On regularity of positive linear functionals

By Atushi INOUE

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§1. Introduction

In this paper we study positive linear functionals f on $*$ -algebras, by investigating the relation between the operator-representation π_f and the vector-representation λ_f generated by the well-known GNS-construction of f . The primary purpose is to answer the following question: When is a positive linear functional f on a $*$ -algebra $\mathcal{A}$ represented as $f(\cdot) = \sum_a (\pi_f(\cdot) \xi_a | \xi_a)$ for some subset $\{\xi_a\}$ of the domain $\mathcal{D}(\pi_f)$ of π_f ?

Most study of positive linear functionals f is for the case that π_f are bounded. However, the operator-representation π_f is in general unbounded. In the paper [22], Powers investigated unbounded representations of $*$ -algebras and pointed out that there exists a pathological relation between subrepresentations and commutants, which does not occur in the bounded case. To remove such a phenomenon he introduced the notion of self-adjoint representation. After that, unbounded representations have been studied by many mathematicians [9, 12, 13, 14, 15, 16, 22, 25]. We here treat positive linear functionals f satisfying a condition weaker than the self-adjointness of π_f .

The nature of the study of positive linear functionals on $*$ -algebras $\mathcal{A}$ depends on whether $\mathcal{A}$ has identity or not. When a $*$ -algebra $\mathcal{A}$ has identity e , our question is solved; in fact, every positive linear functional f on $\mathcal{A}$ is represented as $f(x) = (\pi_f(x) \lambda_f(e) | \lambda_f(e))$ for all $x \in \mathcal{A}$. We remark that by considering the positive invariant sesquilinear form f° defined by $f^\circ(x, y) = f(y^*x)$ for a positive linear functional f , the study of positive invariant sesquilinear forms is seen to be more general than that of positive linear functionals when $*$ -algebras don't possess identity.

With this fact in mind we study positive invariant sesquilinear forms ϕ on $*$ -algebras $\mathcal{A}$ without identity such that the adjoint π_ϕ^* of π_ϕ are self-adjoint representations.

We proceed along the following plan.

In Section 2, we introduce some notions which are used in this paper, for example, O_p^* -algebras, positive invariant sesquilinear forms and so on.

In Section 3, we introduce a T^+ -algebra $\mathfrak{A}_\phi$ for a positive invariant sesquilinear form ϕ , which is an unbounded generalization of the notion of observable algebras established by Tomita [28]. We define a commutant $\mathfrak{A}'_\phi$ of $\mathfrak{A}_\phi$ and show that $\mathfrak{A}'_\phi$ is a $*$ -algebra, and there exist a $*$ -representation π of $\mathfrak{A}'_\phi$ into the Power's commutant $\pi_\phi(\mathcal{A})'$ of π_ϕ and a linear map λ of $\mathfrak{A}'_\phi$ into the Hilbert space $\mathfrak{H}_\phi$ satisfying that $\lambda(SK) = \pi(S)\lambda(K)$ for each $S, K \in \mathfrak{A}'_\phi$. By using the $*$ -algebras $\mathfrak{A}_\phi$ and $\mathfrak{A}'_\phi$ we investigate positive invariant sesquilinear forms ϕ in the latter section.

In Section 4, we investigate under what conditions a positive invariant sesquilinear form ϕ on a $*$ -algebra $\mathcal{A}$ is represented as $\phi(x, y) = (\pi_\phi^*(x)\xi | \pi_\phi^*(y)\xi)$ for some $\xi \in \mathcal{D}(\pi_\phi^*)$. Such a form ϕ is called strongly regular. We show that ϕ is strongly regular if and only if $\pi(\mathfrak{A}'_\phi)$ is a C^* -algebra.

In Section 5, we introduce the notion of regularity (resp. Σ -regularity) of a positive invariant sesquilinear form ϕ on a $*$ -algebra $\mathcal{A}$ as a natural generalization of the notion of strong regularity; that is, ϕ is represented as $\phi(x, y) = \lim_\alpha (\pi_\phi^*(x)\xi_\alpha | \pi_\phi^*(y)\xi_\alpha)$ (resp. $\phi(x, y) = \sum_\alpha (\pi_\phi^*(x)\xi_\alpha | \pi_\phi^*(y)\xi_\alpha)$) for some net $\{\xi_\alpha\}$ in $\mathcal{D}(\pi_\phi^*)$. We show that ϕ is regular if and only if $\pi(\mathfrak{A}'_\phi)$ is nondegenerate, and show that if π_ϕ is selfadjoint and ϕ_E is regular for each non-zero projection E in $\pi_\phi(\mathcal{A})'$, where $\phi_E(x, y) = (E\lambda_\phi(x) | \lambda_\phi(y))$ for $x, y \in \mathcal{A}$, then there exists a subset $\{\xi_\alpha\}$ in $\mathcal{D}(\pi_\phi)$ such that $(\pi_\phi(x)\xi_\alpha | \pi_\phi(y)\xi_\beta) = 0$ for each $x, y \in \mathcal{A}$ and $\alpha \neq \beta$, and $\phi(x, y) = \sum_\alpha (\pi_\phi(x)\xi_\alpha | \pi_\phi(y)\xi_\alpha)$ for each $x, y \in \mathcal{A}$. We remark that if $\mathcal{A}$ is commutative then ϕ is regular if and only if ϕ is Σ -regular.

In Section 6, we give a decomposition theorem for a positive invariant sesquilinear form. We introduce the notion of singularity of positive invariant sesquilinear forms, and show that every positive invariant sesquilinear form ϕ is decomposed into the sum $\phi = \phi_r + \phi_s$ of the regular part ϕ_r and the singular part ϕ_s . In particular, we show that if $\mathcal{A}$ is commutative then the above decomposition is unique.

In Section 7, we apply our study of positive invariant sesquilinear forms to of weights on C^* -algebras. For a weight ϕ on a C^* -algebra $\mathcal{A}$ a positive invariant sesquilinear form ϕ° on $\mathfrak{N}_\phi \times \mathfrak{N}_\phi$ is defined by $\phi^\circ(x, y) = \phi(y^*x)$ for $x, y \in \mathfrak{N}_\phi$, where $\mathfrak{N}_\phi = \{x \in \mathcal{A}; \phi(x^*x) < \infty\}$. By using the results of Combes [4] and Pedersen, Takesaki [21], we show that the regularity and Σ -regularity of ϕ° are equivalent.

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§ 2. Preliminaries

In this section we state some definitions and results concerning unbounded operator algebras and positive invariant sesquilinear forms.

Let $\mathcal{D}$ be a dense subspace in a Hilbert space $\mathfrak{H}$. We denote by $\mathcal{L}(\mathcal{D})$ the set of all linear operators from $\mathcal{D}$ into $\mathcal{D}$ and denote by $\mathcal{L}^+(\mathcal{D})$ the set of all elements A of $\mathcal{L}(\mathcal{D})$ such that $\mathcal{D}(A^*) \supset \mathcal{D}$ and $A^+ = A^*/\mathcal{D} \in \mathcal{L}(\mathcal{D})$, where $X/\mathcal{D}$ denotes the restriction of a linear operator X to $\mathcal{D}$. Then $\mathcal{L}^+(\mathcal{D})$ is a *-algebra under the usual operations and the involution $A \rightarrow A^+$. A subalgebra of $\mathcal{L}(\mathcal{D})$ is said to be an O_p -algebra on $\mathcal{D}$; and a *-subalgebra of $\mathcal{L}^+(\mathcal{D})$ is said to be an O_p^* -algebra on $\mathcal{D}$. Let $\mathcal{M}$ be an O_p -algebra on $\mathcal{D}$ and let $\mathcal{M}_I$ be the O_p -algebra on $\mathcal{D}$ obtained from $\mathcal{M}$ by adjoining the identity operator I . If $\mathcal{D}$ is complete under the induced topology $t_{\mathcal{M}}$ generated by the seminorms $\{\|\cdot\|_A; A \in \mathcal{M}_I\}$, where $\|\xi\|_A = \|A\xi\|$ for $\xi \in \mathcal{D}$, then $\mathcal{M}$ is called closed. It is known that $\mathcal{M}$ is closed if and only if $\tilde{\mathcal{D}}(\mathcal{M}) \equiv \bigcap_{A \in \mathcal{M}} \mathcal{D}(\bar{A}) = \mathcal{D}$, where $\bar{A}$ is the smallest closed extension of A . For each $A \in \mathcal{M}$ and $\xi \in \tilde{\mathcal{D}}(\mathcal{M})$ putting

$$\tilde{A}\xi = \bar{A}\xi \quad \text{and} \quad \tilde{\mathcal{M}} = \{\tilde{A}; A \in \mathcal{M}\},$$

$\tilde{\mathcal{M}}$ becomes a closed O_p -algebra on $\tilde{\mathcal{D}}(\mathcal{M})$, which is said to be the closure of $\mathcal{M}$. Let $\mathcal{M}$ be an O_p^* -algebra on $\mathcal{D}$. If $\mathcal{D}_*(\mathcal{M}) \equiv \bigcap_{A \in \mathcal{M}} \mathcal{D}(A^*) = \mathcal{D}$, then $\mathcal{M}$ is called self-adjoint. We define a commutant of an O_p^* -algebra $\mathcal{M}$ on $\mathcal{D}$ as follows:

$$\begin{aligned} \mathcal{M}' = \{K \in \mathcal{B}(\mathfrak{H}); (KA\xi|\eta) = (K\xi|A^+\eta) \\ \text{for all } A \in \mathcal{M} \text{ and } \xi, \eta \in \mathcal{D}\}, \end{aligned}$$

where $\mathcal{B}(\mathfrak{H})$ denotes the set of all bounded linear operators on $\mathfrak{H}$. If $K \in \mathcal{M}'$, then $K\mathcal{D} \subset \mathcal{D}_*(\mathcal{M})$ and $(A^+)^*K\xi = KA\xi$ for every $A \in \mathcal{M}$ and $\xi \in \mathcal{D}$. The commutant $\mathcal{M}'$ of $\mathcal{M}$ is not necessarily an algebra. However, if $\mathcal{M}$ is self-adjoint, then $\mathcal{M}'$ is a von Neumann algebra and $\mathcal{M}'\mathcal{D} = \mathcal{D}$ [22].

Let $\mathcal{A}$ be a *-algebra. A homomorphism (resp. *-homomorphism) π of $\mathcal{A}$ onto an O_p -algebra (resp. an O_p^* -algebra) on $\mathcal{D}$ is said to be a representation (resp. a *-representation) of $\mathcal{A}$ on $\mathfrak{H}$ with domain $\mathcal{D}(\pi) = \mathcal{D}$. A representation π of $\mathcal{A}$ is called closed if the O_p -algebra $\pi(\mathcal{A})$ is closed; and a *-representation π of $\mathcal{A}$ is called self-adjoint if the O_p^* -algebra $\pi(\mathcal{A})$ is self-adjoint. Let π, ρ be representations of $\mathcal{A}$ on a Hilbert space $\mathfrak{H}$. If $\mathcal{D}(\rho) \subset \mathcal{D}(\pi)$ and $\rho(x)\xi = \pi(x)\xi$ for all $x \in \mathcal{A}$ and $\xi \in \mathcal{D}(\rho)$, then we say that π is an extension of ρ , which is denoted by $\rho \subset \pi$. Let π be a representation of $\mathcal{A}$ on a Hilbert space $\mathfrak{H}$. Then we define closed representations of $\mathcal{A}$ as follows:

$$\mathcal{D}(\tilde{\pi}) = \bigcap_{x \in \mathcal{A}} \mathcal{D}(\overline{\pi(x)}), \quad \tilde{\pi}(x) = \overline{\pi(x)}/\mathcal{D}(\tilde{\pi});$$

$$\mathcal{D}(\pi^*) = \bigcap_{x \in \mathcal{A}} \mathcal{D}(\pi(x)^*), \quad \pi^*(x) = \pi(x^*)^* / \mathcal{D}(\pi^*)$$

for $x \in \mathcal{A}$. If $\mathcal{D}(\pi^*)$ is dense in $\mathfrak{H}$, then the closed representation π^{**} of $\mathcal{A}$ is defined by

$$\mathcal{D}(\pi^{**}) = \bigcap_{x \in \mathcal{A}} \mathcal{D}(\pi^*(x)^*), \quad \pi^{**}(x) = \pi^*(x^*)^* / \mathcal{D}(\pi^{**})$$

for $x \in \mathcal{A}$. Then we see that if π is a $*$ -representation, then $\tilde{\pi}$ and π^{**} are $*$ -representations satisfying that $\pi \subset \tilde{\pi} \subset \pi^{**} \subset \pi^*$ and $\pi^{**}(\mathcal{A})' = \pi(\mathcal{A})'$. However, π^* is not necessarily a $*$ -representation. We remark that there exist $*$ -representations π_1, π_2 such that $\tilde{\pi}_1 \subseteq \pi_1^{**} = \pi_1^*$ and $\tilde{\pi}_2 = \pi_2^{**} \subseteq \pi_2^*$ [9, 15].

Let $\mathcal{A}$ be a $*$ -algebra. A sesquilinear form ϕ of $\mathcal{A} \times \mathcal{A}$ into the complex number field C is called positive invariant if

$$\phi(x, x) \geq 0 \quad \text{and} \quad \phi(xy, z) = \phi(y, x^*z)$$

for each $x, y, z \in \mathcal{A}$. Let ϕ be a positive invariant sesquilinear form on $\mathcal{A} \times \mathcal{A}$. It is easily seen that $N_\phi = \{x \in \mathcal{A}; \phi(x, x) = 0\}$ is a left ideal in $\mathcal{A}$. For each $x \in \mathcal{A}$ we denote by $\lambda_\phi(x)$ the coset of $\mathcal{A}/N_\phi$ which contains x , and define by

$$(\lambda_\phi(x) | \lambda_\phi(y)) = \phi(x, y)$$

an inner product on $\lambda_\phi(\mathcal{A})$. Let $\mathfrak{H}_\phi$ be the Hilbert space which is the completion of the pre-Hilbert space $\lambda_\phi(\mathcal{A})$. We define a $*$ -representation π_ϕ° of $\mathcal{A}$ on $\mathfrak{H}_\phi$ with domain $\lambda_\phi(\mathcal{A})$ by

$$\pi_\phi^\circ(x) \lambda_\phi(y) = \lambda_\phi(xy)$$

for $x, y \in \mathcal{A}$. We denote by π_ϕ the closure $\tilde{\pi}_\phi^\circ$ of π_ϕ° ; that is, π_ϕ is a closed $*$ -representation of $\mathcal{A}$ on $\mathfrak{H}_\phi$ with domain $\mathcal{D}(\pi_\phi) = \bigcap_{x \in \mathcal{A}} \mathcal{D}(\pi_\phi^\circ(x))$. We call $(\pi_\phi, \lambda_\phi, \mathcal{D}(\pi_\phi), \mathfrak{H}_\phi)$ the GNS-construction of ϕ . If π_ϕ is bounded; that is, $\pi_\phi(\mathcal{A}) \subset \mathcal{B}(\mathfrak{H}_\phi)$, then ϕ is called admissible; and if π_ϕ is selfadjoint, then ϕ is said to be a Riesz form. We note that every admissible positive invariant sesquilinear form is a Riesz form. Examples of Riesz forms have appeared in [2, 3, 6, 12, 15, 19, 22].

