Estimation method of current density between laminated thin sheets by inverse analysis of magnetic field (Application to short circuit localization)

Yoshinao KISHIMOTO*, Takahito TOGO*, Yukiyoshi KOBAYASHI*, Toshihisa OHTSUKA* and Kenji AMAYA**

* Department of Mechanical Engineering, Tokyo City University
1-28-1 Tamazutsumi, Setagaya-ku, Tokyo, Japan
E-mail: ykishimo@tcu.ac.jp

** Department of Mechanical and Environmental Informatics, Tokyo Institute of Technology
2-12-1 O-okayama, Meguro-ku, Tokyo, Japan

Received 22 January 2016

Abstract
Analyses of magnetic field help find out internal states noninvasively and there are many applications of the analyses for nondestructive inspections. In the field of power sources, there has been a great discussion about behaviors of an internal short circuit in a lithium-ion battery. Since lithium-ion batteries supply or preserve high energy, the visualization of the current flow in the batteries is an important issue to prevent serious incidents by such short circuits. The basic structure of lithium-ion batteries is a type of laminated constructions of thin sheets. This study has developed a novel estimation method of the current density between laminated thin sheets by the inverse analysis of the magnetic field, and investigated its applicability to the short circuit localization. In the proposed method, the boundary element method (BEM) for thin plate, the Tikhonov regularization and the Kullback-Leibler divergence are applied. The BEM calculates the observation equation relating the current density to the magnetic flux density in high accuracy and high speed. The Tikhonov regularization reduces the influence of the measurement error on the estimated value. By using the Kullback-Leibler divergence as the criterion to determine the Tikhonov regularization parameter, it makes the standard deviation of the magnetic flux density useful to the evaluation of the estimated current density. In order to verify the proposed method, numerical simulations and actual measurements were performed. From the results of the numerical simulations and the actual measurements, it is concluded that the proposed method provides the location of the short circuit between the thin sheets through the estimation of the current density. Moreover, the limitation of the proposed method was observed in terms of the standard deviation of the magnetic flux density.

Key words: Inverse problem, Laminated construction, Shell structure, Short circuit, Electromagnetic measurement, Nondestructive inspection, Boundary element method

1. Introduction
Analyses of magnetic field help find out internal states including an electrical condition of an analysis object. The fact that the magnetic field for the analyses can be measured outside the analysis object makes the estimation of the internal states to be conducted noninvasively. Taking this advantage, the analyses of the magnetic field are mainly applied on nondestructive inspections. Nondestructive inspections enable observations of phenomena difficult to be directly observed. Many applications have been discussed in the field of engineering, for example, damage identification of structure (Nara and Ando, 2007; Suzuki, et al., 2011; Nara, et al., 2014) and monitoring corrosion or electroplating (Ridha, et al., 2005; Kishimoto and Amaya, 2008). Also the concern over biomagnetism measurements has risen from the perspective of analysis techniques (Sarvas, 1987; Cheyne, et al., 2007) and medical applications (Baillet, et al., 2001; Agirre-Arrizubieta, et al., 2009; Liljestrom, et al., 2015).
Our previous research (Kishimoto, et al., 2015) studied the estimation of the current density distribution in a lithium-ion battery by using a magnetic sensor. A lithium-ion battery is a type of rechargeable batteries with high energy density and it is fabricated by laminating electrodes with separator. The electrodes and the separator are thin sheets, and the thickness of each sheet is less than half of a millimeter. Over the past few years, many researchers have shown an interest in the initiation process of the interlaminar short circuit of these thin sheets and behaviors of the battery with the short circuit (Rosso, et al., 2006; Cai, et al., 2011; Greve and Fehrenbach, 2012). It has been presumed that the short circuit is caused mainly by damage in the separator sheet. Even though the visualization of the current flow in the battery is an important issue to prevent such a problem by the interlaminar short circuit, there was difficulty in the direct measurement of the current density distribution between these sheets. To conquer this difficulty, the previous research has developed an estimation method of the current density based on an inverse analysis of the magnetic field induced outside the battery. The validity of the estimation method has been confirmed by using numerical simulations. However, little study has been done to investigate how the estimation method works in actual situation, especially for short circuit localization.

The present study discusses the estimation method of the current density between laminated thin sheets by the inverse analysis of the magnetic field and its applicability to the short circuit localization. The basic structure of the lithium-ion battery is a laminated construction of thin sheets. The main objective is to clarify that the proposed method provides the location of the short circuit between the thin sheets through the estimation of the current density. The first novelty of the proposed method is in the modeling for the inverse analysis. For the modeling of the analysis object, the boundary element method (BEM) for a thin sheet (Kishimoto, et al., 2013) is applied to derive the observation equation relating the current density to the magnetic flux density. Since this computation method calculates the magnetic flux density directly from the result of two-dimensional BEM for the electrostatic field analysis, the computational accuracy and the computational time are improved.

The second objective is to investigate the workability of the proposed method in actual situation. In actual situation, the influence of the measurement error in the magnetic flux density on the estimation is inevitable. The second novelty of the proposed method is in the process of the inverse analysis. In the inverse analysis, the Tikhonov regularization and the Kullback-Leibler divergence are applied. The Tikhonov regularization reduces the influence of the measurement error on the estimated value. Moreover, by using the Kullback-Leibler divergence as the criterion to determine the Tikhonov regularization parameter, it makes the standard deviation of the magnetic flux density useful to the evaluation of the estimated current density. This study also discusses the correlation between the limitation of the proposed method and the standard deviation of the magnetic flux density.

In the present paper, the observation equation for the inverse analysis, the detail of the inverse analysis, the verification of the proposed method and the verification results are described in order. The verification of the proposed method was performed by numerical simulations and actual measurements. In the numerical simulations, the correct distribution of the current density was given by the delta function to simulate short circuits. The magnetic flux density with artificial error was also given, and the current density estimated by the proposed method was compared with the correct distribution of the current density. In the actual measurements, specimens consisting of one or more short circuits placed between two thin sheets were fabricated. Using the data of the magnetic flux density and its measurement error obtained around the specimens, the behavior of the proposed method was examined.

