Interior point method based contact analysis algorithm for structural analysis of electronic device models

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Abstract
In this paper, we present an algorithm for frictionless contact problems of linear elastic bodies with multi-point constraints. Our algorithm is based on an interior point method and is developed for large scale stress analysis of electronic device models. Electronic devices consist of dozens of thin parts such as liquid crystal displays, printed circuit boards and covers and these parts are placed layer by layer. Therefore, the finite element models contain so many discretized contact constraints and multi-point constraints that make convergence of contact states difficult to achieve. In our algorithm, multi-point constraints are removed by a quadratic penalty method at first, then a primal-dual interior point method is applied. We implemented our algorithm into FrontISTR, which is open-source and large scale finite element structural analysis software, and investigated its performance from simple contact models to actual electronic device models. The numerical experiments show that our algorithm is more efficient than an active set method with an penalty method for large models, although the convergency strongly depends on the parameter settings of a primal-dual interior point method.

Key words: Contact problem, Interior point method, Finite element method, Electronic device models

1. Introduction

In a design process of portable electronic devices such as laptop PCs and mobile phones, it is important to satisfy both of lightness and toughness of the whole body. Recently, the design process has become more difficult than ever, mainly because of two following reasons. The first reason is that parts of portable electronic devices such as liquid crystal displays and printed circuit boards have become thin and small. Therefore, each part does not have sufficient stiffness. The second reason is increase of portability. Portable electronic devices encounter various kinds of external loads. A laptop PC pressed in a bag and a mobile phone bended in a pocket are the case examples. Today, structural analysis based on finite element method is widely used in order to improve the design quality and reduce the cost of experimental production. Recent finite element models are usually built by assembled parts to evaluate the stiffness of the whole body. This kind of problems are formulated as static stress analysis with contact constraints and multi-point constraints between various parts.

It should be noted that electronic devices consist of many thin parts such as liquid crystal display, printed circuit board and covers, and other parts are arranged between them. Contacts occur everywhere inside electronic devices when external loads are applied to them. Thus, the finite element models have not only large number of degrees of freedom (DOFs) but also large number of contact constraints.

Major algorithms to solve this kind of contact problems are introduced in Wriggers (2006). An active set method is well known as a widely used method because it can easily be combined with various kinds of solution methods such as a penalty method, a Lagrange multiplier method and an augmented Lagrange method (Wriggers, 2006; Hueber, et al., 2005 and 2008). In this method, a candidate contact node is called active when it is in contact and an active set is the set of all active candidate contact nodes. Active set is given from an initial configuration of model and then iterative
update of active set is done until the complementary conditions between the gap and contact force converge. In each iteration, trial deformation analysis in which current active contact nodes are constrained is done and then active set is updated by the results. It is also called as trial-and-error method. Applying active set method to structural analysis of electronic devices leads to large number of iterations to determine the contact state. Since a matrix solver routine is called once or more times per iteration in computation, increase of the number of iterations directly affects the total computational time.

We focus on interior point methods (Ye, 1997) as an efficient strategy for this kind of problems. Interior point methods are said to be efficient algorithms to solve convex nonlinear problems with the large number of constraints. Christensen et al. (1998) applied primal-dual interior point method to linear elastic contact problem with friction. They also compared interior point method with semismooth Newton method, which is a direct expansion of Newton method to non-differentical functions, and concluded that semismooth Newton method is faster and robust. However, their models have quite small DOFs and they also indicated that interior point method might need less iterations than semismooth Newton method for problems with a huge number of potential contact nodes. Tanoh et al. (2004) also applied primal and primal-dual interior point method, and showed that interior point methods are efficient for large scale problems. Miyamura et al. (2010) proposed the combination of active set method and primal interior point method and showed that their algorithm are faster than simple application of interior point methods in some examples.

In this paper, we propose an interior point method based algorithm to solve the contact problem that contains many contact constraints and multi-point constraints. In our method, multi-point constraints are removed by a quadratic penalty method and then primal-dual interior point method (Kojima, et al., 1989) is applied to the problem. Furthermore, we implement our algorithm into FrontISTR (Okuda, 2012), which is open-source and large scale finite element structural analysis software, and investigate its performance from simple contact models to actual electronic device models.

