Constitutive modeling of cortical bone considering anisotropic inelasticity and damage evolution

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Abstract
Researchers have conducted finite element (FE) analyses with human models to predict injuries due to traffic accidents or falling. In most of their analyses, cortical bone was simply modeled as a general isotropic elastoplastic material. In this study, a constitutive model of cortical bone considering anisotropic inelasticity and damage evolution was developed to predict injuries more accurately. The new model can satisfactorily represent mechanical properties of cortical bone including anisotropy of elastic modulus and yield stress with strain-rate dependency, and asymmetric stress–strain curves in tension and compression. Simultaneously the included damage-evolution equation enables to predict failure stress and strain with rate dependency in bone fracture simulations. The proposed model was verified using experimental data obtained from the literature. We applied the proposed model to a simple cylindrical FE model of the human femur, and performed simulations under loading conditions such as tension, compression, and torsion. The results showed some tendency of characteristic fracture patterns such as transverse fracture in tensile loading, oblique fracture in compression, and spiral fracture in torsion. The proposed constitutive model would have the potential for better injury prediction in the future.

Key words: Cortical bone, Constitutive modeling, Anisotropy, Damage evolution, Fracture, Rate dependency

1. Introduction

In recent years, many researchers have conducted finite element (FE) analyses with human models to predict injuries due to traffic accident or falling (Tanaka et al., 2004; Watanabe et al., 2011). However, compared with the reproduction accuracy of the shapes of bones, the material model used in such analyses is simplified because mechanical properties of biomaterials have not been fully explained and satisfactory constitutive models do not exist. In most FE analyses, the option of isotropic elastoplastic material built in general-purpose FE-analysis software is selected as the mechanical property for cortical bone.

Many researchers studied the actual mechanical properties of cortical bone. For example, Reilly and Burstein (1975) and Yoon and Katz (1976) conducted experiments that found transverse isotropy in elastic modulus of the femur. Moreover, at the point of strength and damage, anisotropy is observed along the circumferential and radial directions; hence, cortical bone could be considered as an orthotropic material (Currey, 2002). McElhaney (1966), Wood (1971), and Carter and Hayes (1977) researched strain-rate dependency in the stress–strain curve of cortical bone; they found that the stiffness and brittleness were proportional to the strain rate. In addition, the strength of cortical bone decreases in the order of compressive, tensile, and torsional loading (Yamada, 1970). As explained previously, the mechanical properties of cortical bone considerably depend on the loading direction, differences in strain rates, and loading patterns such as tension and compression (Cowin et al., 1987).

For long bones, such as the femur, some types of fracture patterns could be qualitatively observed depending on the loading conditions (Gomez and Nahum, 2002; Rich et al., 2005; Piece, 2004). For example, in the case of tensile...
loading, bone tends to break in the transverse direction with respect to the axis direction (transverse fracture). However, in the case of compressive loading, fracture occurs diagonally with respect to the axis direction (oblique fracture) because the shear strength of bones is weaker than the compressive strength. In torsional loading, spiral fracture is observed. The differences in fracture patterns are considered to be due to the anisotropy in the strength of cortical bone, strain-rate dependency, and asymmetric tensile or compressive loading. However, it is difficult to predict fracture patterns and occurrences quantitatively using a simple constitutive model. The fracture pattern strongly influences the duration of a treatment; hence, by accurately predicting the fracture patterns, the possibility of long-term impairment and disability could be considered while evaluating car-crash safety devices and protection instruments.

Hence, we developed a constitutive model of cortical bone satisfactorily describing the risk of fracture and fracture patterns to more accurately predict injuries. Previously, we developed a constitutive model of cortical bone considering transverse isotropy, strain-rate dependency of elastic modulus, anisotropy in yield stress, and asymmetry in tensile or compressive loading, and then applied the model to injury analyses of the foot and ankle regions due to frontal impacts (Iwamoto et al., 2005). However, the model could not fully represent the properties of cortical bone because the frameworks of the viscoelastic constitutive model and anisotropic damage evolution were not considered. Other constitutive models of cortical bone were also proposed by some researchers. Johnson et al. (2010) proposed a viscoelastic, viscoplastic constitutive model which is valid at strain rates ranging over seven orders of magnitude. However, their model does not incorporate anisotropy and failure. Sahoo et al. (2013) also proposed an anisotropic composite human skull model taking into account damage, but the validation for rate dependency was not demonstrated in the material level. Hence, in this study, we developed a new constitutive model considering orthotropic damage of cortical bone including a new framework.

