Restitution properties in direct central collision of three inelastic spheres

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Abstract
The purpose of this paper is to investigate the micromechanical processes during impact as well as the related macro-mechanical restitution properties in a three body multiple impact system. Thereby, the microscale refers to the detailed processes during impact, while the macroscale refers to the overall dynamics of the impact system which is normally evaluated by rigid bodies. Specifically, this paper deals with numerical and experimental investigations of direct central collisions of three identical inelastic spheres. In the experiments the velocities of three spheres during impacts are measured by Laser-Doppler-Vibrometers. In the numerical simulation, finite element analyses are performed, where the material properties obtained from static and dynamic compression tests are used. In order to confirm the validity of the numerical model, the results from the finite element analyses are compared to the experimental results and show good agreement. Afterwards, micromechanical investigations of the impact processes are performed using finite element analysis. Thereby quantities such as impact force, deformation, and kinetic energy loss are investigated in detail. Then, the macroscale rebound properties of multiple collisions are derived using the simulation results on a microscale. In general, a single coefficient of restitution (COR) can be used to evaluate on the macroscale the restitution property in two body impacts. However, in instantaneous impact modelling this cannot be used for general multibody systems where multiple concurrent impacts occur. Therefore, a COR matrix is employed in this paper as measure of the macroscale restitution properties and the overall impact dynamics of the system. Finally, a rigid body simulation with continuous contact law is attempted as an alternative macro-mechanical model. It is shown for this specific impact system that this method can produce practical macroscale behavior of the colliding spheres if a simple single impact COR is used. This is shown by comparison with the COR matrix.

Key words: Direct central collision, Inelastic sphere, Multiple collisions, Coefficient of restitution, Coefficient of restitution matrix, Finite element analysis
1. Introduction

The rebound properties of impacting bodies have been widely described in engineering applications, such as shot peening process, multibody dynamics analyses (Flores et al., 2006), concise identifications of material hardness (Taber, 1951) and yield stress (Cristescu, 2007), and so on. Many investigations on impacts of two bodies with a single contact point have been conducted and they are nowadays fairly well understood, e.g. (Johnson, 1985; Thornton, 1997; Gilardi and Sharf, 2002). More recently, also research on multiple impacts has been advanced e.g. (Pfeiffer and Glocker, 1996; Hinch and Saint-Jean, 1999; Stronge 2000b; Sen and Mancia, 2001; Liu, Zhao and Brogliato, 2009; Hayden and Chiara, 2015). However, multiple impacts possess a much higher complexity and a large variety of possible impact configurations exists. Kinetic energy is lost during impact due to inelastic material behavior or impact induced wave phenomena. For rigid body models, these effects are normally summarized and expressed by the coefficient of restitution (COR). Thus, the COR is the typical macroscopic measure for the impact, which determines the overall motion of the impact system. The COR must be estimated from experience, costly experiments, or can be derived from detailed simulations on a microscale, e.g. using finite element analysis (Seifried, Schiehlen and Eberhard 2010). Such detailed simulations then include all necessary effects of kinetic energy loss such as plasticity and wave propagation. Since the COR can be defined by the ratio of relative velocities between two bodies, it is difficult to apply this concept directly to multiple impacts. Several approaches have been proposed to treat multiple impacts analytically, such as simultaneous contacts (Johnson, 1976), sequential contacts (Glocker and Pfeiffer, 1995), a COR matrix (Frémond, 1955), and the impulse correlation ratio (Hurmuzlu and Ceanga, 2000). Some theoretical and experimental investigations (Stronge, 2000a; Gilardi and Sharf, 2002; Payr et al. 2005; Minamoto et al. 2014, 2015) have been conducted for chain impacts, occurring e.g. in Newton’s cradle. However, a full and detailed analysis is still open, especially for inelastic material behavior.