Let f be a linear functional on a $*$ -algebra $\mathcal{A}$. If $f(x^*x) \geq 0$ for each $x \in \mathcal{A}$, then f is called positive. Let f be a positive linear functional on $\mathcal{A}$. We define a positive invariant sesquilinear form f° on $\mathcal{A} \times \mathcal{A}$ by

$$f^\circ(x, y) = f(y^*x)$$

for $x, y \in \mathcal{A}$, which is said to be the positive invariant sesquilinear form generated by f . We simply denote by $(\pi_f, \lambda_f, \mathcal{D}(\pi_f), \mathfrak{H}_f)$ the GNS-construction of f° . A positive linear functional f on $\mathcal{A}$ is called admissible (resp. a Riesz

functional) if f° is admissible (resp. a Riesz form).

§ 3. Unbounded observable algebras

Let $\mathcal{D}$ be a dense subspace of a Hilbert space $\mathfrak{H}$. For each $\xi \in \mathfrak{H}$ we define a continuous linear functional ξ^* on $\mathfrak{H}$ by $\eta \in \mathfrak{H} \rightarrow (\eta|\xi) \in \mathbb{C}$. For a subspace $\mathfrak{M}$ of $\mathfrak{H}$ the map $\xi \rightarrow \xi^*$ is a conjugate linear isometry of $\mathfrak{M}$ onto $\mathfrak{M}^* = \{\xi^*; \xi \in \mathfrak{M}\}$. We define on the product space $T^+(\mathcal{D}) = \mathcal{L}^+(\mathcal{D}) \times \mathcal{D} \times \mathcal{D}^*$ a multiplication and an involution as follows:

$$\begin{aligned} (S_0, \xi_1, \xi_2^*)(T_0, \eta_1, \eta_2^*) &= (S_0 T_0, S_0 \eta_1, (T_0^+ \xi_2^*)^*), \\ (S_0, \xi_1, \xi_2^*)^+ &= (S_0^+, \xi_2, \xi_1^*). \end{aligned}$$

Then $T^+(\mathcal{D})$ is a $*$ -algebra. For each $S = (S_0, \xi_1, \xi_2^*) \in T^+(\mathcal{D})$ we put

$$\pi(S) = S_0, \quad \lambda(S) = \xi_1, \quad \lambda^*(S) = \xi_2^*.$$

DEFINITION 3.1. A $*$ -subalgebra of $T^+(\mathcal{D})$ is said to be an unbounded observable algebra on $\mathcal{D}$ (simply, a T^+ -algebra on $\mathcal{D}$).

Let $\mathfrak{A}$ be a T^+ -algebra on $\mathcal{D}$. We note that π is a $*$ -representation of $\mathfrak{A}$ on $\mathfrak{H}$ with domain $\mathcal{D}$, λ is a linear map of $\mathfrak{A}$ into $\mathcal{D}$ satisfying $\lambda(AB) = \pi(A)\lambda(B)$ and $\lambda(A^+) = \lambda^*(A)^*$, and λ^* is a conjugate linear map of $\mathfrak{A}$ into $\mathcal{D}^*$ satisfying $\lambda^*(AB) = (\pi(B)^+ \lambda(A^+))^*$ for each $A, B \in \mathfrak{A}$.

DEFINITION 3.2. Let $\mathfrak{A}$ be a T^+ -algebra on $\mathcal{D}$. If the $*$ -representation π is closed (resp. self-adjoint), then $\mathfrak{A}$ is called π -closed (resp. π -self-adjoint).

REMARK. When $\mathcal{D} = \mathfrak{H}$, $T^+(\mathfrak{H})$ is a Banach $*$ -algebra equipped with the norm:

$$\|T\| = \max(\|\pi(T)\|, \|\lambda(T)\|, \|\lambda(T^+)\|).$$

Tomita has called a $*$ -subalgebra of $T^+(\mathfrak{H})$ an observable algebra on $\mathfrak{H}$ (simply, T^* -algebra). In particular, a closed $*$ -subalgebra of $T^+(\mathfrak{H})$ is said to be a CT^* -algebra.

DEFINITION 3.3. Let $\mathfrak{A}$ be a T^+ -algebra on $\mathcal{D}$. We define a commutant of $\mathfrak{A}$ as follows:

$$\begin{aligned} \mathfrak{A}' &= \{K = (K_0, \xi, \eta^*) \in \pi(\mathfrak{A})' \times \mathcal{D}(\pi^*) \times \mathcal{D}(\pi^*)^*; \\ &\quad K_0 \lambda(A) = \pi(A^+)^* \xi, K_0^* \lambda(A) = \pi(A^+)^* \eta \text{ for all } A \in \mathfrak{A}\}. \end{aligned}$$

For each $K = (K_0, \xi, \eta^*) \in \mathfrak{A}'$ we put

$$\pi(K) = K_0, \quad \lambda(K) = \xi, \quad \lambda^*(K) = \eta^*.$$

The commutant $\mathfrak{U}'$ is a vector space, which is invariant under the involution $K=(\pi(K), \lambda(K), \lambda^*(K)) \rightarrow K^*=(\pi(K)^*, \lambda^*(K)^*, \lambda(K)^*)$. But, $\mathfrak{U}'$ is not necessarily a T^* -algebra.

PROPOSITION 3.4. *If $\mathfrak{U}$ is a T^+ -algebra on $\mathcal{D}$ satisfying $\pi^*=\pi^{**}$ (in particular, $\mathfrak{U}$ is π -selfadjoint), then $\mathfrak{U}'$ is a CT^* -algebra and $\pi(\mathfrak{U}')$ is a $*$ -subalgebra of the von Neumann algebra $\pi(\mathfrak{U})'$.*

PROOF. Since π^* is a self-adjoint representation of $\mathfrak{U}$ with domain $\mathcal{D}(\pi^*)$ and $\pi^*(\mathfrak{U})'=\pi^{**}(\mathfrak{U})'=\pi(\mathfrak{U})'$, it follows that $\pi(\mathfrak{U})'$ is a von Neumann algebra, $\pi(\mathfrak{U})'\mathcal{D}(\pi^*)=\mathcal{D}(\pi^*)$ and $C\pi^*(A)\xi=\pi^*(A)C\xi$ for each $A \in \mathfrak{U}$, $C \in \pi(\mathfrak{U})'$ and $\xi \in \mathcal{D}(\pi^*)$, which implies that $\mathfrak{U}'$ is a CT^* -algebra.

Let $\mathcal{A}$ be a $*$ -algebra and ϕ be a positive invariant sesquilinear form on $\mathcal{A} \times \mathcal{A}$. We put

$$\mathfrak{U}_\phi = \{(\pi_\phi(x), \lambda_\phi(x), \lambda_\phi(x^*)^*); x \in \mathcal{A}\}.$$

Then $\mathfrak{U}_\phi$ is a π -closed T^+ -algebra on $\mathcal{D}(\pi_\phi)$, which is said to be the T^+ -algebra generated by ϕ . Let f be a positive linear functional on $\mathcal{A}$. We denote by $\mathfrak{U}_f$ the T^+ -algebra generated by f° .

In this paper we shall study positive invariant sesquilinear forms ϕ by using the T^+ -algebra $\mathfrak{U}_\phi$ and the commutant $\mathfrak{U}'_\phi$.

§ 4. Strongly regular positive invariant sesquilinear forms

Throughout this section let $\mathcal{A}$ be a $*$ -algebra without identity. We investigate under what conditions a positive invariant sesquilinear form ϕ on $\mathcal{A} \times \mathcal{A}$ is represented as $\phi(x, y)=(\pi_\phi^*(x)\xi|\pi_\phi^*(y)\xi)$ for some $\xi \in \mathcal{D}(\pi_\phi^*)$.

DEFINITION 4.1. Let ϕ be a positive invariant sesquilinear form on $\mathcal{A} \times \mathcal{A}$. A positive invariant sesquilinear form ψ on $\mathcal{A} \times \mathcal{A}$ is called ϕ -dominated if $\psi \leq \gamma\phi$ for some positive number γ ; that is, $\psi(x, x) \leq \gamma\phi(x, x)$ for all $x \in \mathcal{A}$. A positive linear functional f on $\mathcal{A}$ is called ϕ -majorized if f° is ϕ -dominated and there is a constant γ such that $|f(x)|^2 \leq \gamma\phi(x, x)$ for all $x \in \mathcal{A}$.

LEMMA 4.2. *Let ϕ and ψ be positive invariant sesquilinear forms on $\mathcal{A} \times \mathcal{A}$ and f be a positive linear functional on $\mathcal{A}$. Then the following statements hold.*

- (1) ψ is ϕ -dominated if and only if there exists a positive operator K_0 in $\pi_\phi(\mathcal{A})'$ such that $\psi(x, y)=(K_0\lambda_\phi(x)|\lambda_\phi(y))$ for all $x, y \in \mathcal{A}$.
- (2) f is ϕ -majorized if and only if there exists an element K of $\mathfrak{U}'_\phi$ such that $\pi(K) \geq 0$ and $f(x)=(\lambda_\phi(x)|\lambda(K))$ for all $x \in \mathcal{A}$.

PROOF. (1) Suppose that ψ is ϕ -dominated. Then the map $\lambda_\phi(x) \rightarrow \lambda_\psi(x)$

is extended to a continuous linear map S_0 of $\mathfrak{S}_\phi$ into $\mathfrak{S}_\psi$. We put $K_0 = S_0^* S_0$. Then it is clear that $K_0 \in \pi_\phi(\mathcal{A})'$ and $\psi(x, y) = (K_0 \lambda_\phi(x) | \lambda_\phi(y))$ for all $x, y \in \mathcal{A}$. The converse is trivial.

(2) Suppose that f is ϕ -majorized. Then the map $\lambda_\phi(x) \rightarrow f(x)$ is extended to a continuous linear functional on $\mathfrak{S}_\phi$. By the Riesz theorem there exists an element ξ of $\mathfrak{S}_\phi$ such that $f(x) = (\lambda_\phi(x) | \xi)$ for all $x \in \mathcal{A}$. We now have

$$|(\pi_\phi(x^*) \lambda_\phi(y) | \xi)|^2 = |f(x^*y)|^2 \leq f(x^*x)f(y^*y) \leq \gamma f(x^*x) \|\lambda_\phi(y)\|^2$$

for all $x, y \in \mathcal{A}$, and hence $\xi \in \mathcal{D}(\pi_\phi^*)$ and $f(x^*y) = (\lambda_\phi(y) | \pi_\phi^*(x)\xi)$ for all $x, y \in \mathcal{A}$. Since f° is ϕ -dominated, it follows from (1) that there exists a positive operator K_0 in $\pi_\phi(\mathcal{A})'$ such that $f(x^*y) = (\lambda_\phi(y) | K_0 \lambda_\phi(x))$ for all $x, y \in \mathcal{A}$. Hence, we have $K_0 \lambda_\phi(x) = \pi_\phi^*(x)\xi$ for all $x \in \mathcal{A}$, which implies that $K = (K_0, \xi, \xi^*) \in \mathfrak{U}'_\phi$ and $f(x) = (\lambda_\phi(x) | \lambda(K))$ for all $x \in \mathcal{A}$.

DEFINITION 4.3. A positive invariant sesquilinear form ϕ on $\mathcal{A} \times \mathcal{A}$ is called strongly regular if there exists a ϕ -majorized positive linear functional f on $\mathcal{A}$ such that $f^\circ = \phi$. A positive linear functional f on $\mathcal{A}$ is called strongly regular if f° is strongly regular.

THEOREM 4.4. Let ϕ be a positive invariant sesquilinear form on $\mathcal{A} \times \mathcal{A}$. Consider the following statements.

- (1) ϕ is strongly regular.
- (2) There exists an element ξ of $\mathcal{D}(\pi_\phi^*)$ such that $\lambda_\phi(x) = \pi_\phi^*(x)\xi$ for all $x \in \mathcal{A}$.
- (3) $I \in \pi(\mathfrak{U}'_\phi)$.
- (4) There exists an element ξ of $\mathcal{D}(\pi_\phi^*)$ such that $\phi(x, y) = (\pi_\phi^*(x)\xi | \pi_\phi^*(y)\xi)$ for all $x, y \in \mathcal{A}$.

Then the following implications hold:

$$\begin{array}{c} (1) \iff (2) \iff (3) \\ \downarrow \\ (4). \end{array}$$

If π_ϕ^* is a self-adjoint representation (in particular, ϕ is a Riesz form), then the above statements (1) ~ (4) are equivalent to the following statements (5), (6) and (7).

- (5) $\pi(\mathfrak{U}'_\phi)$ is a nondegenerate $*$ -algebra and $\pi(K) \in \pi(\mathfrak{U}'_\phi) \rightarrow \lambda(K) \in \mathfrak{S}_\phi$ is uniformly continuous.
- (6) $\pi(\mathfrak{U}'_\phi)$ is a C^* -algebra.
- (7) $\pi_\phi(\mathcal{A})' = \pi(\mathfrak{U}'_\phi)$.

