Phase-field topology optimization model that removes the curvature effects

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Abstract

The conventional phase-field topology optimization (PFTO) models minimize not only the objective function but also the interface energy. In the present study, a new PFTO model, which minimizes only the objective function, is developed by removing the curvature effect from the conventional PFTO model. Simulations of elastic strain energy minimization under a constant-volume constraint condition of a cantilever are performed using the developed and conventional PFTO models. From the simulation results, we confirm that the developed PFTO model that removes the curvature effects can efficiently optimize only the objective function.

Key words : Phase-field method, Finite element method, Topology optimization, Maximum stiffness, Curvature

1. Introduction

The phase-field method has emerged as a powerful numerical tool to simulate the formation process of complicated material microstructures (Chen, 2002). The successful work of Kobayashi to simulate the complicated and beautiful dendrite growth during solidification opened the phase-field study area (Kobayashi, 1993; Kobayashi, 1994). Since then, in the solidification field (Asta et al., 2009; Steinbach, 2009; Takaki, 2014), the phase-field model has been applied to binary alloys (Warren and Boettinger, 1995; Wheeler et al., 1992), polycrystals (Miyoshi and Takaki, 2016; Steinbach and Pezzolla, 1999), quantitative models (Karma and Rappel, 1996; Ohno and Matsuura, 2009), multi-component alloys (Ohno et al., 2012), coupled models with convection (Beckermann et al., 1999; Rojas et al., 2015; Takaki et al., 2015b), and large-scale computations (Sakane et al., 2015; Shibuta et al., 2015; Takaki et al., 2014; Takaki et al., 2013; Yamanaka et al., 2011). The great success of the phase-field method in material science is due to the advantages of the phase-field method: tracking the interface position is not necessary, the curvature effects are included in the model, the evolution equations can be derived from the free-energy functional based on the second law of thermodynamics, the discretization of the time evolution equations is easy, and so on. Even with the exception of material microstructures, the phase-field method is widely used in surface morphology (Takaki et al., 2006; Takaki et al., 2008), dislocation dynamics (Wang and Li, 2010), crack propagation (Ambati et al., 2015; Oshima et al., 2014), biomechanics (Lee et al., 2016; Takaki et al., 2015a), multiphase flow (Takada et al., 2006), and so on.

The phase-field method has also been successfully applied to the topology optimization problem. Here and in the rest of this paper, we call the topology optimization model using the phase-field method as the phase-field topology optimization (PFTO) model. To the best of our knowledge, the first application of the phase-field method to topology optimization was performed by Bourdin and Chambolle (Bourdin and Chambolle, 2003; Bourdin and Chambolle, 2006). For a while since then, the Cahn–Hilliard (CH) model has been widely applied to topology optimization (Burger and Stainko, 2006; Dedè et al., 2012; Wang and Zhou, 2004a; Wang and Zhou, 2004b; Zhou and Wang, 2007; Zhou and Wang, 2006). The CH model can conserve the volume without any operation. However, it needs a very small time increment because the time evolution equation becomes a fourth-order partial differential equation. The first successful application of the Allen–Cahn (AC) equation to topology optimization was performed by Takezawa et al. (Takezawa et al., 2010) in a minimum compliance problem. Because the model is very simple and therefore has wide versatility and applicability, it was employed in many subsequent studies. Choi et al. (Choi et al., 2011) enabled spontaneous nucleation of holes using a reaction–diffusion equation without a double-well potential in the phase-field model. Yamada et al. (Yamada et al., 2010) developed a level set method incorporating a fictitious interface energy, which is a similar idea to

the phase-field method. Lim et al. (Lim et al., 2011) also developed a level set topology optimization method based on the concept of the phase-field method and applied it to a magnetic actuator. Wallin and Ristinmaa (Wallin and Ristinmaa, 2013) applied the Takezawa model to the minimum compliance problem under a constant-volume problem. Recently, the model has been applied to various problems: nonlinear elasticity (Penzler et al., 2012), structure design in electromagnetic fields (Lim et al., 2014), electromagnetic wave propagation problem (Takezawa and Kitamura, 2014), Stokes flow (Garcke and Hecht, 2015), finite strain (Wallin and Ristinmaa, 2015), and multi-materials (Tavakoli, 2014). As a numerical scheme, unstructured polygonal mesh (Gain and Paulino, 2012) and adaptive mesh (Wallin and Ristinmaa, 2015) are employed to the PFTO model.

