A function \( f: [a, b] \rightarrow A \) is said to be \( \mathcal{Z} \)-continuous if for every \( \varepsilon > 0 \)

\[
1 = \sqrt{n} \sqrt{\sum_{x \in [a, b]} \frac{1}{n} (x - y) (x - y)} 
\]

and \( \sup \{ \|f(x)\| : x \in [a, b] \} < \infty \). The set of all \( \mathcal{Z} \)-continuous functions is denoted by \( A[a, b]^\mathcal{Z} \). For \( f, g \in A[a, b]^\mathcal{Z} \), \( z \in \mathcal{Z} \) we define \( f+g \), \( zf \) and \( f^* \) by the following equations:

\[
\begin{align*}
  x \in [a, b] & \implies (f+g)(x) = f(x) + g(x) \\
  x \in [a, b] & \implies (zf)(x) = zf(x) \\
  x \in [a, b] & \implies f^*(x) = (f(x))^*.
\end{align*}
\]

**Proposition 11.** Let \( \hat{A} \) be a \( C^* \)-algebra in \( V^{(s)} \) and \( A \) its bounded part. Let \( a, b \in \mathbb{R} \) and \( a \leq b \). The \( C^* \)-algebra of all continuous functions from \( [\hat{a}, \hat{b}] \) into \( \hat{A} \) in \( V^{(s)} \) is denoted by \( \hat{A}[\hat{a}, \hat{b}] \). If \( \hat{f} \) is an element of the bounded part of \( \hat{A}[\hat{a}, \hat{b}] \) and \( f: [a, b] \rightarrow A \) is defined by the equation

\[
x \in [a, b] \implies f(x) = \hat{f}(\hat{x}),
\]

then \( f \in A[a, b]^\mathcal{Z} \). Conversely every function in \( A[a, b]^\mathcal{Z} \) is obtained in this way.
PROOF. Let \( \hat{f} \) be a function in \( V^\omega \) described in the theorem. Then \( \hat{f} \) is uniformly continuous in \( V^\omega \). Therefore for every \( \varepsilon > 0 \)

\[
[ \exists \delta > 0 \forall x, y \in [\bar{a}, \bar{b}](|x - y| < \delta \implies \|\hat{f}(x) - \hat{f}(y)\| < \varepsilon)] = 1.
\]

Hence there exist a partition \( \{b_i\} \) of unity and a set \( \{\delta_i\} \) of positive numbers such that

\[
[\forall x, y \in [\bar{a}, \bar{b}](|x - y| < \delta \implies \|\hat{f}(x) - \hat{f}(y)\| < \varepsilon)] = 1,
\]

where \( \delta = \sum_i b_i \delta_i \). This implies the statement that for every \( x, y \in [a, b] \)

\[
\|\hat{f}(x) - \hat{f}(y)\| < \varepsilon
\]

i.e. \( |x - y| < \delta \implies b_n \leq \|f(x) - f(y)\| < \varepsilon \)

which is equivalent to

\[
|x - y| < \delta \implies \|b_n(f(x) - f(y))\| < \varepsilon.
\]

From this follows \( f \in A[a, b]^\omega \).

Conversely let \( f \in A[a, b]^\omega \). By tracing the above argument backward, we can prove that \( f: [a, b] \to A \) is uniformly continuous in \( V^\omega \) if \( f \) is considered as a function on \( ([a, b])^\omega \). Since \( ([a, b])^\omega \) is dense in \([a, b]\), \( f \) has a unique continuous extension on \([a, b]\).

COROLLARY. Let \( A \) be a \( \mathcal{D}' \)-C*-algebra, \( a, b \in \mathbb{R} \), and \( a < b \). Then \( A[a, b]^\omega \) is a \( \mathcal{D}' \)-C*-algebra where the norm is defined by the following equation

\[
f \in A[a, b]^\omega \implies \|f\| = \sup \{\|f(x)\| : x \in [a, b]\}.
\]

DEFINITION. Let \( A \) be a \( \mathcal{D}' \)-C*-algebra. A function \( f: \mathbb{R}^n \to A \) is said to be a \( \mathcal{D}' \)-continuous function vanishing at \( \infty \) if for every \( M \) and for every \( \varepsilon > 0 \)

\[
1 = \bigvee_m \bigvee_k \{b \mid \forall x, y \in \mathbb{R}^n \times (|x|, |y| \leq M \land |x - y| < 1/2^m \implies \|b(f(x) - f(y))\| < \varepsilon),
\]

for every \( M > 0 \), \( \sup_{\mathbb{R}^n} \|f(x)\| : \|x\| \leq M < \infty \), and

\[
1 = \bigvee_m \bigvee_k \{b \mid \exists x \in \mathbb{R}^n(x > m \implies \|b(f(x))\| < \varepsilon)\}.
\]

The set of all \( \mathcal{D}' \)-continuous functions vanishing at \( \infty \) is denoted by \( A(\mathbb{R}^\omega)^\omega \). For \( f, g \in A(\mathbb{R}^\omega)^\omega \) and \( z \in \mathcal{D}' \), \( f + g, z \circ f \) and \( f^* \) are defined as before.

In the same way as in Proposition 11 and its corollary, we have the following proposition and its corollary.

PROPOSITION 11'. Let \( \hat{A} \) be a C*-algebra in \( V^\omega \) and \( A \) its bounded part. The C*-algebra of all continuous functions vanishing at \( \infty \) from \( (\mathbb{R}^\omega)^\omega \) into \( A \) is
denoted by $A((R^*)^n)$. If $f$ is an element of the bounded part of $A((R^*)^n)$ and $f: R^* \to A$ is defined by the equation

$$x \in R^* \implies f(x) = \hat{f}(\hat{x}),$$

then $f \in A((R^*)^n)$. Conversely every function in $A((R^*)^n)$ is obtained in this way.

**Corollary.** Let $A$ be a $C^*$-algebra. Then $A((R^*)^n)$ is a $C^*$-algebra where the norm is defined by the following equation

$$\|f\| = \sup \{ |f(x)| : x \in R^n \}.$$

In the same way, define $A[a, b]^*$, $A(a, b]^*$, and $A(a, b]$. Let $f \in A[a, b]$. Then if we divide $[a, b]$ into $N$ equal parts and denote the linear graph connecting $f(a + i(b - a)/N)(0 \leq i \leq N)$ by $g$, then $g$ is a good approximation of $f$ for a large number $N$. If $f$ is represented by a function in $A[a, b]^*$, then $g$ is also represented by a function in $A[a, b]$. We denote the $C^*$-algebra of all continuous functions from $[a, b]$ into $A$ by $A[a, b]$. Now let $N = \sum_n b_i$, where $\{b_i\}$ is a partition of unity in. Then $g$ is represented by a function $\sum b_i g_i$ where $g_i \in A[a, b]$ and $\|g_i\| \leq \|f\|$. This shows that all the functions of the form $\sum b_i g_i$ is dense in $A[a, b]^*$ and that $A[a, b]^*$ is obtained as a completion of the algebra of all the functions of the form $\sum b_i g_i$ where $g_i \in A[a, b]$ and $\|\sum b_i g_i\| = \sup \|b_i g_i\| < \infty$.

**Definition.** Let $A$ be a $C^*$-algebra and $\mathcal{I}$ be its ideal. $\mathcal{I}$ is said to be a $C^*$-ideal if it satisfies the following conditions.

1) $x \in \mathcal{I}, z \in C \implies zx \in \mathcal{I}$.

2) If $\{b_i\}_s$ is a partition of unity in $C$, $\{a_s\} \subseteq \mathcal{I}$, and $\sum b_i a_s \in A$, then $\sum b_i a_s \in \mathcal{I}$.

**Proposition 12.** Let $A$ be a $C^*$-algebra in $V^s$ and $A$ its bounded part. If $\hat{\mathcal{I}}$ is an ideal of $A$ in $V^s$, then the bounded part of $\hat{\mathcal{I}}$ is a $C^*$-ideal of $A$. Furthermore every $C^*$-ideal of $A$ is obtained in this way.

**Proof.** The first part of the proposition is obvious. For the second part, let $\mathcal{I}$ be a $C^*$-ideal of $A$. Now $\hat{A}$ is $\{fx | f \in C^* \text{ and } x \in A\}$. We define $\hat{\mathcal{I}}$ by $\hat{\mathcal{I}} = \{fx | f \in C^* \text{ and } x \in \mathcal{I}\}$. The proposition is now obtained in the same way as in the proof of Theorem 1 in §1.

**Proposition 13.** Let $A$ be a $C^*$-algebra in $V^s$ and $\hat{\mathcal{I}}$ its ideal. Let $A$ and $\mathcal{I}$ be the bounded part of $\hat{A}$ and $\mathcal{I}$ respectively. Then the bounded part of $A/\hat{\mathcal{I}}$ is $A/\mathcal{I}$.

**Proof.** Let $a + \mathcal{I}$ be an element of the bounded part of $\hat{A}/\hat{\mathcal{I}}$. Then there exists $M$ such that $\|a + \mathcal{I}\| \leq M$ holds in $V^s$. Then for every $\varepsilon > 0$
there exists \( a_i \in A \) such \( \|a_i\| \leq (M + \varepsilon)^{1/2} \) and \( a + \mathcal{J} = a_1 + \mathcal{J} \). Then \( a_1 + \mathcal{J} \) is a member of \( A/\mathcal{J} \). If \( a_1, a_2 \in A \) and \( a_1 + \mathcal{J} = a_2 + \mathcal{J} \), then \( a_1 - a_2 \in A \cap \mathcal{J} = \mathcal{J} \). Therefore \( a_1 + \mathcal{J} \) does not depend on the choice of \( a_1 \). This gives a mapping from the bounded part of \( A/\mathcal{J} \) into \( A/\mathcal{J} \). Since \( a_1 + \mathcal{J} = a_2 + \mathcal{J} \) implies \( a_1 + \mathcal{J} = a_2 + \mathcal{J} \), this mapping is injective. The mapping is obviously surjective.

Since \( a_1 + \mathcal{J} = a_2 + \mathcal{J} \) is equivalent to \( a_1 + \mathcal{J} = a_2 + \mathcal{J} \) for \( a_1, a_2 \in A \), the norm of \( a + \mathcal{J} \) is equal to the norm of \( a + \mathcal{J} \) in the bounded part of \( A/\mathcal{J} \).

**Proposition 14.** Let \( A \) be a C*-algebra in \( V(\sigma) \) and \( A \) its bounded part. The bounded part of the algebra of multipliers of \( A \) in \( V(\sigma) \) is the algebra of multipliers of \( A \).

**Proof.** If \((T', T'')\) is a multiplier of \( \hat{A} \) and its norm is bounded, then \((T', T'')\) is obviously a multiplier of \( A \). On the other hand, if \((T', T'')\) is a multiplier of \( A \), then \( T'(fx) = fT'(x) \) and \( T''(fx) = fT''(x) \) for every \( f \in C^\sigma \). Extend \((T', T'')\) to a multiplier of \( A \) in \( V(\sigma) \).

We denote the algebra of multipliers of \( A \) by \( \mathcal{A}(A) \). The proposition implies that \( \mathcal{A}(A) \) is a \( \mathcal{Z} \)-C*-algebra if \( A \) is a \( \mathcal{Z} \)-C*-algebra and that \( A \) is a \( \mathcal{Z} \)-ideal of \( \mathcal{A}(A) \). We denote \( \mathcal{A}(A)/A \) by \( \mathcal{O}(A) \). Then Proposition 13 implies that \( \mathcal{O}(A) \) is the bounded part of \( \mathcal{A}(\hat{A})/\hat{A} \) in \( V(\sigma) \). If \( \mathcal{J} \) is an ideal of \( \hat{A} \) in \( V(\sigma) \) and \( \mathcal{J} \) is the bounded part of \( \mathcal{J} \), then the restriction of the \(*\)-homomorphism \( \mathcal{A}(\hat{A}) \to \mathcal{A}(\mathcal{J}) \) in \( V(\sigma) \) to \( \mathcal{A}(A) \) is the \( \mathcal{Z} \)-*-homomorphism \( \mathcal{A}(A) \to \mathcal{A}(\mathcal{J}) \).

§ 3. \( \mathcal{Z} \)-Hilbert \( A \)-modules

**Definition.** Let \( A \) be a \( \mathcal{Z} \)-C*-algebra and \( E \) be a linear space over \( \mathcal{Z} \) which is a right \( A \)-module at the same time satisfying the following condition

\[
x \in E, \ a \in A, \ z \in \mathcal{Z} \implies z(xa) = (zx)a = x(za).
\]

\( E \) is said to be a \( \mathcal{Z} \)-pre-Hilbert \( A \)-module if an inner product \( E \times E \to A \) is defined which satisfies the following conditions for every \( x, y, u \in E \), every \( a \in A \) and every \( z \in \mathcal{Z} \):

1) \((a, y + u) = (x, y) + (x, u); (x, zy) = z(x, y)\);  
2) \((x, ya) = (x, y)a\);  
3) \((y, x) = (x, y)^*\);  
4) \((x, x) \geq 0 \) and if \((x, x) = 0\), then \( x = 0 \).

We define a \( \mathcal{Z} \)-norm on \( E \) by setting \( \|x\|^* = (\|x(x, x)\|)^{1/2} \). Then we define a norm on \( E \) by setting \( \|x\| = \|x\|^* \).