PROOF. (1) $\Rightarrow$ (3) Let f be a ϕ -majorized positive linear functional on $\mathcal{A}$ with $f^\circ = \phi$. By Lemma 4.2 there exists an element K of $\mathfrak{U}'_\phi$ such that

$\pi(K) \geq 0$ and $f(x) = (\lambda_\phi(x) | \lambda(K))$ for all $x \in \mathcal{A}$. Then we have

$$\begin{aligned} (\lambda_\phi(x) | \lambda_\phi(y)) &= f(y^*x) = (\lambda_\phi(x) | \pi_\phi^*(y)\lambda(K)) \\ &= (\lambda_\phi(x) | \pi(K)\lambda_\phi(y)) \end{aligned}$$

for all $x, y \in \mathcal{A}$, and hence $\pi(K) = I$.

Implications (3) $\Rightarrow$ (2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (4) are almost trivial.

Suppose that π_ϕ^* is self-adjoint; that is, $\pi_\phi^* = \pi_\phi^{**}$.

(4) $\Rightarrow$ (1) Suppose that there exists an element ξ of $\mathcal{D}(\pi_\phi^*)$ such that $\phi(x, y) = (\pi_\phi^*(x)\xi | \pi_\phi^*(y)\xi)$ for all $x, y \in \mathcal{A}$. We put

$$f(x) = (\pi_\phi^*(x)\xi | \xi) \quad \text{for } x \in \mathcal{A}.$$

Since $\xi \in \mathcal{D}(\pi_\phi^*) = \mathcal{D}(\pi_\phi^{**})$, it follows that f is a ϕ -majorized positive linear functional on $\mathcal{A}$ with $f^\circ = \phi$.

(2) $\Rightarrow$ (7) Suppose that there exists an element ξ of $\mathcal{D}(\pi_\phi^*)$ such that $\lambda_\phi(x) = \pi_\phi^*(x)\xi$ for all $x \in \mathcal{A}$. By Proposition 3.4 $\pi(\mathfrak{V}'_\phi)$ is a $*$ -algebra on $\mathfrak{S}_\phi$. Since π_ϕ^* is a self-adjoint representation of $\mathcal{A}$ and $\pi_\phi^*(\mathcal{A})' = \pi_\phi(\mathcal{A})'$, it follows that $\pi_\phi(\mathcal{A})'\mathcal{D}(\pi_\phi^*) = \mathcal{D}(\pi_\phi^*)$ and $C\pi_\phi^*(x)\xi = \pi_\phi^*(x)C\xi$ for each $x \in \mathcal{A}$, $C \in \pi_\phi(\mathcal{A})'$ and $\xi \in \mathcal{D}(\pi_\phi^*)$. Hence, we have $(C, C\xi, (C^*\xi)^*) \in \mathfrak{V}'_\phi$ for each $C \in \pi_\phi(\mathcal{A})'$, which implies that $\pi_\phi(\mathcal{A})' = \pi(\mathfrak{V}'_\phi)$.

(7) $\Rightarrow$ (6) This follows since $\pi_\phi(\mathcal{A})'$ is a von Neumann algebra.

(6) $\Rightarrow$ (5) Since $\pi(\mathfrak{V}'_\phi)$ is a C^* -algebra, it is nondegenerate. Suppose $\pi(K) = 0$ ($K \in \mathfrak{V}'_\phi$). Then, for each $S \in \mathfrak{V}'_\phi$ and $x \in \mathcal{A}$ we have

$$\begin{aligned} (\lambda(K) | \pi(S)\lambda_\phi(x)) &= (\lambda(K) | \pi_\phi^*(x)\lambda(S)) \\ &= (\pi_\phi^*(x^*)\lambda(K) | \lambda(S)) \quad (\text{by } \pi_\phi^* = \pi_\phi^{**}) \\ &= (\pi(K)\lambda_\phi(x^*) | \lambda(S)) \\ &= 0, \end{aligned}$$

and hence $\lambda(K) = 0$. Hence, $\pi(K) \rightarrow \lambda(K)$ is a linear map of $\pi(\mathfrak{V}'_\phi)$ into $\mathfrak{S}_\phi$. Let $\{K_n\}$ be a sequence in $\mathfrak{V}'_\phi$ such that $\{\pi(K_n)\}$ converges uniformly to K_0 and $\{\lambda(K_n)\}$ converges to ξ . Since $\pi(\mathfrak{V}'_\phi)$ is a C^* -algebra, there exists an element K of $\mathfrak{V}'_\phi$ such that $K_0 = \pi(K)$. Furthermore, since

$$\begin{aligned} (\pi_\phi(x^*)\lambda_\phi(y) | \xi) &= \lim_{n \rightarrow \infty} (\pi_\phi(x^*)\lambda_\phi(y) | \lambda(K_n)) \\ &= \lim_{n \rightarrow \infty} (\lambda_\phi(y) | \pi(K_n)\lambda_\phi(x)) \\ &= (\lambda_\phi(y) | \pi(K)\lambda_\phi(x)) \\ &= (\lambda_\phi(y) | \pi_\phi^*(x)\lambda(K)) \end{aligned}$$

for each $x, y \in \mathcal{A}$, we have $\xi \in \mathcal{D}(\pi_\phi^*)$ and $\pi_\phi^*(x)\xi = \pi_\phi^*(x)\lambda(K)$ for all $x \in \mathcal{A}$. Since $\pi(\mathfrak{V}'_\phi)$ is nondegenerate, we have $\xi = \lambda(K)$, so that the linear map $\pi(K)$

$\rightarrow \lambda(K)$ is closed. By the closed graph theorem the map $\pi(K) \rightarrow \lambda(K)$ is uniformly continuous.

(5) $\Rightarrow$ (2) Since $\pi(K) \rightarrow \lambda(K)$ is uniformly continuous, there exists a constant γ such that $\|\lambda(K)\| \leq \gamma \|\pi(K)\|$ for all $K \in \mathfrak{V}'_\phi$. For each $\varepsilon > 0$ and $K_1, \dots, K_n \in \mathfrak{V}'_\phi$ we put

$$\Delta(\varepsilon; K_1, \dots, K_n) = \{\xi \in \mathfrak{H}_\phi; \|\xi\| \leq \gamma, \|\pi(K_i)\xi - \lambda(K_i)\| \leq \varepsilon \text{ for } i=1, \dots, n\}.$$

Then $\Delta(\varepsilon; K_1, \dots, K_n)$ is a weakly compact subset of $\mathfrak{H}_\phi$. Since $\pi(\mathfrak{V}'_\phi)$ is non-degenerate, we have $\Delta(\varepsilon; K_i, \dots, K_n) \neq \emptyset$. Hence, $\bigcap_{\substack{\varepsilon > 0 \\ K \in \mathfrak{V}'_\phi}} \Delta(\varepsilon; K) \neq \emptyset$, which

implies that there exists an element ξ of $\mathfrak{H}_\phi$ such that $\lambda(K) = \pi(K)\xi$ for all $K \in \mathfrak{V}'_\phi$. Since $\pi(\mathfrak{V}'_\phi)$ is nondegenerate, there exists a net $\{K_\alpha\}$ in $\mathfrak{V}'_\phi$ such that $\{\pi(K_\alpha)\}$ converges strongly to I . Then we have

$$\begin{aligned} \lim_\alpha \lambda(K_\alpha) &= \lim_\alpha \pi(K_\alpha)\xi = \xi, \\ (\pi_\phi(x^*)\lambda_\phi(y)|\xi) &= \lim_\alpha (\pi_\phi(x^*)\lambda_\phi(y)|\lambda(K_\alpha)) \\ &= \lim_\alpha (\lambda_\phi(y)|\pi(K_\alpha)\lambda_\phi(x)) \\ &= (\lambda_\phi(y)|\lambda_\phi(x)) \end{aligned}$$

for all $x, y \in \mathcal{A}$, and hence $\xi \in \mathcal{D}(\pi_\phi^*)$ and $\lambda_\phi(x) = \pi_\phi^*(x)\xi$ for all $x \in \mathcal{A}$. This completes the proof.

COROLLARY 4.5. *Let f be a positive linear functional on $\mathcal{A}$. Consider the following statements.*

(1) *f is f° -majorized; that is, there exists a constant γ such that $|f(x)|^2 \leq \gamma f(x^*x)$ for all $x \in \mathcal{A}$.*

(2) *There exists an element (I, ξ, ξ^*) of $\mathfrak{V}'_f$ such that $f(x) = (\pi_f^*(x)\xi|\xi)$ for all $x \in \mathcal{A}$.*

(3) *f is strongly regular.*

(4) *There exists an element ξ of $\mathcal{D}(\pi_f^*)$ such that $\lambda_f(x) = \pi_f^*(x)\xi$ for all $x \in \mathcal{A}$.*

Then the following implications hold:

$$\begin{aligned} (1) &\Longleftrightarrow (2) \\ &\Downarrow \\ (3) &\Longleftrightarrow (4). \end{aligned}$$

PROOF. This is almost similar to the proof of Theorem 4.4.

EXAMPLES. (1) Every positive invariant sesquilinear form on a $*$ -algebra with identity is strongly regular.

(2) A linear functional f on $\mathcal{A}$ is called hermitian if $f(x^*) = \overline{f(x)}$ for each $x \in \mathcal{A}$. A positive hermitian linear functional f on $\mathcal{A}$ satisfies the condition (1) of Corollary 4.5 if and only if f is extendable; that is, f can be extended to a positive linear functional on $\mathcal{A}_e$ [17, 18, 23]. Hence, every extendable positive linear functional is strongly regular.

(3) Let ϕ be an admissible positive invariant sesquilinear form on a $*$ -algebra $\mathcal{A}$. Then we see that ϕ is strongly regular if and only if $\pi_\phi(x) \rightarrow \lambda_\phi(x)$ is uniformly continuous. This is similar to the proof (5) $\Rightarrow$ (2) of Theorem 4.4.

(4) Let $\mathcal{A}$ be a locally convex $*$ -algebra [1]. A set $\{e_\alpha\}$ in $\mathcal{A}$ is said to be an approximate identity of $\mathcal{A}$ if $\lim_\alpha e_\alpha x = \lim_\alpha x e_\alpha = x$ for each $x \in \mathcal{A}$. An approximate identity $\{e_\alpha\}$ of $\mathcal{A}$ is called bounded if $\{e_\alpha\}$ and $\{e_\alpha^* e_\alpha\}$ are bounded subsets of $\mathcal{A}$. We see that every B^* -algebra, or more generally, locally convex GB $*$ -algebra has a bounded approximate identity [3]. Every continuous positive linear functional f on a locally convex $*$ -algebra with a bounded approximate identity satisfies the condition (1) of Corollary 4.5, and hence f is strongly regular.

§ 5. Σ -regular positive invariant sesquilinear forms

In Section 4 we have investigated under what conditions a positive invariant sesquilinear form ϕ is strongly regular; that is, $\phi(x, y) = (\pi_\phi^*(x)\xi | \pi_\phi^*(y)\xi)$ for some $\xi \in \mathcal{D}(\pi_\phi^*)$. As a natural generalization of the strongly regularity, in this section we investigate under what conditions a positive invariant sesquilinear form ϕ is represented as $\phi(x, y) = \sum_\alpha (\pi_\phi^*(x)\xi_\alpha | \pi_\phi^*(y)\xi_\alpha)$ for some set $\{\xi_\alpha\}$ in $\mathcal{D}(\pi_\phi^*)$.

Throughout this section let $\mathcal{A}$ be a $*$ -algebra without identity.

DEFINITION 5.1. A positive invariant sesquilinear form ϕ on $\mathcal{A} \times \mathcal{A}$ is called Σ -regular (resp. regular) if there exists a set $\{f_\alpha\}$ of ϕ -majorized positive linear functionals on $\mathcal{A}$ such that $\phi(x, y) = \sum_\alpha f_\alpha^\circ(x, y)$ (resp. $f_\alpha^\circ \leq \phi$ for each α and $\phi(x, y) = \lim_\alpha f_\alpha^\circ(x, y)$) for all $x, y \in \mathcal{A}$. A positive linear functional f on $\mathcal{A}$ is called Σ -regular (resp. regular) if f° is Σ -regular (resp. regular).

Let ϕ be a positive invariant sesquilinear form on $\mathcal{A} \times \mathcal{A}$. We denote by $P_{\mathfrak{U}'_\phi}$ the projection of $\mathfrak{S}_\phi$ onto the closed subspace $[\pi(\mathfrak{U}'_\phi)\mathfrak{S}_\phi]$ generated by $\pi(\mathfrak{U}'_\phi)\mathfrak{S}_\phi$. We note that $P_{\mathfrak{U}'_\phi} \in \pi_\phi(\mathcal{A})'$ if $\pi_\phi^* = \pi_\phi^{**}$.

PROPOSITION 5.2. *Let ϕ be a positive invariant sesquilinear form on $\mathcal{A} \times \mathcal{A}$. Consider the following statements.*

- (1) ϕ is Σ -regular.

(2) There exists a set $\{K_\alpha\}$ in $\mathfrak{U}'_\phi$ such that $\pi(K_\alpha) \geq 0$ for each α and $\phi(x, y) = \sum_\alpha (\pi_\phi^*(x) \lambda(K_\alpha) | \pi_\phi^*(y) \lambda(K_\alpha))$ for each $x, y \in \mathcal{A}$.

(3) There exists a net $\{\phi_\alpha\}$ of strongly regular positive invariant sesquilinear forms on $\mathcal{A} \times \mathcal{A}$ such that $\phi_\alpha \leq \phi$ for each α and $\lim_\alpha \phi_\alpha(x, y) = \phi(x, y)$ for each $x, y \in \mathcal{A}$.

(4) ϕ is regular.

(5) There exists a net $\{K_\alpha\}$ in $\mathfrak{U}'_\phi$ such that $0 \leq \pi(K_\alpha) \leq I$ for each α and $\{\pi(K_\alpha)\}$ converges weakly to I .

(6) $P_{\mathfrak{U}'_\phi} = I$.

Then the following implications hold:

$$\left. \begin{array}{l} (1) \iff (2) \\ (3) \end{array} \right\} \Rightarrow \begin{array}{c} (4) \\ \Updownarrow \\ (5) \end{array} \Rightarrow (6).$$

In particular, if $\pi_\phi^* = \pi_\phi^{**}$, then the above statements (3) ~ (6) are equivalent.

