Thermal convection inside an oscillating cube analysed with proper orthogonal decomposition

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Abstract

We reveal the fundamental and dominant flow structures of thermal convection in a cubic cavity under forced oscillation heated differentially by analysing the flow field with the proper orthogonal decomposition (referred to as POD). The database is made of consecutive series of three-dimensional results obtained by the direct numerical simulation based on the Boussinesq approximation for a forcedly-oscillating cube under the zero-gravity environment, at vibrational Rayleigh number (the Rayleigh number based on the cavity's acceleration amplitude instead of the gravitational acceleration) $Ra_\eta = 5.0 \times 10^4 - 1.1 \times 10^5$, Prandtl number $Pr = 7.1$ (water) and non-dimensional forced-oscillating frequency $\omega = 1.0 \times 10^0 - 2.0 \times 10^2$. The direction of the forced sinusoidal oscillation is parallel to the temperature gradient. It appears that the most energetic POD modes, or the first POD eigenfunctions with large eigenvalues, account for the transient process on flow structures during one forcing cycle. The first eigenfunctions correspond to the steady and laminar flow structures which appear inside a non-oscillating cube in the terrestrial environments. The POD expansion coefficient is found to be useful for predicting a consecutive series of the dominant flow structures.

Key words : Rayleigh-Bénard convection, Forced vibration, POD, PCA, KL expansion, Laminar flow, Unsteady flow

1. Introduction

Thermal convection has been a key phenomenon of interest in the area of heat and mass transfer. Numerous researchers have studied about thermal convection. —See Bénard (1900), Davis (1967) and Stork & Müller (1972) for a cubic cavity.— Furthermore, thermal convection is treated in various fields such as geology, meteorology, nuclear engineering, material mixings in many appliances. Especially not in the terrestrial-gravity environment but in low-gravity or microgravity environments, we can expect that the reduction/elimination of thermal convection could enhance the properties and the performances of materials such as crystals and composites. However, aboard orbiting spacecraft, all objects experience low-amplitude broad-band perturbed accelerations or *g*-jitters, caused by crew activities, orbiter maneuvers, equipment vibrations, solar drag, space-debris impacts and other sources. Therefore, understanding the effects of these perturbations on the system behaviour is important. —See Kamotani et al. (1981), Biringen & Danabasoglu (1990), Gershuni et al. (1988), Hirata et al. (2001) and Shevtsova et al. (2010).—

In general, it is difficult to examine unsteady three-dimensional flow. A candidate technique to analyse this type of flow is the proper orthogonal decomposition (hereinafter, referred to as POD), which is also called as the principal component analysis (PCA) or Karhunen-Loève (KL) expansion in other fields. Among other linear decompositions, the POD is optimal in such a sense as it provides a set of modes that captures the largest amount of energy contained in the snapshot ensemble. Here, a series of raw data is decomposed into a number of modes which make up an orthonormal basis spanning the entire data set. Based on energy considerations, the POD captures the most energetic and hence largest structures of the flow in the first modes. Thus, if the dynamics of the flow is dominated by a few dominant flow

structures, the data can often be represented satisfactorily using only a few of the first modes. We expect that the most important modes could reflect the dominating flow structures. It is usually not possible to have sufficiently high time resolution to get several snapshots of the concerning flow. Furthermore, a snapshot of a given flow may contain random fluctuations due to turbulence and may be a superposition of several coherent flow structures. Hence, it is necessary for us to use some kind of statistical analysis of the data to extract flow structures effectively. In contrast to methods like conditional sampling, the POD does not need any assumptions about flow. Thus, the analyses using the POD have previously been applied to various experimental and computational outputs (Mayer et al., 2007).

In the present study, we numerically investigate the influence of the forced oscillation upon the three-dimensional thermal convection in a cubic cavity heated from one wall and chilled from its opposite wall in the non-gravitational field with a vibrational Rayleigh number $Ra_\eta = 5.0 \times 10^4 - 1.1 \times 10^5$, a Prandtl number $Pr = 7.1$ (water) and non-dimensional forced-oscillation frequency $\omega = 1.0 \times 10^0 - 2.0 \times 10^2$. The direction of a forced sinusoidal oscillation is parallel to the temperature gradient. The full Navier-Stokes equations in three-dimensional Cartesian coordinates are simplified by introducing the Boussinesq approximation. These simplified equations are solved by using the finite difference method (Hirata et al., 2006, Tanigawa et al., 2009, Hirata et al., 2013). We especially focus upon the identification of the periodically-fluctuating flow during one forcing cycle by means of the use of POD, in order to deduce fundamental and dominant flow structures in transient process of thermal convection.