The proposed model was validated using the experimental data obtained from the literature. In addition, we introduced the new model into the user material subroutine of LS-DYNA, and then performed simulations on a simple cylindrical FE model of the human femur, under loading conditions such as tension, compression, and torsion. The results verified the effectiveness of the constitutive model for general injury prediction.

2. Formulation of the anisotropic inelastic constitutive equation of cortical bone

2.1 Framework

This chapter explains the orthotropic constitutive model, which can accurately describe the deformation properties and damage phenomenon of cortical bone. The proposed model is intended for long bones, such as the femur. Moreover, the model assumed that the orthotropy consisted of longitudinal, circumferential, and radial directions in arbitrary points in the cortical bone. The tensor notation used in this study is based on the summation convention.

First, the proposed constitutive model is a model that combines the viscoelastic and viscoplastic models. The model can be described as a rheology model for illustration in one dimension, indicated in Fig. 1. Based on the infinitesimal deformation theory, we defined the total strain as the sum of the viscoelastic and viscoplastic strains.

\[
\varepsilon = \varepsilon^{ve} + \varepsilon^{vp}
\]

where \(\varepsilon\), \(\varepsilon^{ve}\), and \(\varepsilon^{vp}\) are the total strain tensor, viscoelastic strain tensor, and viscoplastic strain tensor, respectively.

To explain the orthotropy of cortical bone, we introduced two structural tensors \(A_0\) and \(G_0\). They are defined by using each unit vector \(a_0\) in the longitudinal direction and \(g_0\) in the circumferential direction, given in Fig. 2 and Eq. (2).

\[
A_0 = a_0 \otimes a_0, \quad G_0 = g_0 \otimes g_0
\]

The inelastic deformation properties and damage phenomenon in cortical bone is explained based on the theory of the irreversible thermodynamics constitutive equation (Lemaitre and Chaboche, 2002).

We defined the Helmholtz free energy \(\Psi\) as the function of viscoelastic strain tensor \(\varepsilon^{ve}\), the second order damage tensor \(D\), structural tensor \(A_0\), \(G_0\), internal state variable of viscoelasticity \(\gamma_1\), and isotropic hardening variable \(r\). The sum of the viscoelastic part \(\Psi^{ve}\) and viscoplastic part \(\Psi^{vp}\) is considered as \(\Psi\) (Eq. (3)).
\[ \Psi = \Psi^{ve}(\varepsilon^\text{ve}, D, A_0, G_0, Y_1, \ldots, Y_n) + \Psi^{vp}(D, r) \]  \hfill (3)

The following equation expresses the Clausius–Duhem inequality, which is correlated to the second law of thermodynamics. Here, "\( \cdot \)" denotes the dot product of tensors, i.e., \( X : Y = X_{ij}Y_{ij} \).

\[ \sigma : \dot{\varepsilon}^{vp} + \left( \sigma - \frac{\partial \Psi}{\partial \varepsilon^\text{ve}} \right) : \dot{\varepsilon}^\text{ve} - \frac{\partial \Psi}{\partial Y_i} : \dot{Y}_i - \frac{\partial \Psi}{\partial r} \dot{r} \geq 0 \]  \hfill (4)

The flow rate \( J \) and thermodynamic conjugate force \( X \) are defined using the following expressions:

\[ J = \{ \dot{\varepsilon}^{vp}, \dot{r}, \dot{Y}_i \} \]  \hfill (5)

\[ X = \{ \sigma, -R, Q_i \} \]  \hfill (6)

Hence, the energy dissipation is expressed using Eq. (7) (Murakami, 2012).

\[ \Pi = J : X \geq 0 \]  \hfill (7)

Then, by using Eqs. (5) and (6), Eq. (8) can be expressed as follows.

\[ \Pi = \sigma : \dot{\varepsilon}^{vp} - R\dot{r} + Q_i : \dot{Y}_i \geq 0 \]  \hfill (8)

Referring to the study by Voyiadis and Deliktas (2000), energy dissipations due to viscoelasticity \( \Pi^{ve} \) and viscoplasticity \( \Pi^{vp} \) are considered independent of each other, i.e., it can be described as follows.

\[ \Pi = \Pi^{ve} + \Pi^{vp} \geq 0 \]  \hfill (9)

\[ \Pi^{ve} = Q_i : \dot{Y}_i \geq 0 \]  \hfill (10)

\[ \Pi^{vp} = \sigma : \dot{\varepsilon}^{vp} - R\dot{r} \geq 0 \]  \hfill (11)

Fig. 1 A simple 1D rheological model consisting of a generalized Maxwell element and a viscoplastic element connected in series.