5) Let \( \{b_\alpha\}_\alpha \) be a partition of unity in \( \mathcal{B} \) and \( \{x_\alpha\}_\alpha \subseteq E \). If \( \{\|x_\alpha\|\}_\alpha \) has a
bound, then there exists a unique $x \in E$ such that for every $\alpha, b_\alpha x = b_\alpha x$. Such $x$ is denoted by $\sum b_\alpha x_\alpha$.

**Proposition 1.** Let $A$ be a $\mathcal{C}^*$-algebra and $E$ be a $\mathcal{X}$-pre-Hilbert $A$-module. Then $E$ is a normal $\mathcal{X}$-space.

**Proof.** 1) $\|x+y\| \leq \|x\| + \|y\|.$
   
   \[\|x+y\| = \|x+y\|^* \leq \|x\|^* + \|y\|^* \leq \|x\| + \|y\| = \|x\| + \|y\|.
   \]

2) $\|zx\| \leq \|z\| \|x\|.$

\[\|zx\| = \|zx\|^* = \|z\| \|x\|^* \leq \|z\| \|x\| = \|z\| \|x\|.
\]

**Proposition 2.** Let $\hat{A}$ be a $C^*$-algebra in $V(\mathcal{A})$ and $\hat{E}$ be a pre-Hilbert $\hat{A}$-module in $V(\mathcal{A})$. Let $A$ and $E$ be the bounded part of $\hat{A}$ and $\hat{E}$ respectively. Then $E$ is a $\mathcal{X}$-pre-Hilbert $A$-module and every $\mathcal{X}$-pre-Hilbert $A$-module is obtained in this way.

**Proof.** The first part of the proposition is obvious. For the second part, we define

\[\{(f, \hat{x}) | f \in C^* \text{ and } x \in E\} \quad \text{in } V(\mathcal{A}).\]

We denote $(f, \hat{x})$ by $f \hat{x}$. We define $\|f \hat{x}\|^* = \|f\|\|x\|^*$ and $[f \hat{x} = 0] = [\|f\| = 0] \lor [\|x\|^* = 0]$ as in the proof of Theorem 1 in §1. Since $\|x\| = \inf(M\|x\|^* \leq M)$, everything goes through as before.

**Definition.** Let $E$ be a $\mathcal{X}$-pre-Hilbert $A$-module. $E$ is said to be a $\mathcal{X}$-Hilbert $A$-module if it is complete.

**Corollary 1.** Let $\hat{A}$ be a $C^*$-algebra in $V(\mathcal{A})$ and $\hat{E}$ be a Hilbert $\hat{A}$-module in $V(\mathcal{A})$. Let $A$ and $E$ be the bounded part of $\hat{A}$ and $\hat{E}$ respectively. Then $E$ is a $\mathcal{X}$-Hilbert $A$-module and every $\mathcal{X}$-Hilbert $A$-module is obtained in this way.

**Corollary 2.** Let $A$ be a $\mathcal{X}$-$C^*$-algebra and $E$ a $\mathcal{X}$-Hilbert $A$-module. Then $E$ is a Hilbert $A$-module.

**Proof.** Obviously $E$ is a pre-Hilbert $A$-module. The norm of $E$ as a pre-Hilbert $A$-module coincides with the norm $E$ as a $\mathcal{X}$-pre-Hilbert $A$-module as is seen in the following.

\[
\|(x, x)\|^1/2 \leq M \iff \|(x, x)\| \leq M^2 \iff (x, x)(x, x) \leq M^4
\]

\[
\text{iff } (x, x) \leq M^2 \text{ iff } \|(x, x)\|^* \leq M^2
\]

\[
\text{iff } (\|(x, x)\|^*)^{1/2} \leq M \text{ iff } \|(x, x)\|^{1/2} \leq M
\]
**Corollary 3.** Let $A$ be a $\mathcal{D}$-$C^*$-algebra and $E$ a Hilbert $A$-module. Then $E$ is a $\mathcal{D}$-Hilbert $A$-module iff it satisfies the following conditions.

1) $E$ is a linear space over $\mathcal{D}$ and the following condition holds

$$x \in E, a \in A, z \in \mathcal{D} \implies z(xa) = (zx)a = x(za).$$

2) Let $\{b_\alpha\}_\alpha$ be a partition of unity in $\mathcal{D}$ and $\{x_\alpha\}_\alpha \subseteq E$. If $\{\|x_\alpha\|\}_\alpha$ has a bound, then there exists a unique $x \in E$ such that for every $\alpha$, $b_\alpha x_\alpha = b_\alpha x$.

**Definition.** Let $A$ be a $\mathcal{D}$-$C^*$-algebra and $E_i(i \in I)$ be $\mathcal{D}$-Hilbert $A$-modules. The Hilbert direct sum $\bigoplus_{i \in I} E_i$ is, by definition, the $\mathcal{D}$-algebraic direct sum of $E_i$ which is the set of the form $\sum \alpha u_\alpha$, where $\{b_\alpha\}_\alpha$ is a partition of unity in $\mathcal{D}$ and $u_\alpha$ is a finite sum of the form $\sum x_i$. The inner product in the $\mathcal{D}$-algebraic direct sum of $E_i$ is defined by the following equations

$$\left(\bigoplus_i x_i, \bigoplus_i y_i\right) = \sum_{i \in I} \langle x_i, y_i \rangle$$

$$\sum_{\alpha} b_\alpha u_\alpha, \sum_{\beta} b'_\beta v_\beta = \sum_{\alpha, \beta} (b_\alpha b'_\beta)(u_\alpha, v_\beta).$$

The $\mathcal{D}$-Hilbert direct sum $\bigoplus_{i \in I} E_i$ is, by definition, the completion of the $\mathcal{D}$-algebraic direct sum in the norm.

**Proposition 3.** Let $\hat{A}$ be a $C^*$-algebra in $V^{(s)}$ and $\hat{E}_i(i \in I)$ be Hilbert $\hat{A}$-modules in $V^{(s)}$. Let $A$ and $E_i$ be the bounded part of $\hat{A}$ and $\hat{E}_i$ respectively. Then $\bigoplus_i E_i$ is the bounded part of $\bigoplus_i E_i$ in $V^{(s)}$.

**Proof.** This is immediate from Proposition 2 and its corollaries.

**Definition.** Let $A$ be a $\mathcal{D}$-$C^*$-algebra. Then the $\mathcal{D}$-Hilbert space over $A$ is defined as the Hilbert direct sum $\mathcal{H}_A^\mathcal{D} = \bigoplus_{i \in I} A$. The inner product on $A$ is defined by $\langle a_i, a_\alpha \rangle = a_i^* a_\alpha$. Let $\mathcal{H}_\hat{A}$ be the separable Hilbert space over $\hat{A}$ in $V^{(s)}$ and $A$ the bounded part of $\hat{A}$. Then $\mathcal{H}_A^\mathcal{D}$ is the bounded part of $\mathcal{H}_\hat{A}$ as is easily seen as a corollary of Proposition 3.

**Definition.** Let $E$ be a Hilbert $A$-module. As usual by $\mathcal{D}(E)$ we denote the set of mappings $T: E \to E$ for which there exists $T^*: E \to E$ satisfying the condition

$$\langle T(x), y \rangle = \langle x, T^*(y) \rangle \quad \text{for every } x, y \in E.$$  

Every element $T \in \mathcal{D}(E)$ is bounded, linear and $A$-modular and $\mathcal{D}(E)$ is a $C^*$-algebra with the norm induced from the space of bounded linear operators on $E$. (See [4] and [6].)

**Proposition 4.** Let $\hat{A}$ be a $C^*$-algebra in $V^{(s)}$ and $\hat{E}$ a Hilbert $\hat{A}$-module in $V^{(s)}$. Let $A$ and $E$ be the bounded part of $\hat{A}$ and $\hat{E}$ respectively. Then
$\mathcal{L}(E)$ is the bounded part of $\mathcal{L}(\hat{E})$.

**Proof.** Let $T$ be a bounded element of $\mathcal{L}(\hat{E})$. Then the norm of $T^*$ is also bounded and $T: E \to E$ and $T^*: E \to E$ satisfying $(T(x), y) = (x, T^*(y))$ for every $x, y \in E$. Then for every $x \in E$ and $z \in \mathbb{C}$, we have $T(zx) = zT(x)$ and $T^*(zx) = zT^*(x)$ and $T$ and $T^*$ can be uniquely extended to $\hat{T}: \hat{E} \to \hat{E}$ and $\hat{T}^*: \hat{E} \to \hat{E}$ satisfying $\hat{T}(zx) = z\hat{T}(x)$ and $\hat{T}^*(zx) = z\hat{T}^*(x)$ for $x \in \hat{E}$ and $z \in \mathbb{C}$. Since $bx = by$ implies $b\hat{T}(x) = b\hat{T}(y)$ and $b\hat{T}^*(x) = b\hat{T}^*(y)$, $\hat{T}$ and $\hat{T}^*$ can be considered as members of $V(\omega)$. Obviously $\hat{T}$ and $\hat{T}^*$ satisfy $(\hat{T}(x), y) = (x, \hat{T}^*(y))$ for every $x, y \in E$. Now the proposition is clear since the boundedness of $\hat{T}$ and $\hat{T}^*$ follows immediately from the boundedness of $T$ and $T^*$.

**Corollary.** Let $A$ be a $\mathcal{Z}$-$C^*$-algebra and $E$ a $\mathcal{Z}$-Hilbert $A$-module. Then $\mathcal{L}(E)$ is a $\mathcal{Z}$-$C^*$-algebra.

**Definition.** We recall the definition of $\mathcal{K}(E)$. Let $E$ be a Hilbert $A$-module. For $x, y, u \in E$ set $\theta_{x,y}(u) = x(y, u)$. Then $\theta_{x,y} \in \mathcal{L}(E)$. The closure in $\mathcal{L}(E)$ of the linear subspace generated by the operators $\theta_{x,y}$ is an ideal in $\mathcal{L}(E)$, which is denoted by $\mathcal{K}(E)$. Let $A$ be a $\mathcal{Z}$-$C^*$-algebra and $E$ a $\mathcal{Z}$-Hilbert $A$-module. The smallest $\mathcal{Z}$-$C^*$-subalgebra of $\mathcal{L}(E)$ which has all $\theta_{x,y}$ for $x, y \in E$ is denoted by $\mathcal{K}^*(E)$.

**Proposition 5.** Let $A$ be a $C^*$-algebra in $V(\omega)$ and $A$ its bounded part. If $A_1$ is a $C^*$-subalgebra of $A$ in $V(\omega)$, then the bounded part of $A_1$ is a $\mathcal{Z}$-$C^*$-subalgebra of $A$ and every $\mathcal{Z}$-$C^*$-subalgebra of $A$ is obtained in this way.

**Proof.** The first half is obvious. Now let $A_1$ be a $\mathcal{Z}$-$C^*$-subalgebra of $A$. Then there exists a $C^*$-algebra $\hat{A}_1$ in $V(\omega)$ such that the bounded part of $\hat{A}_1$ is isomorphic to $A_1$. Then there exists an embedding of $\hat{A}_1$ into $\hat{A}$ which induces the isomorphism between the bounded part of $\hat{A}_1$ and $A_1$. The image of $\hat{A}_1$ is the $C^*$-subalgebra of $A$ we are looking for.

**Proposition 6.** Let $\hat{A}$ be a $C^*$-algebra in $V(\omega)$ and $\hat{E}$ a Hilbert $\hat{A}$-module in $V(\omega)$. Let $A$ and $E$ be the bounded part of $\hat{A}$ and $\hat{E}$ respectively. Then $\mathcal{K}^*(E)$ is the bounded part of $\mathcal{K}(\hat{E})$.

**Proof.** $\mathcal{K}(\hat{E})$ is generated by $\{\theta_{\hat{x},\hat{x}}| \hat{x} \in \hat{E}\}$. $\theta_{\hat{x},\hat{x}}$ is bounded iff $\hat{x}$ is bounded. Therefore the bounded part of $\mathcal{K}(\hat{E})$ has all linear combinations with the coefficients in $\mathcal{Z}$ of $\theta_{\hat{x},\hat{x}}$ with $x \in E$ and include $\mathcal{K}^*(E)$. On the other hand let $\hat{A}_1$ be the $C^*$-subalgebra of $\mathcal{L}(\hat{E})$ in $V(\omega)$, whose bounded part is $\mathcal{K}^*(E)$. Then $\hat{A}_1$ includes all $\theta_{\hat{x},\hat{x}}$ for $\hat{x} \in \hat{E}$. Therefore $\hat{A}_1$ includes $\mathcal{K}(\hat{E})$. Hence $\mathcal{K}^*(E)$ is the bounded part of $\mathcal{K}(\hat{E})$.

**Corollary.** Let $A$ be a $\mathcal{Z}$-$C^*$-algebra and $E$ a $\mathcal{Z}$-Hilbert $A$-module.
Then $\mathcal{X}^z(E)$ is an $\mathcal{Z}$-ideal of $\mathcal{L}(E)$.