2. Computational method

2.1 Model, governing equation and numerical set-up

Figure 1 shows the present model, namely, the computational domain to be analysed, together with the present coordinate system and boundary conditions. The model is the flow in a cubic cavity with a length scale H^* . We analyse the thermal convection assuming incompressible flow with a constant Prandtl number $Pr = 7.1$ (water) in the non-gravitational field. A pair of opposed walls are taken to be isothermal, and the temperature one of them (hereinafter, referred to as a cold wall) is greater than the other (hereinafter, referred to as a hot wall). The four walls besides the cold and hot walls (hereinafter, referred to as sidewalls) are conductive. The present coordinate system is a 3D Cartesian O-xyz. The origin O is at a corner of the cubic cavity on the hot wall. x and y axes are perpendicular to the temperature gradient. And, the z axis is parallel to the temperature gradient, which is the same as the direction of the forced oscillation.

The governing equations are the Navier-Stokes equations with the Boussinesq approximation and an energy equation, which are dimensionless. They are as follows;

$$\nabla \cdot \mathbf{u} = 0, \quad (1)$$

$$\frac{D\mathbf{u}}{Dt} = -\nabla p + Pr\Delta\mathbf{u} + Ra_\eta Pr T \sin \omega t \mathbf{e}_z \quad (2)$$

and

Fig. 1 Computational domain, together with coordinate system and boundary conditions.

$$\frac{DT}{Dt} = \Delta T, \quad (3)$$

where $\mathbf{u}$, t , p , T , $\mathbf{e}_z$ and ω denote flow velocity vector, time, pressure, temperature, the unit vector in the z direction and (angular) frequency, respectively. We will give the definitions of Ra_η and Pr in the following subsection.

Independent variables are non-dimensionalised as follows;

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{H^*} \begin{pmatrix} x^* \\ y^* \\ z^* \end{pmatrix} \quad (4)$$

and

$$t = \frac{t^*}{H^{*2} / \alpha^*}, \quad (5)$$

where α^* denotes the coefficient of thermal diffusivity. A superscript “*” represents dimensional quantities.

As well, dependent variables are non-dimensionalised as follows;

$$\mathbf{u} = \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \frac{H^*}{\alpha^*} \begin{pmatrix} u^* \\ v^* \\ w^* \end{pmatrix}, \quad T = \frac{T^* - T_c^*}{T_h^* - T_c^*} \quad \text{and} \quad p = \frac{p^* H^{*2}}{\rho^* \alpha^{*2}}, \quad (6)$$

where T_c^* , T_h^* and ρ^* denote a cold-wall temperature, a hot-wall temperature and mean density, respectively.

The governing equations (1) – (3) are solved by a finite difference method based on the MAC scheme for velocity-pressure coupling using a staggered computational grid. As there might exist unknown factors in non-linear problems, we have conducted many preliminary computations, prior to main ones in order to ensure the numerical accuracy, as well as Hirata et al. (2006), Tanigawa et al. (2009) and Hirata et al. (2013). Concerning the mesh size, we adopt 81^3 for the main computations. Concerning the time step Δt , most of the main computations are conducted with $\Delta t = 5.0 \times 10^{-7}$. Incidentally, even in some cases of the main computations, we further check the Δt effect upon numerical accuracy. Concerning the spin-up time, we always discuss the results at $t > 20$. Because, we have confirmed that they are strictly periodic.

2.2 Governing parameter

Non-dimensional governing parameters are as follows.

$$\text{Vibrational Rayleigh number based on the acceleration amplitude } \eta^*: \quad Ra_\eta = \frac{\eta^* \beta^* (T_h^* - T_c^*) H^{*3}}{\nu^* \alpha^*}. \quad (7)$$

$$\text{Prandtl number:} \quad Pr = \frac{\nu^*}{\alpha^*}. \quad (8)$$

$$\text{And, non-dimensional (angular) frequency:} \quad \omega = \frac{\omega^* H^{*2}}{\alpha^*}. \quad (9)$$

Here, β^* , ν^* and η^* denote thermal expansion coefficient, kinetic viscosity and acceleration amplitude, respectively. In the present study, Pr is fixed to 7.1 (water). The test ranges of Ra_η and ω are from 5.0×10^4 to 1.1×10^5 and from 1.0×10^0 to 2.0×10^2 , respectively.

Under a constant gravity like the terrestrial gravity, as governing parameters, we usually assume the Rayleigh number Ra , Pr , ω and a non-dimensional acceleration amplitude η , instead of Ra_η , Pr and ω (Hirata et al., 2006). We should note that zero gravity is represented by $Ra = 0$. And, non-forced oscillation is represented by $\eta = 0$. Ra and η are related with Ra_η by the following equation.

$$Ra \eta = Ra_\eta. \quad (10)$$

2.3 Boundary condition

The boundary conditions for flow velocity are as follows.

$$u = v = w = 0. \quad (\text{on all the walls}) \quad (11)$$

The boundary conditions for pressure are as follows.

$$\frac{\partial p}{\partial x} = Pr \frac{\partial^2 u}{\partial x^2}, \quad (\text{at } x = 0 \text{ and } x = 1) \quad (12)$$

$$\frac{\partial p}{\partial y} = Pr \frac{\partial^2 v}{\partial y^2} \quad (\text{at } y = 0 \text{ and } y = 1) \quad (13)$$

and

$$\frac{\partial p}{\partial z} = Pr \frac{\partial^2 w}{\partial z^2} + Ra_\eta Pr T \sin \omega t. \quad (\text{at } z = 0 \text{ and } z = 1) \quad (14)$$

The boundary conditions for temperature are as follows.

$$T = 1, \quad (\text{at } z = 0) \quad (15)$$

$$T = 0 \quad (\text{at } z = 1) \quad (16)$$

and

$$\frac{\partial T}{\partial z} = -1 \quad (\text{at } x = 0, x = 1, y = 0 \text{ and } y = 1) \quad (17)$$

In Eq. (17), we consider thermally-conductive conditions on the four sidewalls.