2. Observation equation

2.1. Mathematical model

Figure 1 shows the mathematical model used in the proposed method. The mathematical model consists of two thin sheets and the electric conductivity is homogeneous in each sheet. Hereinafter, the sheet attached to the positive electrode is called top sheet, and the sheet attached to the negative electrode is called bottom sheet. These sheets are nonmagnetic and the magnetic permeability of each sheet is the same as that of the air. In Fig. 1, though the space between the sheets is drawn thicker than its original thickness, it is presumed that the space is much thinner than the sheets and the current density in the space flows only along thickness direction. In addition, the magnetic flux density induced by the current density in the space is ignored because the length of the current path is relatively short.

$Z$-axis is defined along the thickness direction of the sheet. $X$-axis and $Y$-axis are defined to be normal to each other as shown in Fig. 1. The magnetic permeability of the entire analytical domain including the sheets is defined as homogeneous and expressed as $\mu$. The magnetic flux density around the sheets is distributed three-dimensionally and expressed as $B = \{B_x, B_y, B_z\}$. Hereafter, parameters with subscript $l$ mean the parameters of the top sheet when $l = 1$, and the parameters of the bottom sheet when $l = 2$. The electric conductivity, the thickness, the electric potential and
the electric field domain on $XY$-plane of each sheet are expressed as $\kappa_l$, $t_l$, $u_l$ and $\Omega_l$, respectively. The boundary of the domain $\Omega_l$ is expressed as $\Gamma_l$ and the normal vector of the boundary $\Gamma_l$ is expressed as $n_l$. The current density flowing from the top sheet to the bottom sheet is defined by $j$. Then the current density flowing out the top sheet is $-j$. The reference point of the electric potential in each sheet is defined to be the point attached to each electrode as shown in Fig. 1, and the electric potential at each reference point is defined as 0 V. This definition is based on the following facts. The electric field domains of the sheets are defined to be independent of each other excluding setting the constraint condition that the absolute value of the current density $j$ is the same value on the same position in the sheets. The magnetic flux density is independent from the value of the electric potential at the reference point and it depends on the gradient of the electric potential by Biot-Savart law. Therefore the magnetic flux density is constant even if the electric potential at the reference point is varied. Although IR drop occurs between the sheets in actual situation, the value of the IR drop appears in the difference between the electric potentials at the reference points of the top sheet and the bottom sheet.

2.2. Electric field analysis

Electric potential in a thin sheet can be approximated to be uniform along thickness direction. Therefore the electric potential in each sheet $u_l$ shown in Fig. 1 is distributed two-dimensionally. Then the electric potential $u_l$ obeys the following Poisson’s equation in each electric field domain $\Omega_l$.

$$\kappa_l t_l \nabla^2 u_l + (-1)^l j = 0 \tag{1}$$

where $\nabla$ denotes the two-dimensional nabla operator. The boundary conditions of Eq. (1) are given on the boundary $\Gamma_l$ as follows.

$$u_l = 0 \quad \text{(at the reference point)}$$

$$\kappa_l \frac{\partial u_l}{\partial n_l} = 0 \quad \text{(at the other point)} \tag{3}$$

where $\partial/\partial n_l$ is the normal derivative along the normal direction of the boundary $\Gamma_l$.

Applying Green theorem after the both sides of Eq. (1) are multiplied by the fundamental solution of Laplace equation $\phi^*$ and integrated in the domain $\Omega_l$, Eq. (1) is transformed into the following integral equation for the boundary $\Gamma_l$ and the domain $\Omega_l$.

$$c_1 u_l = \int_{\Gamma_l} \phi^* \frac{\partial u_l}{\partial n_l} d\Gamma - \int_{\Gamma_l} \frac{\partial \phi^*}{\partial n_l} u_l d\Gamma + \frac{(-1)^l}{\kappa_l t_l} \int_{\Omega_l} \phi^* j d\Omega \tag{4}$$

where $c_1$ is the coefficient depending on the position to compute the electric potential $u_l$ in the left-hand side of Eq. (4) as follows.

$$c_1 = \begin{cases} 1 & \text{(inside the domain } \Omega_l) \\ \frac{\theta}{2\pi} & \text{(on the boundary } \Gamma_l) \\ 0 & \text{(outside the domain } \Omega_l) \end{cases} \tag{5}$$

where $\theta$ is the internal angle of the boundary $\Gamma_l$ at the computing position of the electric potential $u_l$. 

Fig. 1 Mathematical model of thin sheets used in the proposed method. The space between the sheets is drawn thicker than its original thickness.
Setting the computing position of the electric potential \( u_l \) on the boundary \( \Gamma_l \) and discretizing the boundary \( \Gamma_l \) and the domain \( \Omega_l \) by line elements and area elements, respectively, the algebra equation for the electric potential \( u_l \) on the boundary \( \Gamma_l \) and its normal derivative \( \partial u_l / \partial n_l \) is derived as follows.

\[
[H_l][u_l] = [G_l][q_l] + \frac{1}{k_l}[F_l](j)
\]  

(6)

where \( [u_l] \) and \( [q_l] \) are vectors whose components are the values of \( u_l \) and \( \partial u_l / \partial n_l \) at the nodes generated by the element discretization of the boundary \( \Gamma_l \). \( [G_l] \), \( [H_l] \) and \( [F_l] \) are coefficient matrices obtained by the calculation based on Eq. (4) and these matrices depend on only the shape of the boundary \( \Gamma_l \). \( \{j\} \) is a vector whose components are the values of \( j \) at the nodes generated by the element discretization of the domain \( \Omega_l \).