This paper deals with experimental and numerical analysis of direct central collision of three identical inelastic spheres. It is one of the simplest cases of chain impacts but encapsulates basic properties involved in complex multiple impacts. Firstly, in experiments three identical steel spheres are collided by pendular motion, where the velocities of the three spheres are measured by two 1-dimensional Laser-Doppler-Vibrometers (LDVs) and a 3-dimensional LDV. Then, finite element analyses (FEA) are performed to obtain a detailed understanding of the microscopic processes during impact. Thereby inelastic material properties are used which are obtained from static and dynamic compression tests. In order to evaluate the rebound properties of multiple collisions on a macroscale, the concept of a COR matrix (Frémond, 1955) is adapted. The components of the COR matrix are evaluated from the detailed FEA and their dependency on the impact velocity is shown. Finally, as an alternative approach a rigid body model with a continuous contact law is employed, which is based on a single impact COR. This simple model turned out to be an alternative efficient approach for obtaining good first estimations of the overall dynamics and the COR matrix for this impact system.

2. Impact System

The direct central collision of three identical steel spheres, sometimes called Newton’s cradle, is investigated. Thereby, inelastic material behavior is assumed for the spheres. First, an experimental setup of the impact system is presented. Then, a detailed numerical model based on nonlinear finite element analysis is presented. It is shown that this numerical model is very precise and reproduces the experiments very well. This model is capable for detailed analysis of the micromechanical processes during impact, including inelastic deformation and wave phenomena.

2.1 Experimental setup

In Figure 1 a schematic representation of the experimental setup as well as the photograph of the experimental setup are presented. Three steel spheres of diameter 13.49mm (17/32in) are aligned like pendula. Each sphere is suspended by strings in a V-shape. The first sphere is raised and hold at an initial angle by an electromagnet. By changing the magnet height the initial impact velocity of sphere 1 can be adjusted. For first experiments an initial velocity of 0.5m/s is used. The first sphere is released from the magnet to collide with the two resting spheres at the bottom position. The velocities of the three spheres are measured using three Laser-Doppler-Vibrometers (LDV). The velocities of the spheres at both sides are measured by 1D-LDVs (Polytec OFV3000, OFV3100) and the velocity of the middle sphere is measured by a 3D-LDV (Polytec CLV 3D). Since the impact velocity is low enough to limit the
deforming region in the vicinity of the initial contact point, these velocity measurements can be regarded as representative of the entire mass of the sphere. The velocity signal is recorded with a sampling time of one microsecond. For verification purposes the experiments are conducted for elastic and inelastic spheres, respectively. In the experiments commercial steel spheres made of JIS SUJ2 (steel used for journal bearings) are used. The surface of the delivered spheres is hardened by quenching. Therefore, these spheres are regarded as elastic spheres. In order to obtain inelastic spheres full annealing is performed with the delivered spheres.

![Schematic representation of experimental setup](image1)

**Fig. 1** Schematic representation of experimental setup (left) and photograph of experimental setup (right). Three identical spheres are suspended by Kevlar strings like pendula and one sphere collides against the other two resting spheres. The impact velocity is set at 0.5 m/s. The velocities of three spheres during impacts are measured using three Laser Doppler Vibrometers.