2.4 Global indicator

As a global indicator, we consider such a physical quantity as a spatially-averaged kinetic energy $\bar{K}$. Its definition is as follows.

$$\bar{K} = \frac{1}{V} \iiint K dx dy dz, \quad (18)$$

where K is a local kinetic energy defined as

$$K = \frac{1}{2} (u^2 + v^2 + w^2) \quad (19)$$

and V is the volume of the cavity. According to our previous studies (Hirata et al., 2006, Tanigawa et al., 2009), $\bar{K}$ is more useful than other global indicators like Bejan number, Nusselt number, spatially-averaged internal energy (or spatially-averaged temperature) and so on. More specifically, the dominant spectral peaks of $\bar{K}$ tend to be clearer than those of the other global indicators.

2.5 Proper orthogonal decomposition

Among various types of the POD, we use the so-called ‘snapshot POD’ (Sirovich 1987a, 1987b & 1987c; Taira, 2011a & 2011b). Here, computation result at each instant is considered to be one snapshot of the concerning flow. An analysis is performed on 25 snapshots – 400 snapshots taken in the whole space in the cubic cavity during one forcing cycle.

Because the POD generally deals with the fluctuating parts, the first step is to calculate the mean velocity field, which can be considered the zeroth mode of the POD. However, we now skip this step, as fluid is completely stationary everywhere during an interval each forcing cycle. The mean velocity field, which is not complete stationary but convective, does not be removed, as we focus our interest upon POD on the velocity fluctuation in transient flow.

We deal with the velocity components (u_j^l, v_j^l, w_j^l) themselves, where u , v and w denote the three velocity components. Index l runs through the n snapshots, and j runs through the M^3 positions of velocity vectors in a given snapshot (i.e., $u_j = u(x_j, y_j, z_j)$). All fluctuating velocity components from the n snapshots are arranged in a matrix $\mathbf{X}$ as

$$\mathbf{X} = [\chi(t_1), \chi(t_2), \dots, \chi(t_n)] = [\chi^1, \chi^2, \dots, \chi^n] = \begin{bmatrix} u_1^1 & u_1^2 & \cdots & u_1^n \\ \vdots & \vdots & \vdots & \vdots \\ u_{M^3}^1 & u_{M^3}^2 & \cdots & u_{M^3}^n \\ v_1^1 & v_1^2 & \cdots & v_1^n \\ \vdots & \vdots & \vdots & \vdots \\ v_{M^3}^1 & v_{M^3}^2 & \cdots & v_{M^3}^n \\ w_1^1 & w_1^2 & \cdots & w_1^n \\ \vdots & \vdots & \vdots & \vdots \\ w_{M^3}^1 & w_{M^3}^2 & \cdots & w_{M^3}^n \end{bmatrix} \in \mathbb{R}^{m \times n}, \text{ where } m = 3 \times M^3. \quad (20)$$

Table 1 Test cases and their parameters ($Pr = 7.1$).

	$Ra_\eta \left(\equiv \frac{\eta^* \beta^* (T_h^* - T_c^*) H^{*3}}{\nu^* \alpha^*} \right)$	$\omega \left(\equiv \frac{\omega^* H^{*2}}{\alpha^*} \right)$
Case I	7.0×10^4	5.0×10^1
Case II	5.0×10^4	1.0×10^1
Case III	1.1×10^5	1.0×10^0
Case IV	9.0×10^4	2.0×10^2

In this step, we may introduce the weight for time/space un-uniformity into $\mathbf{X}$. However, we do not consider the weight, as we use a consecutive series of snapshots with the same time interval and a spatially-uniform mesh system.

So, we get

$$\mathbf{X}^T \mathbf{X} \mathbf{b}^q = \lambda^q \mathbf{b}^q, \mathbf{b}^q \in \mathbb{R}^n, n \ll m. \quad (21)$$

The solutions of Eq. (22) are ordered according to the size of the eigenvalues, as follows.

$$\lambda^1 > \lambda^2 > \dots > \lambda^n = 0. \quad (22)$$

The eigenvectors $\mathbf{b}^l$ (with $l = 1, 2, \dots, n$) of Eq. (22) make up a basis for constructing the POD modes $\boldsymbol{\phi}^l$, as follows.

$$\boldsymbol{\phi}^l = \frac{\sum_{j=1}^n b_j^l \boldsymbol{\chi}^j}{\left\| \sum_{j=1}^n b_j^l \boldsymbol{\chi}^j \right\|} \in \mathbb{R}^m, l = 1, 2, \dots, n, \quad (23)$$

where b_j^l is the j th component of the eigenvector corresponding to λ^l from Eq. (22), and ' $\| \cdot \|$ ' represents the discrete 2-norm.