Gathering only the component \( u_l \) or \( q_l \) at the reference point into a vector, the vector is described with subscript \( \text{ref} \). Similarly, gathering only the component of \([G_l] \) or \([H_l] \) multiplied with \( u_l \) or \( q_l \) at the reference point into a matrix, the matrix is described with subscript \( \text{ref} \). The other vectors and matrices are described with subscript \( \text{other} \). Then Eq. (6) can be expressed as follows.

\[
\begin{bmatrix}
[H_{l,\text{ref}}] & [H_{l,\text{other}}] \end{bmatrix}
\begin{bmatrix}
[u_{l,\text{ref}}] \\
[u_{l,\text{other}}] 
\end{bmatrix} =
\begin{bmatrix}
[G_{l,\text{ref}}] & [G_{l,\text{other}}] \end{bmatrix}
\begin{bmatrix}
[q_{l,\text{ref}}] \\
[q_{l,\text{other}}] 
\end{bmatrix} + \frac{1}{k_l}[F_l](j)
\]  

(7)

Substituting Eqs (2), (3) into Eq. (7) and solving for \( \{u_{l,\text{other}}\} \), \( \{q_{l,\text{ref}}\} \), the following equation is derived.

\[
\begin{bmatrix}
[q_{l,\text{ref}}] \\
[u_{l,\text{other}}] 
\end{bmatrix} = \frac{1}{k_l}
\begin{bmatrix}
[H_{l,\text{other}}] \\
[H_{l,\text{other}}] 
\end{bmatrix}^{-1}
[F_l](j)
\]  

(8)

Taking account of the fact that the coefficient matrices \([G_l] \), \([H_l] \) and \([F_l] \) depend on only the shape of the boundary \( \Gamma_l \), the matrix in the right-hand of Eq. (8) can be computed independently of the vector \( \{j\} \) and expressed as follows.

\[
\begin{bmatrix}
[H_{l,\text{other}}] \\
[H_{l,\text{other}}] 
\end{bmatrix}^{-1}
\begin{bmatrix}
[L_{l,\text{ref}}] \\
[K_{l,\text{other}}] 
\end{bmatrix} =
\begin{bmatrix}
[O_{l,\text{ref}}] \\
[K_{l,\text{other}}] 
\end{bmatrix}
\]  

(9)

Therefore, by using Eq. (8) and Eq. (9), the algebra equations for the electric potential \( u_l \) on the boundary \( \Gamma_j \) and its normal derivative \( \partial u_l / \partial n_l \) are derived as follows.

\[
[u_l] = \begin{bmatrix}
[u_{l,\text{ref}}] \\
[u_{l,\text{other}}] 
\end{bmatrix} = \frac{1}{k_l}
\begin{bmatrix}
[O_{l,\text{ref}}] \\
[K_{l,\text{other}}] 
\end{bmatrix}(j) = \frac{1}{k_l}[K_l](j)
\]  

(10)

\[
[q_l] = \begin{bmatrix}
[q_{l,\text{ref}}] \\
[q_{l,\text{other}}] 
\end{bmatrix} = \frac{1}{k_l}
\begin{bmatrix}
[L_{l,\text{ref}}] \\
[O_{l,\text{other}}] 
\end{bmatrix}(j) = \frac{1}{k_l}[L_l](j)
\]  

(11)

where

\[
[K_l] =
\begin{bmatrix}
[O_{l,\text{ref}}] \\
[K_{l,\text{other}}] 
\end{bmatrix}
\]  

(12)

\[
[L_l] =
\begin{bmatrix}
[L_{l,\text{ref}}] \\
[O_{l,\text{other}}] 
\end{bmatrix}
\]  

(13)

\([0_{l,\text{ref}}]\) and \([0_{l,\text{other}}]\) are zero vectors whose dimensions are the same as \([u_{l,\text{ref}}]\) and \([q_{l,\text{other}}]\), respectively. \([O_{l,\text{ref}}]\) and \([O_{l,\text{other}}]\) are zero matrices whose sizes are the same as \([L_{l,\text{ref}}]\) and \([K_{l,\text{other}}]\), respectively.

### 2.3. Magnetic field analysis

According to our previous research (Kishimoto, et al., 2013), the following equation enables to compute the magnetic flux density \( B \) induced by the electric field where the electric potential is distributed uniformly along the thickness direction of the electric field.

\[
B = \sum_{l=1}^{2}
\frac{\kappa_l \mu}{4\pi}
\left[\int_{\Gamma_j} \nu^* \frac{\partial u_l}{\partial n_l} \, d\Gamma - \int_{\Omega_j} \nu^* u_l \, d\Omega + \frac{1}{k_l} \int_{\Omega_j} \nu^* j \, d\Omega \right]
\]  

(14)

where \( \nu^* \) and \( w^* \) are three-dimensional vectors depending on the measurement position of the magnetic flux density \( B \) and the position of the integral point. Setting a number of measurement points of the magnetic flux density and discretizing the
boundary $\Gamma_l$ and the domain $\Omega_l$ by the same elements in the electric field analysis, the algebra equation for the magnetic flux density $B$ is derived as follows.

$$
[B] = \frac{\mu_0}{\pi} \left[ -\frac{\kappa_l}{4\pi r_l} \left( [V_\Omega^l][q_l] - [W_\Omega^l][u_l] + \frac{1}{\kappa_l} [M_l^l][j] \right) \right] \tag{15}
$$

where $[B]$ is a vector whose components are the values of $B$ at the measurement points. $[V_\Omega^l]$, $[W_\Omega^l]$ and $[M_l^l]$ are coefficient matrices obtained by the calculation based on Eq. (14) and these matrices depend on only the measurement location and the integration range i.e. the shape of the boundary $\Gamma_l$ or the domain $\Omega_l$.

Therefore, substituting Eq. (10) and Eq. (11) into Eq. (15), the following observation equation that describes the direct relation between the current density between the sheets $[j]$ and the magnetic flux density measured around the sheets $[B]$ is obtained.

$$
[B] = [A][j] \tag{16}
$$

where

$$
[A] = \frac{2}{\pi} \left[ -\frac{\mu_0}{4\pi r_l} \left( [V_\Omega^l][L_j] - [W_\Omega^l][K_j] + [M_j^l] \right) \right] \tag{17}
$$

The electric conductivity $\kappa_l$ is vanished in the process to derive the observation equation. This means that the value of the electric conductivity $\kappa_l$ is unnecessary to estimate the current density between the sheets $[j]$ by using the magnetic flux density $[B]$.