This paper is organized as follows: Formulation of contact problem is briefly explained in section 2. In section 3, a primal-dual algorithm for contact problem given by Tanoh et al. (2004) is introduced and our algorithm is proposed. Numerical results are given in section 4, and conclusion is stated in section 5.

2. Formulation of contact problem

In this paper, we consider three-dimensional, small deformation and frictionless contact problem of elastic bodies that are discretized by finite element method. We use node-to-segment discretization for contact model. It is because mesh of contact surfaces is non-conforming since the mesh is generated in a part-by-part manner for electronic device models.

Let $n$ denote the number of nodes and $n_c$ denote the number of contact candidate pairs of master segment and slave node. The nodal displacement vector, external load vector and stiffness matrix are represented by $u \in \mathbb{R}^{3n}$, $f \in \mathbb{R}^{3n}$ and $K \in \mathbb{R}^{3n\times 3n}$, respectively. Let $r_i$ ($i=1,2,...,n_c$) denote the contact force for $i$-th contact constraint. The equilibrium equations are given by

$$ Ku = f + \sum_{i=1}^{n_c} r_i T_i, $$

where $T_i \in \mathbb{R}^{3n}$ is the vectors that transform contact force $r_i$ to that of global coordinate. Here, compressive contact force is defined to be positive in order to apply an interior point method to contact problem. For the node-to-segment contact model, the components of $T_i$ is calculated from the normal vector and the shape function at a projection point of a slave node to a master segment. A slave node and a master segment are illustrated in Fig. 1.

Furthermore, Let $h_i$ ($i=1,2,...,n_c$) denote initial gap of $i$-th contact constraint, then the gap at current configuration can be represented by $T_i^T u + h_i$. For the gap and contact force, the following conditions must be satisfied:

$$ r_i (T_i^T u + h_i) = 0, (i = 1,2,...,n_c), $$

$$ r_i \geq 0, (i = 1,2,...,n_c), $$

$$ T_i^T u + h_i \geq 0, (i = 1,2,...,n_c). $$


Fig. 1 Slave node and master segment.

A contact candidate node is said to be “free” when the gap is positive and the contact force is zero, and is said to be “in contact” when its gap is zero and contact force is positive. Equations (2), (3) and (4) represent that the state of each contact constraint is either free or in contact.

We also point out that Eqs.(1), (2), (3) and (4) are the Karush Kuhn Tucker (KKT) condition of the minimization problem of the following objective function under the constraint of Eq.(4):

\[ I(u) \equiv \frac{1}{2} u^T K u - f^T u. \] (5)

Thus, it is possible to apply various optimization methods to contact problems.

3. Interior point methods for contact problem

In this section, we describe primal-dual interior point algorithm for contact problem given by Tanoh et al. (2004) and then represent proposed algorithm.

3.1 Primal-dual interior point algorithm for contact problem

The main idea of interior point method is to remove inequality constraints by using a logarithmic barrier function. We consider the following barrier problem:

\[ \min_{u} I(u) - \mu \sum_{i=1}^{nc} \log(T_i^T u + h_i), \] (6)

where \( \mu \) is a positive real parameter. The second barrier term in Eq.(6) diverges at the boundary of inequality (4), so adding this term makes the optimal point of original problem move inside the constrained region. \( \mu \) is the coefficient of the barrier term and it is known that the optimal point of barrier problem converges to that of original problem when \( \mu \) tends to zero. The orbit that optimal point makes when \( \mu \) moves from infinity to zero is called central path and interior point methods follow this path. The basic procedure is as follows: First, initial internal point is set in some way. Second, Newton direction of barrier problem is calculated and the step is taken in the direction keeping the updated point inside the constrained region. Third, \( \mu \) is decreased in some way. Second and third procedures are repeated until current point converges to KKT point of original problem.