Fig. 2 Unit vectors representing longitudinal and circumferential directions.

### 2.2 Formulation using Helmholtz free energy

An actual form of the Helmholtz free energy function is assumed, expressed as Eq. (12).

\[ \Psi = \frac{1}{2} \varepsilon^\text{ve} : C_0(D, A_0, G_0) : \varepsilon^\text{ve} + \sum_{i=1}^{N} \frac{1}{2} (\varepsilon^\text{ve} - Y_i) : C_i(D, A_0, G_0) : (\varepsilon^\text{ve} - Y_i) \]

\[ + R^{ve}(D) \left[ r + \frac{1}{b(D)} \exp(-b r(D)) - \frac{1}{b(D)} \right] \]  \hfill (12)
Here \( A : B = A_{ijkl}B_{kl}e_i \otimes e_j \). In Eq. (12), the first and second viscoelastic terms are the sum of the energy stored in the spring elements of the 3D extended generalized Maxwell model. \( N \) represents the number of spring–dashpot series elements.

The stiffness tensors \( \mathcal{C}_0 \) and \( \mathcal{C}_i \) and the viscosity coefficient tensor \( \mathcal{G}_i \) are the fourth order tensors describing the orthotropy. These tensors are represented in Eqs. (13)-(15), referring to the description in the case of transverse isotropy (Jaunzemis, 1967).

\[
\mathcal{C}_0 = \lambda_0 \mathbb{I} + \mu_0 \frac{1}{2}(\mathbb{I} + \tilde{\mathbb{I}}) + \alpha_0 (\mathbb{I} \otimes A_0 + A_0 \otimes \mathbb{I}) + \beta_0 (\mathbb{I} \otimes A_0 + A_0 \otimes \mathbb{I}) + \gamma_0 A_0 \otimes A_0 + \delta_0 (\mathbb{I} \otimes \mathbb{G}_0 + \mathbb{G}_0 \otimes \mathbb{I}) + \epsilon_0 (\mathbb{I} \otimes \mathbb{G}_0 + \mathbb{G}_0 \otimes \mathbb{I}) + \zeta_0 \mathbb{G}_0 \otimes \mathbb{G}_0 + \eta_0 (\mathbb{A}_0 \otimes \mathbb{G}_0 + \mathbb{G}_0 \otimes \mathbb{A}_0)
\]

\[
\mathcal{C}_i = \lambda_i \mathbb{I} + \mu_i \frac{1}{2}(\mathbb{I} + \tilde{\mathbb{I}}) + \alpha_i (\mathbb{I} \otimes A_0 + A_0 \otimes \mathbb{I}) + \beta_i (\mathbb{I} \otimes A_0 + A_0 \otimes \mathbb{I}) + \gamma_i A_0 \otimes A_0 + \delta_i (\mathbb{I} \otimes \mathbb{G}_0 + \mathbb{G}_0 \otimes \mathbb{I}) + \epsilon_i (\mathbb{I} \otimes \mathbb{G}_0 + \mathbb{G}_0 \otimes \mathbb{I}) + \zeta_i \mathbb{G}_0 \otimes \mathbb{G}_0 + \eta_i (\mathbb{A}_0 \otimes \mathbb{G}_0 + \mathbb{G}_0 \otimes \mathbb{A}_0)
\]

\[
\mathcal{G}_i = \lambda_{gi} \mathbb{I} + \mu_{gi} \frac{1}{2}(\mathbb{I} + \tilde{\mathbb{I}}) + \alpha_{gi} (\mathbb{I} \otimes A_0 + A_0 \otimes \mathbb{I}) + \beta_{gi} (\mathbb{I} \otimes A_0 + A_0 \otimes \mathbb{I}) + \gamma_{gi} A_0 \otimes A_0 + \delta_{gi} (\mathbb{I} \otimes \mathbb{G}_0 + \mathbb{G}_0 \otimes \mathbb{I}) + \epsilon_{gi} (\mathbb{I} \otimes \mathbb{G}_0 + \mathbb{G}_0 \otimes \mathbb{I}) + \zeta_{gi} \mathbb{G}_0 \otimes \mathbb{G}_0 + \eta_{gi} (\mathbb{A}_0 \otimes \mathbb{G}_0 + \mathbb{G}_0 \otimes \mathbb{A}_0)
\]