**Definition.** If $\hat{A}$ is $C^*$, then $\mathcal{H}^z_\hat{A}$ is denoted by $\hat{A}^z$. If $A$ is $\mathcal{Z}$, then $\mathcal{H}^z\hat{A}$ is denoted by $\mathcal{H}^z\hat{\hat{A}}$. Then $\mathcal{H}^z\hat{A}$ is the bounded part of $\mathcal{H}^z\hat{\hat{A}}$. We set $\mathcal{H}^z=\mathcal{H}(\mathcal{H}^z), \mathcal{H}^z\hat{\hat{A}}=\mathcal{H}(\mathcal{H}^z\hat{\hat{A}})$, and $\mathcal{H}^z\hat{\hat{A}}=\mathcal{H}(\mathcal{H}^z\hat{\hat{A}})$. We have $\mathcal{H}(\hat{A})\simeq\hat{A}$ and $\mathcal{H}^z(A)\simeq A$. We also have $\mathcal{H}^z\simeq\mathcal{H}^z\hat{\hat{A}}$ in $\mathcal{V}(\mathcal{O})$ and $\mathcal{H}^z\hat{\hat{A}}\simeq\mathcal{H}^z\hat{\hat{A}}\hat{\hat{A}}$, where $\mathcal{H}^z\hat{\hat{A}}\hat{\hat{A}}$ is the bounded part of $\mathcal{H}\hat{\hat{A}}\hat{\hat{A}}$ in $\mathcal{V}(\mathcal{O})$. If $\mathcal{Z}'$ is $\mathcal{Z}$-separable, then $\mathcal{L}(\mathcal{H}^z)$ is isomorphic to $\mathcal{Z}'$ and $\mathcal{H}^z\hat{\hat{A}}$ is the $\mathcal{Z}$-$C^*$-subalgebra of $\mathcal{Z}'$ generated by all finite projections in $\mathcal{Z}'$.

As a translation of Theorem 1 in [4], we have the following theorem.

**Theorem 1.** Let $A$ be a $\mathcal{Z}$-$C^*$-algebra and $E$ a $\mathcal{Z}$-Hilbert $A$-module. Then the correspondence $T \in \mathcal{L}(E) \rightarrow (T_1, T_2) \in \mathcal{M}(\mathcal{H}(E))$, where $T_1, T_2 \in \mathcal{M}(E)$ and $T_1 x, y = \theta_{x, y}, T_2 x, y = \theta_{x, y} T_1(x, y), x, y \in E$, defines a $\mathcal{Z}$-isomorphism of $\mathcal{L}(E)$ onto $\mathcal{M}(\mathcal{H}(E))$.

**Corollary.** $\mathcal{L}(\mathcal{H}^z)\simeq\mathcal{M}(\mathcal{H}^z\hat{\hat{A}})$.

**Definition.** We state the standard definitions about graded algebras and the corresponding $\mathcal{Z}$-definitions by adding the inside of $(\mathcal{Z})$. A $(\mathcal{Z})C^*$-algebra $A$ is said to be $\mathcal{Z}$-graded if we have a decomposition $A = A^{(0)} \oplus A^{(1)}$ in which $A^{(0)}$ and $A^{(1)}$ are closed self-adjoint $(\mathcal{Z})$-linear subspaces such that $A^{(i)}A^{(j)} \subseteq A^{(i+j)}$ for $i, j \in \mathcal{Z}$. The grading of $A$ is said to be trivial if $A^{(1)} = 0$. Any algebra $A$ can be considered trivially graded by putting $A^{(0)} = A$ and $A^{(1)} = 0$. In particular the algebra $C$ (or $\mathcal{Z}$) of scalars will be considered trivially graded. For a unital algebra $A$ the condition $1 \in A^{(0)}$ is satisfied. By the equality $\deg x = i$ we mean that $x \in A^{(i)}$. The graded commutator $[x, y]$ is defined on homogeneous elements $x, y \in A$ by the formula

$$[x, y] = xy - (-1)^{\deg x \cdot \deg y} yx.$$

This definition can be extended to all $x, y \in A$ by linearity.

A $(\mathcal{Z})$ Hilbert $A$-module $E$ is said to be graded if it admits a decomposition $E = E^{(0)} \oplus E^{(1)}$ into the direct sum of closed $(\mathcal{Z})$-subspaces, where for every $i, j \in \mathcal{Z}$, we have $E^{(i)}A^{(j)} \subseteq E^{(i+j)}$ and $(E^{(i)}, E^{(j)}) \subseteq A^{(i+j)}$ (inner product). The grading of $E$ defines a grading of $\mathcal{L}(E)$ and $\mathcal{H}(E)$ (or $\mathcal{H}^z(E)$) in the following way:

$$\deg T = j \text{ if } T(E^{(i)}) \subseteq E^{(i+j)} (i, j \in \mathcal{Z}).$$

Since $\mathcal{M}(A)$ is isomorphic to $\mathcal{L}(A)$ and $(\mathcal{Z})$-isomorphic to $\mathcal{L}(A)$ if $A$ is a $\mathcal{Z}$-$C^*$-algebra, $\mathcal{M}(A)$ is also graded. In the same way, the $(\mathcal{Z})$-isomorphism $\mathcal{L}(E) \cong \mathcal{M}(\mathcal{H}(E))$ (or $\mathcal{M}(\mathcal{H}^z(E))$) is also graded. The opposite grading of the $(\mathcal{Z})$ Hilbert module can be obtained by interchanging $E^{(0)}$ and $E^{(1)}$. The grading
of $\mathcal{L}(E)$ is preserved under changing the grading of $E$ to the opposite grading. If it is known that the grading of $\mathcal{N}(E)$ (or $\mathcal{N}^*(E)$) is induced by the grading of $E$, then the restoration of the grading of $E$ by means of the grading of $\mathcal{N}(E)$ (or $\mathcal{N}^*(E)$) is possible in two ways: since for every $x, y \in E$ we have $\deg x - \deg y = \deg \theta_{x, y}$, it is sufficient to give $\deg x$ for a single element $x \in E$. If $A$ has the trivial grading, then $E^{(0)}$ and $E^{(1)}$ are orthogonal ($\mathcal{F}$-) $A$-modules and the grading operator $\varepsilon \in \mathcal{L}(E)$ is defined as follows: $\varepsilon = 1$ on $E^{(0)}$ and $\varepsilon = -1$ on $E^{(1)}$.

**Definition.** A canonically graded ($\mathcal{F}$-) Hilbert space is defined as the direct sum $\mathcal{H}_A \oplus \mathcal{H}_A$ (or $\mathcal{H}_A^\mathcal{F} \oplus \mathcal{H}_A^\mathcal{F}$), where the gradings of the two summands are opposite and the grading of the first one $\mathcal{H}_A = \bigoplus_{i=1}^\infty A$ (or $\mathcal{H}_A^\mathcal{F} = \bigoplus_{i=1}^\infty A$) is defined in such a way that $\deg x = \deg a$ if $x = (0, \ldots, 0, a, 0, \ldots)$.

**Definition.** Let $E_1$ and $E_2$ be $\mathcal{F}$-modules. The $\mathcal{F}$-algebraic tensor product of $E_1$ and $E_2$ is obtained from the algebraic tensor product by adding the following rule:

\[ x_1 \in E_1, x_2 \in E_2, z \in \mathcal{Q} \implies z(x_1 \otimes^\mathcal{F} x_2) = (zx_1) \otimes^\mathcal{F} x_2 = x_1 \otimes^\mathcal{F} (zx_2). \]

Let $A_1$ and $A_2$ be graded $\mathcal{F}$-$C^*$-algebras, $E_1$ and $E_2$ graded $\mathcal{F}$-Hilbert modules over $A_1$ and $A_2$ respectively and $\phi: A_1 \to \mathcal{L}(E)$ a graded $\mathcal{F}$-$*$-homomorphism. The $\mathcal{F}$-algebraic tensor product $E_1 \otimes^\mathcal{F} E_2$ is a right $A_2$-module with the grading $\deg(x_1 \otimes^\mathcal{F} x_2) = \deg x_1 + \deg x_2$ and inner product $(x_1 \otimes^\mathcal{F} x_2, y_1 \otimes^\mathcal{F} y_2) = (x_2, \phi((x_1, x_2))y_2)$. Factoring $E_1 \otimes^\mathcal{F} E_2$ over $\mathcal{F}$-$A_2$-module $N = \{u \in E_1 \otimes^\mathcal{F} E_1 | (u, u) = 0\}$, and then completing it in the norm $\|u\| = \|(u, u)\|^{1/2}$, we obtain a $\mathcal{F}$-Hilbert $A_2$-module, which will be denoted by $E_1 \otimes^\mathcal{F}_A E_2$. The correspondence $F \to F \otimes^\mathcal{F} 1$ defines a $\mathcal{F}$-$*$-homomorphism $\phi_*: \mathcal{L}(E_1) \to \mathcal{L}(E_1 \otimes^\mathcal{F}_A E_2)$. If $\phi$ is an embedding, so is $\phi_*$. In particular, every faithful representation $A_1 \to \mathcal{L}(\mathcal{K}^\mathcal{F})$ induces a faithful representation $\mathcal{L}(E_1) \to \mathcal{L}(E_1 \otimes^\mathcal{F}_A \mathcal{K}^\mathcal{F})$. Let $\hat{A}_1$ and $\hat{A}_2$ be $C^*$-algebras in $V(A)$, $\hat{E}_1$ and $\hat{E}_2$ Hilbert modules over $\hat{A}_1$ and $\hat{A}_2$ in $V(A)$, and $\hat{\phi}: \hat{A}_1 \to \mathcal{L}(\hat{E}_2)$ a graded $*$-homomorphism in $V(A)$. Let $A_1, A_2, E_1$, and $E_2$ be the bounded part of $\hat{A}_1, \hat{A}_2, \hat{E}_1$, and $\hat{E}_2$ respectively. If $\phi$ is the restriction of $\hat{\phi}$ to $A_1$, then $E_1 \otimes^\mathcal{F}_A E_2$ is the bounded part of $\hat{E}_1 \otimes^\mathcal{F}_A \hat{E}_2$. $\phi_*$ is also the restriction of $\hat{\phi}_*$ to $\mathcal{L}(E)$.

**Definition.** Let $A$ be a $\mathcal{F}$-$C^*$-algebra. A $\mathcal{F}$-linear functional $\omega: A \to \mathcal{F}$ is called positive if $\omega(x^* x) \geq 0$ for every $x \in A$. A positive $\mathcal{F}$-linear functional of norm one is called a $\mathcal{F}$-state. Let $\hat{A}$ be a $C^*$-algebra in $V(A)$ and $A$ the bounded part of $\hat{A}$. If $\hat{\omega}$ is a state on $\hat{A}$ in $V(A)$, then the restriction of $\hat{\omega}$ to $A$ is a $\mathcal{F}$-state on $A$ and every $\mathcal{F}$-state on $A$ is obtained in this way. Let $A$ be graded. A $\mathcal{F}$-state $\omega$ on $A$ is said to be graded, if $\omega = 0$ on $A^{(1)}$. 

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Let $A$ and $B$ be graded $\mathcal{Z}$-$C^*$-algebras. The skew-commutative $\mathcal{Z}$-algebraic tensor product of $A$ and $B$ is the algebraic tensor product of $A$ and $B$ as $\mathcal{Z}$-linear spaces with the following grading, product, involution and $\mathcal{Z}$-norm.

$$\deg(a \otimes \sigma b) = \deg a + \deg b,$$

$$(a \otimes \sigma b)(a \otimes \sigma b) = (-1)^{(\deg b_1)(\deg a_2)}(a_1 \otimes \sigma b_1 b_2),$$

$$(a \otimes \sigma b)^* = (-1)^{(\deg a)(\deg b)}(a^* \otimes \sigma b^*),$$

\[
\left( \left\| \sum_{i=1}^{m} a_i \otimes \sigma b_i \right\| \right)^2
\]

\[
= \sup \left( \rho \otimes \lambda \right) \left\{ \left( \sum_{i=1}^{m} x_i \otimes \sigma y_i \right)^* \left( \sum_{i=1}^{m} a_i \otimes \sigma b_i \right) \left( \sum_{i=1}^{m} x_i \otimes \sigma y_i \right) \right\}
\]

where sup is taken over all strictly non-zero elements $\sum_{j=1}^{n} x_j \otimes \sigma y_j$ in the following sense and all graded $\mathcal{Z}$-states $\rho$ and $\lambda$. The element $\sum_{j=1}^{n} x_j \otimes \sigma y_j$ is said to be strictly non-zero, if for every $b \in \mathcal{B}$, $b(\sum_{j=1}^{n} x_j \otimes \sigma y_j) = 0$ implies $b = 0$. $A \hat{\otimes} B$ is the set of all members of the form $\sum_{i} b_i u_i$ where $\{b_i\}$ is a partition of unity in $\mathcal{B}$ and $u$ is of the form $\sum_{i} a_i \otimes \sigma b_i$ and $\sum_{i} b_i u_i \|u_i\|^2$ is a member of $\mathcal{Z}$. The $\mathcal{Z}$-norm of $\sum_{i} b_i u_i$ is, by definition, $\sum_{i} b_i \|u_i\|^2$. The grading, product, and involution in $A \hat{\otimes} B$ is defined by a natural manner e.g.

\[
(\sum_{i} b_i u_i) \cdot (\sum_{j} b_j v_j) = \sum_{i,j} (b_i b_j) u_i \cdot v_j.
\]

The norm of $A \hat{\otimes} b$ is defined by the following equation $\|u\| = \|u\|$. The completion of $A \hat{\otimes} b$ in this norm will be denoted by $A \hat{\otimes} \sigma b$ and called the skew-commutative $\mathcal{Z}$-tensor product of $A$ and $B$. Let $A$ and $B$ be graded $C^*$-algebras in $V^{(x)}$ and $A$ and $B$ the bounded part of $A$ and $B$ respectively. Then $A \hat{\otimes} B$ and $A \hat{\otimes} B$ are the bounded part of $A \hat{\otimes} B$ and $A \hat{\otimes} B$ respectively. If $A$ and $B$ are not graded, we define $A \otimes \sigma b$ to be $A \hat{\otimes} b$ by considering $A$ and $B$ with trivial grading. If $A$ and $B$ are $C^*$-algebras in $V^{(x)}$ and $A$ and $B$ are the bounded part of $A$ and $B$ respectively, then $A \otimes \sigma b$ is the bounded part of $A \otimes B$ in $V^{(x)}$, where $A \otimes B$ denote the tensor product with the minimal $C^*$-norm in $V^{(x)}$.