We can straightforwardly get the first order optimal condition of barrier problem in Eq.(6) by differentiating object function of Eq.(6):

\[ Ku - f - \mu \sum_{i=1}^{nc} \frac{T_i}{(T_i^T u + h_i)} = 0. \] (7)
Let \( r_i = \mu/(T_i^Tu + h_i) \), then we can see that Eq.(7) satisfies following equations:

\[
Ku - f - \sum_{i=1}^{n_c} r_i T_i = 0, \tag{8}
\]

\[
r_i(T_i^Tu + h_i) = \mu, (i = 1, 2, ..., n_c). \tag{9}
\]

Here, Eq.(8) is obtained by replacing \( \mu T_i/(T_i^Tu + h_i) \) in Eq.(7) with \( r_i T_i \) and Eq.(9) is the definition of \( r_i \) itself. The barrier problem is defined in the interior of inequality (4) and inequality (3) holds since \( r_i = \mu/(T_i^Tu + h_i) \) and \( \mu > 0 \). Thus, the optimal condition of the barrier problem is equivalent to (3), (4), (8) and (9). We note that this condition is the same as the KKT condition of original minimization problem except that zero in the RHS of Eq.(2) is replaced with \( \mu \).

In primal-dual algorithm, both \( u \) and \( r_i \) are treated as unknown variables. Let \( \Delta u \) and \( \Delta r_i \) denote the Newton direction of Eqs.(8) and (9), respectively, then the linear equations that \( \Delta u \) and \( \Delta r_i \) should satisfy are as follows:

\[
K\Delta u - \sum_{i=1}^{n_c} T_i \Delta r_i = -\left( Ku - f - \sum_{i=1}^{n_c} r_i T_i \right), \tag{10}
\]

\[
r_i T_i^T \Delta u + \Delta r_i (T_i^Tu + h_i) = \mu - r_i (T_i^Tu + h_i), (i = 1, 2, ..., n_c). \tag{11}
\]

The coefficient matrices of Eqs.(10) and (11) are not symmetric, but we can get positive definite symmetric coefficient matrix by removing \( \Delta r_i \) from Eq.(10). First,

\[
\Delta r_i = \frac{\mu - r_i T_i^T \Delta u}{T_i^Tu + h_i} - r_i, (i = 1, 2, ..., n_c)
\]

is obtained by solving Eq. (11) for \( \Delta r_i \). Then, substituting Eq.(12) for Eq.(10),

\[
\left( K + \sum_{i=1}^{n_c} \frac{r_i}{T_i^Tu + h_i} T_i T_i^T \right) \Delta u = -\left( Ku - f - \sum_{i=1}^{n_c} \frac{\mu}{T_i^Tu + h_i} T_i \right)
\]

follows. Thus, we obtain the Newton direction \( \Delta u \) and \( \Delta r_i \) of Eqs.(8) and (9) by solving Eqs.(12) and (13). Finally, the residual of Eqs.(8) and (9) is defined as

\[
\phi(u, r, \mu) = \left( \left\| Ku - f - \sum_{i=1}^{n_c} r_i T_i \right\|^2 + \sum_{i=1}^{n_c} \left( r_i T_i^Tu + h_i - \mu \right)^2 \right)^{\frac{1}{2}}, \tag{14}
\]

where \( r = (r_1, r_2, ..., r_{n_c})^T \). The primal-dual interior point algorithm is depicted in Fig. 2. The step length \( \sigma \) is defined as follows. Let \( \delta_1 \) denote \( \min_i \{-T_i^T u + h_i\}/T_i^T \Delta u \), \( \delta_2 \) denote \( \min_i \{-r_i/\Delta r_i \} \), and \( \delta \) denote \( \min \{0.99\delta_1, 0.99\delta_2, 1\} \). Let \( \beta \in (0, 1) \) be a constant value and set \( \sigma = \delta \beta^k \). We used \( \beta = 0.9 \) here. \( \tau \) and \( \eta \) are constant parameters. Then, let us define \( \tilde{k} \) as the minimum non-negative integer that satisfies Armijo rule. The final value of \( \sigma \) is given by \( \tilde{k} \beta^k \). In this algorithm, matrix solver routine is called once per iteration of the main loop in computation.

### 3.2 Proposed algorithm

We describe proposed algorithm here. Multi-point constraints are considered in the algorithm and some minor modifications are applied to primal-dual interior point algorithm for the analysis of electronic device models.