3. Inverse analysis

3.1. Tikhonov regularization

If the size of the vector $[B]$ is the same as that of the vector $[j]$, the matrix $[A]$ is a square matrix and the solution is $[j] = [A]^{-1}[B]$. The sizes of the vectors can be set arbitrarily depending on the element discretization, and the matrix $[A]$ is not always a square matrix. Moreover, the condition number of the matrix $[A]$ is equal to the maximum singular value divided by the minimum singular value in the matrix $[A]$, and the relationship between the disturbance ratios is expressed as follows.

$$
\sqrt{\frac{\|\Delta j\|^2}{\| j \|^2}} \leq (\text{cond}[A]) \sqrt{\frac{\| \Delta B \|^2}{\| B \|^2}} \tag{18}
$$

where $\|\Delta j\|$, $\|\Delta B\|$ and $\text{cond}[A]$ are the disturbance of the vector $[j]$, the disturbance of the vector $[B]$ and the condition number of the matrix $[A]$, respectively. The superscript $T$ denotes matrix transposition. The condition number of the matrix $[A]$ is equal to the maximum singular value divided by the minimum singular value in the matrix $[A]$.

In this study, after using the information of the total current flowing in the sheets, the Tikhonov regularization is applied to reduce the influence of the above problems. The estimation procedure is as follows. Taking account of the fact that the total current is equal to the current supplied by the electrodes, the total current can be measured. The average of the current density which is the total current divided by the junction area of the sheets can be treated as known value in the inverse analysis. Therefore the difference between the current density $[j]$ and the average current density $j_0$.

$$
x = [j] - [j_0] \tag{19}
$$

is estimated in the proposed method. In Eq. (19), the size of the vector $[j_0]$ is the same as that of the vector $[j]$ and all of the components of the vector $[j_0]$ are the average current density $j_0$. Then Eq. (16) is transformed as follows.

$$
[b] = [A][x] \tag{20}
$$

where

$$
[b] = [B] - [A][j_0] \tag{21}
$$

The Tikhonov regularization for Eq. (20) selects the solution of the vector $[x]$ which minimizes the following evaluation function $\Pi([x])$.

$$
\Pi([x]) = ((b) - [A][x])^T ([b] - [A][x]) + \alpha [x]^T [x] \tag{22}
$$
Eq. (20) when the singular value decomposition, the coefficient matrix $[A]$ is transformed as follows.

$$[A] = [U][S][V]^T$$  \hspace{1cm} (23)

where the matrices $[U]$ and $[V]$ are orthonormal matrices ($[U]^{-1} = [U]^T$ and $[V]^{-1} = [V]^T$). The matrix $[S]$ is composed of the singular values of the matrix $[A]$ and expressed as follows. When the dimension of the matrix $[B]$ is larger than that of the vector $[x]$,

$$[S] = \begin{bmatrix}
\lambda_1 & 0 & \cdots & 0 \\
\lambda_2 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & \lambda_m \\
\end{bmatrix}, \quad (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m > 0)$$ \hspace{1cm} (24)

Otherwise,

$$[S] = \begin{bmatrix}
\lambda_1 & 0 & 0 & \cdots & 0 \\
\lambda_2 & 0 & \cdots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & \cdots & \lambda_m & 0 \\
\end{bmatrix}, \quad (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m > 0)$$ \hspace{1cm} (25)

where the dimension of the vector $[B]$ and $[x]$ are expressed as $m$ and $n$, respectively. Then the size of the matrix $[A]$ or $[S]$ is $m$ by $n$. By using Eq. (23), the vector $[x]$ which minimizes the evaluation function $\Pi$ is derived as follows.

$$[x] = [V][S]^T([S] + \alpha[I]^{-1})^T[U]^T([B] - [A][j_0])$$ \hspace{1cm} (26)

where the matrix $[I]$ is an unit matrix. Therefore, from Eqs (19), (21) and Eq. (26), the following equation to obtain the current density $[j]$ from the magnetic flux density $[B]$ is derived as follows.

$$[j] = [j_0] + [V][S]^T([S] + \alpha[I]^{-1})^T[U]^T([B] - [A][j_0])$$ \hspace{1cm} (27)

The solution by Eq. (27) is equivalent to the solution obtained by substituting $(\lambda_i^2 + \alpha)/\lambda_i$ into the singular values $\lambda_i$ ($i = 1, 2, \cdots, \min(m,n)$) in the coefficient matrix $[A]$. Then the relationship between the disturbance of the vector $[B]$ and that of the vector $[j]$ is expressed as follows.

$$\frac{\sqrt{\sum_{i}(A_i^T[A_i])}}{\sqrt{\sum_{i}(j_i^T[j_i])}} \leq \frac{\lambda_{\min(m,n)}(\lambda_1^2 + \alpha)}{\lambda_1}(\frac{\lambda_{\min(m,n)}^2}{\lambda_1^2} + \alpha) \sqrt{\frac{\sum_{i}(B_i^T[B_i])}{\sum_{i}(B_i^T[B_i])}}$$ \hspace{1cm} (28)

Thus the enlargement factor of the disturbance ratio is equivalent to $[\lambda_{\min(m,n)}(\lambda_1^2 + \alpha)]/[\lambda_1](\lambda_{\min(m,n)}^2 + \alpha)]$.

The Tikhonov regularization is workable regardless of whether the matrix $[A]$ is a square matrix. If the value of $\alpha$ is set as zero and the matrix $[A]$ is a square matrix, the solution of Eq. (22) is equivalent to the solution obtained by directly solving Eq. (20). On the other hand, if the value of $\alpha$ is set as zero, the solution of Eq. (22) is the least square solution of Eq. (20) when $m > n$, and the solution of Eq. (22) is the least norm solution of Eq. (20) when $m < n$.