In the phase-field method, the time evolution equation is reduced to a reaction–diffusion equation, where the interface migration is derived by the balance of the reaction and diffusion terms. Takezawa et al. (Takezawa et al., 2010) insisted that a structure with reasonable complexity can be computed by adjusting the gradient coefficient, which controls the contribution of the Laplacian operator, in the phase-field equation because the structures computed by the conventional optimization methods such as the homogenization (Bendsøe and Kikuchi, 1988; Bourdin, 2001) and level-set (Allaire et al., 2004; Osher and Santosa, 2001; Sethian and Wiegmann, 2000; Wang et al., 2003) methods are too fine, which cannot be manufactured. When the phase-field method is employed in topology optimization, an objective to reduce the interface perimeter is added to the objective function, which is caused by the curvature driving of the interface migration due to the presence of Laplacian operator in the phase-field equation. The curvature effect is very important in material microstructures. However, the curvature effect should be omitted from the phase-field time evolution equation in the PFTO, and only the original objective function should be minimized. Recently, a conserved AC equation that removes the curvature effects from a conventional AC equation has been well studied in the area of computational fluid dynamics (Chiu and Lin, 2011; Sun and Beckermann, 2007; Takada et al., 2014).

In the present study, we propose a PFTO model which has no curvature effect. Specifically speaking, we introduce a new time evolution law by removing the interface motion driven by the curvature from the conventional PFTO model. As far as we know, this is the first attempt in the PFTO model and we believe it to be the novelty of the present study. Here, we treat a strain energy minimization problem as a sample problem and employ our previous PFTO model (Takaki, 2011). Through a series of simulations of a cantilever using the new and previous PFTO models, the fundamental characteristics of the developed model are investigated in detail. This model was derived so that the free-energy functional is reduced depending on the second law of thermodynamics, which is the same as the phase-field method in material science (Takaki, 2014).

2. Model

Here, we first derive our previous PFTO model to aim for maximum stiffness while maintaining a constant volume (Takaki, 2011). Then, we propose a new PFTO model by removing the curvature effect from the conventional model. In this study, we consider an optimization problem where the elastic strain energy of a system is minimized under a constant volume. The objective function is the elastic strain energy and the design variable is the phase-field variable. In our problem setting, the minimization of elastic strain energy as the objective function is equivalent to the maximization of stiffness problem.

2.1 PFTO model

Phase-field variable ϕ is defined as $\phi = 1$ in a solid (material) and $\phi = 0$ in gas (hole), and it smoothly changes at the solid–gas interface. Using the phase-field variable, the solid density is defined as $p(\phi) = \phi^3(10 - 15\phi + 6\phi^2)$. The free-energy functional F is expressed as

$$F = F_m + F_p - F_f, \quad (1)$$

where F_m is the free-energy functional of a material, F_p is the penalty energy to maintain a constant volume, and F_f is the potential of the external force. F_m is expressed by

$$F_m = \int_V f dV, \quad (2)$$

where f is the free-energy density expressed as

$$f = f_{grad} + f_{doub} - f_{elast}. \quad (3)$$

Here, the gradient energy density f_{grad} , the double-well potential f_{doub} , and the elastic strain energy density f_{elast} are expressed by the following equations:

$$f_{grad} = \frac{a^2}{2} |\nabla \phi|^2, \quad (4)$$

$$f_{doub} = Wq(\phi) = W\phi^2(1-\phi)^2, \quad (5)$$

$$f_{elast} = \frac{1}{2} C_{ijkl} \varepsilon_{ij} \varepsilon_{kl} p(\phi), \quad (6)$$

where a is the gradient coefficient, W is the height of the double-well potential, C_{ijkl} is the elastic coefficient tensor, and ε_{ij} is the strain tensor. To maintain a constant volume, F_p , which is expressed as

$$F_p = k|V - V_0|, \quad (7)$$

is introduced into the free-energy functional, namely, Eq. (1), where k is the penalty coefficient, V_0 is the predetermined constant solid volume, and V is the current volume of the solid computed by $V = \int_V p(\phi) dV$. When traction vector t_i is applied to the structure, F_f , which is expressed as

$$F_f = - \int_S u_i t_i dS, \quad (8)$$

is added to the free-energy functional in Eq. (1), where u_i is the displacement vector and S is the outer surface of the material.