**DEFINITION.** Let $V = \mathcal{Z}^{\delta} \oplus \mathcal{Z}^{\delta}$. For $x=(x_1, \ldots, x_p, y_1, \ldots, y_q) \in V$. We set $x^*=(x_1^*, \ldots, x_p^*, -y_1^*, \ldots, -y_q^*)$. Let $\varepsilon_1, \ldots, \varepsilon_p, e_1, \ldots, e_q$ be the coordinate basis over $\mathcal{Z}$ in $V$. We define $e_i^* = 1$, $e_i = \varepsilon_i$, and $\varepsilon_i e_j = -\varepsilon_i e_i (i \neq j)$, $i, j \leq p$ and $e_j^* = -1$, $e_j = -e_j$, and $\varepsilon_i e_j = -\varepsilon_i e_i (i \neq j)$, $i, j \leq q$ and $\varepsilon_i e_j = -\varepsilon_j e_i (i \leq p, j \leq q)$. The we have the $\mathcal{Z}$-Clifford algebra $C_{p,q}^\sigma$ over $\mathcal{Z}$. The $\mathcal{Z}$-norm $\|\|_\sigma$ of $C_{p,q}^\sigma$, whose value is in $\mathcal{Z}$, is defined as the usual $\mathcal{Z}$-operator norm. The
norm of $C_{p,q}^\varepsilon$ is defined by $\|x\| = \|x\|^\varepsilon$.

An orientation of $C_{p,q}^\varepsilon$ is a homogeneous element $\omega \in C_{p,q}^\varepsilon$ such that

1) $\omega^\varepsilon = \pm \omega$; $\omega^\varepsilon \omega = 1$.

2) $\forall x \in C_{p,q}^\varepsilon$, $x \omega = (-1)^{\deg x (\deg \omega + 1)} \omega x$.

$C_{p,q}^\varepsilon$ with a fixed orientation $\bar{\omega}$ is called an oriented $\mathcal{Z}$-Clifford algebra. If $\phi$ is a graded automorphism of $C_{p,q}^\varepsilon$, then $\phi(\omega) = \pm \omega$. Let $V = (C_{p,q}^\varepsilon, \omega_1)$ and $W = (C_{p,q}^\varepsilon, \omega_2)$. We say that an isomorphism $\phi: C_{p,q}^\varepsilon \rightarrow C_{p,q}^\varepsilon$ preserves orientation if $\phi(\omega_1) = \omega_2$.

If $V = (C_{p_1,q_1}^\varepsilon, \omega_1)$ and $W = (C_{p_2,q_2}^\varepsilon, \omega_2)$, then the orientation of $V \otimes^\varepsilon W \simeq C_{p_1+p_2,q_1+q_2}^\varepsilon$ will be the element $\omega_1 \otimes^\varepsilon \omega_2$. The standard orientation on $C_{p,q}^\varepsilon$ is defined by $\omega_{p,q} = (-1)^{\varepsilon_1} \varepsilon_1 \cdots \varepsilon_p e_1 \cdots e_q$.

So

$$C_{p,q}^\varepsilon \otimes C_{p',q'}^\varepsilon \simeq C_{p+p',q+q'}^\varepsilon;$$

$$\varepsilon_i \otimes^\varepsilon 1 \rightarrow \varepsilon_i, \quad i \leq p,$$

$$e_j \otimes^\varepsilon 1 \rightarrow e_j, \quad j \leq q,$$

$$1 \otimes^\varepsilon \varepsilon_i \rightarrow (-1)^{\varepsilon_i+1} \varepsilon_{i+p}, \quad i \leq p',$$

$$1 \otimes^\varepsilon e_j \rightarrow e_{j+q}, \quad j \leq q'.$$

We fix the orientation-preserving graded isomorphisms $f: C_{p,q}^\varepsilon \rightarrow C_{p',q'}^\varepsilon$ as follows:

$$f(\varepsilon_i) = \varepsilon_i \quad (\varepsilon_i \leq p); 
\hat{f}(e_i) = i \varepsilon_{p+1}; 
\hat{f}(e_j) = e_{j-1} \quad (2 \leq j \leq q + 1).$$

Obviously $C_{p,q}^\varepsilon$ is the bounded part of $C_{p,q}^\varepsilon$ in $V^{(\varepsilon)}$.

**Definition.** Let $A$, $B$ and $D$ be trivially graded $\mathcal{Z}$-$C^*$-algebras, and let $\phi: A \rightarrow D$ and $\psi: B \rightarrow D$ be $\mathcal{Z}$-$*$-homomorphisms. The $\mathcal{Z}$-subalgebra $A \oplus B = \{(a, b) \in A \oplus B | \phi(a) = \psi(b)\} \subseteq A \oplus B$ is called the fibered sum of $A$ and $B$ over $D$ with projections $\phi$ and $\psi$. In particular, for $\alpha = 0$ and $\alpha = 1$, we define the $\alpha$-cylinder $Z_{\alpha}^\varepsilon(A, D, \phi)$ of the $\mathcal{Z}$-$*$-homomorphism $\phi: A \rightarrow D$ as the fibered sum $A \oplus D[0, 1]^\varepsilon$ with projections $\phi: A \rightarrow D$ and $D[0, 1]^\varepsilon \rightarrow D[\alpha] = D$. If $\phi$ is an embedding or projection onto a quotient algebra, we denote the $\alpha$-cylinder briefly by $Z_{\alpha}^\varepsilon(A, D)$. The $\mathcal{Z}$-cone $S_{\alpha}^\varepsilon(A, D, \phi)$ of a $\mathcal{Z}$-$*$-homomorphism $\phi: A \rightarrow D$ is defined as the fibered sum $A \oplus D[0, 1]^\varepsilon$ with projections $\phi: A \rightarrow D$ and $D[0, 1]^\varepsilon \rightarrow D[0]$. The abbreviated notation is $S_{\alpha}^\varepsilon(A, D)$. Let $A$, $B$ and $D$ be the bounded part of $C^*$-algebras $\hat{A}$, $\hat{B}$ and $\hat{D}$ in $V^{(\varepsilon)}$ respectively, and let $\phi$ be the restriction of $\hat{\phi}: \hat{A} \rightarrow \hat{D}$ to $A$. Then $Z_{\alpha}^\varepsilon(A, D, \phi)$, $S_{\alpha}^\varepsilon(A, D, \phi)$ are the bounded part of $Z_{\alpha}(\hat{A}, \hat{D}, \hat{\phi})$ and $S(\hat{A}, \hat{D}, \hat{\phi})$ in $V^{(\varepsilon)}$ respectively. Therefore it is easy to see that the presence of $\mathcal{Z}$-strictly positive elements in $B$ and a $\mathcal{Z}$-ideal $J \subseteq B$ implies the presence of $\mathcal{Z}$-strictly positive elements in $Z_{\alpha}(J, B)$, $Z_{\alpha}(B, B/J)$, $S_{\alpha}(J, B)$ and $S_{\alpha}(B, B/J)$. For the $\mathcal{Z}$-separability or $\mathcal{Z}$-nuclearity of these algebras, it is sufficient that $B$ have the corresponding properties.
§ 4. $\mathcal{Z}$-K-functors

Definition. Let $A$ and $B$ be $\mathcal{Z}$-$C^*$-algebras. We denote by $\mathcal{E}^z(A, B)$ the set of triples $(\varepsilon, \phi, F)$ where $\varepsilon$ is a canonical grading of the space $\mathcal{H}^z_0$, $\phi: A \to \mathcal{L}(\mathcal{H}^z_0)$ is a $\mathcal{Z}$-$C^*$-homomorphism, $F \in \mathcal{L}(\mathcal{H}^z_0)$ is an operator of degree 1 and the elements

$$[\phi(a), F], \quad (F^*-1)\phi(a), \quad (F-F^*)\phi(a) \quad (1)$$

belong to $\mathcal{H}^z_n=\mathcal{H}^z(\mathcal{H}^z_n)$ for every $a \in A$. By $\mathcal{D}^z(A, B)$ we denote the set of degenerate triples, i.e., those for which all elements (1) are equal to 0.

Let $(\varepsilon, \phi, \hat{F}) \in \mathcal{E}(\hat{A}, \hat{B})$ in $V(\sigma)$. $(\varepsilon, \phi, \hat{F})$ is, by definition, bounded if $\hat{F}$ is bounded. If $\hat{A}$ and $\hat{B}$ are $C^*$-algebras in $V(\sigma)$ and $A$ and $B$ are the bounded part of $\hat{A}$ and $\hat{B}$ respectively, then $\mathcal{E}^z(A, B)$ and $\mathcal{D}^z(A, B)$ are the bounded part of $\mathcal{E}(\hat{A}, \hat{B})$ and $\mathcal{D}(\hat{A}, \hat{B})$ in $V(\sigma)$ respectively.

Definition. The triples $(\varepsilon_1, \phi_1, F_1), (\varepsilon_2, \phi_2, F_2) \in \mathcal{E}^z(A, B)$ are said to be unitarily equivalent if there exists a unitary element $u \in \mathcal{L}(\mathcal{H}^z_0)$ which converts $(\varepsilon_1, \phi_1, F_1)$ into $(\varepsilon_2, \phi_2, F_2)$ i.e., $\deg_u(u(x)) = \deg(x)$ for $x \in \mathcal{H}^z_0$, $\phi_1(a) = u\phi_2(a)u^{-1}$ for $a \in A$, and $F_2 = uF_1u^{-1}$. Let $\hat{A}$ and $\hat{B}$ be $C^*$-algebras in $V(\sigma)$ and $A$ and $B$ the bounded part of $\hat{A}$ and $\hat{B}$ respectively. For $(\varepsilon_i, \phi_i, F_i) \in \mathcal{E}^z(A, B) (i=1, 2), (\varepsilon_i, \phi_i, F_i)$ and $(\varepsilon_i, \phi_i, F_i)$ are unitarily equivalent in $\mathcal{E}^z(A, B)$ iff they are unitarily equivalent in $\mathcal{E}(\hat{A}, \hat{B})$ in $V(\sigma)$, since the unitary elements are absolute.

Definition. A $\mathcal{Z}$-homotopy connecting the triples $x_0=(\varepsilon_0, \phi_0, F_0)$ and $x_1=(\varepsilon_1, \phi_1, F_1) \in \mathcal{E}^z(A, B)$ is, by definition, a triple

$$[x_i]=([\varepsilon_i], [\phi_i], [F_i])_{i \in [0, 1]} \in \mathcal{E}^z(A, B[0, 1])$$

the restriction of which to the endpoints of the interval $[0, 1]$ coincides with the given triples $x_0$ and $x_1$.

Proposition 1. Let $\hat{A}$ and $\hat{B}$ be $C^*$-algebras in $V(\sigma)$ and $A$ and $B$ the bounded part of $\hat{A}$ and $\hat{B}$ respectively. Let $x_0=(\varepsilon_0, \phi_0, F_0)$ and $x_1=(\varepsilon_1, \phi_1, F_1)$ be members of $\mathcal{E}^z(A, B)$. $x_0$ and $x_1$ are $\mathcal{Z}$-homotopic in $\mathcal{E}^z(A, B)$ iff they are homotopic in $\mathcal{E}(\hat{A}, \hat{B})$ in $V(\sigma)$.