Each snapshot can be expanded in a series of the POD modes with expansion coefficients a_l for each POD mode l . The expansion coefficients, or POD coefficients, are determined by projecting the fluctuating part of the velocity field onto the POD modes as follows

$$\mathbf{a}^l = \boldsymbol{\Psi}^T \boldsymbol{\chi}^l \in \mathbb{R}^n, \quad (24)$$

where $\boldsymbol{\Psi} = [\boldsymbol{\phi}^1, \boldsymbol{\phi}^2, \dots, \boldsymbol{\phi}^n]$. The expansion of the fluctuating part of a snapshot l is

$$\boldsymbol{\chi}^l = \sum_{j=1}^n a_j^l \boldsymbol{\phi}^j = \boldsymbol{\Psi} \mathbf{a}^l \in \mathbb{R}^m. \quad (25)$$

The amount of the total kinetic energy from velocity fluctuations in the snapshots that is associated with a given POD mode is proportional to the corresponding eigenvalue (Fukunaga, 1990). The ordering of the eigenvalues and eigenvectors in Eq. (23) therefore ensures that the most important modes in terms of energy are the first modes. This usually means that the first modes will be associated with large-scale flow structures. If a flow has dominant flow structures, these are therefore reflected in the first several POD modes and hence a given snapshot can often be reconstructed satisfactorily using only the first few modes.

3. Results and discussion

3.1 Time history of $\overline{K}$

In the present study, we examine four test cases. Table 1 shows these test cases, namely, Cases I – IV, together with their governing parameters Ra_η and ω . The four test cases are in a range of $Ra_\eta = 5.0 \times 10^4 - 1.1 \times 10^5$ and in a range of $\omega = 1.0 \times 10^0 - 2.0 \times 10^2$. The lower limit of the Ra_η range is related with the occurrence of convective flow; the occurrence condition for a non-forcing cavity under the terrestrial gravity is that $Ra \gtrsim 6.8 \times 10^3$. The upper limit of the Ra_η range is related to both the appearances of spontaneous unsteadiness and turbulence; that is, the Hopf bifurcation and the laminar-to-turbulence transition for a non-forcing cavity under the terrestrial gravity happen at $Ra \approx 2.6 \times 10^5$ and 3.7×10^5 , respectively. The lower limit of the ω range should be discussed in terms of the validity of quasi-steady

approximation. The minimum ω tested in the present study is 1.0×10^0 in Case III, which is not regarded to be sufficiently low (e.g., see Fig. 2(c) and Fig. 10(c)) because of limited computational resource. The upper limit of the ω range is related to the suppression of convective flow at high forcing frequency, because fluid cannot follow a faster acceleration change (Hirata et al., 2001).

Figure 2 shows the time history of $\bar{K}$, together with $Ra_\eta \sin \omega t$. Figures (a), (b), (c) and (d) correspond to Cases I, II, III and IV, respectively. Blue and red lines in each figure denote $\bar{K}$ and $Ra_\eta \sin \omega t$, respectively. The abscissa represents a reduced time ωt .

In all the four test cases, we have confirmed that $\bar{K}$ becomes non-zero (convective state) at least in some instant in each forcing cycle (or each forced-oscillation period), and that the periodicity of $\bar{K}$ is strict one forcing cycle after the beginning of computation. To be exact, the periodicity is definitely strict in not one forcing cycle but several forcing cycles after the beginning of computation. Therefore, all the time histories in Fig. 2 represent those during only one forcing cycle, which are in strictly-periodic states obtained enough after the beginnings of computations.

We should note that, in all the four test cases, there exist the time-intervals where $\bar{K}$ becomes zero in each forcing cycle. In addition, because we have confirmed that thermal distribution is conductive in the time intervals, the perfectly-conductive state which means linear thermal distribution with $\bar{K} = 0$ can appear once each forcing cycle. This suggests that the convective motion in each forcing cycle is free from the hysteresis in preceding forcing cycles. Of course, this freedom depends upon the choice of governing parameters Ra_η and ω , namely, the perfectly-conductive state in each forcing cycle might not appear by else choice of governing parameters than Cases I – IV.

We can see that the wave form of $\bar{K}$ in Case I (in figure (a)) is the simplest among the four. That is, there exists only one solitary wave at $\omega t = 0 - \pi$. The wave forms in the other three test cases are more complicated than that in Case I. More specifically, in Case II, remarkable three peaks appear near the top of the solitary wave. In Case III, both the slopes of the solitary wave is not smooth but jagged. In Case IV, there exists another solitary wave at ωt of π to 2π in addition to the first one at ωt of 0 about to π .