The point of this PFTO model is that the signs of F_f and f_{elast} in Eqs. (1) and (3), respectively, are opposite to those of the conventional phase-field model used to simulate the material microstructure evolution, where the material microstructures evolve to reduce the elastic strain energy (Chen, 2002; Takaki et al., 2008; Tsukada et al., 2014; Tsukada et al., 2011; Wang and Li, 2010; Yamanaka et al., 2010). By substituting Eqs. (2), (7), and (8) into Eq. (1), we can obtain the following equation:

$$F = \int_V \left(\frac{a^2}{2} |\nabla \phi|^2 + W\phi^2(1-\phi)^2 \right) dV + \left(\int_S u_i t_i dS - \int_V \frac{1}{2} C_{ijkl} \varepsilon_{ij} \varepsilon_{kl} p(\phi) dV \right) + k|V - V_0|. \quad (9)$$

This is the objective function in the formulation and is minimized by solving the following phase-field equation. Meanwhile, actual objective function is the second term in the right side of Eq. (9). This means that the displacement u_i is minimized in a condition applying an external force, which indeed corresponds to the stiffness maximization, or the elastic strain energy is minimized. Consequently, it is a key on how to remove the contribution of the interface energy from the objective function in the formulation. However, the first term at the right-hand side of Eq. (9) is needed to derive the following time evolution equation of the phase-field variable.

By substituting the free-energy functional, Eq. (1), into the AC equation

$$\frac{\partial \phi}{\partial t} = -M_\phi \frac{\delta F}{\delta \phi}, \quad (10)$$

and considering the following relationship (Takaki, 2014):

$$a = \sqrt{\frac{3\delta\gamma}{b}}, \quad W = \frac{6\gamma b}{\delta}, \quad \text{and} \quad M_\phi = \frac{\sqrt{2W}}{6a} M = \frac{b}{3\delta} M, \quad (11)$$

the phase-field equation can be obtained, i.e.,

$$\frac{\partial \phi}{\partial \tilde{t}} = \frac{\delta^2}{8b^2} \nabla^2 \phi + \phi(1-\phi) \left(\phi - \frac{1}{2} + \frac{5\delta}{4\gamma b} \frac{1}{2} C_{ijkl} \varepsilon_{ij} \varepsilon_{kl} \phi(1-\phi) \pm \frac{5\delta}{4\gamma b} k\phi(1-\phi) \right). \quad (12)$$

Here, $\delta F/\delta \phi$ is the functional derivative of F , δ is the interface thickness, γ is the interface energy, M is the interface mobility, and $b = 2 \tanh^{-1}(1-2\phi_0)$ is a constant that determines the interface region and is usually set as $\phi_0 = 0.1$. Time $\tilde{t}$ is a nondimensional time expressed as $\tilde{t} = t\delta^2/(8b^2\gamma M)$, and $\tilde{t}$ is expressed by t here and in the rest of this paper.

Equation (12) determines the morphology due to the balance between the elastic strain energy and the interface energy under a constant volume. In other words, the structure changes to reduce the elastic strain energy and the interface energy.