### 2.2. Finite element analysis

Finite element analyses (FEA) are used for the detailed micromechanical analysis of the impact process. The finite element software LS-DYNA (Livermore Software Technology Corporation, 2006 and 2007) is used. Considering the symmetry of the impact system, 4-node axisymmetric elements are employed. The axis through the centers of the spheres is taken as symmetry axis. Therefore, the spheres can move freely along the symmetry axis, but the motion perpendicular to the symmetry axis is constrained. The major deformation of the spheres is close to the contact area. Therefore, the finite element mesh is refined in the contact area as shown in Fig. 2. The numbers of elements are 7890 for sphere 2 and 4015 for spheres 1 and 3. The spheres have Young's modulus 217 GPa, Poisson ratio 0.3 and the mass density is 7752 kg/m$^3$. The penalty method is chosen for the contact modelling and frictionless conditions are assumed in tangential direction. In accordance to the experiments, an initial velocity is given to the sphere 1, while the other two spheres are initially at rest. Hereby, the same initial velocity is assigned to all nodes of sphere 1, resulting in a pure rigid body motion before impact.
For the elastic spheres a perfectly elastic material model is employed. In the inelastic regime the steel is rate sensitive, and therefore, it shows elastic-viscoplastic material behavior. This strain rate dependency has significant influence on the COR (Minamoto and Kawamura, 2008; Seifried, Minamoto and Eberhard, 2010). Therefore, for the inelastic spheres a rate dependent bi-linear elastic-plastic material model is employed. In order to obtain the stress-strain curves and to identify the material properties of the inelastic spheres, static and dynamic compression tests are conducted. A hydraulic servo type material testing machine (Shimadzu Servopulser) is used for the static compression tests and a split Hopkinson pressure bar (Chen and Song, 2011) is used for the dynamic compression tests. Cylinder specimens are cut from the spheres by wire cut electric discharge machining and both end faces are finished by lathe work. Specimens with different sizes are used, e.g. 8mm in diameter and 12mm in length for static test and 5mm in diameter and 7.5mm in length for dynamic compression test. In order to obtain quasi-static loading the static tests are conducted with a sufficient low displacement speed, e.g. 0.5mm/min. The dynamic compression tests are conducted using a Hopkinson pressure bar for strain rates which range from 500/s to 3000/s. From the static stress-strain curve, a static yield stress $\sigma_{y\text{static}}$ is identified as 570 MPa. Further, a linear stress-strain relation is assumed after yielding and the tangent modulus is identified as 4 GPa. From the split Hopkinson pressure bar tests, dynamic stress-strain curves are obtained. It is observed that they also can be approximated by the bi-linear relation, although the yield stress is increased with strain rate. Then, the ratio of dynamic yield stress $\sigma_y$ to the static one $\sigma_{y\text{static}}$, which is called scale factor, is derived from the dynamic stress-strain curves as shown in Fig. 2.

### 2.3 Validation of the numerical model

The results of the FEA are validated by comparison with the experimental results. The velocity changes of the three spheres during impact are shown in Fig. 3 for elastic (E) and elastic-viscoplastic (EVP) spheres, respectively. The origin of the abscissa is taken as the starting time of the first impact, which occurs between sphere 1 and sphere 2. The velocities in Fig. 3 are taken at the center of the 3 spheres in the FEA, while the velocities of spheres 1 and 3 are measured at the free ends of the spheres as shown in Fig. 1. However, it is confirmed by the FEA results that the velocities at the centers and the free ends of spheres 1 and 3 are almost same since the deformation of the spheres are limited near the contact area. Therefore, the velocities of the FEA and the experiments can be compared.

From Fig. 3 it is seen clearly that the FEA results agree very well with the experimental results for both the elastic and inelastic impacts. In the elastic case it is observed that a slight rebound velocity is visible for sphere 1, while in the elastic-viscoplastic case a small forward motion remains for sphere 1. For the elastic case the post-impact velocity of sphere 3 is nearly the same as the initial velocity of sphere 1. However, in the elastic-viscoplastic case the velocity of sphere 3 is significantly lower.
3. Rigid body models