In Eq. (27), an objective criterion to determine the value of the Tikhonov regularization parameter $\alpha$ is necessary. This study selects the Kullback-Leibler divergence (Kullback and Leibler, 1951; Burnham and Anderson, 2002) as the criterion. The Kullback-Leibler divergence $D(\alpha)$ for Eq. (20) is described as follows.

$$D(\alpha) = \log \left(\frac{(b - [A][x])^T(b - [A][x])}{(b - [A][x])^T(b - [A][x])}\right) + \frac{2\sigma^2}{\min(\alpha, \lambda_i^2)} \sum_{i=1}^{\min(m,n)} \lambda_i^2$$ \hspace{1cm} (29)

where $\sigma$ is the standard deviation of the vector $[b]$ which is the same as the standard deviation of the measured magnetic flux density $[B]$ because of Eq. (21). The first term means the residual sum of squares when the solution by the inverse analysis (Eq. (26)) has been substituted into the observation equation (Eq. (20)). The second term means the penalty for
decreasing the value of $\alpha$. The value of $\alpha$ minimizing Eq. (29) is determined as the estimation result. Using Eq. (19) and Eq. (21), Eq. (29) is also expressed as follows.

$$D(\alpha) = \log ((B - [A]/j)^T (B - [A]/j)) + \frac{2\alpha^2}{(|B| - [A]/j)^T (|B| - [A]/j)} \sum_{i=1}^{\min(m,n)} \frac{A_i^2}{\lambda_i^2 + \alpha}$$

(30)

3.2. Estimation procedure

First, applying the element discretization on the models of the thin sheets and setting the measurement points of the magnetic flux density, the matrix $[A]$ is obtained by Eq. (17). Next, the magnetic flux density at each measurement point and the total current are measured. Then, using the calculated matrix $[A]$, the measured magnetic flux density $[B]$ and the average current density $[j_0]$ which is calculated by the total current, the current density $[j]$ and the Kullback-Leibler divergence $D(\alpha)$ are calculated for a certain value of $\alpha$ by Eq. (27) and Eq. (30), respectively. Finally, the current density $[j]$ when the value of $\alpha$ minimizes $D(\alpha)$ is determined as the estimation result by the inverse analysis.

4. Verification

4.1. Specimen

Figure 2 shows the schematic of the specimen for the verification of the proposed method. The specimen consisted of two copper sheets ($60 \text{ mm} \times 50 \text{ mm} \times 0.5 \text{ mm}$). The sheets were attached to electrodes and current was supplied to the sheets by the electrodes. The junction area of the sheets was $60 \text{ mm} \times 20 \text{ mm}$ and the distance between the sheets was $0.5 \text{ mm}$. In Fig. 2, the space between the sheets is drawn thicker than its original thickness. PVC sheet ($60 \text{ mm} \times 20 \text{ mm} \times 0.5 \text{ mm}$) was inserted as an insulator into the space between the sheets. Short circuits were prepared between the sheets by using copper foil tape. The thickness of the copper foil tape was $0.08 \text{ mm}$ and both ends of the tape were attached to the copper sheets.

![Fig. 2 Schematic of specimen. The specimen consists of two copper sheets, PVC sheet and copper foil tape. The copper foil tape assumes a short circuit between the copper sheets.](image)

Figure 3 shows the configurations of the short circuits. The shapes of the short circuits were two kinds; point-like short circuit and linear short circuit. As shown in Fig. 3, the point-like short circuit was fabricated by drilling a micro hole in the PVC sheet and passing the copper foil tape shreeded in $1 \text{ mm}$ width approximately through the hole. The linear short circuit was fabricated by creating a slit in the PVC sheet and passing the copper foil tape through the slit.

As shown in Fig. 2 and Fig. 3, the Cartesian coordinate system $(X, Y, Z)$ was defined. The origin of the coordinates $(X, Y)$ was defined at the center of the junction area of the sheets. The origin of the $Z$-coordinate was defined at the top sheet. Hereafter, the locations of the short circuits are described as $(X, Y) [\text{mm}]$ and the $Z$-coordinates of the short circuits are undescribed for brevity. The alignments of the point-like short circuits were three patterns; (i) one short circuit was located at $(0, 0) [\text{mm}]$, (ii) one short circuit was located at $(20, 0) [\text{mm}]$ and (iii) two short circuits were located at $(10, 0) [\text{mm}]$ and $(-10, 0) [\text{mm}]$. The linear short circuits were two patterns and these short circuits were located (iv) from $(-5, 0)$ to $(5, 0) [\text{mm}]$ and (v) from $(-10, 0)$ to $(10, 0) [\text{mm}]$.

4.2. Experimental setup

Figure 4 illustrates the schematic of the experimental setup. Direct current 1 A was supplied to the specimen by the constant-current generator (DP-3003S made by CUSTOM corporation). Then the magnetic flux density around the specimen was measured by the magnetic sensor (475 DSP gaussmeter and probe No. HMNT-4E04-VR made by Lake Shore Cryotronics, Inc.). Figure 5 shows the measurement location of the magnetic flux density. The distance between
Fig. 3 Configurations of short circuits. Both ends of the copper foil tape as the short circuit are attached to the copper sheets. The shapes of the short circuits are point-like or linear.

Fig. 4 Schematic of experimental setup

Fig. 5 Measurement location of magnetic flux density

the measurement points and the specimen was 5 mm. The measurement points were located at intervals of 2 mm on the plane paralleled to the $XY$-plane and the number of the measurement points was 341.

According to our previous research (Kishimoto, et al., 2013), the $X$-component of the magnetic flux density would tend to be relatively large value compared to the other components in the experimental setup shown in Fig. 4. Therefore only the $X$-component of the magnetic flux density was measured in this study. Moreover, the magnetic flux density induced by the wiring between the specimen and the constant-current generator was comparable to that induced by the specimen. The magnetic flux density by the wiring could be calculated by Biot-Savart law shown as Eq. (31).

$$B(r) = \frac{\mu}{4\pi} \int \frac{I(r') \times (r - r')}{|r - r'|^3} ds(r')$$

where $r$ and $r'$ are the position vectors of the measurement point of the magnetic flux density and the integral point, respectively. $B(r)$ is the magnetic flux density at $r$, and $I(r')$ is the current in the wiring at $r'$. Equation (31) is a line integral for the wiring path $s$. The difference by subtracting the value calculated by Eq. (31) from the measured magnetic flux density was determined as the magnetic flux density induced by the specimen.