The coefficients \(\lambda_0, \mu_0, \lambda_i, \mu_i, \lambda_{gi}, \) and \(\mu_{gi}\) are isotropic elastic modulus components; \(\alpha_0, \beta_0, \gamma_0, \delta_0, \epsilon_0, \zeta_0, \) \(\lambda_{gi}, \beta_{gi}, \gamma_{gi}, \delta_{gi}, \) and \(\gamma_{gi}\) are the anisotropic elastic modulus components correlated to the structural tensor \(A_0\) in the longitudinal direction; \(\delta_0, \epsilon_0, \zeta_0, \delta_i, \epsilon_i, \zeta_i, \delta_{gi}, \) and \(\zeta_{gi}\) are the anisotropic elastic modulus components correlated to the structural tensor \(G_0\) in the circumferential direction; \(\eta_0, \eta_i, \) and \(\eta_{gi}\) are the elastic modulus components describing the interaction between \(A_0\) and \(G_0\) in the stiffness-tensor equations.

The fourth order unit tensors in Eqs. (13)-(15) are defined below.

\[
\mathbb{I} = \delta_{ik} \delta_{jl} e_i \otimes e_j \otimes e_k \otimes e_l, \quad \tilde{\mathbb{I}} = \delta_{il} \delta_{jk} e_i \otimes e_j \otimes e_k \otimes e_l
\]

\[
B = \tilde{\mathbb{I}} : B, \quad B^T = \tilde{\mathbb{I}} : B
\]

In addition, in the equation given below, “\( \otimes \)” denotes the product of the arbitrary second-order tensor \( \theta \) and \( \Phi \).

\[
\theta \otimes \phi = \frac{1}{2} (\theta_{ijk} \phi_{kl} + \theta_{kl} \phi_{jk}) e_i \otimes e_j \otimes e_k \otimes e_l
\]

For the proposed constitutive model, we do not introduce the kinematic hardening rule because the model does not consider the routine repetitive loads acting on cortical bone. Hence, to describe the general isotropic hardening satisfactorily, the proposed model adopts the form used in the study by Lemaitre and Chaboche (2002) for the third term of the Helmholtz free energy function.

By using Eqs. (4), (8), and (12) the viscoelastic constitutive equation and thermodynamic conjugate forces correlated to the internal state variable for viscoelasticity and the isotropic hardening variable are derived, as expressed in Eqs. (19)-(21), respectively.

\[
\sigma \equiv \frac{\partial \psi}{\partial e^{ve}} = \mathcal{C}_0 (D) : e^{ve} + \sum_{i=1}^{N} \mathcal{C}_i (D) : (e^{ve} - \gamma_i)
\]

\[
Q_i \equiv -\frac{\partial \psi}{\partial \gamma_i} = \mathcal{C}_i (D) : (e^{ve} - \gamma_i)
\]

\[
R = \frac{\partial \psi}{\partial r} = R^{ve}(D)[1 - \exp(-b(D)r)]
\]

Moreover, by considering the equilibrium conditions between the spring and dashpot components in the generalized
Maxwell model, Eqs. (22) and (23) are derived.

\[
\begin{align*}
Q_i &= G_i : \gamma_i = C_i : (\dot{\epsilon}^{ve} - \gamma_i) \quad (i: \text{no sum}) \\
Q_i + C_i : (G_i^{-1} : Q_i) &= C_i : \dot{\epsilon}^{ve} \quad (i: \text{no sum})
\end{align*}
\] (22) (23)

As described previously, as the Helmholtz free energy function is satisfactorily represented, the viscoelastic constitutive equation was derived. In addition, from these equations, the energy-dissipation inequality Eq. (10) for viscoelasticity is fully satisfied.