PROOF. When $\pi_\phi^* = \pi_\phi^{**}$, we show the implication (6) $\Rightarrow$ (3). The other implications follow immediately from Lemma 4.2. Since $P_{\mathfrak{U}'_\phi} = I$, it follows that $\pi(\mathfrak{U}'_\phi)$ is a nondegenerate *-algebra on $\mathfrak{S}_\phi$, so that there exists a net $\{K_\alpha\}$ in $\mathfrak{U}'_\phi$ such that $\|\pi(K_\alpha)\| \leq 1$ for each α and $\{\pi(K_\alpha)\}$ converges strongly to I . We now put

$$f_\alpha(x) = (\pi(K_\alpha) \lambda_\phi(x) | \lambda(K_\alpha))$$

for $x \in \mathcal{A}$. Then we have

$$\begin{aligned} f_\alpha(x^*x) &= \|\pi(K_\alpha) \lambda_\phi(x)\|^2 \leq \phi(x, x), \\ |f_\alpha(x)|^2 &\leq \|\lambda(K_\alpha)\|^2 \|\pi(K_\alpha) \lambda_\phi(x)\|^2 = \|\lambda(K_\alpha)\|^2 f_\alpha(x^*x) \end{aligned}$$

for all $x \in \mathcal{A}$. Hence, f_α is a strongly regular positive linear functional on $\mathcal{A}$ with $f_\alpha^\circ \leq \phi$. Furthermore, we have

$$\begin{aligned} \lim_\alpha f_\alpha^\circ(x, y) &= \lim_\alpha (\pi(K_\alpha) \lambda_\phi(x) | \pi(K_\alpha) \lambda_\phi(y)) \\ &= (\lambda_\phi(x) | \lambda_\phi(y)) \\ &= \phi(x, y) \end{aligned}$$

for all $x, y \in \mathcal{A}$. Hence, the statement (3) holds.

REMARK. We note by the equivalence of (3) and (4) in Proposition 5.2 that if for a positive invariant sesquilinear form ϕ on $\mathcal{A} \times \mathcal{A}$ there exists a net $\{\phi_\alpha\}$ of regular Riesz forms on $\mathcal{A}$ such that $\phi_\alpha \leq \phi$ for each α and $\lim_\alpha \phi_\alpha(x, y) = \phi(x, y)$ for each $x, y \in \mathcal{A}$, then ϕ is regular.

We next investigate when a regular positive invariant sesquilinear form is Σ -regular.

Let ϕ be a positive invariant sesquilinear form on $\mathcal{A} \times \mathcal{A}$. For each positive operator K_0 in $\pi_\phi(\mathcal{A})'$ we put

$$\phi_{K_0}(x, y) = (K_0 \lambda_\phi(x) | \lambda_\phi(y))$$

for $x, y \in \mathcal{A}$. Then ϕ_{K_0} is a ϕ -dominated positive invariant sesquilinear form on $\mathcal{A} \times \mathcal{A}$. By Lemma 4.2 we see that every ϕ -dominated positive invariant sesquilinear form is of this form.

THEOREM 5.3. *Suppose that ϕ is a positive invariant sesquilinear form on $\mathcal{A} \times \mathcal{A}$ satisfying $\pi_\phi^* = \pi_\phi^{**}$. Consider the following statements.*

- (1) ϕ_E is Σ -regular for each non-zero projection E in $\pi_\phi(\mathcal{A})'$.
- (2) ϕ_E is regular for each non-zero projection E in $\pi_\phi(\mathcal{A})'$.
- (3) There exists a set $\{E_\alpha\}$ in $\mathcal{A}'$ such that $\{\pi(E_\alpha)\}$ is of mutually orthogonal projections and $\sum_\alpha \pi(E_\alpha) = I$.
- (4) There exists a set $\{\xi_\alpha\}$ in $\mathcal{D}(\pi_\phi^*)$ such that
 - (4.1) $(\pi_\phi^*(x)\xi_\alpha | \pi_\phi^*(y)\xi_\beta) = 0$ for each $x, y \in \mathcal{A}$ and $\alpha \neq \beta$;
 - (4.2) $\lambda_\phi(x) = \sum_\alpha \pi_\phi^*(x)\xi_\alpha$ for each $x \in \mathcal{A}$.
- (5) There exists a set $\{\xi_\alpha\}$ in $\mathcal{D}(\pi_\phi^*)$ such that
 - (5.1) $(\pi_\phi^*(x)\xi_\alpha | \pi_\phi^*(y)\xi_\beta) = 0$ for each $x, y \in \mathcal{A}$ and $\alpha \neq \beta$;
 - (5.2) $\sum_\alpha \|\pi_\phi^*(x)\xi_\alpha\|^2 < \infty$ for each $x \in \mathcal{A}$;
 - (5.3) $\{\sum_\alpha \pi_\phi^*(x)\xi_\alpha; x \in \mathcal{A}\}$ is dense in $\mathfrak{H}_\phi$;
 - (5.4) $\phi(x, y) = \sum_\alpha (\pi_\phi^*(x)\xi_\alpha | \pi_\phi^*(y)\xi_\alpha)$ for each $x, y \in \mathcal{A}$.
- (6) There exists a set $\{\xi_\alpha\}$ in $\mathcal{D}(\pi_\phi^*)$ such that
 - (6.1) $(\pi_\phi^*(x)\xi_\alpha | \pi_\phi^*(y)\xi_\beta) = 0$ for each $x, y \in \mathcal{A}$ and $\alpha \neq \beta$;
 - (6.2) $\phi(x, y) = \sum_\alpha (\pi_\phi^*(x)\xi_\alpha | \pi_\phi^*(y)\xi_\alpha)$ for each $x, y \in \mathcal{A}$.
- (7) ϕ is Σ -regular.

Then the following implications hold;

$$\begin{array}{c}
 (3) \\
 \Updownarrow \\
 (1) \Rightarrow (2) \quad \text{and} \quad (4) \Rightarrow (6) \Rightarrow (7). \\
 \Updownarrow \\
 (5)
 \end{array}$$

In particular, if ϕ is a Riesz form on $\mathcal{A}$, then the implication $(2) \Rightarrow (3)$ holds.

PROOF. $(1) \Rightarrow (2)$ This follows since the Σ -regularity implies the regularity.

$(3) \Rightarrow (4) \Rightarrow (5)$ This is trivial.

$(5) \Rightarrow (3)$ We put

$$U_0 \lambda_\phi(x) = \sum_\alpha \pi_\phi^*(x) \xi_\alpha$$

for $x \in \mathcal{A}$. Then U_0 is extended to a unitary operator U on $\mathfrak{H}_\phi$ which is contained in $\pi_\phi(\mathcal{A})'$. Since π_ϕ^* is a self-adjoint representation of $\mathcal{A}$, we have

$$\lambda_\phi(x) = \sum_\alpha \pi_\phi^*(x) U^* \xi_\alpha$$

for each $x \in \mathcal{A}$. We now denote by E_α the projection of $\mathfrak{H}_\phi$ onto $[\pi_\phi^*(\mathcal{A}) U^* \xi_\alpha]$. Then it follows since U is a unitary operator that $\{E_\alpha\}$ is a set of mutually orthogonal projections in $\pi_\phi(\mathcal{A})'$, $(E_\alpha, U^* \xi_\alpha, (U^* \xi_\alpha)^*) \in \mathfrak{V}'_\phi$ for each α and $\sum_\alpha E_\alpha = I$.

(5) $\Rightarrow$ (6) This is trivial.

(6) $\Rightarrow$ (7) We put

$$f_\alpha(x) = (\pi_\phi^*(x) \xi_\alpha | \xi_\alpha)$$

for $x \in \mathcal{A}$. Since π_ϕ^* is a $*$ -representation of $\mathcal{A}$, we have

$$\begin{aligned} f_\alpha(x^*x) &= \|\pi_\phi^*(x) \xi_\alpha\|^2 \leq \sum_\alpha \|\pi_\phi^*(x) \xi_\alpha\|^2 = \phi(x, x), \\ |f_\alpha(x)|^2 &\leq \|\xi_\alpha\|^2 \|\pi_\phi^*(x) \xi_\alpha\|^2 \leq \|\xi_\alpha\|^2 \phi(x, x) \end{aligned}$$

for all $x \in \mathcal{A}$, and hence $\{f_\alpha\}$ is a set of ϕ -majorized positive linear functionals on $\mathcal{A}$ and $\sum_\alpha f_\alpha^\circ(x, y) = \phi(x, y)$ for all $x, y \in \mathcal{A}$, which means that ϕ is Σ -regular.

(2) $\Rightarrow$ (3) Suppose that ϕ is a Riesz form on $\mathcal{A}$. Let $\{E_\lambda\}_{\lambda \in \mathcal{A}}$ be a maximal family in $\mathfrak{V}'_\phi$ satisfying that $\{\pi(E_\lambda)\}_{\lambda \in \mathcal{A}}$ is of mutually orthogonal projections. We show $\sum_{\lambda \in \mathcal{A}} \pi(E_\lambda) = I$. Suppose that $E_A \equiv I - \sum_{\lambda \in \mathcal{A}} \pi(E_\lambda) \neq 0$. Then ϕ_{E_A} is a non-zero positive invariant sesquilinear form on $\mathcal{A} \times \mathcal{A}$. Since ϕ_{E_A} is regular by the assumption, there exists a ϕ_{E_A} -majorized positive linear functional f on $\mathcal{A}$ with $f^\circ \neq 0$. Since $\phi_{E_A} \leq \phi$, f is also ϕ -majorized. Hence, by Lemma 4.2 there exists an element K of $\mathfrak{V}'_\phi$ such that $\pi(K) \geq 0$ and $f(x) = (\lambda_\phi(x) | \lambda(K))$ for all $x \in \mathcal{A}$. Since f is ϕ_{E_A} -majorized, there is a constant γ such that $|f(x)|^2 \leq \gamma \phi_{E_A}(x, x)$ for all $x \in \mathcal{A}$. Hence, we have

$$\begin{aligned} |(\lambda_\phi(y) | \pi(K) \lambda_\phi(x))|^2 &= |f(x^*y)|^2 \leq \gamma \phi_{E_A}(x^*y, x^*y) \\ &= \gamma (\pi_\phi(x^*) E_A \lambda_\phi(y) | \pi_\phi(x^*) \lambda_\phi(y)) \end{aligned}$$

for each $x, y \in \mathcal{A}$. Since ϕ is a Riesz form on $\mathcal{A}$, it follows that

$$|(\xi | \pi(K) \lambda_\phi(x))|^2 \leq \gamma (\pi_\phi(x^*) E_A \xi | \pi_\phi(x^*) \xi)$$

for each $x \in \mathcal{A}$ and $\xi \in \mathcal{D}(\pi_\phi)$, which implies that $((I - E_A) \lambda_\phi(y) | \pi(K) \lambda_\phi(x)) = 0$ for each $x, y \in \mathcal{A}$. Hence, $\pi(K) E_A = \pi(K)$. Since $f^\circ \neq 0$, we have $\pi(K) \neq 0$. Let $\pi(K) = \int_0^{\|\pi(K)\|} \lambda dE(\lambda)$ be the spectral resolution of $\pi(K)$. We now put

$$E_t^\circ = \int_t^{\|\pi(K)\|} dE(\lambda) \quad \text{and} \quad X_t = \int_t^{\|\pi(K)\|} \frac{1}{\lambda} dE(\lambda)$$

for $0 < t < \|\pi(K)\|$. Then, it follows that $E_t^\circ, X_t \in \pi_\phi(\mathcal{A})'$ and $E_t^\circ = \pi(K)X_t^2\pi(K)$. Since $\pi(K) \neq 0$, we have $E_{t_0}^\circ \neq 0$ for some $t_0 \in (0, \|\pi(K)\|)$. Furthermore, we have

$$\begin{aligned} E_{t_0}^\circ \lambda_\phi(x) &= \pi(K)X_{t_0}^2\pi(K)\lambda_\phi(x) \\ &= \pi(K)X_{t_0}^2\pi_\phi(x)\lambda(K) \\ &= \pi_\phi(x)\pi(K)X_{t_0}^2\lambda(K) \end{aligned}$$

for all $x \in \mathcal{A}$. It hence follows that $E_{t_0} = (E_{t_0}^\circ, \pi(K)X_{t_0}^2\lambda(K), (\pi(K)X_{t_0}^2\lambda(K))^*) \in \mathfrak{V}'_\phi$ and $E_{t_0}^\circ E_\lambda = \pi(K)X_{t_0}^2\pi(K)E_\lambda = \pi(K)X_{t_0}^2\pi(K) = E_{t_0}^\circ$, which contradicts the maximality of $\{E_\lambda\}_{\lambda \in \mathcal{A}}$. Hence, $\sum_{\lambda \in \mathcal{A}} \pi(E_\lambda) = I$. This completes the proof.

We next investigate when a positive invariant sesquilinear form ψ on $\mathcal{A} \times \mathcal{A}$ with $\psi \leq \phi$ is Σ -regular (resp. regular), where ϕ is a Σ -regular (resp. regular) positive invariant sesquilinear form on $\mathcal{A} \times \mathcal{A}$.

LEMMA 5.4. *Let ϕ be a positive invariant sesquilinear form on $\mathcal{A} \times \mathcal{A}$ satisfying $\pi_\phi^* = \pi_\phi^{**}$. Suppose that ϕ is regular (resp. Σ -regular); that is, there exists a set $\{K_\alpha\}$ in $\mathfrak{V}'_\phi$ such that $\phi(x, y) = \lim_\alpha (\pi_\phi^*(x)\lambda(K_\alpha)|\pi_\phi^*(y)\lambda(K_\alpha))$ (resp. $\phi(x, y) = \sum_\alpha (\pi_\phi^*(x)\lambda(K_\alpha)|\pi_\phi^*(y)\lambda(K_\alpha))$) for each $x, y \in \mathcal{A}$. Then the following statements hold.*

(1) *If E is an element of $\mathfrak{V}'_\phi$ such that $\pi(E)$ is a projection, then $\phi_{\pi(E)}$ is regular (resp. Σ -regular).*

(2) *If K_0 is a positive operator in $\pi_\phi(\mathcal{A})'$ such that $K_0\pi(K_\alpha) = \pi(K_\alpha)K_0$ for each α , then ϕ_{K_0} is regular (resp. Σ -regular).*

PROOF. Suppose that ϕ is regular.