4.3. Analysis model

Figure 6 illustrates the element discretization of the specimen model for the inverse analysis. In Fig. 6, the space between the sheets is drawn thicker than its original thickness. Each boundary of the sheets was discretized by 3-node quadratic line elements at intervals of 1 mm. 440 nodes and 220 line elements were applied to model each sheet. Each junction area in the sheets was discretized by 8-node quadratic square elements at intervals of 4 mm. 266 nodes and 75 square elements were applied to define the junction area. Therefore the number of the calculation points of the current density flowing from the top sheet to the bottom sheet was 266, and the size of the matrix $[A]$ in Eq. (17) was 341 rows and 266 columns. The maximum singular value is the first singular value $\lambda_1$ and the minimum singular value is the 266th
Fig. 6  Element discretization of specimen. Each boundary of the sheets is discretized by 3-node quadratic line element with 440 nodes and 220 elements at intervals of 1 mm. Each junction area in the sheets is discretized by 8-node quadratic square element with 266 nodes and 75 elements at intervals of 4 mm.

singular value $\lambda_{266}$.

The nodes on the 2 elements located at the positive electrode ($X = -1 \sim 1$ mm, $Y = -40$ mm) were defined as the reference points of the electric potential in the top sheet $u_{1,\text{ref}} (= 0 \text{ V})$. Similarly, the nodes on the 2 elements located at the positive electrode ($X = -1 \sim 1$ mm, $Y = 40$ mm) were defined as the reference points of the electric potential in the bottom sheet $u_{2,\text{ref}} (= 0 \text{ V})$. On the other boundary elements, the normal component of the current density was set as zero, i.e. $\partial u_1/\partial n_1 = 0 \text{ V/m}$ and $\partial u_2/\partial n_2 = 0 \text{ V/m}$. The magnetic permeability of the entire analytical domain was set as $\mu = 1.26 \text{ H/m}$, whose value is the same as that of the air.

4.4. Verification procedure

The proposed method was verified by numerical simulations and actual measurements. The magnetic flux density and its standard deviation $\sigma$ in Eq. (30) were obtained by the following procedure. The numerical simulations were performed to confirm that the proposed method worked properly when the magnetic flux density with artificial error was given. The correct distribution of the current density $j(X, Y)$ [A/m$^2$] under each condition as shown in Fig. 3 was given by the delta function $\delta$ as follows.

\begin{align*}
\text{Condition (i)} & \quad j(X, Y) = \delta(X) \times \delta(Y) \\
\text{Condition (ii)} & \quad j(X, Y) = \delta(X - 20) \times \delta(Y) \\
\text{Condition (iii)} & \quad j(X, Y) = 0.5 \times [\delta(X - 10) \times \delta(Y) + \delta(X + 10) \times \delta(Y)] \\
\text{Condition (iv)} & \quad j(X, Y) = \begin{cases} 0.1 \times \delta(Y) & (-5 \leq X \leq 5) \\ 0 & (X < -5, 5 < X) \end{cases} \\
\text{Condition (v)} & \quad j(X, Y) = \begin{cases} 0.05 \times \delta(Y) & (-10 \leq X \leq 10) \\ 0 & (X < -10, 10 < X) \end{cases}
\end{align*}

where the total current of each distribution was set to 1 A. The magnetic flux density was calculated based on Eq. (14). The first term and the second term in Eq. (14) were calculated by using the element discretization shown in Fig 6. Only the third term in Eq. (14) was calculated analytically, because the current density $j$ was given by the delta function. Then the artificial error was added to the calculated magnetic flux density. In general, the artificial error can be given arbitrarily. The artificial error was given by the truncation error generated by rounding off the magnetic flux density in this study. The value of $\sigma$ in Eq. (30) was set to several values in the estimation of the current density.

The actual measurements were performed to investigate how the proposed method worked when the magnetic flux density included measurement error in the actual situation. The magnetic flux density was obtained by using the specimen and the experimental setup described in the previous sections. The standard deviation of the magnetic flux density was obtained by repeating 5 times with the same instrument. Then the average of the standard deviations at the measurement points (341 points) was defined as the value of $\sigma$ as follows.

$$\sigma = \frac{1}{341} \sum_{i=1}^{341} \sigma_i$$
where $\sigma_i$ is the standard deviation of the magnetic flux density at the $i$-th measurement point.

Using the obtained magnetic flux density and its standard deviation, the current density was estimated by the proposed method. To obtain the value of $\alpha$ minimizing the Kullback-Leibler divergence (Eq. (30)), the following values of $\alpha$ were substituted each into Eq. (30).

$$
\alpha_i = \begin{cases} 
0 & (i = 1) \\
\lambda_{266}^2 \times \left( \frac{\lambda_1^2}{\lambda_{266}^2} \right)^{(i-2)/100} & (i = 2, 3, \ldots, 102)
\end{cases}
$$

(38)

where $\alpha_2, \alpha_3, \ldots, \alpha_{102}$ are the geometrical progression, in which the first term is the square of the minimum singular value $\lambda_{266}^2$ and the 101st term is the square of the maximum singular value $\lambda_1^2$.

5. Results and discussion

5.1. Numerical simulation

Figure 7 shows the calculation results of the magnetic flux density and the current density under the condition (i). The magnetic flux density was rounded off to the whole number in units of microtesla ($\mu$T), and the value of $\sigma$ in Eq. (30) was set to 0, 0.1, 0.333 (= 1/3), 0.5 or 1 $\mu$T, as an example. In Fig. 7, the black dot in each estimated current density indicates the position of the peak in the correct current density. The estimated current density had no peak at (0, 0) [mm] when the value of $\sigma$ was set to 0 $\mu$T or 0.1 $\mu$T, whereas the correct current density had a peak at (0, 0) [mm]. The estimation of the current density failed when the value of $\sigma$ was set to 0 $\mu$T or 0.1 $\mu$T. When the value of $\sigma$ was set to 0.333 $\mu$T or 0.5 $\mu$T, the estimated current density had a peak at (0, 0) [mm]. Although the current density was not necessarily zero in the other areas, the current density was relatively low. While the current density had a peak at (0, 0) [mm] also when the value of $\sigma$ was set to 1 $\mu$T, the distribution of the current density was oversmoothed. Therefore it seemed that the estimation of the current density was successful at the time when the value of $\sigma$ was set to 0.333 $\mu$T or 0.5 $\mu$T.