Fig. 2 Time history of $\bar{K}$, together with $Ra_\eta \sin \omega t$. Blue and red lines denote $\bar{K}$ and $Ra_\eta \sin \omega t$, respectively.

3.2 Eigenvalue

In order to confirm the influence of the number of snapshots n , Fig. 3 shows the eigenvalues of the first ten POD modes in Case III. The abscissa and the ordinate denote the modal number k and its eigenvalue λ^k , respectively. Strictly speaking, the ordinate is a normalised eigenvalues $\lambda^k / \sum_{i=1}^n \lambda^i$, which indicates the total-energy ratio of the ensemble. At each n , the normalised λ^k monotonically decreases with increasing k . We can see that this decreasing manner at $n \geq 50$ almost coincide with one another, but slightly differ from that at $n = 25$. So, we can suppose that POD at $n \geq 50$ could be sufficiently accurate. Thus, we hereinafter carry out POD at $n = 100$.

Figure 4 shows the eigenvalues λ^k of the first ten POD modes plotted against the modal number k . As well as Fig. 2, figures (a), (b), (c), and (d) represent Cases I, II, III and IV, respectively. On the basis of the results in Fig. 4, Table 2 quantitatively summarises the normalised eigenvalues of the first five POD modes.

From Fig. 4, we can see that only the first one POD mode is dominant concerning the value of λ^k in Case I. As well, the first two, three and one POD modes are dominant concerning the value of λ^k in Cases II, III and IV, respectively. From a quantitative point of view, these dominant POD modes keep more than about 5 % in the normalised eigenvalue as shown in Table 2, while the normalized eigenvalue of the first one POD mode is much larger than those of other first POD modes in all the test cases.

3.3 POD mode

Now, we discuss the spatial structures of the POD modes. Figures 5, 6, 7 and 8 show the first four POD modes in Cases I, II, III and IV, respectively. Figures (a), (b), (c) and (d) represent the first one, two, three and four POD modes, respectively. These POD modes are visualised as the perspective views of eigenfunctions of vertical velocity component w on three horizontal planes at $z = 0.25, 0.5$ and 0.75 . In addition, the POD modes are visualized by an iso-kinetic-energy surface, on the upper high hands in each figure. The colour on the horizontal planes and on the iso-kinetic-energy surface denote the value of w , as shown by a legend on the lower right hands in each figure.

Fig. 3 Eigenvalues of the first ten POD modes in Case III: influence of the number n of snapshots.

(a) Case I

(b) Case II

(c) Case III

(d) Case IV

Fig. 4 Eigenvalues λ^k of the first ten POD modes (for $n = 100$).Table 2 Normalised eigenvalues $\lambda^k / \sum_{i=1}^n \lambda^i$ of the first five POD modes (for $n = 100$).

Modal number k	1	2	3	4	5
Case I	97.43 %	1.77 %	0.60 %	0.13 %	0.04 %
Case II	70.14 %	26.39 %	1.62 %	1.22 %	0.24 %
Case III	81.66 %	11.47 %	4.87 %	1.04 %	0.45 %
Case IV	99.13 %	0.48 %	0.22 %	0.10 %	0.06 %

In Figs. 5 – 8, we can find out some similar spacial structures with those observed under the terrestrial gravity. For reference, Table 3 summarises the definitions of flow structures S1 – S7, which have been reported as steady and laminar ones in a non-forcing cubic cavity under the terrestrial gravity. The S1 – S7 are according to Pallares et al. (1996, 1999 & 2002). Gray zones of schematic diagrams on the mid-plane (at $z = 0.5$) represent ascending (z -ward) fluid, and white zones represent descending ($-z$ -ward) fluid. Chained lines of the schematic diagrams denote the roll axes of flow structures, and solid thin lines denote vertical (z -ward) symmetry planes. It should be noted that, because of the symmetry of the problem, the roll axes, the symmetry elements and the ascending and descending regions shown in schematic diagram can be rotated by $\pi/2$, π or $3\pi/2$ with respect to the vertical symmetry axis of the cubic cavity. In addition, the toroidal roll of the S4 can also be obtained with the descending central current and ascending peripheral current near the sidewalls. Referring to Table 3, we see that Mode 1 in Case I (in Fig. 5(a)) is similar with S4. As well, Mode 1 in Case II (in Fig. 6(a)), Mode 2 in Case II (in Fig. 6(b)), Mode 1 in Case III (in Fig. 7(a)), Mode 2 in Case III (in Fig. 7(b)), Mode 3 in Case III (in Fig. 7(c)), Mode 4 in Case III (in Fig. 7(d)) and Mode 1 in Case IV (in Fig. 8(a)) are similar with S2, S4, S6, S5, S2, S4 and S4, respectively. We call these POD modes as Pseudo-S2, Pseudo-S4, Pseudo-S5 and Pseudo-S6.