Proof. It is obvious that if $x_0$ and $x_1$ are $\mathcal{Z}$-homotopic in $\mathcal{E}^z(A, B)$, then they are homotopic in $\mathcal{E}(\hat{A}, \hat{B})$ in $V(\sigma)$. Now let $x_0$ and $x_1$ be homotopic in $\mathcal{E}(\hat{A}, \hat{B})$ in $V(\sigma)$. Let $M \in \mathbb{R}$ satisfy the following conditions: $\|F_0\| \leq \hat{M}$, $\|F_1\| \leq \hat{M}$ and $1 \leq M$. Define $f: \mathcal{L}(\mathcal{H}_0) \to \mathcal{L}(\mathcal{H}_0)$ in $V(\sigma)$ as follows.
Let \( \hat{x}_t = (\hat{\epsilon}_t, \hat{\phi}_t, \hat{F}_t) \) be a homotopy connecting \( x_0 \) and \( x_1 \) in \( \mathcal{C}(A, B) \) in \( V(\sigma) \). Let \( \hat{\epsilon}_t = \hat{\epsilon} | \mathcal{H}^n, \hat{\phi}_t = \hat{\phi}_t | A \), and \( \hat{F}_t \) be the restriction of \( f(\hat{F}_t) \) to \( \mathcal{L}(\mathcal{H}^n) \). Then \( (\hat{\epsilon}_t, \hat{\phi}_t, \hat{F}_t) \) is a \( \mathcal{L} \)-homotopy connecting \( x_0 \) and \( x_1 \).

**Definition.** A \( \mathcal{L} \)-homotopy \( \{x_t\} = (\{\epsilon_t\}, \{\phi_t\}, \{F_t\})_{t \in [0, 1]} \) connecting the triples \( x_0 = (\epsilon_0, \phi_0, F_0) \) and \( x_1 = (\epsilon_1, \phi_1, F_1) \) is called a \( \mathcal{L} \)-operator homotopy if \( \epsilon_t = \epsilon_0 = \epsilon_1 \) and \( \phi_t = \phi_0 = \phi_1 \) for every \( t \in [0, 1] \). Let \( \hat{A} \) and \( \hat{B} \) be \( \mathcal{C}^* \)-algebras in \( V(\sigma) \) and \( A \) and \( B \) the bounded part of \( \hat{A} \) and \( \hat{B} \) respectively. Let \( x_0 \) and \( x_1 \in \mathcal{C}(A, B) \) and \( \hat{x}_t \) be a bounded operator homotopy connecting \( x_0 \) and \( x_1 \) in \( V(\sigma) \). Then the restriction of \( \{\hat{x}_t\} \) to \([0, 1]\) is a \( \mathcal{L} \)-homotopy and every \( \mathcal{L} \)-operator homotopy connecting \( x_0 \) and \( x_1 \) is obtained in this way.

**Definition.** Let \( \mathcal{D}^g(A, B) \) be the set of \( \mathcal{L} \)-homotopy classes of triples and \( \overline{\mathcal{D}}^g(A, B) \) the image of \( \mathcal{D}^g(A, B) \) in \( \mathcal{D}^g(A, B) \). We identify \( \mathcal{H}^n \oplus \mathcal{H}^n \) with \( \mathcal{H}^n \) by means of a \( \mathcal{L} \)-isometry of degree 0 and introduce the following condition on \( \mathcal{D}^g(A, B) \):

\[
(\epsilon_1, \phi_1, F_1) \oplus (\epsilon_0, \phi_0, F_0) = (\epsilon_1 \oplus \epsilon_0, \phi_1 \oplus \phi_0, F_1 \oplus F_0).
\]

We denote by \( \mathcal{L} \)-\( \text{KK}(A, B) \) the factor group \( \mathcal{D}^g(A, B) / \overline{\mathcal{D}}^g(A, B) \).

**Proposition 2.** Let \( \hat{A} \) and \( \hat{B} \) be \( \mathcal{C}^* \)-algebras in \( V(\sigma) \) and \( A \) and \( B \) the bounded part of \( \hat{A} \) and \( \hat{B} \) respectively. If \( x_0 \in \mathcal{D}^g(A, B) \) and \( \hat{x}_t \in \mathcal{D}(\hat{A}, \hat{B}) \) and \( x_0 \) and \( \hat{x}_t \) are homotopic in \( \mathcal{D}(\hat{A}, \hat{B}) \), then \( x_0 \) is \( \mathcal{L} \)-homotopic to some \( y \) in \( \mathcal{D}(\hat{A}, \hat{B}) \).

**Proof.** Let \( \{\hat{x}_t\} \subseteq (\{\hat{\epsilon}_t\}, \{\hat{\phi}_t\}, \{\hat{F}_t\})_{t \in [0, 1]} \) be a homotopy connecting \( x_0 \) and \( \hat{x}_t \) in \( \mathcal{C}(\hat{A}, \hat{B}) \). Let \( M \in \mathbb{R} \) satisfy \( ||F_0|| \leq M \) and \( M \geq 1 \). Let \( f \) be the function defined in proof of Proposition 1. Let \( \epsilon_1 = \epsilon_t | \mathcal{H}^n, \phi_1 = \phi_t | A \) and \( F_t \) be the restriction to \( \mathcal{L}(\mathcal{H}^n) \). Obviously \( y = (\epsilon_1, \phi_1, F_1) \in \mathcal{D}^g(A, B) \) and the proposition is proved.

**Definition.** A member of \( \text{KK}(\hat{A}, \hat{B}) \) in \( V(\sigma) \) is said to be bounded if it is the image of a bounded member of \( \mathcal{C}(\hat{A}, \hat{B}) \).

**Proposition 3.** Let \( \hat{A} \) and \( \hat{B} \) be \( \mathcal{C}^* \)-algebras in \( V(\sigma) \) and \( A \) and \( B \) the bounded part of \( \hat{A} \) and \( \hat{B} \). Then the bounded part of \( \text{KK}(\hat{A}, \hat{B}) \) in \( V(\sigma) \) is \( \mathcal{L} \)-\( \text{KK}(A, B) \).

As a translation of Theorem 1 and Remark 2 in §4 of [5], we have the following theorem and remark.
**Theorem 1.** If $A$ and $B$ are $\mathcal{Z}$-$C^*$-algebras, then $\mathcal{Z}$-$KK(A, B)$ is a group.

**Remark.** If $A$ is unital and $B$ has a $\mathcal{Z}$-countable approximate identity, then in the definition of $\mathcal{Z}$-$\ast(A, B)$ it can be required additionally that the $\mathcal{Z}$-$\ast$-homomorphism $\phi$ be unital. This does not change the group $\mathcal{Z}$-$KK(A, B)$. If $\phi$ is unital, then we have $F^* - 1 \in \mathcal{K}_B^*$ and $F - F^* \in \mathcal{K}_B^*$ for triple $(\varepsilon, \phi, F) \in \mathcal{Z}$-$\ast(A, B)$. In the case $A = \mathcal{Z}$ we shall usually assume that $\phi$ is unital and omit $\phi$ in the notation of the element $(\varepsilon, \phi, F) \in \mathcal{Z}$-$\ast(\mathcal{Z}, B)$.

As the translation of Theorem 2 and its corollary in §4 of [5], we have the following theorem and corollary.

**Theorem 2.** Let $A_1$ and $B_1$ be $\mathcal{Z}$-$C^*$-algebras and $A$ and $B$ a $\mathcal{Z}$-ideal of $A_1$ and $B_1$ respectively. If $A$, $B$, $A_1$, and $B_1$ have $\mathcal{Z}$-countable approximate identities, then the following hold.

1. The group $\mathcal{Z}$-$KK(A, B)$ does not change if in the definition of $\mathcal{Z}$-$KK(A, B)$, $\mathcal{Z}(\mathcal{K}_B^*)$ is replaced by $\mathcal{Z}(\mathcal{K}_B^*)$ and the condition that the elements (1) belong to the ideal $\mathcal{K}_B^*$ is preserved (note that $\mathcal{K}_B^* \subseteq \mathcal{K}_B^* \subseteq \mathcal{Z}(\mathcal{K}_B^*)$).

2. The group $\mathcal{Z}$-$KK(A, B)$ does not change if in its definition it is required that the $\mathcal{Z}$-$\ast$-homomorphism $\phi$ be extended to $A_1$. Both of these changes are permissible at the same time.

**Corollary.** The following properties hold.

1. $\mathcal{Z}$-$KK(A, B) \otimes \mathcal{Z}$-$KK(A_1, B_1) \simeq \mathcal{Z}$-$KK(A, B) \otimes \mathcal{Z}$-$KK(A_1, B_1)$.

2. If $A_1$, $A_2$, and $B$ have $\mathcal{Z}$-countable approximate identities, then $\mathcal{Z}$-$KK(A_1 \oplus A_2, B) \simeq \mathcal{Z}$-$KK(A_1, B) \oplus \mathcal{Z}$-$KK(A_2, B)$.

**Definition.** (Functorial properties). A $\mathcal{Z}$-$\ast$-homomorphism $f: A \rightarrow A_1$ induces a group homomorphism $f^*: \mathcal{Z}$-$KK(A, B) \rightarrow \mathcal{Z}$-$KK(A_1, B)$ by the formula

\[ f^*(\varepsilon, \phi, F) = (\varepsilon, f \ast \phi, F). \]

Let $\mathcal{Z}$-$C^*$-algebras $B_1$ and $B_2$ have $\mathcal{Z}$-countable approximate identities. Let $g: B_1 \rightarrow B_2$ be a $\mathcal{Z}$-$\ast$-homomorphism. It induces $g_*: \mathcal{Z}(\mathcal{K}_{B_1}^*) \rightarrow \mathcal{Z}(\mathcal{K}_{B_2}^*)$, where $B_1$ and $B_2$ are the $\mathcal{Z}$-$C^*$-algebras obtained from $B_1$ and $B_2$ by adjunction of 1 respectively. The group homomorphism $g_*: \mathcal{Z}$-$KK(A, B) \rightarrow \mathcal{Z}$-$KK(A, B)$ is defined by the formula $g_*(\varepsilon, \phi, F) = (\varepsilon, g_* \ast \phi, g_*(F))$, where $\varepsilon$ is the grading of $\mathcal{K}_{B_1}^* \otimes \mathcal{K}_{B_2}^* B_2$ corresponding to $\varepsilon$.

For any $\mathcal{Z}$-$C^*$-algebra $D$ the homomorphism

\[ \tau^D_0: \mathcal{Z}$-$KK(A, B) \rightarrow \mathcal{Z}$-$KK(A \otimes^D B, B \otimes^D D) \]

is defined by the formula $\tau^D_0(\varepsilon, \phi, F) = (\varepsilon, \phi \otimes^D 1, F \otimes^D 1)$, where $\varepsilon$ is the grading
of $\mathcal{K}_{p}^\pi \hat{\otimes} \pi D$ corresponding to $\varepsilon$.

Let $\hat{A}_1, \hat{A}_2, \hat{B}_1, \hat{B}_2$, and $\hat{D}$ be $C^*$-algebras in $V^{(\pi)}$ and $A_1, A_2, B_1, B_2$, and $D$ the bounded part of $\hat{A}_1, \hat{A}_2, \hat{B}_1, \hat{B}_2$, and $\hat{D}$ respectively. Let $\hat{f}: \hat{A}_1 \to \hat{A}_2$ and $\hat{g}: \hat{B}_1 \to \hat{B}_2$ be $\pi$-homomorphisms in $V^{(\pi)}$. Let $\hat{B}_1$ and $\hat{B}_2$ have countable approximate identities in $V^{(\pi)}$. Then $\hat{f}^*, \hat{g}^*$, and $\tau_{\hat{D}}^\pi$ are the restriction of $\hat{f}^*$, $\hat{g}^*$, and $\tau_{\hat{D}}^\pi$ to the bounded part respectively.

The following theorem is a translation of Theorem 3 in § 4 of [5].

**Theorem 3.** (Homotopy invariance). If the $\pi$-homomorphisms $f_0$ and $f_1: A_1 \to A_2$, are $\pi$-homotopic, then $f_0^*$ and $f_1^*: \pi$-KK$(A_2, B) \to \pi$-KK$(A_1, B)$ coincide. If the $\pi$-homomorphisms $g_0$ and $g_1: B_1 \to B_2$, are $\pi$-homotopic, then $(g_0)_{\pi}$ and $(g_1)_{\pi}: \pi$-KK$(A, B_1) \to \pi$-KK$(A, B_2)$ coincide.

The following theorem is the central theorem in this section.