In this paper, we consider the multi-point constraints that tie two surfaces. Each slave node is forced to coincide with its projection point on master surface. In this case, the multi-point constraints are presented as the following linear equations:

\[
B_i^T u = 0, (i = 1, 2, ..., n_M) \tag{15}
\]
where \( n_M \) is the number of multi-point constraints and \( B_i \) is coefficient vector.

\[
\phi_M(u, r, \mu) = \left( \frac{1}{2} \| K u - f \|_2^2 + \sum_{i=1}^{n_M} u B_i B_i^T u \right)^{1/2}.
\]

Then we show the present algorithm in Fig. 3. The step length \( \sigma \) is defined as follows. Let \( \delta r_i \) denote \( \min_i(-T_i^T u + h_i)/T_i^T | T_i^T u < 0 |) \), \( \sigma_2 \) denote \( \min(\min_i(-r_i/\Delta r_i | \Delta r_i < 0, 1) \), and \( \bar{\sigma} \) denote \( \min(0.99 \delta r_i, 1) \). Let \( \beta \in (0,1) \) be a constant value and set \( \sigma_1 = \bar{\sigma} \beta^k \). We used \( \beta = 0.9 \) here. Then, let us define \( \tilde{k} \) as the minimum non-negative integer that satisfies Armijo rule. The final value of \( \sigma \) is given by \( \delta \beta^k \). The present algorithm differs in two points from original primal-dual interior point algorithm. First, we adopt different step factors \( \sigma_1 \) and \( \sigma_2 \) for \( u \) and \( r \), respectively. It implies that an interior point method for nonlinear functions is applied to the quadratic function \( I_M \). In structural analysis of electronic device models, compared with the same step factors, the different step factors can accelerate convergence. Next, we introduce ‘decrease ratio’ of \( \mu \) that is equivalent to the interior point method for quadratic programming. The primal-dual algorithm in Tanoh et al. (2004) decreases the value of \( \mu \) regardless of the step length, so it tends to decrease \( \mu \) excessively.

Fig. 2 Primal-dual algorithm.
Set \( \mu > 0, \epsilon > 0, \eta > 0, \tau \in (0,1) \).
Choose initial \( u, r \) from interior point of (3) and (4)

Main loop
If \( \phi_M(u, r, \mu) < \epsilon \) then stop
Get \( u, r \) from (17) and (13)
Use backtracking to find \( \sigma_1, \sigma_2 \in (0,1] \) that satisfies following conditions:
\[
\phi(u + \sigma_1 \Delta u, r + \sigma_2 \Delta r, \mu) - \phi(u, r, \mu) < \tau \sigma_1 \nabla \phi(u, r, \mu) \text{ (Armijo rule)},
\]
\[
T_i^r(u + \sigma_1 \Delta u + h_i) > 0 \quad (i = 1, 2, \ldots, n_c),
\]
\[
\sigma_1 + \sigma_2 \Delta r_i > 0 \quad (i = 1, 2, \ldots, n_c).
\]
Update \( u \leftarrow u + \sigma_1 \Delta u, r \leftarrow r + \sigma_2 \Delta r \)
Decrease \( \mu \) as follows: \( \mu \leftarrow (\sigma \eta + (1 - \sigma))\mu, \) where \( \sigma = (\sigma_1 + \sigma_2)/2 \)
Continue loop.

Fig. 3 Proposed algorithm.

4. Numerical Experiments

In this section, we present the results of the numerical experiment. We implemented proposed algorithm into FrontISTR (Okuda, 2012). Since FrontISTR supports contact analysis function by an active set method and an augmented Lagrange method, we could easily implement our algorithm by using its data structure in the contact analysis module. In this numerical experiment, we compare accuracy and computational speed performance of an interior point method (IPM) and an active set method (ASM). We also tested an original primal-dual algorithm (Ori-PD) for the reference of present algorithm.