(1) We now put

$$f_\alpha(x) = (\pi(K_\alpha)^*\pi(K_\alpha)\pi(E)\lambda_\phi(x)|\lambda(E))$$

for $x \in \mathcal{A}$. Then it follows that f_α is a $\phi_{\pi(E)}$ -majorized positive linear functional on $\mathcal{A}$ satisfying that $f_\alpha^\circ \leq \phi_{\pi(E)}$ and $\lim_\alpha f_\alpha^\circ(x, y) = \phi_{\pi(E)}(x, y)$ for each $x, y \in \mathcal{A}$. Hence, $\phi_{\pi(E)}$ is regular.

(2) A set $\{f_\alpha\}$ of positive linear functionals on $\mathcal{A}$ defined by

$$f_\alpha(x) = (K_0\pi(K_\alpha)\lambda_\phi(x)|\lambda(K_\alpha))$$

for $x \in \mathcal{A}$ implies our assertion.

We can similarly show that $\phi_{\pi(E)}$ and ϕ_{K_0} are Σ -regular if ϕ is Σ -regular. This completes the proof.

LEMMA 5.5. *If ϕ is a Σ -regular (resp. regular) Riesz form on a commuta-*

tive $*$ -algebra $\mathcal{A}$, then every positive invariant sesquilinear form ψ on $\mathcal{A} \times \mathcal{A}$ with $\psi \leq \phi$ is Σ -regular (resp. regular).

PROOF. Suppose that ϕ is regular. By Proposition 5.2, it follows that $\pi(\mathfrak{U}'_\phi)$ is a nondegenerate $*$ -algebra on $\mathfrak{H}_\phi$, so that there exists a set $\{K_\alpha\}$ in $\mathfrak{U}'_\phi$ such that $\|\pi(K_\alpha)\| \leq 1$ for each α and $\{\pi(K_\alpha)\}$ converges strongly to I . Since

$$\begin{aligned} (\lambda_\phi(x^*) | \lambda_\phi(y^*)) &= \lim_{\alpha} (\pi(K_\alpha) \lambda_\phi(x^*) | \lambda_\phi(y^*)) \\ &= \lim_{\alpha} (\pi_\phi(x^*) \lambda(K_\alpha) | \lambda_\alpha(y^*)) \\ &= \lim_{\alpha} (\lambda(K_\alpha) | \lambda_\phi(xy^*)) \\ &= \lim_{\alpha} (\lambda(K_\alpha) | \lambda_\phi(y^*x)) \\ &= \lim_{\alpha} (\pi(K_\alpha) \lambda_\phi(y) | \lambda_\phi(x)) \\ &= (\lambda_\phi(y) | \lambda_\phi(x)) \end{aligned}$$

for each $x, y \in \mathcal{A}$, the map $\lambda_\phi(x) \rightarrow \lambda_\phi(x^*)$ is extended to a reflexive conjugate linear isometry J_ϕ on $\mathfrak{H}_\phi$. Furthermore, we have

$$\begin{aligned} (J_\phi \lambda(K) | \lambda_\phi(xy)) &= (\lambda_\phi(y^*x^*) | \lambda(K)) \\ &= (\pi(K)^* \lambda_\phi(x^*) | \lambda_\phi(y)) \\ &= (\lambda(K^+) | \lambda_\phi(xy)) \end{aligned}$$

for each $x, y \in \mathcal{A}$ and $K \in \mathfrak{U}'_\phi$. Since $[\pi(\mathfrak{U}'_\phi) \lambda_\phi(\mathcal{A})] = [\pi_\phi(\mathcal{A}) \lambda(\mathfrak{U}'_\phi)] \subset [\pi_\phi(\mathcal{A}) \mathcal{D}(\pi_\phi)] = [\lambda_\phi(\mathcal{A}^2)]$, we have $J_\phi \lambda(K) = \lambda(K^+)$ for each $K \in \mathfrak{U}'_\phi$. Take arbitrary $C \in \pi_\phi(\mathcal{A})'$ and $K \in \mathfrak{U}'_\phi$. Since $\lambda(K) \in \mathcal{D}(\pi_\phi)$ by the self-adjointness of π_ϕ , there exists a net $\{x_\beta\}$ in $\mathcal{A}$ such that $\{\lambda_\phi(x_\beta)\}$ converges to $\lambda(K)$ with respect to the $t_{\pi_\phi(\mathcal{A})}$ -topology. Then, by the above argument we have

$$\lim_{\beta} \lambda_\phi(x_\beta^*) = \lim_{\beta} J_\phi \lambda_\phi(x_\beta) = J_\phi \lambda(K) = \lambda(K^+).$$

Since

$$\begin{aligned} (C \pi(K) \lambda_\phi(x) | \lambda_\phi(y)) &= \lim_{\beta} (C \pi_\phi(x) \lambda_\phi(x_\beta) | \lambda_\phi(y)) \\ &= \lim_{\beta} (C \lambda_\phi(x) | \pi_\phi(y) \lambda_\phi(x_\beta^*)) \\ &= (\pi_\phi(y)^* C \lambda_\phi(x) | \lambda(K^+)) \\ &= (\pi(K) C \lambda_\phi(x) | \lambda_\phi(y)) \end{aligned}$$

for each $x, y \in \mathcal{A}$, we have $C \pi(K) = \pi(K) C$. It hence follows from Lemma 5.4 that a positive invariant sesquilinear form ψ on $\mathcal{A} \times \mathcal{A}$ with $\psi \leq \phi$ is regular (resp. Σ -regular) if ϕ is regular (resp. Σ -regular).

THEOREM 5.6. Let ϕ be a Riesz form on a commutative $*$ -algebra $\mathcal{A}$. Consider the following statements.

- (1) ϕ is regular.
- (2) ϕ is Σ -regular.
- (3) There exists a net $\{\xi_\alpha\}$ in $\mathcal{D}(\pi_\phi)$ such that
 - (3.1) $(\pi_\phi(x)\xi_\alpha|\pi_\phi(y)\xi_\beta)=0$ for each $x, y \in \mathcal{A}$ and $\alpha \neq \beta$;
 - (3.2) $\sum_\alpha \|\pi_\phi(x)\xi_\alpha\|^2 < \infty$ for each $x \in \mathcal{A}$;
 - (3.3) $\{\sum \pi_\phi(x)\xi_\alpha; x \in \mathcal{A}\}$ is dense in $\mathfrak{H}_\phi$;
 - (3.4) $\phi(x, y) = \sum_\alpha (\pi_\phi(x)\xi_\alpha|\pi_\phi(y)\xi_\alpha)$ for each $x, y \in \mathcal{A}$.
- (4) ϕ is nondegenerate; that is, $\lambda_\phi(\mathcal{A}^2)$ is total in $\mathfrak{H}_\phi$.
- (5) $\pi_\phi(\mathcal{A})'$ is an abelian von Neumann algebra.

Then the following implications hold:

$$\begin{array}{ccc} (1) & \Longleftrightarrow & (2) \Longleftrightarrow (3) \\ & \swarrow & \searrow \\ & (4) & (5). \end{array}$$

In particular, if π_ϕ is standard; that is, $\pi_\phi(x)^* = \overline{\pi_\phi(x^*)}$ for each $x \in \mathcal{A}$, then the statements (1)~(4) are equivalent. In this case, $\pi_\phi(\mathcal{A})'$ is a maximal abelian von Neumann algebra.

PROOF. The equivalence of (1)~(3) follows from Theorem 5.3 and Lemma 5.5.

(3) $\Rightarrow$ (4) This is proved in the proof of Lemma 5.5.

(1) $\Rightarrow$ (5) Since ϕ is regular, there exists a net $\{K_\alpha\}$ in $\mathfrak{U}'_\phi$ such that $\|\pi(K_\alpha)\| \leq 1$ for each α and $\{\pi(K_\alpha)\}$ converges strongly to I . In the proof of Lemma 5.5 we have seen that $\pi(K)C = C\pi(K)$ for each $K \in \mathfrak{U}'_\phi$ and $C \in \pi_\phi(\mathcal{A})'$. Furthermore, we note that $\pi(S)C\pi(K) \in \mathfrak{U}'_\phi$ for each $S, K \in \mathfrak{U}'_\phi$ and $C \in \pi_\phi(\mathcal{A})'$. Hence, we have

$$\begin{aligned} (C_1 C_2 \lambda_\phi(x) | \lambda_\phi(y)) &= \lim_{\alpha} (\pi(K_\alpha)^* C_1 \pi(K_\alpha) C_2 \lambda_\phi(x) | \lambda_\phi(y)) \\ &= \lim_{\alpha} (C_2 \pi(K_\alpha)^* C_1 \pi(K_\alpha) \lambda_\phi(x) | \lambda_\phi(y)) \\ &= (C_2 C_1 \lambda_\phi(x) | \lambda_\phi(y)) \end{aligned}$$

for each $C_1, C_2 \in \pi_\phi(\mathcal{A})'$ and $x, y \in \mathcal{A}$, and so $C_1 C_2 = C_2 C_1$. Hence, $\pi_\phi(\mathcal{A})'$ is an abelian von Neumann algebra.

(2) $\Rightarrow$ (3) Suppose that π_ϕ is standard and ϕ is non-degenerate. Take arbitrary $x \in \mathcal{A}$. Let $\overline{\pi_\phi(x)} = U |\pi_\phi(x)|$ be the polar decomposition and $|\pi_\phi(x)| = \int_0^\infty \lambda dE(\lambda)$ be the spectral resolution of $|\pi_\phi(x)|$. We now put

$$X_n = \int_{-n}^n \lambda dE(\lambda) \quad \text{and} \quad E_n = \int_{-n}^n dE(\lambda)$$

for $n=1, 2, \dots$. Since $\overline{\pi_\phi(x)}$ is affiliated with $\pi_\phi(\mathcal{A})'$ and $\pi_\phi(\mathcal{A})'' \subset \pi_\phi(\mathcal{A})'$ ([22] Theorem 7.1), we have $U, E_n, X_n \in \pi_\phi(\mathcal{A})'' \cap \pi_\phi(\mathcal{A})'$ for $n=1, 2, \dots$, which implies that $(UX_n, UE_n\lambda_\phi(x), (UE_n\lambda_\phi(x^*))^*) \in \mathfrak{A}'_\phi$ and $\lim_{n \rightarrow \infty} UX_n\lambda_\phi(y) = \lambda_\phi(xy)$ for each $y \in \mathcal{A}$. Hence, it follows since $\lambda_\phi(\mathcal{A}^2)$ is total in $\mathfrak{S}_\phi$ that ϕ is regular, so that $\pi_\phi(\mathcal{A})'$ is maximal abelian by ([22] Theorem 7.1) and (3) $\Rightarrow$ (5). This completes the proof.

EXAMPLES. (1) Every admissible continuous positive invariant sesquilinear form on a commutative locally convex $*$ -algebra with an approximate identity is Σ -regular.

(2) Let $\mathcal{A}$ be a locally convex $*$ -algebra with an approximate identity $\{e_\alpha\}$. Every continuous positive invariant sesquilinear form ϕ on $\mathcal{A} \times \mathcal{A}$ satisfying that $\phi(xe_\alpha, xe_\alpha) \leq \phi(x, x)$ for each α and $x \in \mathcal{A}$ is regular.

(3) Let $\mathcal{A}$ be a $*$ -algebra and ϕ be a positive invariant sesquilinear form on $\mathcal{A} \times \mathcal{A}$. If there exists a net $\{\phi_\alpha\}$ of admissible positive invariant sesquilinear forms on $\mathcal{A} \times \mathcal{A}$ such that $\phi_\alpha \leq \phi$ for each α and $\lim_\alpha \phi_\alpha(x, y) = \phi(x, y)$ for all $x, y \in \mathcal{A}$, then ϕ is called approximately admissible. Suppose that ϕ is an approximately admissible positive invariant sesquilinear form on $\mathcal{A} \times \mathcal{A}$ satisfying $\pi_\phi^* = \pi_\phi^{**}$. Then we see that there exists a set $\{E_\lambda\}_{\lambda \in A}$ of mutually orthogonal projections in $\pi_\phi(\mathcal{A})'$ such that $\sum_{\lambda \in A} E_\lambda = I$ and $\pi_\phi^*(x)E_\lambda$ is bounded for each $x \in \mathcal{A}$ and $\lambda \in A$. A locally convex topology on the closed O_p^* -algebra $\pi_\phi(\mathcal{A})$ generated by a family of seminorms $\{\|\cdot\|_\lambda; \lambda \in A\}$, where $\|\pi_\phi(x)\|_\lambda$ denotes the uniform norm $\|\pi_\phi^*(x)E_\lambda\|$ of $\pi_\phi^*(x)E_\lambda$, is said to be a locally uniform topology t_{lu} . If the map $\pi_\phi(x) \in (\pi_\phi(\mathcal{A}), t_{lu}) \rightarrow \lambda_\phi(x) \in \mathfrak{S}_\phi$ is continuous, then ϕ is Σ -regular. In fact, since $\pi_\phi(x) \rightarrow \lambda_\phi(x)$ is locally uniformly continuous, there exist a constant γ and a finite subset $\{\lambda_1, \dots, \lambda_n\}$ of A such that $\|\lambda_\phi(x)\| \leq \gamma \sum_{i=1}^n \|\pi_\phi^*(x)E_{\lambda_i}\|$ for all $x \in \mathcal{A}$. Take arbitrary $\lambda \in A$. Putting $E = \sum_{i=1}^n E_{\lambda_i} + E_\lambda$, we have $\|E\lambda_\phi(x)\| \leq (n+1)\gamma \|\pi_\phi^*(x)E\|$ for all $x \in \mathcal{A}$. Similarly to the proof (5) $\Rightarrow$ (2) of Theorem 4.4, we can show that there exists an element ξ of $E\mathfrak{S}_\phi$ such that $E\lambda_\phi(x) = \pi_\phi^*(x)\xi$ for all $x \in \mathcal{A}$. Hence, it follows since $\xi_\lambda = E_\lambda\xi \in E_\lambda\mathfrak{S}_\phi$ and $E_\lambda\lambda_\phi(x) = \pi_\phi^*(x)\xi_\lambda$ for all $x \in \mathcal{A}$ that $\lambda_\phi(x) = \sum_{\lambda \in A} \pi_\phi^*(x)\xi_\lambda$ for all $x \in \mathcal{A}$, which implies that ϕ is Σ -regular.