Because the actual objective function is only the elastic strain energy, Eq. (12) is modified as follows to increase the contribution of the strain energy as much as we possibly can (Takaki, 2011):

$$\frac{\partial \phi}{\partial t} = \frac{\delta^2}{8b^2} \nabla^2 \phi + \phi(1-\phi) \left(\phi - \frac{1}{2} + \beta \right), \quad (13)$$

with

$$\beta = \begin{cases} \frac{e(\phi)}{\|e(\phi)\|} \beta_1 + 4\phi(1-\phi)\beta_2' & \cdots & e(\phi) < \|e(\phi)\| \\ \beta_1 + 4\phi(1-\phi)\beta_2' & \cdots & e(\phi) \geq \|e(\phi)\| \end{cases}, \quad (14)$$

where

$$e(\phi) = \frac{1}{2} C_{ijkl} \varepsilon_{ij} \varepsilon_{kl} \phi(1-\phi), \quad (15)$$

$$\|e(\phi)\| = \sqrt{\frac{1}{N} \sum^N [e(\phi)^2]}. \quad (16)$$

Here, $\|e(\phi)\|$ is the root mean square of $e(\phi)$ in the interface region, and N is the number of lattices in $\phi_0 \leq \phi \leq (1-\phi_0)$. β is the driving force to determine the rate and direction of the interface migration, and should be in the range $-0.5 \leq \beta \leq 0.5$ to stably compute Eq. (13). β_1 and β_2 are positive constants expressing the contribution of the elastic strain energy and volume conservation, respectively, in β with constraint $\beta_1 + \beta_2 \leq 0.5$. β_2' is determined at in every time step to accurately maintain a constant volume (or $V = V_0$) in $-\beta_2 < \beta_2' < \beta_2$. Equation (14) means that, in the case of a high elastic strain energy condition with $e(\phi) \geq \|e(\phi)\|$, the interface migrates so as to increase the solid with a maximum driving force, and, in the case of $e(\phi) < \|e(\phi)\|$, the interface migration direction is determined depending on the amount of $e(\phi)$. The other details are explained in (Takaki, 2011).

2.2 Curvature effects

In Eq. (13), we have tried to maximize the contribution of the objective function (or the elastic strain energy) within the condition in which the computation of Eq. (13) can be stably performed by properly designing the driving force β expressed by Eq. (14). However, even if we make an increase the contribution of the strain energy, the curvature-driven interface migration cannot be removed. In other words, the objective function includes the contribution of the interface energy. Therefore, to minimize only the actual objective function in the PFTO method, the curvature effect should be removed from the phase-field equation.

In the phase-field equation, the curvature effect is included in the diffusion term or Laplacian. Considering a local coordinate on the interface, the Laplacian can be expressed as

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial r^2} + \kappa \frac{\partial \phi}{\partial r}, \quad (17)$$

where r is the coordinate in the normal direction of the interface and κ is the curvature of the interface. Here, the derivation with respect to the tangential direction is neglected. To remove the curvature effect, we must remove the second term in the right-hand side of Eq. (17) from the Laplacian. By considering the unit interface normal vector $\mathbf{n} = -\nabla \phi / |\nabla \phi|$, $\partial \phi / \partial r$ and κ can be expressed as follows:

$$\frac{\partial \phi}{\partial r} = \frac{\partial \phi}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial \phi}{\partial y} \frac{\partial y}{\partial r} = \nabla \phi \cdot \mathbf{n} = \nabla \phi \cdot \left(-\frac{\nabla \phi}{|\nabla \phi|} \right) = -|\nabla \phi|, \quad (18)$$

$$\kappa = \nabla \cdot \mathbf{n} = -\nabla \cdot \left(\frac{\nabla \phi}{|\nabla \phi|} \right). \quad (19)$$

Consequently, the second term in the right-hand side of Eq. (17) is expressed as

$$\kappa \frac{\partial \phi}{\partial r} = |\nabla \phi| \nabla \cdot \left(\frac{\nabla \phi}{|\nabla \phi|} \right). \quad (20)$$

2.3 Development of new PFTO model

For the rest of this paper, the following two models, namely, Models A and B, are compared by simulations.