**Theorem 4.** Let $\mathcal{L}$-C*-algebras $A_1$ and $A_2$ be $\mathcal{L}$-separable. Let $\mathcal{L}$-C*-algebras $D$, $B_1$, and $B_2$ have $\mathcal{L}$-strictly positive elements. The following bilinear (distributive) coupling $\otimes_{\pi}^\mathcal{L}$ is defined:

$$\mathcal{L}$-KK$(A_1, B_1 \hat{\otimes}^\mathcal{L} D) \otimes_{\pi} \mathcal{L}$-KK$(D \hat{\otimes}^\mathcal{L} A_2, B_2) \to \mathcal{L}$-KK$(A_1 \hat{\otimes}^\mathcal{L} A_2, B_1 \hat{\otimes}^\mathcal{L} B_2)$. (2)

This coupling is contravariant in $A_1$ and $A_2$, and covariant in $B_1$ and $B_2$, and for any $\mathcal{L}$-homomorphism $f: D_1 \to D_2$, $f_*(x) \otimes_{\pi}^\mathcal{L} y = x \otimes_{\pi}^\mathcal{L} f_*(y)$. Moreover, $\otimes_{\pi}^\mathcal{L}$ is associative i.e.,

$$(x_1 \otimes_{\pi}^\mathcal{L} x_2) \otimes_{\pi}^\mathcal{L} x_3 = x_1 \otimes_{\pi}^\mathcal{L} (x_2 \otimes_{\pi}^\mathcal{L} x_3).$$

For $x_1 \in \mathcal{L}$-KK$(A_1, B_1 \hat{\otimes}^\mathcal{L} D_1)$, $x_2 \in \mathcal{L}$-KK$(D_2 \hat{\otimes}^\mathcal{L} A_2, B_2 \hat{\otimes}^\mathcal{L} D_1)$ and $x_3 \in \mathcal{L}$-KK$(D_1 \hat{\otimes}^\mathcal{L} A_1, B_1)$, where $\mathcal{L}$-C*-algebras $A_1$, $A_2$, $A_3$, and $D_i$ have to be $\mathcal{L}$-separable and the remaining ones have to have $\mathcal{L}$-strictly positive elements. Moreover $\otimes_{\pi}^\mathcal{L}$ commutes with the homomorphism $\tau_{\pi}^\mathcal{L}$ i.e.,

1) $\tau_{D_1}^\pi(x_1) \otimes_{\pi}^\mathcal{L} \otimes_{\pi}^\mathcal{L} x_2 \otimes_{\pi}^\mathcal{L} \otimes_{\pi}^\mathcal{L} \tau_{D_2}^\pi(x_3) = x_1 \otimes_{\pi}^\mathcal{L} x_2 \otimes_{\pi}^\mathcal{L} \tau_{D_1}^\pi(x_3)$

for $x_1 \in \mathcal{L}$-KK$(A_1, B_1 \hat{\otimes}^\mathcal{L} D_1)$, $x_2 \in \mathcal{L}$-KK$(D_2 \hat{\otimes}^\mathcal{L} A_2, B_2 \hat{\otimes}^\mathcal{L} D_1)$, $x_3 \in \mathcal{L}$-KK$(D_1 \hat{\otimes}^\mathcal{L} A_1, B_1)$, where $\mathcal{L}$-C*-algebras $A_1$, $A_2$, and $D_i$ have to be $\mathcal{L}$-separable and the remaining ones have to have $\mathcal{L}$-strictly positive elements; and

2) $\tau_{D_3}^\pi(x_1) \otimes_{\pi}^\mathcal{L} x_2 \otimes_{\pi}^\mathcal{L} \tau_{D_3}^\pi(x_3) = \tau_{D_1}^\pi(x_1) \otimes_{\pi}^\mathcal{L} x_2 \otimes_{\pi}^\mathcal{L} \tau_{D_2}^\pi(x_3)$

for $x_1 \in \mathcal{L}$-KK$(A_1, B_1 \hat{\otimes}^\mathcal{L} D_1)$, $x_2 \in \mathcal{L}$-KK$(D_2 \hat{\otimes}^\mathcal{L} A_2, B_2)$, where $\mathcal{L}$-C*-algebras $A_1$, $A_2$, and $D_i$ have to be $\mathcal{L}$-separable, and the remaining ones have to have $\mathcal{L}$-strictly positive elements.

**Proof.** We prove this as a translation of Theorem 4 in § 4 of [5]. So let $A_1, A_2, B_1, B_2$, and $D$ be the bounded part of $\hat{A}_1, \hat{A}_2, \hat{B}_1, \hat{B}_2$, and $\hat{D}$ respec-
tively. Let \( x_n \in \mathcal{X}-KK(A_n, B_n \hat{\otimes} D) \) and \( x_n \in \mathcal{X}-KK(D \hat{\otimes} \epsilon_n A_n, B_n) \). Then it suffices to show that \( x_n \otimes_{\mathcal{B}} x_i \) is a bounded element of \( KK(\hat{A}_n \hat{\otimes} \hat{A}_i, \hat{B}_n \hat{\otimes} \hat{B}_i) \). Let \( x_i = (\epsilon_i, \phi_i, F_i) \) and \( x_i = (\epsilon_i, \phi_i, F_i) \). Then \( x_i \otimes_{\mathcal{B}} x_i \) is of the form \( (\epsilon_i, \phi_i, F_i \# F_i) \). Therefore what to prove is the boundedness of \( F_i \# F_i \). However \( F_i \# F_i \) is of the form \( \sqrt{M_i} \phi_i(F_i \otimes 1) + \sqrt{M_i} (1 \otimes F_i) \), where \( M_i \geq 0, M_k \geq 0 \) and \( M_i + M_k = 1 \). Since \( \phi_i \) is a *-homomorphism in \( \mathcal{V}(\mathcal{A}) \) and \( F_i \) and \( F_k \) are bounded, the boundedness of \( F_i \# F_i \) is obvious.

As translations of Theorem 5 and Theorem 6 in §4 of [5], we have the following theorems.

**Theorem 5.** Let \( \mathcal{H}^{(0)} \simeq \mathcal{H}^{(1)} \simeq \mathcal{H}^{(2)} \) and \( \mathcal{H}^{(2)} = \mathcal{H}^{(0)} \oplus \mathcal{H}^{(1)} \). Let \( T_1: \mathcal{H}^{(0)} \rightarrow \mathcal{H}^{(1)} \) be a \( \mathcal{Z} \)-linear operator satisfying \( T_1 T_1^* = 1 \) and \( T_1^* T_1 = 1 - p \), where \( p \) is the projection onto a one-dimensional \( \mathcal{Z} \)-subspace of \( \mathcal{H}^{(0)} \). In \( \mathcal{H}^{(2)} \), consider the operator

\[
T = \begin{pmatrix}
0 & T_1^*
\end{pmatrix}
\]

and denote by \( c_1^* \) the element \( (\epsilon_1, T) \in \mathcal{X}-KK(\mathcal{G}, \mathcal{X}) \) where \( \epsilon_1 \) is the grading of \( \mathcal{X} \). This element is the identity for the operation of \( \otimes_{\mathcal{H}} \) i.e., \( x \otimes_{\mathcal{H}} c_1^* x = x \) for every \( x \in \mathcal{X}-KK(\mathcal{A}, \mathcal{B}) \).

**Theorem 6.** Let \( \mathcal{Z} \)-C*-algebra \( \mathcal{A} \) be \( \mathcal{Z} \)-separable, and assume that \( \mathcal{Z} \)-C*-algebras \( \mathcal{B}, \mathcal{D} \) and \( \mathcal{E} \) have \( \mathcal{Z} \)-strictly positive elements.

1) Assume that there exist elements \( \alpha \in \mathcal{Z}-KK(\mathcal{D}, \mathcal{E}) \) and \( \beta \in \mathcal{Z}-KK(\mathcal{E}, \mathcal{D}) \) such that

\[
\alpha \otimes_{\mathcal{Z}} \beta = \pm \tau_{\mathcal{D}}(c_1^*) \in \mathcal{Z}-KK(\mathcal{D}, \mathcal{D}),
\]

\[
\beta \otimes_{\mathcal{Z}} \alpha = \pm \tau_{\mathcal{E}}(c_1^*) \in \mathcal{Z}-KK(\mathcal{E}, \mathcal{E}).
\]

Then the homomorphisms

\[
\otimes_{\mathcal{Z}} \alpha: \mathcal{Z}-KK(\mathcal{A}, B \hat{\otimes} \mathcal{D}) \rightarrow \mathcal{Z}-KK(\mathcal{A}, B \hat{\otimes} \mathcal{E}),
\]

\[
\otimes_{\mathcal{Z}} \beta: \mathcal{Z}-KK(\mathcal{A}, B \hat{\otimes} \mathcal{E}) \rightarrow \mathcal{Z}-KK(\mathcal{A}, B \hat{\otimes} \mathcal{D})
\]

are isomorphisms. If \( \mathcal{D} \) and \( \mathcal{E} \) are \( \mathcal{Z} \)-separable, then

\[
\beta \otimes_{\mathcal{Z}}: \mathcal{Z}-KK(\mathcal{A} \hat{\otimes} \mathcal{D}, B) \rightarrow \mathcal{Z}-KK(\mathcal{A} \hat{\otimes} \mathcal{E}, B),
\]

\[
\alpha \otimes_{\mathcal{Z}}: \mathcal{Z}-KK(\mathcal{A} \hat{\otimes} \mathcal{E}, B) \rightarrow \mathcal{Z}-KK(\mathcal{A} \hat{\otimes} \mathcal{D}, B)
\]

are also isomorphisms.

2) Assume that \( \mathcal{Z} \)-C*-algebras \( \mathcal{D} \) and \( \mathcal{E} \) are \( \mathcal{Z} \)-separable and there exist elements \( \alpha \in \mathcal{Z}-KK(D \hat{\otimes} \mathcal{D}, \mathcal{Z}) \) and \( \beta \in \mathcal{Z}-KK(D \hat{\otimes} \mathcal{E}, \mathcal{Z}) \), such that
\( \beta \otimes \alpha = \pm \tau_\beta (c_i) \in \mathcal{L} \text{-} KK(E, E) \),
\( \beta \otimes_\beta \alpha = \pm \tau_\beta (c_i) \in \mathcal{L} \text{-} KK(D, D) \).

Then the homomorphisms
\[
\begin{align*}
\beta \otimes_\beta : \mathcal{L} \text{-} KK(A \otimes E, B) & \to \mathcal{L} \text{-} KK(A, B \otimes E), \\
\beta \otimes_\beta : \mathcal{L} \text{-} KK(A \otimes E, B) & \to \mathcal{L} \text{-} KK(A, B \otimes E), \\
\otimes_\beta \alpha : \mathcal{L} \text{-} KK(A, B \otimes E) & \to \mathcal{L} \text{-} KK(A \otimes E, B), \\
\otimes_\beta \alpha : \mathcal{L} \text{-} KK(A, B \otimes E) & \to \mathcal{L} \text{-} KK(A \otimes E, B)
\end{align*}
\]
are isomorphisms.

§ 5. Periodicity

In this section, we always assume that \( A \) is a \( \mathcal{L} \)-separable graded \( \mathcal{L} \text{-} C^* \)-algebra and \( B \) is a graded \( \mathcal{L} \text{-} C^* \)-algebra with \( \mathcal{L} \)-countable approximate identities.

**DEFINITION.** We set
\[
\mathcal{L} \text{-} K_{p,q} K^{\mathcal{L},\mathcal{L}}(A, B) = \mathcal{L} \text{-} KK(A \otimes C^{\mathcal{L}}_{p,q}, B \otimes C^{\mathcal{L}}_{p,q}),
\]
\[
\mathcal{L} \text{-} K_{p,q} K(A, B) = \mathcal{L} \text{-} KK(A \otimes C^{\mathcal{L}}_{p,q}, B),
\]
\[
\mathcal{L} \text{-} KK^{\mathcal{L},\mathcal{L}}(A, B) = \mathcal{L} \text{-} KK(A, B \otimes C^{\mathcal{L}}_{p,q}),
\]
\[
\mathcal{L} \text{-} K_{p,q}(A) = \mathcal{L} \text{-} K_{p,q} K(A, \mathcal{L}),
\]
\[
\mathcal{L} \text{-} K^{\mathcal{L},\mathcal{L}}(B) = \mathcal{L} \text{-} KK^{\mathcal{L},\mathcal{L}}(\mathcal{L}, B).
\]

The \( \mathcal{L} \text{-} K \)-functor for unital \( A \) and arbitrary \( B \) is defined in the following way.
\[
\begin{align*}
\mathcal{L} \text{-} \tilde{K}_{p,q} K^{\mathcal{L},\mathcal{L}}(A, B) & = \text{Ker} \{ f^* : \mathcal{L} \text{-} K_{p,q} K^{\mathcal{L},\mathcal{L}}(A, B) \to \mathcal{L} \text{-} K_{p,q} K^{\mathcal{L},\mathcal{L}}(\mathcal{L}, B) \} \\
\mathcal{L} \text{-} \tilde{K}_{p,q} K^{\mathcal{L},\mathcal{L}}(A, \tilde{B}) & = \text{Ker} \{ g^* : \mathcal{L} \text{-} K_{p,q} K^{\mathcal{L},\mathcal{L}}(A, \tilde{B}) \to \mathcal{L} \text{-} K_{p,q} K^{\mathcal{L},\mathcal{L}}(A, \mathcal{L}) \}
\end{align*}
\]
where \( \tilde{B} \) is the \( \mathcal{L} \text{-} C^* \)-algebra obtained from \( B \) adjunction of 1, \( f : \mathcal{L} \to A \) is a unital \( \mathcal{L} \text{-} \ast \)-homomorphism, and \( g : \tilde{B} \to \mathcal{L} \) is a unital \( \mathcal{L} \text{-} \ast \)-homomorphism.

As translations of Theorems 1-7 in § 5 of [5], we have the following theorems.