Fig. 5 The first four POD modes in Case I; perspective views of eigenfunctions of vertical velocity component w on three horizontal planes at $z = 0.25, 0.5$ and 0.75 , together with iso-kinetic-energy surface.

Fig. 6 The first four POD modes in Case II; perspective views of eigenfunctions of vertical velocity component w on three horizontal planes at $z = 0.25, 0.5$ and 0.75 , together with iso-kinetic-energy surface.

Fig. 7 The first four POD modes in Case III; perspective views of eigenfunctions of vertical velocity component w on three horizontal planes at $z = 0.25, 0.5$ and 0.75 , together with iso-kinetic-energy surface.

Fig. 8 The first four POD modes in Case VI; perspective views of eigenfunctions of vertical velocity component w on three horizontal planes at $z = 0.25, 0.5$ and 0.75 , together with iso-kinetic-energy surface.

Table 3 Definitions of flow structures S1 – S7 (Pallares et al., 1996, 1999 & 2002). Gray zones of schematic diagrams represent ascending (z -ward) fluid, and white zones represent descending ($-z$ -ward) fluid. Chained lines of schematic diagrams denote the roll axes of flow structures, and solid thin lines denote vertical (z -ward) symmetry planes.

Flow Structure	Definitions	Schematic diagram on the mid plane (at $z = 0.5$)
S1 (including S3 and S7 with twisted axis of rotation)	A single roll. (For the difference among the S1, S3 and S7, we have to consider the hysteresis of Nu with increasing Ra . (See Pallares et al. (2002)))
S2	A single diagonally-oriented roll.
S4	A nearly-toroidal roll.
S5	Four rolls. Each one is with its axis perpendicular to one sidewall.
S6	Four rolls. Each one is with its axis perpendicular to one vertical (z -ward) edge.

3.4 Time history on spatial structure

Figure 9 shows expansion coefficients a of the first four POD modes plotted against the reduced time ωt in one forcing cycle. As well as Figs. 2 and 4, figures (a), (b), (c) and (d) represent Cases I, II, III and IV, respectively. First, let us consider Fig. 9(a). In Case I, only Mode 1 is dominant concerning a as well as the eigenvalue λ (see Fig. 4 and Table 2). Second, let us consider Fig. 9(b). In Case II, Modes 1 and 2 are dominant concerning a as well as λ , while Modes 3 and 4 are not negligible especially at $\omega t \approx 0.5\pi$. It should be noted that a in Mode 2 is remarkable with negative value at ωt of 0.25π to 0.4π . Thirdly, let us consider Fig. 9(c). In Case III, only Mode 1 is dominant concerning a , while Modes 2 – 4 are not negligible especially at ωt of 0 to 0.4π and of 0.8π to π . Finally, let us consider Fig. 9(d). In Case VI, only Mode 1 is dominant concerning a , as well as Cases I and III. It should be noted that a in Mode 1 is not zero even at $\omega t > \pi$, and has notably negative value which is in contrast to that at $\omega t < \pi$.

In order to discuss the correspondence of a to $\bar{K}$, we consider a^2 instead of a . Because, a^2 can be directly related to $\bar{K}$ (see Eq. (25)). Figure 10 shows a^2 of the first four POD modes plotted against ωt , as well as Fig. 9. At first, we see Fig. 10(a). In Case I, the wave form of a^2 in Mode 1 is well similar with that of $\bar{K}$ in Fig. 2(a). Second, we see Fig. 10(b). In Case II, the first peak near the top of the solitary wave of $\bar{K}$ (in Fig. 2(b)) corresponds to the peak of a^2 in Mode 2. As well, the second and third peaks of $\bar{K}$ (in Fig. 2(b)) correspond to the first and second peaks of a^2 in Mode 1, respectively. Thirdly, we see Fig. 10(c). In Case III, the jaggies on both slopes the shape curve of the solitary wave of $\bar{K}$ at ωt of 0 to 0.4π and of 0.8π to π (in Fig. 2(c)) corresponds to the peaks of a^2 in not only Mode 2 and 3 but also Mode 4. Finally, we see Fig. 10(d). In Case IV, the wave form of a^2 in Mode 1 is well similar with that of $\bar{K}$ in Fig. 2(d), as well as Case I.

Figure 11 shows the POD mode with the largest a^2 at each ωt than the other POD modes, referring to Fig. 10. Figures (a), (b), (c) and (d) represent Cases I, II, III and IV, respectively. In order to avoid numerical errors, we do not regard the instant with $\bar{K} < 1.0 \times 10^0$. So, black zones in Fig. 11 denotes to be in the state with $\bar{K} < 1.0 \times 10^0$.