Figure 4, 5 and 6 show sample models which we use for the numerical experiment. Two beams models shown in Fig. 4 to test the performance of two methods for simple shape models. A smart phone case model in Fig. 5 is used to evaluate strain at the liquid crystal display when the smart phone is bended by external force. For the sake of simplicity, inner parts such as electronic components and circuit boards are not included to this models. A laptop PC model is shown in Fig. 6. This analysis aims to evaluate stress at the liquid crystal display when the rear cover is pressed by a jig. Here, a jig is a tool to give a external pressure to a product in stiffness evaluation test. Number of DOFs, contact constraints and MPCs of sample models are shown in Table. 1.

Fig. 4 Two beams to which a load is applied by the jig. Left : Small model, Right : Large model.
As for the contact conditions of which penetration is within the interior point. For an active set method, a penalty method is used to deal with equality constraints which stem from non-penetration constraints Eqn. (4). Table 2 shows the maximum contact stress obtained by an active set method and penalty method with various penalty values that can be solved. From these results, we decided to use $10^7$ as a penalty value. We use $10^6$ for the penalty value of MPCs.

<table>
<thead>
<tr>
<th>Model</th>
<th>Total # of DOFs</th>
<th># of contact constraints</th>
<th># of MPCs</th>
</tr>
</thead>
<tbody>
<tr>
<td>Two beams small</td>
<td>1 215</td>
<td>63</td>
<td>9</td>
</tr>
<tr>
<td>Two beams large</td>
<td>11 283</td>
<td>909</td>
<td>25</td>
</tr>
<tr>
<td>Smart phone case</td>
<td>142 596</td>
<td>2 563</td>
<td>535</td>
</tr>
<tr>
<td>Laptop PC</td>
<td>523 426</td>
<td>50 658</td>
<td>2 492</td>
</tr>
</tbody>
</table>

Table 2 The maximum values of the contact stress by an active set method and penalty method with various penalty values.

<table>
<thead>
<tr>
<th>Penalty value</th>
<th>1E+4</th>
<th>1E+5</th>
<th>1E+6</th>
<th>1E+7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Two beams small</td>
<td>0.02835</td>
<td>0.02827</td>
<td>0.02827</td>
<td>0.02826</td>
</tr>
<tr>
<td>Two beams large</td>
<td>3.579E-06</td>
<td>5.596E-05</td>
<td>5.582E-05</td>
<td>5.582E-05</td>
</tr>
<tr>
<td>Smart phone case</td>
<td>5.248</td>
<td>5.264</td>
<td>5.265</td>
<td>5.265</td>
</tr>
<tr>
<td>Laptop PC</td>
<td>75.81</td>
<td>81.75</td>
<td>82.66</td>
<td>82.75</td>
</tr>
</tbody>
</table>

We used the following parameters: initial $\mu = 0.5$, $\epsilon = 10^{-6}$, $\eta = 0.3, 0.4, 0.5, 0.6, 0.7$ and $\tau = 0.9$. We used $\eta = 0.54$ for original primal-dual algorithm according to Tanoh et al. (2004). As for the contact conditions of which initial gap is to be zero, we set $h_i = 10^{-8}$ so that an initial condition of $u = 0$ is within the interior point.

The maximum nodal displacements and the maximum contact stress of the three methods are shown in Table 3 and 4. As for the smart phone case model and the laptop PC model, we show the nodal displacement contour and the contact stress contour of these models in Fig. 7, 8, 9 and 10. The results are almost the same for two methods. Although the potential contact area is wide, the active contact area is small for these models. This is because board shaped parts that are placed layer by layer contact with other parts at their edges.

![Fig. 7 Nodal z displacements of the smart phone case model. Left: active set method, right: proposed method.](image1)

![Fig. 8 Contact stress of the smart phone case model. Left: active set method, right: proposed method.](image2)

![Fig. 9 Nodal z displacements of the smart phone case model. Left: active set method, Right: proposed method.](image3)
Table 3  The maximum values of the nodal displacements.