Model A:

$$\frac{\partial \phi}{\partial t} = \bar{\delta}^2 \nabla^2 \phi + \phi(1-\phi) \left(\phi - \frac{1}{2} + \beta \right) \quad (21)$$

Model B:

$$\frac{\partial \phi}{\partial t} = \bar{\delta}^2 \left(\nabla^2 \phi - |\nabla \phi| \nabla \cdot \left(\frac{\nabla \phi}{|\nabla \phi|} \right) \right) + \phi(1-\phi) \left(\phi - \frac{1}{2} + \beta \right) \quad (22)$$

Here, we use $\bar{\delta} = \delta / (2\sqrt{2}b)$. Model A, expressed by Eq. (21), is the conventional model identical to Eq. (13). Model B is the proposed model in this paper and is obtained by subtracting Eq. (20) from the Laplacian in Eq. (21). In other words, Eq. (22) can be considered as one-dimensional phase-field equation in the interface normal direction. Therefore, except for the elimination of the curvature effect, Eq. (22) can retain the characteristics of the phase-field method. Equation (14) is used as β for Eqs. (21) and (22).

The following simulations are performed in two dimensions. Equations (21) and (22) are discretized using a normal finite-difference method, or a forward difference is used for the time and a second-order central difference is used for the Laplacian. Although Eq. (22) includes singular diffusivity (Kobayashi and Giga, 1999), $|\nabla \phi|$ is multiplied to the term and we can omit the term when $|\nabla \phi| = 0$. The stress fields are computed by finite-element method using a four-node isoparametric element. The material is assumed to have an isotropic linear elasticity with Young's modulus E . The Young's modulus varies as $E(\phi) = p(\phi)E$ using the solid density $p(\phi)$ in the interface region. In the gas phase or hole, $E(0)$ is set to $E/1000$. The finite-difference and finite-element meshes have the same size and share the same nodes.

3. Computational conditions

In the simulations presented in the next section, stiffness maximization simulations of a cantilever subjected to a concentration force are performed. Figure 1 shows the initial morphology and boundary conditions. The displacements in the x - and y -directions of the nodes along the left end are constrained, and concentrated force P is applied in the y -direction of the node at the center of the right end. The color indicates ϕ , blue represents the material with $\phi = 1$, and the hole represents that with $\phi = 0$. The computational domain size is $400\Delta x \times 200\Delta x$ with mesh size Δx . The initial solid fraction is set to 0.5, and it is maintained during the simulation. A plane stress problem with unit thickness is assumed for the stress field. Time increment Δt is set to $\Delta t = \Delta x^2 / (5\bar{\delta}^2)$ to stably and explicitly solve Eqs. (21) and (22), and a 10,000-step computation is performed. The boundary conditions for the phase field are set to zero Neumann for all boundaries. The Poisson's ratio is set to 0.3.

Fig. 1 Initial morphology and boundary conditions for the stiffness maximization simulations of a cantilever where the displacements in the left side are constrained and nodal force P is applied at the center in the right side. The blue region is inside a material with $\phi = 1$, and the white regions are holes with $\phi = 0$.

4. Results and discussion

4.1 Changes in morphology and stiffness

Figure 2 shows the morphological changes in the simulations for Models A and B under the conditions $\delta = 3\Delta x$, $\beta_1 = 0.25$, and $\beta_2 = 0.05$. Figure 3 shows the variation in the stiffness ratio K/K_{ini} , where stiffness K is defined as $K = P/v$ and K_{ini} is the K for the initial morphology shown in Fig. 1. Here, v is the displacement in the y -direction at the point where P is applied. In Fig. 3, the maximum stiffness during the simulations is shown by the dashed lines. Note that stiffness K is a scalar here. In the computational condition shown in Fig. 1, Eq. (8) can be expressed as $F_f = -\int_S u_i t_i dS = -vP = -P^2/K$. Therefore, minimizing v corresponds to maximizing K . Figure 3 shows some non-smooth cusp points. These cusp points are caused by the elimination of members. In the present model, the morphological change simply occurs from a fine to coarse one.

Fig. 2 Morphological changes during the simulations from the initial condition shown in Fig. 1 for Models A and B. (a) 250, (b) 1,000, (c) 2,000, (d) 6,000, and (e) 10,000 steps.

Fig. 3 Variations in stiffness ratio K/K_{ini} during the simulations shown in Fig. 2.