In order to discuss Fig. 11 more phenomenologically, Fig. 12 shows the time history of the dominant flow structure which is observed directly from raw data such as velocity components. As well as Fig. 11, subplots (a), (b), (c) and (d) represent Cases I, II, III and IV, respectively. Black zones denote to be in conductive state. Blue shaded zones illustrate the transitional states. We can see a good agreement between Figs. 11 and 12, even for Mode 4 in Case III with rather small values of a^2 . This agreement suggests a possibility of POD in order to deduce fundamental and dominant flow structures even in such transient process as an unsteady and laminar three-dimensional thermal convection. In the present study, the expansion coefficient a is useful in particular, as we can predict a consecutive series of flow structures only by means of a pure mathematical treatment without any physical considerations. Of course, we should note that the present success presumes the orthogonality of basis; fortunately, the steady and laminar flow structures S2 – S6 for a non-oscillating cavity under the terrestrial gravity which appear in Cases I – IV. This makes possible to give the obtained POD modes physical interpretations. So, if S1 appears together with S2 which seems not orthogonal to S1 in the same test case, it will be more difficult to interpret the obtained POD modes from a physical viewpoint. —From a phenomenological viewpoint, the developing process of such a diversity of convective flow structures is amazing, especially in Case III. This diversity has not been suggested directly, but implied in Hirata et al. (2013).—

Complementarily speaking, we examine only single forcing cycle. On the other hand, the POD over multiple forcing cycles is another interesting topic owing to plural symmetric modes for each flow structure. For example, concerning S1, there exist other three symmetric modes in addition to that shown by the schematic diagram in Table 3. One of symmetric modes is randomly selected and appears each forcing cycle, because the perfectly-conductive state appears once each forcing cycle in all the test cases. The POD could be theoretically effective to capture this random appearance of symmetric modes over a large number of forcing cycles, if we ignore numerical stability. We do not examine multiple forcing cycles, because the random appearance does not simulate by the present method in which small numerical perturbation always brings only one mode among its symmetry modes.

(a) Case I

(b) Case II

(c) Case III

(d) Case IV

Fig. 9 Expansion coefficients $a_1 - a_4$ of the first four POD modes.

(a) Case I

(b) Case II

(c) Case III

(d) Case IV

Fig. 10 Square of expansion coefficients $a_1^2 - a_4^2$ of the first four POD modes.

Fig. 11 Time history of the POD mode with the largest a^2 at each ωt , in each case. Black zones represent the state with $K < 1.0 \times 10^0$.

Fig. 12 Time history of flow structure in each case. Black zones denote to be in conductive state. Blue shaded zones denote the transitional states. S4r indicates the reversed S4 which has the central and peripheral currents with opposite directions to the S4.

4. Summary

To reveal fundamental and dominant flow structures of thermal convection in a differentially-heated cubic cavity, we have performed POD analyses on the flow field. The database is made of consecutive series of three-dimensional results obtained by the direct numerical simulation based on the Boussinesq approximation for a forcedly-oscillating cube under the zero-gravity environment, at vibrational Rayleigh number $Ra_\eta = 5.0 \times 10^4 - 1.1 \times 10^5$, Prandtl number $Pr = 7.1$ (water) and non-dimensional forced-oscillating frequency $\omega = 1.0 \times 10^0 - 2.0 \times 10^2$. The direction of the forced sinusoidal oscillation is parallel to the temperature gradient.

As a result, it appears that the most energetic POD modes, or the first POD eigenfunctions with large eigenvalues, account for the transient process on spacial flow structures during one forcing cycle. The first eigenfunctions captures the steady and laminar flow structures which appear inside a non-oscillating cube under the terrestrial environments. The POD expansion coefficient is useful in particular, as we could predict a consecutive series of the dominant flow structures. By the present analyses, we have successfully exhibited the effectivity of POD for fully-unsteady (transient) and three-dimensional phenomena.

NOMENCLATURE

a	expansion coefficient
$\mathbf{b}$	eigenvector
$\mathbf{e}_z$	unit vector in the z direction
g^*	gravitational acceleration [m/s ²]
H^*	height of a cavity [m]
K	kinetic energy (non-dimensional)
M^3	mesh size
m	$3 \times M^3$
n	the number of snapshots
p	pressure (non-dimensional)
Pr	Prandtl number
Ra	Rayleigh number
Ra_η	vibrational Rayleigh number
T	temperature (non-dimensional)
t	time (non-dimensional)
Δt	time step (non-dimensional)
u, v	horizontal components of velocity (non-dimensional)
$\mathbf{u}$	velocity vector (non-dimensional)
V	volume of a cavity (non-dimensional)
w	vertical component of velocity (non-dimensional)
x, y	coordinates in the horizontal directions (non-dimensional)
z	coordinate in the vertical direction (non-dimensional)

Greek Letters

α^*	thermal diffusivity [m ² /s]
β^*	thermal expansion coefficient [1/K]
η	acceleration amplitude (non-dimensional)
λ	eigenvalue
ν^*	kinetic viscosity [m ² /s]
ρ^*	density [kg/m ³]
ϕ	POD mode, $\in \mathbb{R}^m$
$\mathbf{X}$	all fluctuating velocity components from the n snapshots, $\in \mathbb{R}^{m \times n}$
χ	all velocity components at each snapshot, $\in \mathbb{R}^m$
ω	(angular) frequency (non-dimensional)