The smooth motion of a dynamical system might be interrupted by impacts. The numerical FEA is suitable for the investigation of the detailed micromechanical processes during impact. However, they are relatively time consuming and often not suitable for analysis of the overall behavior of dynamical systems. Thereby, impacts are only short events in a longer time period where the overall motion takes place. For the analysis of the overall motion, rigid body models, such as in multibody systems, are often most suitable. In multibody systems impacts can be modeled using instantaneous or continuous impact models. These are macro-mechanical models, where the kinetic energy loss due to impact, e.g. due to inelastic deformation or wave phenomena, is summarized and expressed using the coefficient of restitution (COR). For single impacts different definitions for the COR exist, namely the kinematic COR, the kinetic COR and the energetic COR (see e.g. Seifried, Schiehlen and Eberhard 2010). These definitions are equivalent for many cases, such as frictionless impacts (Stronge, 2000a). The COR of a single impact, often denoted as \( e \), can take a value between 0 and 1. A value of \( e=1 \) indicates no kinetic energy loss and the impact is called elastic. A value \( e=0 \) indicates maximal kinetic energy loss and the impact is called plastic. A value \( 0<e<1 \) is called elastic-plastic. It should be noted that the terms elastic and plastic relate to the impact behavior and not necessarily to the material behavior at the impact point. The COR cannot be evaluated within the rigid multibody system approach but has to be estimated from experiments or experience. Alternatively, it might be computed from the results of detailed micromechanical simulations such as FEA, yielding a multi-scale simulation approach (Seifried, Schiehlen and Eberhard 2010). These numerical evaluations of the COR have been used so far for single impacts and are extended in this paper to multiple impact systems.

3.1 Instantaneous impact models: COR matrix for the case of three sphere impact

Instantaneous impact modeling is based on classical impact theory. The smooth motion of the system is interrupted by collision and the velocities change instantaneously while the positions remain unchanged. The post-impact velocities are computed using the COR. However, in simultaneous multiple impacts the definitions of the COR originating from two sphere impacts cannot be used directly. In the following, the COR matrix is employed for multiple impacts (Frémond, 1995; Payr and Glocker, 2005; Aeberhard, et al, 2006 Payr, 2008). This can be seen as a generalization of the kinematic COR of single impacts, which is defined as the negative ratio of the relative velocities after and before impact. Using the COR matrix in multiple impacts, the relative velocities at each contact point are summarized in the velocity vector \( v_r \). Then, the post impact relative velocities \( v_r^+ \) of the spheres are related to the relative velocities before impact \( v_r^- \) by

\[
v_r^+ = -\varepsilon \cdot v_r^- , \tag{1}
\]

---

Fig. 3 Velocity changes of three spheres during impact. Experimental and FEA results are compared for the cases of elastic spheres (left) and elastic viscoplastic spheres (right). The lines show the results by FEA and the symbols show the experimental results. The red, blue and green line indicate the spheres 1, 2 and 3, respectively. The FEA results agree well with the experimental results in both cases.
where \( \mathbf{e} \) is the COR matrix. Hereby, \( \mathbf{e} \) instead of \( \mathbf{c} \) is used in order to distinguish between single COR and the COR matrix. The components of the COR matrix \( \mathbf{e} \) give the relationship between the relative velocities of the spheres before and after impacts. However, in the case of one single impact \( \mathbf{e} \) reduces to \( \mathbf{c} \). The COR matrix can be fully populated in order to allow a mutual influence of the relative velocities between the different impact points (Payr and Glocker, 2005; Payr, 2008). In the case of \( n \)-body collisions, Eq. (1) can be written as

\[
\mathbf{v}_i^+ = -\left( e_{11} v_{r1}^+ + e_{12} v_{r2}^+ + \cdots + e_{n-1} v_{rn-1}^+ \right).
\]

Therefore, each coefficient \( e_{ij} \) is considered as the contribution of the pre-impact relative velocity \( v_{ij} \) at contact point \( j \) to the post impact relative velocity \( v_{ij}^+ \) at contact point \( i \).