### 2.3 Formulation using dissipation potential function

Now, suppose the dissipation for viscoplasticity \( \Pi^{vp} \) is described as Eq. (11), the dissipation potential function for viscoplasticity \( F \) is defined as a scalar function composed of \( \sigma, D, A_0, \) and \( G_0 \).

\[
F = F(\sigma, D, A_0, G_0, R)
\]

(24)

This dissipation potential function \( F \) is a non-negative convex function correlated to the stress tensor and isotropic hardening variable. Referring to the general viscoplastic model proposed in the study by Lemaitre and Chaboche (2002), \( F \) is described as the sum of the flow potential \( F^F \) and the recovery potential \( F^R \).

\[
F(\sigma, D, A_0, G_0, R) = F^F(\sigma, D, A_0, G_0, R) + F^R(D, R)
\]

(25)

The following equation gives the concrete form of \( F \).

\[
F = \frac{K}{N_1 + 1} \left( \frac{f(\sigma, D, A_0, G_0, R)}{K} \right)^{N_1+1} + \frac{C}{b_e + 1} \left( R^b_e \right) \]

(26)

Here, \( f \) represents a dissipation potential surface.

\[
f(\sigma, D, A_0, G_0, R) = g(\sigma, D, A_0, G_0) - R - \sigma_y
\]

(27)

In addition, “( )” denotes a Macaulay’s bracket, as defined below.

\[
(x) = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}
\]

(28)

To describe the orthotropic yielding phenomenon and asymmetry in tension and compression loadings, we applied the Tsai–Wu type yield function (Tsai and Wu, 1971) to \( g \).

\[
g(\sigma, D, A_0, G_0) = \sigma_y (g_1 \text{tr} \sigma + g_2 \text{tr} A_0 \sigma + g_3 \text{tr} G_0 \sigma + g_4 (\text{tr} \sigma)^2 + g_5 (\text{tr} A_0 \sigma)^2 + g_6 (\text{tr} G_0 \sigma)^2 + g_7 \text{tr} \sigma^2 \\
+ g_8 \text{tr} A_0 \sigma^2 + g_9 \text{tr} G_0 \sigma^2 + g_{10} \text{tr} \sigma A_0 \sigma + g_{11} \text{tr} \sigma G_0 \sigma + g_{12} \text{tr} A_0 \sigma G_0 \sigma)^{1/2}
\]

(29)

where \( g_1, \ldots, g_{12} \) are the material parameters calculated from the yield stress in each axial direction. From the normality rule for dissipation potential surface, the viscoplastic strain rate \( \dot{\epsilon}^{vp} \) and evolution equation for isotropic hardening variable \( \dot{r} \) are derived as follows.

\[
\dot{\epsilon}^{vp} = \frac{\partial F}{\partial \sigma} = \left( \frac{f}{K} \right)^{N_1} \frac{\partial f}{\partial \sigma}
\]

(30)

\[
\dot{r} = - \frac{\partial F}{\partial R} = \left( \frac{f}{K} \right)^{N_1} - \left( \frac{R}{C} \right)^{b_e}
\]

(31)
From Eqs. (30) and (31), the energy dissipation inequality for viscoplasticity of Eq. (11) is fully satisfied. Consequently, by adequately describing the Helmholtz free energy function $\Psi$ and dissipation potential function $F$, a constitutive equation of cortical bone is formulated.

### 2.4 Application of the energy equivalence hypothesis

In the previous sections, we formulated the constitutive equations including the damage tensor explicitly. In this section, we applied the energy equivalence hypothesis (Saanouni et al., 1994) between the current damage configuration and fictitious undamaged configuration to describe the expansion of stress effect due to the damage evolution with effective stress. This hypothesis states that if the state variables in the current damage configuration are transposed with effective variable in the fictitious undamaged configuration with "\(\bar{\cdot}\)" the total energy is assumed to be equivalent. Hence, by applying this hypothesis to the Helmholtz free energy and dissipation potential function, the formulation including the damage tensor implicitly is possible. Figure 3 illustrated the hypothesis.