(4) Let $\mathcal{M}$ be a self-adjoint O_p^* -algebra on a dense subspace $\mathcal{D}$ in a Hilbert space $\mathfrak{H}$. A linear functional f on $\mathcal{M}$ is called weakly continuous (resp. σ -weakly continuous) if $|f(x)| \leq \sum_{i=1}^n (x\xi_i|\eta_i)$ (resp. $|f(x)| \leq \sum_{i=1}^\infty (x\xi_i|\eta_i)$) for all $x \in \mathcal{A}$, where $\{\xi_i\}, \{\eta_i\} \subset \mathcal{D}_\infty(\mathcal{M}) = \{\{\xi_i\} \subset \mathcal{D}; \sum_{i=1}^\infty \|\xi_i\|^2 < \infty \text{ and } \sum_{i=1}^\infty \|x\xi_i\|^2 < \infty \text{ for each } x \in \mathcal{M}\}$. Then we see that every weakly continuous (resp. σ -weakly continuous) linear functional on $\mathcal{M}$ is a linear combination of strongly regular (resp. Σ -regular) positive linear functionals on $\mathcal{M}$.

(5) Let ϕ be a weight on a C^* -algebra $\mathcal{A}$. Then a positive invariant sesquilinear form ϕ° on $\mathfrak{N}_\phi \times \mathfrak{N}_\phi$ is defined by $\phi^\circ(x, y) = \phi(y^*x)$ for $x, y \in \mathfrak{N}_\phi$,

where $\mathfrak{N}_\phi = \{x \in \mathcal{A}; \phi(x^*x) < \infty\}$. With the aid of Combes [4] and Pedersen, Takesaki [21] we shall show in Section 7 that ϕ is uniformly lower semi-continuous on $\mathcal{D}(\phi) = \{x \in \mathcal{A}; x \geq 0, \phi(x) < \infty\}$ if and only if ϕ° is regular if and only if ϕ° is Σ -regular. In particular, if ϕ is a positive C^* -integral on a C^* -algebra, then ϕ° is Σ -regular [20]. We also see by Haagerup [11] that if ϕ is a normal weight on a von Neumann algebra, then ϕ° is Σ -regular.

§ 6. Decomposition of positive invariant sesquilinear forms.

In this section we show that every positive invariant sesquilinear form ϕ satisfying $\pi_\phi^* = \pi_\phi^{**}$ is decomposed into the sum of the regular part and the singular part. Throughout this section let $\mathcal{A}$ be a $*$ -algebra without identity.

We first introduce the notion of singularity of positive invariant sesquilinear forms.

DEFINITION 6.1. Let ϕ be a positive invariant sesquilinear form on $\mathcal{A} \times \mathcal{A}$. If there does not exist any ϕ -majorized positive linear functional f such that $f^\circ \neq 0$, then ϕ is called singular. If $\phi(xy, z) = 0$ for each $x, y, z \in \mathcal{A}$; that is, $\pi_\phi(\mathcal{A}) = \{0\}$, then ϕ is called nilpotent. A positive linear functional f is called singular (resp. nilpotent) if f° is singular (resp. nilpotent).

PROPOSITION 6.2. Let ϕ be a positive invariant sesquilinear form on $\mathcal{A} \times \mathcal{A}$. Consider the following statements.

- (1) ϕ is nilpotent.
- (2) $\pi_\phi^*(x) \in (\pi_\phi^*(\mathcal{A}), t_w) \rightarrow \lambda_\phi(x) \in \mathfrak{S}_\phi$ is singular, where the weak topology t_w on $\pi_\phi^*(\mathcal{A})$ is a locally convex topology generated by the seminorms: $P_{\xi, \eta}(\pi_\phi^*(x)) = (\pi_\phi^*(x)\xi | \eta)$ for $\xi, \eta \in \mathcal{D}(\pi_\phi^*)$.
- (3) $P_{\mathfrak{S}'_\phi} = 0$.
- (4) ϕ is singular.

Then the following implications hold:

$$(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4).$$

In particular, if $\pi_\phi^* = \pi_\phi^{**}$, then the statements (3) and (4) are equivalent.

PROOF. (1) $\Rightarrow$ (2) This is trivial.

(2) $\Rightarrow$ (3) Since $\pi_\phi^*(x) \rightarrow \lambda_\phi(x)$ is singular, for each $x \in \mathcal{A}$ there exists a net $\{x_\alpha\}$ in $\mathcal{A}$ such that $\lim_\alpha (\pi_\phi^*(x_\alpha)\xi | \eta) = 0$ for each $\xi, \eta \in \mathcal{D}(\pi_\phi^*)$ and $\lim_\alpha \lambda_\phi(x_\alpha) = \lambda_\phi(x)$. Hence, we have

$$\begin{aligned} (\pi(K)\lambda_\phi(x) | \lambda_\phi(y)) &= \lim_\alpha (\pi(K)\lambda_\phi(x_\alpha) | \lambda_\phi(y)) \\ &= \lim_\alpha (\pi_\phi^*(x_\alpha)\lambda(K) | \lambda_\phi(y)) \\ &= 0 \end{aligned}$$

for each $K \in \mathfrak{U}'_\phi$ and $x, y \in \mathcal{A}$, which implies that $P_{\mathfrak{U}'_\phi} = 0$.

(3) $\Rightarrow$ (4) This follows from Lemma 4.2.

(4) $\Rightarrow$ (3) Suppose $\pi_\phi^* = \pi_\phi^{**}$. Take arbitrary $K \in \mathfrak{U}'_\phi$. We put

$$f(x) = (\pi_\phi^*(x)\lambda(K)|\lambda(K))$$

for $x \in \mathcal{A}$. Then it follows from $\pi_\phi^* = \pi_\phi^{**}$ that f is a ϕ -majorized positive linear functional on $\mathcal{A}$. Since ϕ is singular, we have $f^\circ = 0$. Hence, $\pi(K) = 0$, and so $P_{\mathfrak{U}'_\phi} = 0$.

Let $(\mathcal{M}, \mathcal{D})$ be a closed O_p^* -algebra with the identity operator. Then, a positive invariant sesquilinear form ϕ on a T^+ -algebra $\mathfrak{A} = \mathcal{M} \times \mathcal{D} \times \mathcal{D}^*$ defined by $\phi(A, B) = (\lambda(A)|\lambda(B))$ for $A, B \in \mathfrak{A}$ is nondegenerate and singular, and the restriction ϕ_0 of ϕ to $\mathfrak{U}_0$ is nilpotent, where $\mathfrak{U}_0 = \{0\} \times \mathcal{D} \times \mathcal{D}^*$. Tomita has given in [28] an example of a singular positive invariant sesquilinear form on the Schwartz space $\mathcal{S}(\mathbf{R}^n \times \mathbf{R}^n)$ which is regarded as the topological $*$ -algebra of continuous operators in $L^2(\mathbf{R}^n)$.

We proceed to a decomposition theorem of positive invariant sesquilinear forms. Let ϕ be a positive invariant sesquilinear form on $\mathcal{A} \times \mathcal{A}$. We denote by $P(\phi)$ the set of all positive invariant sesquilinear forms ψ on $\mathcal{A} \times \mathcal{A}$ with $\psi \leq \phi$. Then $(P(\phi), \leq)$ is an ordered set. We denote by $P_r(\phi)$ (resp. $P_s(\phi)$, $P_{ns}(\phi)$, $P_n(\phi)$) the set of all regular (resp. singular, nondegenerate and singular, nilpotent) positive invariant sesquilinear forms ψ on $\mathcal{A} \times \mathcal{A}$ with $\psi \leq \phi$.

THEOREM 6.3. *If ϕ is a positive invariant sesquilinear form on $\mathcal{A} \times \mathcal{A}$ satisfying $\pi_\phi^* = \pi_\phi^{**}$, then there exists a decomposition $\phi = \phi_r + \phi_s$ of ϕ such that $\phi_r \in P_r(\phi)$, $\phi_s \in P_s(\phi)$ and ϕ_r is maximal in the ordered set $(P_r(\phi), \leq)$. In particular, if ϕ is a Riesz form, then the above ϕ_s is decomposed into the sum $\phi_s = \phi_{ns} + \phi_n$, where $\phi_{ns} \in P_{ns}(\phi)$ and $\phi_n \in P_n(\phi)$. Furthermore, the above ϕ_r , ϕ_s , ϕ_{ns} and ϕ_n are Riesz forms.*

To prove the above theorem we prepare the following lemma.

LEMMA 6.4. *Let ϕ, ψ be positive invariant sesquilinear forms on $\mathcal{A} \times \mathcal{A}$ such that $\psi \leq \phi$. Put*

$$K_{\phi, \psi} \lambda_\phi(x) = \lambda_\psi(x)$$

for $x \in \mathcal{A}$. Then the following statements hold.

(1) $K_{\phi, \psi}$ is extended to a continuous linear transformation of $\mathfrak{S}_\phi$ into $\mathfrak{S}_\psi$, which is also denoted by $K_{\phi, \psi}$.

(2) $T_{\phi, \psi} = K_{\phi, \psi}^* K_{\phi, \psi} \in \pi_\phi(\mathcal{A})'$.

(3) $K_{\phi, \psi}^* \mathcal{D}(\pi_\psi^*) \subset \mathcal{D}(\pi_\phi^*)$ and $\pi_\phi^*(x) K_{\phi, \psi}^* \eta = K_{\phi, \psi}^* \pi_\psi^*(x) \eta$ for each $x \in \mathcal{A}$ and $\eta \in \mathcal{D}(\pi_\psi^*)$.

(4) $K_{\phi, \psi} \mathcal{D}(\pi_\phi^{**}) \subset \mathcal{D}(\pi_\psi^*)$ and $\pi_\psi^*(x) K_{\phi, \psi} \xi = K_{\phi, \psi} \pi_\phi^{**}(x) \xi$ for each $x \in \mathcal{A}$ and $\xi \in \mathcal{D}(\pi_\phi^{**})$.

(5) If ψ is regular, then $P_{\mathfrak{U}_\phi'} T_{\phi, \psi} = T_{\phi, \psi}$.

PROOF. (1), (2) This is trivial.

(3) This follows from the equality:

$$\begin{aligned} (\pi_\phi(x) \lambda_\phi(y) | K_{\phi, \psi}^* \eta) &= (\pi_\psi(x) \lambda_\psi(y) | \eta) \\ &= (\lambda_\phi(y) | K_{\phi, \psi}^* \pi_\psi^*(x)^* \eta) \end{aligned}$$

for each $x, y \in \mathcal{A}$ and $\eta \in \mathcal{D}(\pi_\psi^*)$.

(4) This follows immediately from the statement (3).

(5) Since ψ is regular, there exists a net $\{K_\alpha\}$ in $\mathfrak{U}_\psi'$ such that $0 \leq \pi(K_\alpha) \leq I$ and $\{\pi(K_\alpha)\}$ converges weakly to I . For each α and $x, y \in \mathcal{A}$ it follows from the statement (3) that

$$\begin{aligned} K_{\phi, \psi}^* \pi(K_\alpha) K_{\phi, \psi} \pi_\phi(x) \lambda_\phi(y) &= K_{\phi, \psi}^* \pi(K_\alpha) \pi_\psi(x) \lambda_\psi(y) \\ &= K_{\phi, \psi}^* \pi_\psi^*(x) \pi(K_\alpha) \lambda_\psi(y) \\ &= \pi_\phi^*(x) K_{\phi, \psi}^* \pi(K_\alpha) K_{\phi, \psi} \lambda_\phi(y) \end{aligned}$$

and

$$\begin{aligned} K_{\phi, \psi}^* \pi(K_\alpha) K_{\phi, \psi} \lambda_\phi(x) &= K_{\phi, \psi}^* \pi(K_\alpha) \lambda_\psi(x) \\ &= K_{\phi, \psi}^* \pi_\psi^*(x) \lambda(K_\alpha) \\ &= \pi_\phi^*(x) K_{\phi, \psi}^* \lambda(K_\alpha), \end{aligned}$$

which means that $(K_{\phi, \psi}^* \pi(K_\alpha) K_{\phi, \psi}, K_{\phi, \psi}^* \lambda(K_\alpha), (K_{\phi, \psi}^* \lambda(K_\alpha))^*) \in \mathfrak{U}_\phi'$. Hence, we have

$$\begin{aligned} (P_{\mathfrak{U}_\phi'} T_{\phi, \psi} \lambda_\phi(x) | \lambda_\phi(y)) &= \lim_{\alpha} (P_{\mathfrak{U}_\phi'} K_{\phi, \psi}^* \pi(K_\alpha) K_{\phi, \psi} \lambda_\phi(x) | \lambda_\phi(y)) \\ &= \lim_{\alpha} (K_{\phi, \psi}^* \pi(K_\alpha) K_{\phi, \psi} \lambda_\phi(x) | \lambda_\phi(y)) \\ &= (T_{\phi, \psi} \lambda_\phi(x) | \lambda_\phi(y)) \end{aligned}$$

for each $x, y \in \mathcal{A}$. Hence, $P_{\mathfrak{U}_\phi'} T_{\phi, \psi} = T_{\phi, \psi}$.

PROOF OF THEOREM 6.3. We put

$$\begin{aligned} \phi_r(x, y) &= (P_{\mathfrak{U}_\phi} \lambda_\phi(x) | \lambda_\phi(y)), \\ \phi_s(x, y) &= ((I - P_{\mathfrak{U}_\phi'}) \lambda_\phi(x) | \lambda_\phi(y)) \end{aligned}$$

for $x, y \in \mathcal{A}$. Since $P_{\mathfrak{U}_\phi'} \in \pi_\phi(\mathcal{A})'$, it is clear that $\phi_r, \phi_s \in P(\phi)$ and $\phi = \phi_r + \phi_s$.