In the following, the COR matrix is applied to the collinear impact of three spheres shown in Fig. 4. The relative velocity at impact point one (between spheres 1 and 2) is given by \( v_{r1} = v_1 - v_2 \) and at impact point two (between spheres 2 and 3) is given by \( v_{r2} = v_2 - v_3 \), where \( v_1, v_2 \) and \( v_3 \) are the velocities of spheres 1, 2 and 3, respectively. Then, for a three body impact system Eq. (1) reduces to,

\[
\begin{bmatrix}
  v_{r1}^+ \\
  v_{r2}^+
\end{bmatrix}
= \begin{bmatrix}
  e_{11} & e_{12} \\
  e_{21} & e_{22}
\end{bmatrix}
\begin{bmatrix}
  v_{r1}^- \\
  v_{r2}^-
\end{bmatrix}.
\]

Further, in the case of direct central impact of three identical spheres, as investigated in this research, it holds that \( e_{11} = e_{22} \) and \( e_{12} = e_{21} \) which is due to the symmetry of the impact system, see (Payr and Glocker, 2005; Payr, 2008). Therefore, only two components of the COR matrix \( e_{11} \) and \( e_{22} \) are independent and have to be identified. Since the initial conditions are in the considered case given as \( v_1 = v_0 \) and \( v_2 = v_3 = 0 \), it follows \( v_{r1} = v_0 \) and \( v_{r2} = 0 \) which yield the entries of the COR matrix as,

\[
e_{11} = e_{22} = -\frac{v_{r1}^+}{v_0} \quad \text{and} \quad e_{21} = e_{12} = -\frac{v_{r2}^+}{v_0}.
\]

### 3.2 Continuous impact model

Using the continuous impact modeling, the impact is modeled as a short contact of finite duration, thus both position and velocity of the bodies change continuously. A nonlinear unilateral force-law is used for the computation of this continuous short contact. Thereby, a small local penetration \( \delta \) of the colliding bodies occurs at the impact point, which is then used in a contact force law. For purely elastic impacts, i.e. without any kinetic energy loss, the Hertzian contact law \( F = k \delta^n \) can be used for spherical bodies. Thereby, \( k \) is a constant which depends on the elastic material properties of the bodies and the radius of the bodies at the contact point. The index \( n=1.5 \) is used for spherical contact (Johnson, 1985).

For elastic-plastic impacts the force-law is brought into relation to the COR to account for the kinetic energy loss during impact. In the following a contact force law is presented, which is based on the definition of the energetic COR (Stronge 2000a). The impact is decomposed into a compression phase and a restitution phase. In the compression phase, the two impacting bodies are slowed down until the relative velocity in normal direction vanishes. During the compression phase the bodies absorb the deformation energy \( E_c \). Then, in the restitution phase, the stored elastic deformation energy \( |E_d| < |E_c| \) is recovered and the two bodies are accelerated in opposite direction. This yields the energetic definition of the COR,
Hereby, $\delta_s=0$ is the penetration at the beginning of the contact, $\delta_c$ is the maximal penetration which occurs at the end of the compression phase, and $\delta_e$ is the remaining compression at the end of the contact. It can be shown, that for many cases, also for the direct central impact investigated here, the kinematic COR and the energetic COR are equivalent and yield exactly the same value (Stronge, 2000b). Therefore, the same notation for the different definitions of the single COR are used in the remaining of this paper.

For the continuous contact model presented in this paper different force laws are used for compression and restitution phase, see Fig. 5. For the compression phase the Hertzian contact law might be used. Then, the force in the restitution phase can be computed as (Lankarani and Nikravesh 1994)

$$F_r = \frac{k\delta_c^n}{(\delta_c - \delta_e)}(\delta - \delta_c)$$

where $\delta_c \leq \delta \leq \delta_e$. (5)

For applying this contact law the final deformation $\delta_e$ must be computed. In contrast to the mentioned approach (Lankarani and Nikravesh 1994) the energetic COR is used here and thus $\delta_e$ can be computed independently of the masses of the impact system. By integrating the force laws for compression and restitution phase over the penetration $\delta$ and inserting in Eq. (4), the remaining compression at the end of the contact is computed as $\delta_e = \delta_c(1-e^2)$. It is noted, that $\delta_e$ is independent of the used exponent $n$ since the same exponent is used for compression and restitution phase. Then, the force law for the restitution phase reduced to

$$F_r = \frac{k}{e^n}[(\delta - \delta_c)(1-e^2)]$$

where $\delta_c \leq \delta \leq \delta_e$. (6)