From these results, the followings are derived. Taking account of the fact that the magnetic flux density in this simulation was varied up to 1 $\mu$T from its exact value by the rounding off, it can be presumed that the standard deviation of the magnetic flux density was from 0.333 $\mu$T (3$\sigma$ = 1 $\mu$T) to 0.5 $\mu$T (2$\sigma$ = 1 $\mu$T). In the above results, the feature of the correct current density could be found without being oversmoothed when the value of $\sigma$ in Eq. (30) agreed with the standard deviation of the magnetic flux density. Therefore it is believed that the proposed method successfully estimates the distribution of the current density if the data of the magnetic flux density and its standard deviation are given.

Figure 8 shows the Kullback-Leibler divergence $D(\alpha)$ and the enlargement factor of the disturbance ratio $\{[\lambda_{266}(\lambda_1^2 + \alpha)]/[\lambda_1(\lambda_{266}^2 + \alpha)]\}$ in Fig. 7 relative to the Tikhonov regularization parameter $\alpha$. Table 1 shows the detail of the parameter setting in the inverse analysis. In Fig. 8, the Kullback-Leibler divergence when $\sigma = 0$ $\mu$T increases with the increase in the Tikhonov regularization parameter. This means that the first term in Eq. (22) and Eq. (30) increases with the increase in the Tikhonov regularization parameter. In addition, if the standard deviation of the magnetic flux density is zero, the Tikhonov regularization parameter should be set to zero in terms of the Kullback-Leibler divergence. When $\sigma = 0.1, 0.333, 0.5$ or 1 $\mu$T, the Kullback-Leibler divergence has the minimum value and the value of the Tikhonov regularization parameter $\alpha$ can be selected uniquely in the inverse analysis. It also seems that the selected value of the Tikhonov regularization parameter $\alpha$ increases with the increase in the value of $\sigma$ as shown in Fig. 8 and Table 1.

Focusing on the enlargement factor of the disturbance ratio, its limit values can be calculated analytically as follows.

$$
\lim_{\alpha \to 0} \frac{\lambda_{266}(\lambda_1^2 + \alpha)}{\lambda_1(\lambda_{266}^2 + \alpha)} = \frac{\lambda_1}{\lambda_{266}} = 6.31 \times 10^6
$$

(39)

$$
\lim_{\alpha \to \infty} \frac{\lambda_{266}(\lambda_1^2 + \alpha)}{\lambda_1(\lambda_{266}^2 + \alpha)} = \lim_{\alpha \to \infty} \frac{\lambda_{266}(\lambda_1^2/\alpha + 1)}{\lambda_1(\lambda_{266}^2/\alpha + 1)} = \frac{\lambda_{266}}{\lambda_1} = 1.59 \times 10^{-7}
$$

(40)

As shown in Fig. 8, it is evident that the calculation result of the enlargement factor of the disturbance ratio also converges with these limit values. It seems that the enlargement factor of the disturbance ratio is almost constant when the Tikhonov regularization parameter $\alpha$ is less than $\lambda_{266}^2$ or more than $\lambda_1^2$. The enlargement factor of the disturbance ratio, i.e. the condition number in the inverse analysis is unable to be improved even if the value of $\alpha$ is set to less than $\lambda_{266}^2$ or more than $\lambda_1^2$. In this connection, it indicates that the search range of the Tikhonov regularization parameter $\alpha$ confined from $\lambda_{266}^2$ to $\lambda_1^2$ is workable from the perspective of the condition number in the inverse analysis. Furthermore, the Kullback-Leibler divergence is also almost constant when the Tikhonov regularization parameter $\alpha$ is less than $\lambda_{266}^2$ or more than $\lambda_1^2$. Bearing in mind that the value of $\alpha$ can be uniquely determined by the value of $\sigma$, it is probable that the value of $\sigma$
has a lower limit and an upper limit. The same can be said about the actual measurements. It is presumed that the standard deviation of the magnetic flux density in the actual measurements should fall within the range between the lower limit and the upper limit to enable the inverse analysis to work properly.

![Fig. 7 Calculation results of magnetic flux density and current density under condition (i)](image)

![Fig. 8 Kullback-Leibler divergence and enlargement factor of disturbance ratio in Fig. 7](image)

![Table 1 Detail of parameter setting in inverse analysis in Fig. 7](image)
Figures 9-12 show the calculation results of the magnetic flux density and the current density under the condition (ii)-(v), respectively. Each of the magnetic flux density was rounded off to the whole number in units of microtesla (μT), as an example. The value of $\sigma$ was set to 0.333 ($= 1/3$) μT in all the inverse analyses by reference to the results under the condition (i). The black dot or the black line in each estimated current density indicates the position of the peak in the correct current density. The distribution of the magnetic flux density was varied by the correct distribution of the current density. The estimated current density had one or more peaks under each condition. The location of the peaks almost agreed with those of the correct current density; (ii) one peak was located at (20, 0) [mm] and (iii) two peaks were located at (10, 0) [mm] and (−10, 0) [mm]. Under the conditions (iv) and (v), the current density was high (iv) from (−5, 0) to (5, 0) [mm] and (v) from (−10, 0) to (10, 0) [mm], respectively. The current density was relatively low in the other areas. From these results including the results under the condition (i), the proposed method enabled the estimation of the approximate distribution of the current density given by the delta function. It is believed that the location of the short circuit between the sheets can be judged from the distribution of the current density estimated by the proposed method. Table 2 shows the detail of the parameter setting in the inverse analysis under each condition. In Table 2, the followings are confirmed. The selected value of the Tikhonov regularization parameter $\alpha$ was between $\lambda_266^2(= 1.01 \times 10^{-17})$ and $\lambda_1^2(= 4.01 \times 10^{-4})$. The enlargement factor of the disturbance ratio was about $10^{-2}$. It is inferred that the inverse analysis works properly in the actual measurements if the standard deviation of the magnetic flux density is about 0.3 μT.