We show that ϕ_r is regular. Since $\pi(\mathfrak{U}'_\phi)$ is a *-algebra on $\mathfrak{S}_\phi$, there exists a net $\{K_\alpha\}$ in $\mathfrak{U}'_\phi$ such that $\|\pi(K_\alpha)\| \leq 1$ for each α and $\{\pi(K_\alpha)\}$ converges strongly to $P_{\mathfrak{U}'_\phi}$. For each $x, y \in \mathcal{A}$ and $K \in \mathfrak{U}'_\phi$ we have

$$\begin{aligned} (\pi(K)P_{\mathfrak{U}'_\phi}\lambda_\phi(x)|\lambda_\phi(y)) &= (\lambda_\phi(x)|P_{\mathfrak{U}'_\phi}\pi(K^+)\lambda_\phi(y)) \\ &= (\lambda_\phi(x)|\pi(K^+)\lambda_\phi(y)) \\ &= (\pi(K)\lambda_\phi(x)|\lambda_\phi(y)), \end{aligned}$$

and hence $\pi(K)P_{\mathfrak{U}'_\phi} = \pi(K)$. We now put

$$f_\alpha(x) = (\pi(K_\alpha)\lambda_\phi(x)|\lambda(K_\alpha))$$

for $x \in \mathcal{A}$. Then it follows from $\pi_\phi^* = \pi_\phi'^*$ that

$$\begin{aligned} f_\alpha(x^*x) &= \|\pi(K_\alpha)\lambda_\phi(x)\|^2 = \|\pi(K_\alpha)P_{\mathfrak{U}'_\phi}\lambda_\phi(x)\|^2 \\ &\leq \|P_{\mathfrak{U}'_\phi}\lambda_\phi(x)\|^2 \\ &= \phi_r(x, x), \\ |f_\alpha(x)|^2 &\leq \|\lambda(K_\alpha)\|^2 \phi_r(x, x) \end{aligned}$$

for all $x \in \mathcal{A}$, which means that f_α is a ϕ_r -majorized positive linear functional on $\mathcal{A}$. Furthermore, since

$$\begin{aligned} \lim_\alpha f_\alpha^\circ(x, y) &= \lim_\alpha (\pi(K_\alpha)\lambda_\phi(x)|\pi(K_\alpha)\lambda_\phi(y)) \\ &= (P_{\mathfrak{U}'_\phi}\lambda_\phi(x)|\lambda_\phi(y)) \\ &= \phi_r(x, y) \end{aligned}$$

for all $x, y \in \mathcal{A}$, it follows that ϕ_r is regular.

We next show that ϕ_s is singular. Suppose that ϕ_s is not singular; that is, there exists a ϕ_s -majorized positive linear functional f on $\mathcal{A}$ such that $f^\circ \neq 0$. Since $\phi_s \leq \phi$, f is also ϕ -majorized. Hence, there exists an element K of $\mathfrak{U}'_\phi$ such that $\pi(K) \neq 0$ and $f(x) = (\lambda_\phi(x)|\lambda(K))$ for all $x \in \mathcal{A}$. Since f° is ϕ_s -dominated, we have

$$\|\pi(K)\lambda_\phi(x)\|^2 = f(x^*x) \leq \phi_s(x, x) = \|(I - P_{\mathfrak{U}'_\phi})\lambda_\phi(x)\|^2$$

for all $x \in \mathcal{A}$, and hence

$$\|\pi(K)\xi\| \leq \|(I - P_{\mathfrak{U}'_\phi})\xi\|$$

for all $\xi \in \mathfrak{S}_\phi$. Hence, $\pi(K)P_{\mathfrak{U}'_\phi} = 0$, and so $\pi(K) = 0$. This is a contradiction. Hence, ϕ_s is singular.

Take arbitrary $\psi \in P_r(\phi)$. It follows from Lemma 6.4, (5) that

$$\begin{aligned} \psi(x, x) &= (T_{\phi, \psi}\lambda_\phi(x)|\lambda_\phi(x)) = \|K_{\phi, \psi}P_{\mathfrak{U}'_\phi}\lambda_\phi(x)\|^2 \\ &\leq \|P_{\mathfrak{U}'_\phi}\lambda_\phi(x)\|^2 \\ &= \phi_r(x, x) \end{aligned}$$

for all $x \in \mathcal{A}$, which implies that ϕ_r is maximal in the ordered set $(P_r(\phi), \leq)$.

Suppose that ϕ is a Riesz form on $\mathcal{A}$. We denote by $P_{\mathfrak{A}_\phi}$ the projection of $\mathfrak{S}_\phi$ onto $[\pi_\phi(\mathcal{A})\mathcal{D}(\pi_\phi)]$. Then we see that $P_{\mathfrak{A}_\phi} \in \pi_\phi(\mathcal{A})' \cap \pi_\phi(\mathcal{A})''$ and $P_{\mathfrak{A}_\phi} \leq P_{\mathfrak{A}_\phi}$. We now put

$$\begin{aligned}\phi_{ns}(x, y) &= (P_{\mathfrak{A}_\phi}(I - P_{\mathfrak{A}_\phi})\lambda_\phi(x) | \lambda_\phi(y)), \\ \phi_n(x, y) &= ((I - P_{\mathfrak{A}_\phi})\lambda_\phi(x) | \lambda_\phi(y))\end{aligned}$$

for $x, y \in \mathcal{A}$. Then it is clear that $\phi_{ns}, \phi_n \in P(\phi_s)$ and $\phi_s = \phi_{ns} + \phi_n$. As for the case of ϕ_r, ϕ_s , we can prove that ϕ_{ns} is nondegenerate and singular, and ϕ_n is nilpotent. We see immediately that $\phi_r, \phi_s, \phi_{ns}$ and ϕ_n are Riesz forms. This completes the proof.

COROLLARY 6.5. *Let ϕ_1 and ϕ_2 be regular positive invariant sesquilinear forms on $\mathcal{A} \times \mathcal{A}$. Suppose that $\phi = \phi_1 + \phi_2$ satisfies $\pi_\phi^* = \pi_\phi^{**}$. Then ϕ is regular. Hence, if ϕ_1 and ϕ_2 are regular admissible positive invariant sesquilinear forms, then $\phi_1 + \phi_2$ is regular.*

PROOF. By Lemma 6.4 there exist positive operators T_1, T_2 in $\pi_\phi(\mathcal{A})'$ such that $T_i P_{\mathfrak{A}_\phi} = T_i$ and $\phi_i(x, y) = (T_i \lambda_\phi(x) | \lambda_\phi(y))$ for all $x, y \in \mathcal{A}$ ($i=1, 2$). Since $\phi = \phi_1 + \phi_2$, we have $T_1 + T_2 = I$, which implies that

$$\begin{aligned}\phi(x, y) &= \phi_1(x, y) + \phi_2(x, y) \\ &= (T_1 P_{\mathfrak{A}_\phi} \lambda_\phi(x) | \lambda_\phi(y)) + (T_2 P_{\mathfrak{A}_\phi} \lambda_\phi(x) | \lambda_\phi(y)) \\ &= (P_{\mathfrak{A}_\phi} \lambda_\phi(x) | \lambda_\phi(y)) \\ &= \phi_r(x, y)\end{aligned}$$

for all $x, y \in \mathcal{A}$. Hence, $\phi = \phi_r$, and so ϕ is regular.

We don't know whether the decomposition $\phi = \phi_r + \phi_s$ of Theorem 6.3 is unique, or not. However, we next show that the decomposition is unique in the case of commutative $*$ -algebras.

THEOREM 6.6. *Every Riesz form ϕ on a commutative $*$ -algebra $\mathcal{A}$ is uniquely decomposed into the sum $\phi = \phi_r + \phi_s$ of the regular Riesz form ϕ_r and the singular Riesz form ϕ_s .*

PROOF. We show the uniqueness of such a decomposition. Suppose that $\phi = \phi_1 + \phi_2$, where $\phi_1 \in P_r(\phi)$ and $\phi_2 \in P_s(\phi)$. Since $\phi_1 \leq \phi_r$ by Theorem 6.3, it follows that $\phi_r - \phi_1 \in P(\phi_r)$, $\phi_2 - \phi_s \in P(\phi_2)$ and $\phi_r - \phi_1 = \phi_2 - \phi_s$. By Lemma 5.5, $\phi_r - \phi_1$ is regular and $\phi_2 - \phi_s$ is singular. Hence, $\phi_r = \phi_1$ and $\phi_s = \phi_2$. This completes the proof.

REMARK. A positive linear functional f on a $*$ -algebra satisfying $\pi_f^* =$

π_f^{**} is decomposed into the sum $f=f_r+f_s$ of a regular positive linear functional f_r and a singular positive linear functional. If f is a Riesz functional on a commutative $*$ -algebra, then the above decomposition is unique. In fact, since $\pi(\mathfrak{A}_f)$ is a $*$ -algebra on $\mathfrak{S}_f$, there exists a net $\{K_\alpha\}$ in $\mathfrak{A}_f$ such that $\|\pi(K_\alpha)\| \leq 1$ for each α and $\{\pi(K_\alpha)\}$ converges strongly to $P_{\mathfrak{A}_f}$. We now put

$$f_r(x) = \lim_{\alpha} (\pi(K_\alpha) \lambda_f(x) | \lambda(K_\alpha)), \quad f_s(x) = f(x) - f_r(x)$$

for $x \in \mathcal{A}$. Then we have

$$f_r(y^*x) = (P_{\mathfrak{A}_f} \lambda_f(x) | \lambda_f(y)), \quad f_s(y^*x) = ((I - P_{\mathfrak{A}_f}) \lambda_f(x) | \lambda_f(y))$$

for all $x, y \in \mathcal{A}$. Hence, it follows from the proof of Theorem 6.3 that f_r is a regular positive linear functional and f_s is a singular positive linear functional.

§ 7. Applications.

In this section we investigate weights on C^* -algebras by applying our study.

Let $\mathcal{A}$ be a C^* -algebra and $\mathcal{A}_+$ be the positive part of $\mathcal{A}$. A function ϕ of $\mathcal{A}_+$ into $[0, \infty]$ is said to be a weight on $\mathcal{A}$ if

- (1) $\phi(x+y) = \phi(x) + \phi(y)$ for $x, y \in \mathcal{A}_+$,
- (2) $\phi(\lambda x) = \lambda \phi(x)$ for $\lambda \geq 0$ and $x \in \mathcal{A}_+$,

where $0 \cdot \infty = 0$. Let ϕ be a weight on $\mathcal{A}$. We denote by $\mathfrak{N}_\phi$ the set $\{x \in \mathcal{A}; \phi(x^*x) < \infty\}$ and denote by $\mathfrak{M}_\phi$ the $*$ -algebra $\mathfrak{N}_\phi^* \mathfrak{N}_\phi$. It is well known in the theory of weights that $\mathfrak{N}_\phi$ is a left ideal in $\mathcal{A}$ and $\mathfrak{M}_\phi$ is a $*$ -subalgebra spanned by $\mathcal{D}(\phi) = \{x \in \mathcal{A}_+; \phi(x) < \infty\}$. The weight ϕ has a unique extension, which we still denote by ϕ , to a linear functional on $\mathfrak{M}_\phi$. We now put

$$\phi^\circ(x, y) = \phi(y^*x)$$

for $x, y \in \mathfrak{N}_\phi$. Then ϕ° is an admissible positive invariant sesquilinear form on $\mathfrak{N}_\phi \times \mathfrak{N}_\phi$. We show later on that for the positive invariant sesquilinear form ϕ° generated by a weight ϕ on a C^* -algebra the regularity and the Σ -regularity are equivalent. We denote by $(\pi_\phi, \lambda_\phi, \mathfrak{S}_\phi)$ the GNS-construction of ϕ° . Since $\mathfrak{N}_\phi$ is a left ideal in $\mathcal{A}$, the representation π_ϕ of $\mathfrak{N}_\phi$ implies a representation of $\mathcal{A}$ on $\mathfrak{S}_\phi$ (again denoted by π_ϕ) by

$$\pi_\phi(a) \lambda_\phi(x) = \lambda_\phi(ax)$$

for $a \in \mathcal{A}$ and $x \in \mathfrak{N}_\phi$.

DEFINITION 7.1. Let ϕ_1, ϕ_2 be weights on a C^* -algebra $\mathcal{A}$. If $\phi_1(x) \leq \phi_2(x)$

for each $x \in \mathcal{A}_+$, then we say that ϕ_1 is dominated by ϕ_2 , which is denoted by $\phi_1 \leq \phi_2$. If $\mathcal{D}(\phi_2) \supset \mathcal{D}(\phi_1)$ and $\phi_1(x) = \phi_2(x)$ for all $x \in \mathcal{D}(\phi_1)$, then we say that ϕ_2 is an extension of ϕ_1 , which is denoted by $\phi_1 \subset \phi_2$.

Let $\{f_\alpha\}$ be a set of positive linear functionals of a C^* -algebra $\mathcal{A}$. We note that $\sum_\alpha f_\alpha$ and $\sup_\alpha f_\alpha$ are weights on $\mathcal{A}$.

DEFINITION 7.2. Let ϕ be a weight on a C^* -algebra $\mathcal{A}$. If there exists a positive linear functional f on $\mathcal{A}$ such that $f \supset \phi$, then ϕ is called strongly regular. If there exists a set $\{f_\alpha\}$ of positive linear functionals on $\mathcal{A}$ such that $\sum_\alpha f_\alpha \supset \phi$ (resp. $\sup_\alpha f_\alpha \supset \phi$), then ϕ is Σ -regular (resp. regular). If there does not exist any positive linear functional f on $\mathcal{A}$ such that the restriction $f|_{\mathcal{D}(\phi)}$ of f to $\mathcal{D}(\phi)$ is non-zero and $f \leq \phi$, then ϕ is called singular.

We note that a weight ϕ on a C^* -algebra is strongly regular if and only if the positive invariant sesquilinear form ϕ° on $\mathfrak{N}_\phi \times \mathfrak{N}_\phi$ is strongly regular. For the regularity and the Σ -regularity of a weight on a C^* -algebra we have the following result.