Now, we assume the equivalence to the Helmholtz free energy and dissipation potential function, as shown below.

$$
\Psi(\varepsilon^{ve}, A_0, G_0, D, \gamma_1, \ldots, \gamma_m, r) = \Psi(\varepsilon^{ve}, \bar{A}_0, \bar{G}_0, \bar{D} = O, \bar{\gamma}_1, \ldots, \bar{\gamma}_m, \bar{r})
$$

$$
F(\sigma, D, A_0, G_0, R) = F(\bar{\sigma}, \bar{D} = O, \bar{A}_0, \bar{G}_0, \bar{R})
$$

The variables in the current damaged configuration are converted to the effective variables in the fictitious undamaged configuration via the damage effective tensor $\mathbb{M}(D)$, as given in Eq. (34).

$$
\bar{\sigma} = \mathbb{M}(D) : \sigma, \bar{Q}_i = \mathbb{M}(D) : Q_i, \bar{R} = |\mathbb{M}(D)| R
$$

$$
\bar{\varepsilon}^{ve} = \mathbb{M}(D)^{-T} : \varepsilon^{ve}, \bar{\gamma}_i = \mathbb{M}(D)^{-T} : \gamma_i, \bar{r} = |\mathbb{M}(D)|^{-1} r
$$

The damage effective tensor refers to the form proposed by Cordebois and Sidoroff (1982) as follows.

$$
\mathbb{M}(D) = (I - D)^{-1/2} \bigotimes (I - D)^{-1/2}
$$

$$
|\mathbb{M}(D)| = \frac{1}{3} M_{iiij}
$$

Then, by representing the stiffness tensor as follows, the equivalence of the Helmholtz free energy is satisfied.

$$
\mathcal{C}_0(A_0, G_0) = \mathbb{M}(D) : \mathcal{C}_0(D, A_0, G_0) : \mathbb{M}^T(D)
$$

$$
\mathcal{C}_1(A_0, G_0) = \mathbb{M}(D) : \mathcal{C}_1(D, A_0, G_0) : \mathbb{M}^T(D)
$$

As described above, the application of the energy equivalence hypothesis enables to formulate the constitutive equation including the damage tensor implicitly.

![Diagram of energy equivalence hypothesis](image-url)
2.5 Formulation of the damage evolution

In the field of continuum damage mechanics, damage state is modeled via the proper macroscopic mechanical damage variable, while the mechanical behavior and damage evolution of the material is already described. Davy and Jepsen (2001) concluded that, if repeated loads or a simple load with relatively large strain rate is given to cortical bone, the accumulated damage, such as voids inside cortical bone, significantly affect the inelastic behavior of the stress–strain curve. In other words, the threshold of the accumulated damage is more appropriate than the yield stress or ultimate stress for the criterion of fracture. In addition, they showed the basic mechanism of the damage accumulation and mentioned that the growth property of the microcracks in cortical bone depends on the loading direction. Moreover, they said that the cortical bone fails when the microcracks developed into irrecoverable state in the inelastic phase.

To describe a three-dimensional anisotropic damage property due to microvoids occurring inside cortical bone, we defined the second order symmetrical damage tensor (Chow and Lu, 1989), as given below.

\[
D = \sum_{i=1}^{3} D_i n_i \otimes n_i \tag{38}
\]

In Eq. (38), \(D_i\) and \(n_i\) are the principal value and the principal unit vector of the damage tensor, respectively. The principal value \(D_i\) describes the area density of the internal microvoids on the three principal surfaces (Murakami and Ohno, 1981), i.e., \(D_i = 0\) implies intact state, or \(D_i = D_{Cri}\) implies ultimate failure state.

Assuming that the effect of the initial anisotropy is predominant on the damage property, we defined the axial direction of the damage tensor as same as that of the material one. By introducing the representation theorem for the tensor function (Boehler, 1987), the damage tensor is described with the identity tensor \(I\) and structural tensors \(A_0\) and \(G_0\), as given in Eq. (39).

\[
D = D_L A_0 + D_C G_0 + D_R (I - A_0 - G_0) \tag{39}
\]

The suffixes \(C\), \(R\), and \(L\) denote the circumferential, radial, and longitudinal directions, respectively. Now, with the damage \(D\), stress \(\sigma\), structural tensors \(A_0\) and \(G_0\), and the conjugate variable of the isotropic hardening \(R\) as arguments, the damage evolution can be generally represented as given below.

\[
\dot{D} = f(D, \sigma, A_0, G_0, R) \tag{40}
\]

Based on the representation theorem for the tensor function and some physical considerations, Eq. (40) is approximated as,

\[
\dot{D} = \dot{D}_L A_0 + \dot{D}_C G_0 + \dot{D}_R (I - A_0 - G_0) \tag{41}
\]

where \(\dot{D}_L\), \(\dot{D}_C\), and \(\dot{D}_R\) are the damage evolutions in each axial direction. The arguments of each coefficient function are described with some invariants, as given below.