![Fig. 9 Calculation results of magnetic flux density and current density under condition (ii)](image)

![Fig. 10 Calculation results of magnetic flux density and current density under condition (iii)](image)

![Fig. 11 Calculation results of magnetic flux density and current density under condition (iv)](image)

![Fig. 12 Calculation results of magnetic flux density and current density under condition (v)](image)
Table 2: Detail of parameter setting in inverse analysis under conditions (ii)-(v) in numerical simulations

<table>
<thead>
<tr>
<th>Short circuit condition No.</th>
<th>Standard deviation of magnetic flux density ( \sigma ) [( \mu T )]</th>
<th>Selected value of Tikhonov regularization parameter ( \alpha )</th>
<th>Minimum value of Kullback-Leibler divergence ( \min(D(\alpha)) )</th>
<th>Enlargement factor of disturbance ratio ( \frac{\lambda_2(\lambda_1^2 + \alpha)}{\lambda_1(\lambda_2^2 + \alpha)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(ii)</td>
<td>0.333</td>
<td>1.99 \times 10^{-9}</td>
<td>3.55</td>
<td>3.19 \times 10^{-2}</td>
</tr>
<tr>
<td>(iii)</td>
<td></td>
<td>5.10 \times 10^{-9}</td>
<td>3.36</td>
<td>1.25 \times 10^{-2}</td>
</tr>
<tr>
<td>(iv)</td>
<td></td>
<td>1.99 \times 10^{-9}</td>
<td>3.53</td>
<td>3.19 \times 10^{-2}</td>
</tr>
<tr>
<td>(v)</td>
<td></td>
<td>3.73 \times 10^{-9}</td>
<td>3.46</td>
<td>1.71 \times 10^{-2}</td>
</tr>
</tbody>
</table>

5.2. Actual measurement

Figures 13-17 show examples of the measured magnetic flux density and the estimated current density. The black dot or the black line in each estimated current density indicates the position of the short circuit in the specimen. From these figures, it seems that the magnetic flux density was varied by the configuration of the short circuits in the specimen because the current path in the specimen was changed. Although the measurement error of the magnetic flux density affected the estimation of the current density, the estimated current density was high near the locations of the short circuits and relatively low in the other areas under all the conditions. Even though the linear short circuit was located from \((-10, 0)\) to \((10, 0)\) [mm] under the condition (v), the estimated current density was high between about \((0, 0)\) and \((10, 0)\) [mm] and low between \((-10, 0)\) and \((0, 0)\) [mm]. However, only in this case, the resistance in the short circuit and the junction resistance between the short circuit and the sheets might affect the current path because the short circuit was comparatively wide in \(X\)-direction. In this respect, there is a possibility that the correct distribution of the current density was high between about \((0, 0)\) and \((10, 0)\) [mm]. Moreover it can be said that the estimated current density is enough to judge the existence and the approximate position of the short circuit under the condition (v).

Table 3 shows the detail of the parameter setting in the inverse analysis under each condition. The standard deviation of the magnetic flux density \( \sigma \) was about 0.2 to 0.5 \( \mu T \), and comparable to that given in the numerical simulations. The selected value of the Tikhonov regularization parameter \( \alpha \) was about \( 10^{-9} \), and it was located between \( \lambda_2(\lambda_1^2 + \alpha) = 1.01 \times 10^{-17} \) and \( \lambda_1(\lambda_2^2 + \alpha) = 4.01 \times 10^{-4} \). The enlargement factor of the disturbance ratio was about \( 10^{-2} \). Therefore the selected value of the Tikhonov regularization parameter and the enlargement factor of the disturbance ratio were also comparable to those derived in the numerical simulations.

From the above results, it is concluded that, as far as the observation of the relationships between the estimated current density and the location of the short circuit in the actual measurements is concerned, the short circuit can be localized by the estimated current density. It leads to the possibility that the proposed method is functional on the subject of the short circuit localization also in other actual measurement. In the numerical simulations and the actual measurements, the estimated current density was not necessarily zero at the place except the short circuit part. Including the resolution of this problem, the enhancement of the estimation accuracy of the proposed method is an issue in the near future.

(a) Magnetic flux density (Maximum value = 12.7 \( \mu T \))
(b) Estimated current density (Maximum value = 2.81 \( \times 10^{4} \) A/m²)

Fig. 13 Measurement result of magnetic flux density and estimation result of current density under condition (i)

(a) Magnetic flux density (Maximum value = 13.1 \( \mu T \))
(b) Estimated current density (Maximum value = 2.13 \( \times 10^{4} \) A/m²)

Fig. 14 Measurement result of magnetic flux density and estimation result of current density under condition (ii)
6. Conclusion

This paper described the estimation method of the current density between laminated thin sheets by the inverse analysis of the magnetic field and its applicability to the short circuit localization. In order to verify the proposed method, the numerical simulations and the actual measurements were performed. From the results of the numerical simulations and the actual measurements, the followings were summarized.

(1) The proposed method worked properly in the numerical simulations where the five configurations of the short circuits were given and the total current was set to 1 A. The tendency of the correct current density given by the delta function to simulate the short circuits could be found in the estimated current density.

(2) The proposed method could be workable also in the actual measurements, because the locations of the peaks in the estimated current density almost agreed with those of the short circuits. In addition, the distribution of the current density estimated by the proposed method was enough to judge the existences and the approximate positions of the short circuits.

(3) The Kullback-Leibler divergence and the enlargement factor of the disturbance ratio, i.e. the condition number, in...
the inverse analysis were also investigated. As a result, the limitation to the use of the proposed method was observed in terms of the standard deviation of the magnetic flux density. The standard deviation in the actual measurements was about 0.2 to 0.5 $\mu$T, and it fell within the range to enable the inverse analysis to work properly.

(4) The estimated current density was not necessarily zero at the place where the correct value of the current density was zero such as the insulated area. The enhancement of the estimation accuracy of the proposed method should be studied further.

References