**Theorem 1.** Let \( \mathcal{H} \) be a \( \mathcal{L} \)-separable \( \mathcal{L} \)-Hilbert space (possibly finite-dimensional over \( \mathcal{L} \)). The \( \mathcal{L} \text{-} C^* \)-algebra \( \mathcal{H}^\mathcal{L} (\mathcal{H}) \) will be considered as graded by some decomposition \( \mathcal{H} = \mathcal{H}^{(0)} + \mathcal{H}^{(1)} \) (it is allowed that \( \mathcal{H}^{(0)} \) or \( \mathcal{H}^{(1)} = \{0\} \)). There exist canonical generators \( \alpha \in \mathcal{L} \text{-} K_0 (\mathcal{H}^{\mathcal{L}} (\mathcal{H})) \) (i.e., \( \mathcal{L} \text{-} KK (\mathcal{H}^{\mathcal{L}} (\mathcal{H}), \mathcal{L}) \))
and $\beta \in \mathcal{L}_K(\mathcal{H}(\mathcal{H}))$ (i.e., $\mathcal{L}_K(\mathcal{H}(\mathcal{H}))$), with which induces the following isomorphisms:

$$\mathcal{L}_K(\mathcal{H}(\mathcal{H})) \simeq \mathcal{L}_K(\mathcal{H}, \mathcal{H}(\mathcal{H})).$$

**Theorem 2.** The following isomorphisms hold.

$$\mathcal{L}_K(A \otimes \mathcal{H}(\mathcal{H}), B) \simeq \mathcal{L}_K(A, B),$$
$$\mathcal{L}_K(A, B \otimes \mathcal{H}(\mathcal{H})) \simeq \mathcal{L}_K(A, B).$$

**Theorem 3.** Let $\eta: C_{p,q}^* \to C_{p,q}^*$ be a graded automorphism. If $\eta$ is orientation preserving, then $(1 \otimes \mathcal{H})^*(x) = x$ for every $x \in \mathcal{L}_K(A, B)$, and if $\eta$ changes orientation, then $(1 \otimes \mathcal{H})^*(x) = -x$. An analogous assertion holds for $\mathcal{L}_K(A, B)$.

**Theorem 4.** For a fixed difference $(p-q)-(p'-q')$ all groups $\mathcal{L}_K(A, B)$ are canonically isomorphic.

**Definition.** For any integers $p$ and $q$ we set

$$\mathcal{L}_K(A, B) = \mathcal{L}_K(A, B) = \mathcal{L}_K^q(A, B)$$
$$= \begin{cases} \mathcal{L}_K(A, B), & \text{if } p \geq q, \\ \mathcal{L}_K(A, B), & \text{if } p \leq q. \end{cases}$$

**Theorem 5 (Formal periodicity).** The groups $\mathcal{L}_K(A, B)$ are periodic in $n$ with period 2.

**Theorem 6.** Let $D$ be a $\mathcal{L}_K$-algebra. Then $\otimes^0_\gamma$ induces the coupling

$$\mathcal{L}_K(A_1, B_1 \otimes \mathcal{H} D) \otimes^0_\gamma \mathcal{L}_K(D \otimes \mathcal{H} A_2, B_2)$$
$$\longrightarrow \mathcal{L}_K(A_1 \otimes \mathcal{H} A_2, B_1 \otimes \mathcal{H} B_2),$$

which commutes with the periodicity isomorphism and is skew-commutative for $D=\mathcal{H}$ (the case of product): $x_1 \otimes^0_\gamma x_2 = (-1)^{ij} x_i \otimes^0_\gamma x_j$.

**Theorem 7.** (Bott periodicity). The following isomorphisms hold.

$$\mathcal{L}_K(A(\mathcal{H}))^q, B) \simeq \mathcal{L}_K(A(\mathcal{H}), B(\mathcal{H})) \simeq \mathcal{L}_K(A(\mathcal{H}), B(\mathcal{H})).$$

§ 6. Other definitions of $\mathcal{L}_K$-functors

In this section let $A$ be a $\mathcal{L}_K$-separable $\mathcal{L}_K$-algebra and a $\mathcal{L}_K$-algebra $B$ have $\mathcal{L}_K$-countable approximate identities.
As a translation of Theorem 1 in § 6 of [5], we have the following theorem.

**Theorem 1.** Let $A$ and $B$ be graded. The group $\mathcal{X} \text{-} KK(A,B)$ does not change if in the definition of § 4 $\mathcal{X} \text{-} \text{homotopy equivalence}$ is replaced by the combination of unitary equivalence and $\mathcal{X} \text{-} \text{operator homotopy}$. Moreover, if the triples $x_0 = (\varepsilon_0, \phi_0, F_0)$ and $x_1 = (\varepsilon_1, \phi_1, F_1)$ are connected by the $\mathcal{X} \text{-} \text{homotopy}$ $\{x_t\} = \{(\varepsilon_t), \{\phi_t\}, \{F_t\}\}$, then there exist $x'_0, x'_1 \in \mathcal{D}^\alpha(A,B)$ such that $x_0 \otimes x'_0$ is $\mathcal{X} \text{-} \text{operator homotopic}$ to $x_1 \otimes x'_1$. Conversely, unitarily equivalent triples are $\mathcal{X} \text{-} \text{homotopic}.$

**Definition.** Let $A$, $B$ and $\mathcal{X} \otimes^\alpha$ have trivial grading. We denote by $\mathcal{E}^\alpha_{p,q}(A,B)$ the set of pairs $(\phi, F)$, where $\phi : A \otimes^\alpha C_{p+1,q} \rightarrow \mathcal{M}(\mathcal{X} \otimes^\alpha B)$ is a $\mathcal{X} \text{-} \text{homomorphism}$ (not graded), $F$ is an element in $\mathcal{M}(\mathcal{X} \otimes^\alpha B)$ and for every $a \in A$ and $b \in C_{p+1,q}$ the element

$$\begin{align*}
\phi(a \otimes^\alpha b) \cdot F - F \cdot \phi(a \otimes^\alpha b), \\
(F^* - 1)\phi(a \otimes^\alpha b), \ (F - F^*)\phi(a \otimes^\alpha b)
\end{align*}$$

belong to $\mathcal{X} \otimes^\alpha B$. Let $\mathcal{D}^\alpha_{p,q}(A,B)$ be the set of (degenerate) pairs for which the indicated elements are equal to 0, and denote by $\mathcal{E}^\alpha_{p,q}(A,B)$ and $\mathcal{D}^\alpha_{p,q}(A,B)$ the set of pairs $(\phi : \tilde{A} \otimes^\alpha C_{p+1,q} \rightarrow \mathcal{M}(\mathcal{X} \otimes^\alpha B), F)$, satisfying the same conditions for $a \in A$. $\mathcal{X} \text{-} \text{homotopy}$ and $\mathcal{X} \text{-} \text{operator homotopy}$ are defined in the same way as in § 4 (for $\mathcal{E}^\alpha_{p,q}(A,B)$ and $\mathcal{D}^\alpha_{p,q}(A,B)$ homotopy is assumed to be defined on $\tilde{A} \otimes^\alpha C_{p+1,q}$). Pairs $(\phi, F)$ and $(\phi', F')$ are unitarily equivalent if there exists a unitary element $u \in \mathcal{M}(\mathcal{X} \otimes^\alpha B)$ such that

$$\forall a \in A \otimes^\alpha C_{p+1,q}, \ \phi(a) = u\phi(a)u^{-1} \quad \text{and} \quad F_1 = uFu^{-1}$$

(for $\mathcal{E}^\alpha_{p,q}(A,B)$ we assume that $a \in \tilde{A} \otimes^\alpha C_{p+1,q}$).

We denote by $\mathcal{E}^\alpha_{p,q}(A,B)$ the set of equivalence classes of $\mathcal{E}^\alpha_{p,q}(A,B)$ modulo $\mathcal{X} \text{-} \text{homotopy}$, and by $\mathcal{D}^\alpha_{p,q}(A,B)$ the image of $\mathcal{D}^\alpha_{p,q}(A,B)$ in $\mathcal{E}^\alpha_{p,q}(A,B)$. We define $\mathcal{E}^\alpha_{p,q}(A,B)$ and $\mathcal{D}^\alpha_{p,q}(A,B)$ analogously. Moreover, let $\mathcal{E}^\alpha_{p,q}(A,B)$ be the set of equivalence classes of $\mathcal{E}^\alpha_{p,q}(A,B)$ modulo unitary equivalence and $\mathcal{X} \text{-} \text{operator homotopy}$, and let $\mathcal{D}^\alpha_{p,q}(A,B)$ be the image of $\mathcal{D}^\alpha_{p,q}(A,B)$ in $\mathcal{E}^\alpha_{p,q}(A,B)$. The direct sum turns these set of equivalence classes into semigroups.

As a translation of Theorem 2 of § 6 of [5], we have the following theorem.

**Theorem 2.** If $A$ and $B$ are trivially graded, then the following isomorphisms hold.
§ 7. Extensions

In this section, we assume that $A$ is a $\mathcal{Z}$-separable $\mathcal{Z}$-$C^*$-algebra and $B$ is a $\mathcal{Z}$-$C^*$-algebra with $\mathcal{Z}$-countable approximate identities. Let $A$, $B$, $D$ and be graded trivially. We consider the group $\text{Ext}(\mathcal{Z}, A, B)$ constructed from extension of the form

$$0 \to \mathcal{X} \otimes \mathcal{Y} \to D \to A \to 0$$

(1)

where $\to$ stand for $\mathcal{Z}$-$*$-homomorphisms. Let $\hat{A}$, $\hat{B}$ and $\hat{D}$ be $C^*$-algebras in $V^{(q)}$ and $\mathcal{X}^z$ is the $C^*$-algebra $\mathcal{X}$ in $V^{(q)}$. Then there exists a 1-1 correspondence between

$$0 \to \mathcal{X} \otimes \hat{B} \to \hat{D} \to \hat{A} \to 0$$

in $V^{(q)}$ and the extension of the form (1). The extension (1) unambiguously defines a $\mathcal{Z}$-$*$-homomorphism $D \to \mathcal{M}(\mathcal{X} \otimes \mathcal{Y}, B)$, which gives, upon factorization with respect to $\mathcal{X} \otimes \mathcal{Y}$

$$\phi: A \to \mathcal{M}(\mathcal{X} \otimes \mathcal{Y}, B), \mathcal{X} \otimes \mathcal{Y} = \mathcal{M}(\mathcal{X} \otimes \mathcal{Y}, B).$$

Conversely, if $\phi: A \to \mathcal{M}(\mathcal{X} \otimes \mathcal{Y}, B)$ is a $\mathcal{Z}$-$*$-homomorphism then $\phi$ is a restriction of $\hat{\phi}: \hat{A} \to \mathcal{M}(\mathcal{X} \otimes \hat{B})$ in $V^{(q)}$ and corresponds to an extension $0 \to \mathcal{X}^z \otimes \hat{B} \to \hat{D} \to \hat{A} \to 0$ in $V^{(q)}$ and induces an extension $0 \to \mathcal{X}^z \otimes \mathcal{Y} \to D \to A \to 0$. Under this correspondence the subset of decomposable extensions corresponds to the set of $\mathcal{Z}$-$*$-homomorphisms $\phi$ admitting a lifting $A \to \mathcal{M}(\mathcal{X} \otimes \mathcal{Y}, B)$. We shall often identify an extension with the homomorphism $\phi$ corresponding to it.

**Definition.** For fixed $A$ and $B$ we denote the set of extensions of type (1) by $\mathcal{Z} \cdot \text{Ext}(A, B)$ and the subset of decomposable extensions by $\mathcal{Z} \cdot \text{Ext}(A, B)$. Two elements $\phi_1$ and $\phi_2$ of $\mathcal{Z} \cdot \text{Ext}(A, B)$ are said to be unitarily equivalent if there exists a unitary operator $u \in \mathcal{M}(\mathcal{X} \otimes \mathcal{Y}, B)$ such that $\phi_2(a) = u \phi_1(a) u^{-1}$ for every $a \in A$. Let $\mathcal{Z} \cdot \text{Ext}(A, B)$ be the set of classes of unitarily equivalent extensions and let $\mathcal{Z} \cdot \text{Ext}(A, B)$ be the image of $\mathcal{Z} \cdot \text{Ext}(A, B)$ in $\mathcal{Z} \cdot \text{Ext}(A, B)$.

The addition $\phi_1 \oplus \phi_2$ in $\mathcal{Z} \cdot \text{Ext}(A, B)$ is defined as the direct sum $(\mathcal{X}^z (\mathcal{H}^l_\mathcal{H}^r) \oplus \mathcal{X}^z (\mathcal{H}^r_\mathcal{H}^l))$ and is identified with $\mathcal{X}^z (\mathcal{H}^l_\mathcal{H}^r)$ by means of an isometry of $\mathcal{H}^l_\mathcal{H}^r \oplus \mathcal{H}^r_\mathcal{H}^l$ onto $\mathcal{H}^l_\mathcal{H}^r$.

We set

$$\mathcal{Z} \cdot \text{Ext}(A, B) = \mathcal{Z} \cdot \text{Ext}(A, B) | \mathcal{Z} \cdot \text{Ext}(A, B).$$

The direct sum of two extensions
242 GAISI TAKEUTI

\[ 0 \rightarrow \mathcal{X}^x \otimes ^x B \rightarrow D_i \xrightarrow{p_i} A \rightarrow 0 \quad (i=1, 2) \]

represents an extension

\[ 0 \rightarrow M_2^* \otimes (\mathcal{X}^x \otimes ^x B) \rightarrow D_\otimes \xrightarrow{p} A \rightarrow 0, \]

where

\[ D_\otimes = \left\{ \left( \begin{array}{cc} x_i & a_i \\ a_i & x_i \end{array} \right) \right\} | x_i \in D_i, \ a_i \in \mathcal{X}^x \otimes ^x B \text{ and } p_i(x_i) = p_i(x_i), \]

and \( M_2^* \) is the \( \mathcal{X} \)-C*-algebra of \( 2 \times 2 \) matrices of \( \mathcal{X} \). Obviously \( \mathcal{X} \cdot \mathcal{E} \text{xt}(A, B) \), \( \mathcal{X} \cdot \mathcal{E} \text{xt}(A, B) \) and \( \mathcal{X} \cdot \mathcal{E} \text{xt}(A, B) \) are isomorphic with \( \text{Ext}(\hat{A}, \hat{B}), \mathcal{E} \text{xt}(\hat{A}, \hat{B}) \) and \( \mathcal{E} \text{xt}(\hat{A}, \hat{B}) \) in \( V(\otimes) \).