At the 250th step shown in Fig. 2(a), the topologies in Models A and B are almost the same, although rounded holes are observed in Model A. At the 250th step, the stiffness is almost the same as that shown in Fig. 3. However, at around 400 steps, the differences in the stiffness ratio K/K_{ini} between Models A and B are apparent. Specifically, in Model A, sometimes K/K_{ini} decreases rapidly until 1,000 steps. In Model B, reductions in K/K_{ini} also occur at around 1,000 steps. However, the amount of K/K_{ini} reduction is much smaller in Model B than that in Model A, and the step where the K/K_{ini} reduction starts is delayed for Model B compared with Model A. As a result, the topology of Model A is coarser than that of Model B at the 1,000th step, as shown in Fig. 2(b). From 1,000 to 2,000 steps, the topology of Model B also becomes coarser. Figure 2(c) shows that the topologies of Models A and B become the same at the 2,000th step. On the other hand, Fig. 3 shows the presence of a large difference in K/K_{ini} even for the same topology. In Model A, a small cusp can be seen at around 3,000 steps due to the disappearance of small holes, and temporal large reductions are observed after 6,000 steps due to the disappearance of some holes. Figures 2(c)–(e) show that the morphology of Model A continuously and gradually coarsens. The maximum stiffness of Model A is observed at around 6,000 steps. In Model B, the topologies after 2,000 steps remain the same, as shown in Figs. 2(c)–(e), and the stiffness monotonically increases.

From the results shown in Figs. 2 and 3, we conclude that Model B, proposed in this paper, can efficiently increase the stiffness compared with conventional Model A. The differences in the variations in morphology and stiffness between Models A and B are determined by whether the interface migration is driven by the curvature or not. In Model A, the interface migration is driven by the interface curvature, in addition to the strain energy and volume constraint in Model B.

4.2 Effects of β_1

Here, we present the investigation of the effects of the amount of β_1 on the stiffness change. β_1 determines the contribution of strain energy to the value that drives the interface migration. β_1 changes from 0.05 to 0.45 with an increment of 0.1 under constant $\beta_2 = 0.05$. $\beta_2 = 0.05$ is sufficient to keep the volume within an error of 0.1%.

Figure 4 shows the variations in the stiffness ratio K/K_{ini} . Figure 4(a), which shows Model A, illustrates that the maximum stiffness indicated by the filled circles is obtained at different time steps, and the amount of maximum stiffness is different depending on β_1 . The morphologies of Model A at the maximum stiffness point are shown in the left side of Fig. 5. The figures shown in Fig. 5 are the step numbers that indicate the maximum stiffness. From the figures, we can see that the morphologies that indicate the maximum stiffness are significantly different depending on β_1 . As for Model B, on the other hand, Fig. 4 shows that the stiffness monotonically increases with time after some perturbations. Therefore, the maximum stiffness for each β_1 is obtained at the final 10,000 steps in the present simulations. The maximum stiffness values are the same for large β_1 , i.e., $\beta_1 = 0.25, 0.35$, and 0.45 . This phenomenon is also observed from the morphologies at 10,000 steps shown in the right side of Fig. 5. According to the final morphologies, the same morphologies are observed at $\beta_1 = 0.25, 0.35$, and 0.45 . In addition, the topology at $\beta_1 = 0.15$ is the same as those at $\beta_1 = 0.25, 0.35$, and 0.45 . Therefore, if the simulation at $\beta_1 = 0.15$ is continued much longer, the stiffness and morphology will be the same as those at $\beta_1 = 0.25, 0.35$, and 0.45 . The topology obtained at $\beta_1 = 0.05$ is coarser than those obtained at other β_1 values because the contributions of the strain energy and volume constraint that drive the interface are the same.

From the above discussion, we can conclude that stable results that are independent of β_1 can be obtained using Model B if we use β_1 , which is larger than a certain value. From the perspective of computational cost, we can obtain the desired result with a small computational cost by setting β_1 larger. In conventional Model A, although we can obtain a high stiffness structure similar to that of Model B when we use larger β_1 , the results are not stable and largely dependent on the amount of β_1 .

Fig. 4 Variations in stiffness ratio K/K_{ini} during the simulations for Models A and B when β_1 is changed.