Fig. 5 Model for the energetic coefficient of restitution (left). Here, $E_c$ is the deformation energy stored during compression, $E_r$ is the deformation energy released during restitution phase. Continuous force-penetration model with Hertzian contact law for compression phase and force law Eq. (5) for restitution phase (right).

In the following the direct central collisions of three identical inelastic spheres are investigated in detail by using the FEA. Figure 6 shows on the left side the variations of the contact forces over time for $V_0=0.5m/s$. In three sphere collisions two impact forces arise, the first impact is between sphere 1 and 2, the second impact occurs between sphere 2 and 3. The maximum contact force of the first impact is greater than that of the second impact. The beginning of the second impact $t_2$ coincides approximately with the traveling time of stress wave through the sphere diameter. The right plot of Fig. 6 shows the variations of sphere compression during impact. Since impact forces act from two sides on the middle sphere, the compression of the middle sphere becomes largest. Since the spheres are identical, and the behavior in the contact region can be considered as quasi-static, the deformations in the contact areas of both spheres are
equivalent. Thus, the compression of sphere 2 equals the sum of compression of spheres 1 and 3. The permanent deformation after impact is due to the inelastic deformation of the spheres. From both the forces and the deformations, it is seen that the first impact is stronger than the second impact. Due to the inelastic deformation at the first impact, kinetic energy is dissipated before the second impact starts, “weakening” the second impact.

![Graph showing variations of contact forces over time](image1)

**Fig. 6** The left plot shows variations of contact forces over time ($V_0=0.5\text{m/s}$). The two lines show the contact forces obtained by the FEA. $F_{12}$ indicates the first impact between spheres 1 and 2, $F_{23}$ indicates the second impact between spheres 2 and 3. The maximum value of $F_{12}$ is larger than that of $F_{23}$. The right plot shows corresponding compression. The red, blue and green lines indicate the spheres 1, 2 and 3, respectively. The permanent compression is due to inelastic deformation.

Figure 7 on the left side shows the variations of maximum contact forces with impact velocity. Thereby, $F_{12\text{max}}$ and $F_{23\text{max}}$ indicate the maximum contact forces between spheres 1 and 2 and spheres 2 and 3, respectively. Both maximal forces $F_{12\text{max}}$ and $F_{23\text{max}}$ increase with impact velocity and $F_{12\text{max}}$ is in the whole velocity range greater than $F_{23\text{max}}$. It is observed that $F_{23\text{max}}$ increases almost linearly with impact velocity. However, $F_{12\text{max}}$ shows a higher increase rate than $F_{23\text{max}}$. Since steeper increase and larger compression are observed in the first impact as shown in Fig. 6, effects of work hardening and strain rate dependency of yield stress are more predominant in the first impact than in the second one.

Figure 7 on the right side shows the variations of permanent deformations with impact velocity. The values $\delta_1$, $\delta_2$ and $\delta_3$ indicate the permanent deformations of spheres 1, 2 and 3, respectively. For the impact velocity $V_0=0.5\text{m/s}$ these correspond to the final values shown in right plot of Fig. 6. Since the three spheres are identical, the relationship $\delta_1 + \delta_3 = \delta_2$ holds over the entire velocity range. This relationship can also be predicted from the law of action-reaction regarding the both sides of the middle sphere. Since it is based on the static equilibrium, the relationship holds only in low speed impact in which the effect of wave propagation can be negligible. Initially the slopes of the curves increase with impact velocity up to about 1m/s, while they increase almost linearly with impact velocity after that. The plastic deformation increases rapidly by the impact of spherical surface since high contact pressure applies on the small contact area. However, once the spherical surface deforms into a more flat surface, the development of plastic deformation slows down because lower contact pressure applies on the larger contact area.
Figure 8 shows on the left side the variation of the contact time with impact velocity. Thereby, $T_{c12}$ is the contact time for the first impact and $T_{c23}$ is for the second impact. It is found that $T_{c12}$ is longer than $T_{c23}$ at any impact velocity. Both contact times decrease with impact velocity. They decrease steeply in low impact velocity. After that, however, they decrease slowly with increasing impact velocity, and tend to approach constant values.