<table>
<thead>
<tr>
<th></th>
<th>IPM</th>
<th>Ori-PD</th>
<th>ASM</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>η = 0.3</td>
<td>0.4</td>
<td>0.5</td>
</tr>
<tr>
<td>Two beams small</td>
<td>0.3224</td>
<td>0.3224</td>
<td>0.3224</td>
</tr>
<tr>
<td>Two beams large</td>
<td>0.2432</td>
<td>0.2432</td>
<td>0.2432</td>
</tr>
<tr>
<td>Smart phone case</td>
<td>1.496</td>
<td>1.496</td>
<td>1.496</td>
</tr>
</tbody>
</table>

Table 4  The maximum values of the contact stress.

<table>
<thead>
<tr>
<th></th>
<th>IPM</th>
<th>Ori-PD</th>
<th>ASM</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>η = 0.3</td>
<td>0.4</td>
<td>0.5</td>
</tr>
<tr>
<td>Two beams small</td>
<td>0.02827</td>
<td>0.02827</td>
<td>0.02827</td>
</tr>
<tr>
<td>Two beams large</td>
<td>5.582E-5</td>
<td>5.582E-5</td>
<td>5.582E-5</td>
</tr>
<tr>
<td>Smart phone case</td>
<td>5.265</td>
<td>5.265</td>
<td>5.265</td>
</tr>
<tr>
<td>Laptop PC</td>
<td>82.79</td>
<td>82.79</td>
<td>82.79</td>
</tr>
</tbody>
</table>

We show the iteration number of the main loop of two methods and original primal-dual algorithm in Table. 5. Since the objective function is quadratic and constraint equations are linear, an active set method needs only one matrix solver execution per iteration of the main loop. Therefore, this iteration number is the same as the total number of matrix solver executions for both methods and determines the total computational time.

The least iteration number of the present algorithm is larger than that of an active set method for two beams small model and smaller than that of an active set method for other models. This implies that the presented algorithm can be faster than an active set method if an analytic model has certain number of contact constraints. We consider this is because an active set method can determine contact states easily in a simple model such as the two beams small model but spends iteration to determine contact states in a complex model. The iteration number of the present algorithm is the smallest when η = 0.5 for the laptop PC model and η = 0.3 for other models. It means that the convergence of the present algorithm become better if we use small η, but it can become worse if we use too small η, that is, decrease µ too excessively for a complicated model such as the laptop PC model. The iteration number of an original primal-dual algorithm is almost same as that of the present algorithm for η = 0.5 except the laptop PC model. From this result, our modification is considered to be effective in complicated models.
Table 5  Iteration number of active set method and interior point method.

<table>
<thead>
<tr>
<th></th>
<th>IPM 0.3</th>
<th>IPM 0.4</th>
<th>IPM 0.5</th>
<th>IPM 0.6</th>
<th>IPM 0.7</th>
<th>Ori-PD</th>
<th>ASM</th>
</tr>
</thead>
<tbody>
<tr>
<td>Two beams small</td>
<td>14</td>
<td>17</td>
<td>21</td>
<td>28</td>
<td>39</td>
<td>22</td>
<td>7</td>
</tr>
<tr>
<td>Two beams large</td>
<td>22</td>
<td>27</td>
<td>35</td>
<td>47</td>
<td>65</td>
<td>33</td>
<td>40</td>
</tr>
<tr>
<td>Smart phone case</td>
<td>21</td>
<td>21</td>
<td>24</td>
<td>29</td>
<td>40</td>
<td>24</td>
<td>36</td>
</tr>
<tr>
<td>Laptop PC</td>
<td>67</td>
<td>55</td>
<td>48</td>
<td>51</td>
<td>58</td>
<td>84</td>
<td>60</td>
</tr>
</tbody>
</table>

5. Conclusion

We proposed an interior point method based contact algorithm. It is intended to solve large scale contact problems with multi-point constraints which derive from stress analysis of electronic device models. We also implemented this algorithm into the open-source FrontISTR and investigated the performance. From the numerical results, we have confirmed the fast convergence of our algorithm but it strongly depends on the decrease ratio of parameter $\mu$. Additionally, we have to use a direct linear solver because of the strong penalty effect of the barrier function. Therefore, large memory machines are necessary in order to apply this method to large scale analysis. To resolve the penalty value issue and more efficient and robust updating strategy of $\mu$ remain to be open.

References