Superscripts

*	dimensional
-	spatially-averaged

Subscripts

c	on cold wall
h	on hot wall
max	maximum

References

- Bénard, M. H. , Étude expérimentale des courants de convection dans une nappe liquide. – régime permanent: tourbillons cellulaires, *J. de Phys.*, 3^e Série (1900), pp. 513 – 524. (in French)
- Biringen, S. and Danabasoglu, G., Computation of convective flow with gravity modulation in rectangular cavities, *Journal of Thermophysics and Heat Transfer*, Vol. 4 (1990), pp. 357 – 365.
- Davis, S. H., Convection in a box: linear theory, *Journal of Fluid Mechanics*, Vol. 30 (1967), pp.465 – 478.
- Fukunaga, K., Introduction to Statistical Pattern Recognition, 2nd Edn, (1990), Academic Press.
- Gershuni, G. Z. and Lyubimov, D. V., Thermal Vibrational Convection (1988), Wiley.
- Hirata, K., Sasaki, T. and Tanigawa, H., Vibrational effects on convection in a square cavity at zero gravity, *Journal of Fluid Mechanics*, Vol. 445 (2001), pp. 327 – 344.
- Hirata, K., Hakui, R., Ishihara, K., Tanigawa, H. and Funaki, J., On frequency response of fluid in a cubic cavity heated below, *Transactions of the Japan Society of Mechanical Engineers. B*, Vol. 72 (2006), pp. 279 – 284. (in Japanese)
- Hirata, K., Fujita, S., Okaji, A. and Tanigawa, H., Thermal convection in an oscillating cube at various frequencies and amplitudes, *Journal of Thermal Science and Technology*, Vol. 8 (2013), pp. 309 – 322.
- Kamotani, Y. Prasad, A. and Ostrach, S., Thermal convection in an enclosure due to vibrations aboard spacecraft, *AIAA Journal*, Vol. 19 (1981), pp. 511 – 516.
- Meyer, K. E., Pedersen, J. M., Özcan, O., A turbulent jet in crossflow analysed with proper orthogonal decomposition, *Journal of Fluid Mechanics*, Vol. 583 (2007), pp. 199 – 227.
- Pallares, J., Cuesta, Grau, I. F. X. and Giralt, F., Natural convection in a cubical cavity heated from below at low Rayleigh numbers, *International Journal of Heat and Mass Transfer*, Vol. 39 (1996), pp. 3233 – 3247.
- Pallares, J., Grau, F. X. and Giralt, F., Flow transitions in laminar Rayleigh-Benard convection in a cubic cavity at moderate Rayleigh numbers, *International Journal of Heat and Mass Transfer*, Vol. 42 (1999), pp. 753 – 769.
- Pallares, J., Cuesta, I. and Grau, F. X., Laminar and turbulent Rayleigh-Bénard convection in a perfectly conducting cubical cavity, *International Journal of Heat and Fluid Flow*, Vol. 23 (2002), pp. 346 – 358.
- Puragliesi, R., Leriche, E., Proper orthogonal decomposition of a fully confined cubical differentially heated cavity flow at Rayleigh number $Ra = 10^9$, *Computers & Fluids*, Vol. 61 (2012), pp. 14 – 20.
- Shevtsova, V., Gaponenko, Yu., Melnikov, D., Ryzhkov, I. and Mialdum, A., Study of thermoconvective flows induced by vibrations in reduced gravity, *Acta Astronautica*, Vol. 66 (2010), pp. 166 – 173.
- Sirovich, L., Turbulence and the dynamics of coherent structures Part I: coherent structures, *Quarterly of Applied Mathematics*, Vol. 45 (1987a), pp. 561 – 571.
- Sirovich, L., Turbulence and the dynamics of coherent structures Part II: symmetries and transformations, *Quarterly of Applied Mathematics*, Vol. 45 (1987b), pp. 573 – 582.
- Sirovich, L., Turbulence and the dynamics of coherent structures Part III: dynamics and scaling, *Quarterly of Applied Mathematics*, Vol. 45 (1987c), pp. 583 – 590.
- Stork, K. and Müller, U., Convection in a box: experiments, *Journal of Fluid Mechanics*, Vol. 54 (1972), pp.599 – 611.
- Taira, K., Proper orthogonal decomposition in fluid flow analysis: 1. Introduction, *Journal of Japan Society of Fluid Mechanics*, Vol. 30 (2011a), pp. 115 – 123. (in Japanese)
- Taira, K., Proper orthogonal decomposition in fluid flow analysis: 2. Applications, *Journal of Japan Society of Fluid Mechanics*, Vol. 30 (2011b), pp. 263 – 271. (in Japanese)
- Tanigawa, H., Nakamura, N., Fujita, S., Funaki, J. and Hirata, K., On the effects of forced-oscillation Amplitude upon the flow in a cubic cavity heated below”, *Transactions of the Japan Society of Mechanical Engineers. B*, Vol. 75 (2009), pp. 2106 – 2114. (in Japanese)