Fig. 5 Morphologies at maximum stiffness ratio K/K_{ini} in Fig. 4 for Models A and B. Here, $\beta_1 =$ (a) 0.05, (b) 0.15, (c) 0.25, (d) 0.35, and (e) 0.45. The figures shown in Model A are the step numbers corresponding to the filled circles in Fig. 4(a).

4.3 Effects of interface thickness δ

In many AC-type PFTO models, equations similar to Eq. (21) are used. Then, the complexities of a structure are considered to be controlled by the gradient coefficient or $\bar{\delta}^2$. However, changing $\bar{\delta}^2$ in Eq. (21) means that both interface thickness and contribution of the curvature in the driving force increase. From the above results, to perform topology optimization using the phase-field method with high accuracy, we can conclude that the phase-field equation should be used, where the curvature contribution is removed from the diffusion term. Here, we show that the structure complexities can be controlled by changing $\bar{\delta}^2$ even in Eq. (22).

Figure 6 shows the stiffness variations in Model B when the interface thickness is changed, i.e., $\delta = 2\Delta x$, $3\Delta x$, $4\Delta x$, $5\Delta x$, and $6\Delta x$. Here, we use the conditions $\beta_1 = 0.45$ and $\beta_2 = 0.05$. In all curves, the stiffness values rapidly increase at the beginning of the simulations and approach an almost constant region with very small but monotonous increase through some preservation because of the topology changes. The final stiffness clearly increases with decreasing interface thickness. Figure 7 shows the morphologies at the final 1.0×10^4 th step shown in Fig. 6. The structures become more complicated as the interface thickness decreases. As shown here, the structure complexities can be controlled by changing the interface thickness in Eq. (22). The morphological change for $\delta = 2\Delta x$ cannot be observed after 900 steps, and the stiffness becomes constant, as shown in Fig. 6. This means that the computation with $\delta = 2\Delta x$ is a little bit unstable. In our experience, $\delta = 3\Delta x$ is the smallest interface thickness to stably compute the PFTO model. Although we can change $\bar{\delta}^2$ in Eqs. (21) and (22) to control the complexities of the structure, we believe that changing mesh size Δx while keeping $\delta = 3\Delta x$ is better for the computational cost and accuracy. If we fix $\delta = 3\Delta x$, we do not need any other parameter setting except for mesh size Δx .

Fig. 6 Variations in stiffness ratio K/K_{ini} during the simulations using Model B with different interface thicknesses δ for $\beta_1 = 0.45$ and $\beta_2 = 0.05$.

Fig. 7 Morphologies at 1.0×10^4 th step in the simulations shown in Fig. 6: $\delta =$ (a) $2\Delta x$, (b) $3\Delta x$, (c) $4\Delta x$, (d) $5\Delta x$, and (e) $6\Delta x$.

5. Conclusions

In this paper, we have proposed a new phase-field topology optimization (PFTO) model where the curvature effect is removed from the conventional model. By simulating of elastic strain energy minimization under a constant-volume constraint of a cantilever, we have shown that the new model can stably compute the optimum shapes. In other words, the new model can efficiently minimize only the objective function. We can also conclude that by increasing the contribution of the strain energy (or β_1 in Eq. (13)), we can reduce the computational cost. In addition, by changing the interface thickness in the new model, we can control the complexities of the final structure.

The present method is very simple, but we consider that it is essential and important in the PFTO method. Removing the curvature effects from all previous PFTO models is possible using the method proposed in this study. Therefore, the present method can be considered to be a highly versatile method. If we want to design a rounded-shape structure, we can control the curvature reduction term or Eq. (20) in Eq. (22). In the future, we will use the PFTO model to design the material microstructures in multi-scale and multi-material problems (Kato et al., 2009; Kato et al., 2014) because the phase-field method is excellent at simulating the material microstructures. Therefore, successive simulations of an optimum material microstructure using the PFTO model and phase-field simulations to determine the process to make the optimum material microstructures should be performed. However, in successive simulations of the material microstructures, the conventional PFTO model that includes the curvature effects may be better because curvature effects are naturally included in the material microstructure formations. The PFTO models should be properly used depending on the purposes.

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