Figure 9 shows on the left side the variations of internal energies of each sphere with time in the case of impact velocity $V_0=0.5m/s$. The internal energies of the spheres 1 and 2 start increasing due to the first impact. Then, the internal energy of the sphere 3 starts increasing due to the second impact. Since elastic and inelastic deformation occurs, some of the internal energy is recuperated. After the impact, the internal energy of the spheres becomes constant. This value shows the final
energy loss of the spheres due to inelastic deformation. Similar to the deformation, it is observed that the energy loss in sphere 2 equals approximately the summed energy loss of spheres 1 and 3.

On the right side of Fig. 9 the variation of energy loss ratio with impact velocity is presented. The energy loss ratio $\beta$ is defined as the ratio of total energy loss in all three spheres in relation to the initial kinetic energy of the system. The total energy loss is calculated by summation of the energy losses of the three spheres. For example for $V_0=0.5\text{m/s}$ this is the sum of the final values shown in the left plot of Fig. 9. It is seen from the right plot of Fig. 9 that the energy loss ratio increases with impact velocity. It increases steeply in the low velocity range, while after that it increases gradually and tends to approach to around 55% at $V_0=5\text{m/s}$.

Figure 10 left shows the portion of energy loss due to the two impacts. Thereby, the variation of energy loss fraction due to each impact with velocity is presented. The energy loss fraction is defined as the ratio of energy loss of contacts 1 and 2 to the total energy loss of the system. Although the total energy loss ratio $\beta$ of the system increases with impact velocity as shown in Fig. 9 right, the energy loss fraction for each contact does not simply increases with impact velocity, but remains largely constant. It is found from Fig. 10 that the first impact accounts for approximately 60% of the kinetic energy loss, while the second impact accounts for approximately 40% kinetic energy loss.

In the following the FEA analysis is used to identify the components of the COR matrix, as given by Eqs. (2) and (3). This is a macro-mechanical measure for rigid body impact modeling. Figure 10 shows on the right side the variations of the components $\epsilon_{11}$ and $\epsilon_{12}$ of the COR matrix with impact velocity. For comparison, also the COR $e$ of a single impact of two identical spheres is added, where the spheres of both impact systems are identical. The values $\epsilon_{11}$ and $\epsilon_{12}$ are lower than $e$ of a single impact system. Thereby, $\epsilon_{12}$ is higher than $\epsilon_{11}$ at any impact velocity. The $\epsilon_{11}$ expresses the rebound property for the first impact which occurs between spheres 1 and 2, and the $\epsilon_{12}$ expresses the rebound property for the second impact which occurs between sphere 2 and 3. Both components $\epsilon_{11}$ and $\epsilon_{12}$ decrease with impact velocity, which is also observed for the COR $e$ of a single impact. Especially, the component $\epsilon_{11}$ approaches to zero with increase of impact velocity. This means that post impact relative velocity between spheres 1 and 2 tends to vanish with increase of impact velocity.
5. Analysis of macro-mechanical behavior using rigid bodies and continuous impact model