**DEFINITION.** We say that an extension \( \phi \in \mathcal{X} \cdot \mathcal{E} \text{xt}(A, B) \) is absorbing if for every \( \psi \in \mathcal{X} \cdot \mathcal{E} \text{xt}(A, B) \) the elements \( \phi \oplus \psi \) and \( \phi \) are unitarily equivalent. There exists a 1–1 correspondence between the absorbing elements of \( \mathcal{X} \cdot \mathcal{E} \text{xt}(A, B) \) and the absorbing elements of \( \mathcal{E} \text{xt}(\hat{A}, \hat{B}) \) in \( V(\otimes) \). We denote by \( \mathcal{X} \cdot \mathcal{E} \text{xt}_a(A, B) \) the set of absorbing extensions and by \( \mathcal{E} \text{xt}_a(A, B) \) the set of classes of unitarily equivalent absorbing extensions. Addition on \( \mathcal{X} \cdot \mathcal{E} \text{xt}_a \)

\((A, B)\) is defined as the direct sum.

**DEFINITION.** Let \( A \) and \( B \) be trivially graded and assume that \( \mathcal{X}^x \) is also graded trivially. We denote by \( \mathcal{X} \cdot \mathcal{E} \) \((A, B)\) the set of paris \((\phi, P)\), where \( \phi : A \to \mathcal{M}(\mathcal{X}^x \otimes ^x B) \) is a \( \mathcal{X} \)-\( \sigma \)-homomorphism, \( P \in \mathcal{M}(\mathcal{X}^x \otimes ^x B) \), and for every \( a \in A \) the elements

\[ \phi(a)P - P\phi(a), (P^*-P)\phi(a), (P^*-P)\phi(a) \quad (2) \]

belong to \( \mathcal{X}^x \otimes ^x B \). Let \( \mathcal{X} \cdot \mathcal{E} \) \((A, B)\) be the set of (degenerate) paris which all elements (2) are equal to 0. Two pairs \((\phi_1, P_1)\) and \((\phi_2, P_2)\) are unitarily equivalent if there exists a unitary element \( u \in \mathcal{M}(\mathcal{X}^x \otimes ^x B) \) such that \( \phi_2(a) = u\phi_1(a)u^{-1} \) and \( P_2 = uP_1u^{-1} \) for every \( a \in A \). Two pairs \((\phi_1, P_1)\) and \((\phi_2, P_2)\) are said to be homological if \( P_1\phi_1(a) - P_2\phi_2(a) \in \mathcal{X}^x \otimes ^x B \) for every \( a \in A \). Let \( \mathcal{X} \cdot \mathcal{E} \) \((A, B)\) be the set of equivalence classes of \( \mathcal{X} \cdot \mathcal{E} \) \((A, B)\) modulo unitary equivalence and homology, and let \( \mathcal{X} \cdot \mathcal{E} \) \((A, B)\) be the image of \( \mathcal{X} \cdot \mathcal{E} \) \((A, B)\) in \( \mathcal{X} \cdot \mathcal{E} \) \((A, B)\). The addition of equivalence classes is defined to be the direct sum. We set

\[ \mathcal{X} \cdot \mathcal{E} (A, B) = \mathcal{X} \cdot \mathcal{E} (A, B) | \mathcal{X} \cdot \mathcal{E} (A, B). \]

Let \((\phi, P) \in \mathcal{E}(\hat{A}, \hat{B}) \) in \( V(\otimes) \). \((\phi, P)\) is said to be bounded if \( P \) is bounded.
Then $\mathcal{E}(A, B)$, $\mathcal{D}(A, B)$, $\mathcal{E}(A, B)$, $\mathcal{D}(A, B)$ and $E'(A, B)$ are the bounded part of $\mathcal{E}(A, B)$, $\mathcal{D}(A, B)$, $\mathcal{E}(A, B)$, $\mathcal{D}(A, B)$ and $E'(A, B)$ respectively.

As a translation of Theorem 1 in §7 of [5], we have the following theorem.

**Theorem 1.** If $A$ is $\mathcal{Z}$-nuclear, then the following isomorphisms hold.

$$
\mathcal{Z}\text{-Ext}_0(A, B) \simeq \mathcal{Z}\text{-Ext}(A, B) \simeq \mathcal{Z}\text{-Ext}(A, B) \simeq \mathcal{Z}\text{-Ext}(A, B) \simeq \mathcal{Z}\text{-Ext}(A, B).
$$

**Corollary.** $E'(A, B)$ and $\mathcal{KK}'(A, B)$ in $V^{(\phi)}$ are isomorphic to $\mathcal{Z}\text{-Ext}(A, B)$ and $\mathcal{Z}\text{-Ext}(A, B)$.

As a translation of Theorem 2 and 3 in §7 of [5], we have the following theorem.

**Theorem 2.** Let $A$ and $B$ be trivially graded, let $A$ be $\mathcal{Z}$-nuclear, and let $J$ be a $\mathcal{Z}$-ideal with a $\mathcal{Z}$-strictly positive element in $B$. Denote the embedding $J \subseteq B$ by $i$ and the projection $B \to B/J$ by $q$. We have the following doubly infinite exact sequence

$$
\cdots \to \mathcal{Z}\text{-Ext}(A, B) \xrightarrow{i_0} \mathcal{Z}\text{-Ext}(A, B/J) \xrightarrow{q_0} \mathcal{Z}\text{-Ext}(A, B) \to \cdots.
$$

Here, the coboundary homomorphism $\delta$ is defined as the composition

$$
\mathcal{Z}\text{-Ext}(A, B/J) \simeq \mathcal{Z}\text{-Ext}(A, B/J)(0, 1)^\forall \\
\to \mathcal{Z}\text{-Ext}(A, S^\forall(B, B/J)) \\
\simeq \mathcal{Z}\text{-Ext}(A, J)
$$

where $\mathcal{Z}\text{-Ext}(A, B/J) \simeq \mathcal{Z}\text{-Ext}(A, B/J)(0, 1)^\forall$ is the isomorphism described by the Bott periodicity and

$$
\mathcal{Z}\text{-Ext}(A, J) \simeq \mathcal{Z}\text{-Ext}(A, S^\forall(B, B/J))
$$

is induced by $J = J[0] \to S^\forall(B, B/J)$.

**Theorem 3.** Let $A$ and $B$ be trivially graded and assume that $A$ is $\mathcal{Z}$-nuclear and $I$ is a $\mathcal{Z}$-ideal in $A$. Denote by $i$ the embedding $I \to A$ and by $q$ the projection $A \to A/I$. We have the following two-sided exact sequence:

$$
\cdots \to \mathcal{Z}\text{-Ext}_{n+1}(A, B) \xrightarrow{i_0} \mathcal{Z}\text{-Ext}_{n+1}(I, B) \xrightarrow{q_0} \mathcal{Z}\text{-Ext}_{n+1}(A, B) \xrightarrow{i_0} \cdots.
$$
Here the boundary homomorphism $\partial$ is defined as the composition
\[
\mathcal{Z}\text{-}\mathcal{K}_{\pi+1}(I, B) \simeq \mathcal{Z}\text{-}\mathcal{K}_{\pi+1}(\mathcal{S}(A, A/I), B) \\
\rightarrow \mathcal{Z}\text{-}\mathcal{K}_{\pi+1}(A/I(0, 1)\pi, B) \simeq \mathcal{Z}\text{-}\mathcal{K}_{\pi}(A/I, B)
\]
where $\mathcal{Z}\text{-}\mathcal{K}_{\pi+1}(A/I(0, 1)\pi, B) \simeq \mathcal{Z}\text{-}\mathcal{K}_{\pi}(A/I, B)$ is the isomorphism described by the Bott periodicity and the isomorphism
\[
\mathcal{Z}\text{-}\mathcal{K}_{\pi+1}(\mathcal{S}(A, A/I), B) \simeq \mathcal{Z}\text{-}\mathcal{K}_{\pi+1}(I, B)
\]
is induced by $I=I[0] \rightarrow \mathcal{S}(A, A/I)$.

**Corollary.** Let $A$ and $B$ be trivially graded, and assume that $A$ is $\mathcal{Z}$-nuclear. Then the following isomorphisms hold.
\[
\mathcal{Z}\text{-}\mathcal{K}_{\pi}(A, B) \simeq \mathcal{Z}\text{-}\mathcal{K}_\pi(A, B) \\
\simeq \mathcal{Z}\text{-}\mathcal{K}_\pi(A, \tilde{B}) \\
\simeq \mathcal{Z}\text{-}\mathcal{K}_\pi(\tilde{A}, \tilde{B}).
\]

**Postscript**

We always assumed that $\mathcal{Z}$ is an abelian von Neumann algebra and $\mathcal{B}$ the complete Boolean algebra of all the projections in $\mathcal{Z}$. However the whole theory here can be carried out in the much more general situation without much change.

Let $\mathcal{B}$ be a complete Boolean algebra. Let $C^\pi_\mathcal{B}$ be the set of all complex numbers in $V(\mathcal{B})$. The bounded part $\mathcal{Z}$ of $C^\pi_\mathcal{B}$ is defined as follows.
\[
\mathcal{Z} = \langle Z, +, \cdot, \| \cdot \|angle
\]

1) $Z = \{ z \in C^\pi_\mathcal{B} \mid \exists M \in \mathcal{R} \| z \| \leq M \}$ and for every pair $z_1, z_2$ in $\mathcal{Z}$, $z_1 = z_2$ iff $[z_1 = z_2] = 1$.

2) For every pair $z_1, z_2$ in $\mathcal{Z}$, the sum $z$ of $z_1$ and $z_2$ is defined by $[z = z_1 + z_2] = 1$. $z$ is uniquely defined as a member of $\mathcal{Z}$. The sum of $z_1$ and $z_2$ in $\mathcal{Z}$ is also denoted by $z_1 + z_2$.

3) For every $z \in \mathcal{Z}$, $\| z \|$ is defined by $\| z \| = \inf \{ M \in \mathcal{R} \mid \| z \| \leq M \}$ hold in $V(\mathcal{B})$.

After this change, everything goes through as before.

**Added in November, 1983.** M. Ozawa pointed out that the following condition of $\mathcal{Z}$-$C^\pi_\mathcal{B}$-algebra is redundant.

For every $x \in A$ and $z \in \mathcal{Z}$,
\[
x^*x \leq z \text{ if } xx^* \leq z
\]
holds in the $\mathcal{I}$-extension of $A$. Therefore every C*-algebra which is a normal $\mathcal{I}$-module is a $\mathcal{I}$-C*-algebra. His proof goes as follows:

Let $\mathcal{A}$ be the $\mathcal{I}$-extension of $A$. Then $A$ is a C*-subalgebra of $A$ and $\mathcal{I}$ is a C*-subalgebra of the center of $A$. Thus we have only to prove the following general statement: “Let $A$ be a C*-algebra and $\mathcal{I}$ the center of $A$. Then for any $x$ in $A$ and $z$ in $\mathcal{I}$ with $z \geq 0$, we have $x^*x \leq z$ if and only if $xx^* \leq z$”. We shall prove this using the Boolean valued analysis developed in [10] (G. Takeuti, Von Neumann algebras and Boolean valued analysis). Let $(H, \pi)$ be a faithful representation of $A$. We have $\pi(\mathcal{I}) \subseteq \pi(\mathcal{I})'' \subseteq \pi(A)'' \subseteq \pi(\mathcal{I})'$, so that we can assume without any loss of generality that $\mathcal{I} = \pi(\mathcal{I})'$ and that $A = \pi(A)$. Let $\mathcal{B}$ be the complete Boolean algebra of all projections in $\mathcal{I}$. Then we have the embeddings $\mathcal{I} \to \mathcal{B}$ and $A \to \mathcal{B}$ of $\mathcal{I}$ and $A$ into $V(\mathcal{B})$ such that $[\mathcal{B} = C] = 1$ and $[\mathcal{A}$ is a factor$] = 1$ as shown in [10]. By the direct translation of a theorem on C*-algebras, we have

$$([(\forall z \in C . ) (\forall x \in \mathcal{A}) x^* x \leq z 1 \iff xx^* \leq z 1] = 1$$

so that for any $z$ in $\mathcal{B}$, and $x$ in $A$, we have $[x^* x \leq z 1] = 1$ if and only if $[xx^* \leq z 1] = 1$. It follows from the property of the embeddings that for any $z$ in $\mathcal{I}$, and $x$ in $A$, $x^* x \leq z$ if and only if $xx^* \leq z$. QED

The statement will also be proved by some techniques in the direct integrals.

References


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