THEOREM 7.3. *Let ϕ be a weight on a C^* -algebra $\mathcal{A}$. Then the following statements are equivalent.*

- (1) ϕ is regular.
- (2) ϕ° is a regular positive invariant sesquilinear form on $\mathfrak{N}_\phi \times \mathfrak{N}_\phi$.
- (3) ϕ is Σ -regular.
- (4) ϕ° is a Σ -regular positive invariant sesquilinear form on $\mathfrak{N}_\phi \times \mathfrak{N}_\phi$.
- (5) ϕ is uniformly lower semicontinuous on $\mathcal{D}(\phi)$.

PROOF. (1) $\Rightarrow$ (2) It is sufficient to prove $P_{\mathfrak{A}'_\phi} = I$, where $\mathfrak{A}'_\phi$ denotes the commutant of the T^+ -algebra $\mathfrak{A}_\phi$ generated by ϕ° . Suppose that $(I - P_{\mathfrak{A}'_\phi})\lambda_\phi(a) \neq 0$ for some $a \in \mathfrak{N}_\phi$. Since ϕ is regular, for each $\varepsilon > 0$ there exists a positive linear functional f on $\mathcal{A}$ such that $f \leq \phi$ and $f(a^*a) + \varepsilon > \phi(a^*a)$. Then we note that f is ϕ° -majorized. Hence, there exists an element K of $\mathfrak{A}'_\phi$ such that $0 \leq \pi(K) \leq I$ and $f(x) = (\lambda_\phi(x) | \lambda(K))$ for each $x \in \mathfrak{N}_\phi$. Furthermore, since $\pi(K)P_{\mathfrak{A}'_\phi} = \pi(K)$, it follows that

$$\begin{aligned} (P_{\mathfrak{A}'_\phi}\lambda_\phi(a) | \lambda_\phi(a)) + \varepsilon &\geq (\pi(K)P_{\mathfrak{A}'_\phi}\lambda_\phi(a) | \lambda_\phi(a)) + \varepsilon \\ &= (\pi(K)\lambda_\phi(a) | \lambda_\phi(a)) + \varepsilon \\ &= f(a^*a) + \varepsilon \\ &> \phi(a^*a) = \|\lambda_\phi(a)\|^2, \end{aligned}$$

so that $\|(I - P_{\mathfrak{A}'_\phi})\lambda_\phi(a)\| < \varepsilon$. Hence, $(I - P_{\mathfrak{A}'_\phi})\lambda_\phi(a) = 0$, which is a contradiction. Hence, $P_{\mathfrak{A}'_\phi} = I$.

(2) $\Rightarrow$ (3) By Proposition 5.2 there exists a net $\{K_\alpha\}$ in $\mathfrak{A}'_\phi$ such that

$\|\pi(K_\alpha)\| \leq 1$ for each α and $\{\pi(K_\alpha)\}$ converges strongly to I . Then we have

$$\phi(x) = \sup_{\alpha} f_{\alpha}(x)$$

for each $x \in \mathcal{D}(\phi)$, where $f_{\alpha}(a) = (\pi_{\phi}(a)\lambda(K_{\alpha})|\lambda(K_{\alpha}))$ for $a \in \mathcal{A}$. We denote by $\tilde{\mathcal{A}}$ the universal enveloping von Neumann algebra of $\mathcal{A}$. If f is a positive linear functional on $\mathcal{A}$, then there exists a unique normal positive linear functional $\tilde{f}$ on $\tilde{\mathcal{A}}$ which is an extension of f . We now put

$$\tilde{\phi}(x) = \sup_{\alpha} \tilde{f}_{\alpha}(x)$$

for $x \in \tilde{\mathcal{A}}$. Then $\tilde{\phi}$ is a normal weight on the von Neumann algebra $\tilde{\mathcal{A}}$ which is an extension of ϕ . By ([21] Theorem 7.3) there exists a set $\{\omega_{\lambda}\}$ of normal positive linear functionals on $\tilde{\mathcal{A}}$ such that $\tilde{\phi}(x) = \sum_{\lambda} \omega_{\lambda}(x)$ for each $x \in \tilde{\mathcal{A}}_+$. Hence, we have

$$\phi(x) = \tilde{\phi}(x) = \sum_{\lambda} \omega_{\lambda}(x)$$

for each $x \in \mathcal{D}(\phi)$, which implies that ϕ is Σ -regular.

(3) $\Rightarrow$ (4) Suppose $\sum_{\alpha} f_{\alpha} \supset \phi$, where $\{f_{\alpha}\}$ is a set of positive linear functionals on $\mathcal{A}$. It is well known that every positive linear functional on a C^* -algebra is strongly regular. It hence follows that f_{α} is a ϕ° -majorized positive linear functional on $\mathfrak{N}_{\phi}$ and $\sum_{\alpha} f_{\alpha}^{\circ}(x, y) = \phi^{\circ}(x, y)$ for each $x, y \in \mathfrak{N}_{\phi}$, so that ϕ° is Σ -regular.

(4) $\Rightarrow$ (5) Since ϕ° is Σ -regular, there exists a subset $\{\xi_{\alpha}\}$ of $\mathfrak{F}_{\phi}$ such that $\phi^{\circ}(x, y) = \sum_{\alpha} (\pi_{\phi}(x)\xi_{\alpha}|\pi_{\phi}(y)\xi_{\alpha})$ for each $x, y \in \mathfrak{N}_{\phi}$. We now put

$$f_{\alpha}(x) = (\pi_{\phi}(x)\xi_{\alpha}|\xi_{\alpha})$$

for $x \in \mathcal{A}$. Then f_{α} is a positive linear functional on $\mathcal{A}$ and $\sum_{\alpha} f_{\alpha} \supset \phi$. We note that the weight $\sum_{\alpha} f_{\alpha}$ on $\mathcal{A}$ is uniformly lower semicontinuous. Hence, ϕ is uniformly lower semicontinuous on $\mathcal{D}(\phi)$.

(5) $\Rightarrow$ (1) This follows from ([4] Proposition 1.7).

The following corollary follows from the equivalence of (1) and (5) in Theorem 7.3.

COROLLARY 7.4. *If ϕ_1 and ϕ_2 are regular weights on a C^* -algebra $\mathcal{A}$, then $\phi_1 + \phi_2$ is a regular weight on $\mathcal{A}$.*

PROPOSITION 7.5. *Let ϕ be a weight on a C^* -algebra $\mathcal{A}$. Then ϕ is singular if and only if ϕ° is a singular positive invariant sesquilinear form on $\mathfrak{N}_{\phi} \times \mathfrak{N}_{\phi}$.*

PROOF. Suppose that ϕ° is not singular. Then there exists a ϕ° -

majorized positive linear functional g on $\mathfrak{N}_\phi$ such that $g/\mathcal{D}(\phi) \neq 0$. By Lemma 4.2 there exists an element K of $\mathfrak{A}'_\phi$ such that $\pi(K) \neq 0$ and $g(x) = (\lambda_\phi(x) | \lambda(K))$ for all $x \in \mathfrak{N}_\phi$, where $\mathfrak{A}_\phi$ is a T^+ -algebra generated by ϕ° . We now define a positive linear functional f on $\mathcal{A}$ by

$$f(x) = \frac{1}{\|\pi(K)\|^2} (\pi_\phi(x) \lambda(K) | \lambda(K))$$

for $x \in \mathcal{A}$. Then we have

$$\begin{aligned} f(x^*x) &= \frac{1}{\|\pi(K)\|^2} \|\pi_\phi(x) \lambda(K)\|^2 \\ &= \frac{1}{\|\pi(K)\|^2} \|\pi(K) \lambda_\phi(x)\|^2 \\ &\leq \phi(x^*x) \end{aligned}$$

for all $x \in \mathfrak{N}_\phi$ and $f/\mathcal{D}(\phi) \neq 0$ by $\pi(K) \neq 0$, which implies that ϕ is not singular.

Suppose that ϕ is not singular; that is, there exists a positive linear functional f on $\mathcal{A}$ such that $f/\mathcal{D}(\phi) \neq 0$ and $f \leq \phi$. Then we see that $f/\mathfrak{N}_\phi$ is a ϕ° -majorized positive linear functional on $\mathfrak{N}_\phi$ and $(f/\mathfrak{N}_\phi)^\circ \neq 0$, which implies that ϕ° is not singular. This completes the proof.

We finally show a decomposition theorem for a weight ϕ on a C^* -algebra by making use of the decomposition theorem (Theorem 6.3) for the positive invariant sesquilinear form ϕ° .

THEOREM 7.6. *Let ϕ be a weight on a C^* -algebra $\mathcal{A}$. Then there exist a regular weight ϕ_r on $\mathcal{A}$ and a singular weight ϕ_s on $\mathcal{A}$ such that $\phi = \phi_r + \phi_s$ and ϕ_r is maximal in the ordered set $(W_r(\phi), \leq)$, where $W_r(\phi)$ denotes the set of all regular weights ψ on $\mathcal{A}$ with $\psi \leq \phi$.*

PROOF. We denote by $\mathfrak{A}_\phi$ the T^+ -algebra generated by the positive invariant sesquilinear form ϕ° on $\mathfrak{N}_\phi \times \mathfrak{N}_\phi$. Since $\pi(\mathfrak{A}'_\phi)$ is a $*$ -algebra on $\mathfrak{H}_\phi$, there exists a set $\{K_\alpha\}$ in $\mathfrak{A}'_\phi$ such that $\|\pi(K_\alpha)\| \leq 1$ for each α and $\{\pi(K_\alpha)\}$ converges strongly to $P_{\mathfrak{A}'_\phi}$. For each $x \in \mathcal{A}_+$ we put

$$\phi_r(x) = \begin{cases} \lim_{\alpha} (\pi(K_\alpha) \lambda_\phi(x) | \lambda(K_\alpha)), & x \in \mathcal{D}(\phi) \\ \infty & , x \notin \mathcal{D}(\phi). \end{cases}$$

For each $x \in \mathcal{D}(\phi)$ we have

$$\begin{aligned} \phi_r(x) &= \lim_{\alpha} (\pi(K_\alpha) \lambda_\phi(x) | \lambda(K_\alpha)) \\ &= \lim_{\alpha} \|\pi(K_\alpha) \lambda_\phi(x^{1/2})\|^2 \\ &= \|P_{\mathfrak{A}'_\phi} \lambda_\phi(x^{1/2})\|^2. \end{aligned}$$

We show that $\phi_r(x+y) = \phi_r(x) + \phi_r(y)$ for each $x, y \in \mathcal{A}_+$. In fact, since $\phi(x+y) = \phi(x) + \phi(y)$, it follows that $(x+y)^{1/2} \in \mathfrak{N}_\phi$ if and only if $x^{1/2} \in \mathfrak{N}_\phi$ and $y^{1/2} \in \mathfrak{N}_\phi$. Hence we have that if $(x+y)^{1/2} \in \mathfrak{N}_\phi$ then

$$\begin{aligned}\phi_r(x+y) &= \lim_a (\pi(K_a)\lambda_\phi(x+y)|\lambda(K_a)) \\ &= \lim_a \{(\pi(K_a)\lambda_\phi(x)|\lambda(K_a)) + (\pi(K_a)\lambda_\phi(y)|\lambda(K_a))\} \\ &= \phi_r(x) + \phi_r(y);\end{aligned}$$

and if $(x+y)^{1/2} \notin \mathfrak{N}_\phi$, then $\phi_r(x+y) = \phi_r(x) + \phi_r(y) = \infty$. It is clear that $\phi_r(\lambda x) = \lambda \phi_r(x)$ for each $\lambda \geq 0$ and $x \in \mathcal{A}_+$. Hence, ϕ_r is a weight on $\mathcal{A}$ with $\mathfrak{N}_{\phi_r} = \mathfrak{N}_\phi$. For each $x, y \in \mathfrak{N}_\phi$ we have

$$\begin{aligned}\phi_r^\circ(x, y) &= \lim_a (\pi(K_a)\lambda_\phi(x)|\pi(K_a)\lambda_\phi(y)) \\ &= (P_{\mathfrak{N}_\phi}\lambda_\phi(x)|\lambda_\phi(y)).\end{aligned}$$

It hence follows from Theorem 6.3 that ϕ_r° is a regular positive invariant sesquilinear form on $\mathfrak{N}_\phi \times \mathfrak{N}_\phi$, and so ϕ_r is regular by Theorem 7.3. For each $x \in \mathcal{A}^+$ we put

$$\phi_s(x) = \begin{cases} \phi(x) - \phi_r(x), & x \in \mathcal{D}(\phi) \\ \infty & , \quad x \notin \mathcal{D}(\phi). \end{cases}$$

It follows since $\phi_r \leq \phi$ that ϕ_s is a weight on $\mathcal{A}$ satisfying that $\mathfrak{N}_{\phi_s} = \mathfrak{N}_\phi$ and $\phi_s^\circ(x, y) = ((I - P_{\mathfrak{N}_\phi})\lambda_\phi(x)|\lambda_\phi(y))$ for each $x, y \in \mathfrak{N}_\phi$. Hence, it follows from Theorem 6.3 and Proposition 7.5 that ϕ_s is singular and $\phi = \phi_r + \phi_s$.

Suppose that ψ is a regular weight on $\mathcal{A}$ with $\psi \leq \phi$. Since $\mathfrak{N}_\phi \subset \mathfrak{N}_\psi$, we see that the restriction $\psi^\circ/\mathfrak{N}_\phi \times \mathfrak{N}_\phi$ of the positive invariant sesquilinear form ψ° on $\mathfrak{N}_\psi \times \mathfrak{N}_\psi$ is a regular positive invariant sesquilinear form on $\mathfrak{N}_\phi \times \mathfrak{N}_\phi$. It hence follows from Theorem 6.3 that $\psi^\circ/\mathfrak{N}_\phi \times \mathfrak{N}_\phi \leq \phi_r^\circ$, which implies $\psi \leq \phi_r$. This completes the proof.

We remark that if ϕ is a weight on a C^* -algebra $\mathcal{A}$, then $\sup_{f \in \mathcal{F}} f \supset \phi_r$ by Theorem 7.6, where $\mathcal{F}$ denotes the set of all positive linear functionals f on $\mathcal{A}$ with $f \leq \phi$.

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ATSUSHI INOUE
DEPARTMENT OF APPLIED MATHEMATICS
FACULTY OF SCIENCE
FUKUOKA UNIVERSITY
FUKUOKA 814-01
JAPAN