The COR matrix given by Eqs. (2) and (3) and their identified values shown in Fig. 10 are good macro-mechanical measures for describing the overall dynamical behavior of the impact system. These values can be evaluated from micromechanical FEA, however, this might be numerically burdensome. Therefore, in the following some first results using an alternative approach are presented. A simple rigid body model consisting of three masses and continuous contact laws is used. Such models have been used in combination with the Hertzian contact law to investigate the overall dynamics of chains of purely elastic spheres (Pfeiffer and Glocker, 1996; Stronge 2000a). In this section, such a model is extended to inelastic spheres. Thereby, the individual impacts are described by the continuous contact law presented in Section 3.2. For the compression phase the Hertzian contact law is used. In order to account for the kinetic energy loss during impact, the COR $e$ of a single impact system, which is independent of the used definition of the COR, is used in the force law Eq. (6) of the restitution phase. Since for such single impact systems many data for corresponding CORs are available in literature, this might be an appealing ad-hoc approach. For the investigations in this paper the values of the single impact COR $e$, which are shown in Fig. 10, are used. In Table 1 the components $\varepsilon_{11}$ and $\varepsilon_{12}$ of the COR matrix obtained by this simple rigid body model are presented and compared with the corresponding values of the micromechanical FEA for the velocity range up to 2m/s. Especially in this mid-velocity range a very good agreement is seen. This shows that with this ad-hoc approach good initial estimations of the overall impact dynamic behavior can be achieved with a very simple model. At the lowest velocity a relatively large discrepancy of 8% in $\varepsilon_{12}$ is observed. However, this is mainly due to error amplification in the evaluation of the COR. Considering the error of the post impact velocity of sphere 3, and then the error is less than 4%. The contact model depends mainly on two parameters, namely the single impact COR $e$ and the contact stiffness $k$ of the Hertzian contact law which is derived for elastic contact. For contacts involving plasticity the contact stiffness of the loading phase is significantly lower than in the elastic case (Minamoto et al. 2011), which is important for the micromechanical behavior during impact. However, it should be noted that the obtained results for the overall dynamics are fairly insensitive to the used contact stiffness $k$ in the contact model Eq. (6). For demonstration purposes the components of the COR matrix are added in Table 1, which are obtained when the stiffness $k$ is reduced by 33% in the continuous contact model Eq. (6). Thus, it is shown that for this impact system very similar macro-mechanical results can be obtained in the medium velocity range using a simple rigid body model. The macro-mechanical results are fairly insensitive to the used contact stiffness.
Table 1 Comparison of components of COR matrix resulting from detailed FEA and macro-mechanical rigid body model with contact stiffness $k$ from Hertzian contact law and contact stiffness reduced by 33%.

<table>
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<th>initial impact velocity [m/s]</th>
<th>finite element analysis</th>
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6. Conclusions

This paper dealt with direct central collisions of three identical inelastic spheres. The steel spheres collided by pendular motion and the velocities of the three spheres during impacts were measured by Laser-Doppler-Vibrometers. Finite element analyses were performed for the impact system. Since inelastic, rate-dependent material behavior occurred, static and dynamic compression tests were conducted for the used steel spheres. Two successive impacts were observed among three spheres, and the results from the experiments and the FEA agreed very well. Then, detailed micromechanical process during impact was investigated by results from the FEA. It was seen that the first impact between spheres 1 and 2 is significantly stronger than the second impact between spheres 2 and 3. It means that higher contact force, larger permanent deformation and kinetic energy loss were seen in the first impact. The ratio of total energy loss to the initial kinetic energy increased with impact velocity. However, the ratio of energy loss fractions in respect of impacts 1 and 2 remain nearly constant. Then, the overall dynamics of the impact system was evaluated using the COR matrix. Using the FEA the entries of the COR matrix were evaluated and it was shown that they depend on the impact velocity. Finally a rigid body model with continuous contact law for describing the overall dynamics was developed. Thereby, the single impact COR was used to account for the kinetic energy loss. It was shown that with such a relatively simple model very similar behavior of the overall impact dynamics was achieved compared to the detailed FEA.

References


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