\[
\begin{align*}
\dot{D}_L &= \dot{D}_L (\text{tr} A_0 \sigma, \text{tr} A_0 \sigma^2, \text{tr} \sigma^3, R) \tag{42} \\
\dot{D}_C &= \dot{D}_C (\text{tr} G_0 \sigma, \text{tr} G_0 \sigma^2, \text{tr} \sigma^3, R) \tag{43} \\
\dot{D}_R &= \dot{D}_R (\text{tr} H_0 \sigma, \text{tr} H_0 \sigma^2, \text{tr} \sigma^3, R) \tag{44}
\end{align*}
\]

For simplicity, we consider \(H_0 = (I - A_0 - G_0)\) in Eq. (44). \(H_0\) is the structural tensor in the radial direction. The concrete forms of the damage evolution function are defined, as given below.

\[
\dot{D}_L = a_L \left( \frac{g - R - \sigma^3}{K} \right)^{N_2} \left( \frac{\text{tr} A_0 \sigma^2}{\sigma^2} \right)^m \left( \frac{\text{tr} A_0 \sigma^2}{\text{tr} \sigma^2} \right) \exp \left[ b_L \left( \frac{\text{tr} A_0 \sigma^3}{(\text{tr} \sigma^2)^{3/2}} + c_L \text{tr} A_0 D \right) \right] \tag{45}
\]
\[ \dot{D}_C = a_c \left( \frac{g - R - \sigma_y}{K} \right)^{N_2} \left( \frac{\text{tr}G_0\sigma^2}{\sigma_y^2} \right)^m \left( \frac{\text{tr}G_0\sigma^2}{\sigma_y^2} \right)^{\frac{3}{2}} \exp \left\{ b_c \frac{\text{tr}G_0\sigma^3}{(\sigma_y^2)^{3/2}} + c_c \text{tr}G_0D \right\} \]  
(46)

\[ \dot{D}_R = a_r \left( \frac{g - R - \sigma_y}{K} \right)^{N_2} \left( \frac{\text{tr}H_0\sigma^2}{\sigma_y^2} \right)^m \left( \frac{\text{tr}H_0\sigma^2}{\sigma_y^2} \right)^{\frac{3}{2}} \exp \left\{ b_r \frac{\text{tr}H_0\sigma^3}{(\sigma_y^2)^{3/2}} + c_r \text{tr}H_0D \right\} \]  
(47)

In the damage evolution equations, the term \( \left( \frac{g - R - \sigma_y}{K} \right)^{N_2} \) represents the onset of crack advancement at the same time of occurring plasticity. Some literature (McElhaney, 1966; Wood, 1971; Kemper et al., 2007) shows that, as the stress increased, the damage evolution is faster. Hence, we described this property using \( \left( \frac{\text{tr}A_0\sigma^2}{\sigma_y^2} \right)^m \), \( \left( \frac{\text{tr}G_0\sigma^2}{\sigma_y^2} \right)^m \), and \( \left( \frac{\text{tr}H_0\sigma^2}{\sigma_y^2} \right)^m \). In addition, by introducing the non-dimensionalized third invariants \( \text{tr}A_0\sigma^3/(\sigma_y^3)^{3/2} \), \( \text{tr}G_0\sigma^3/(\sigma_y^3)^{3/2} \), and \( \text{tr}H_0\sigma^3/(\sigma_y^3)^{3/2} \), the damage evolutions depending on the loading condition are represented. These values equal 1 in the uniaxial tension, -1 in the uniaxial compression, and 0 in the pure torsion. \( a_c, a_r, b_c, b_r, c_c, c_r, N_2, \) and \( m \) are the material parameters.

With regard to the fracture criterion based on the damage mechanics, it is considered as a break when any component in the damage tensor reaches the threshold \( D_i = D_{\text{cri}} \).

Consequently, a constitutive model of cortical bone describing the viscoelasticity, viscoplasticity, orthotropy, asymmetry in tension and compression, and orthotropic damage evolution is developed. Moreover, the methods of identifying the material parameters are described in the Appendix.

3. Validation of the constitutive model

In this chapter, we identified the material parameters and validated the proposed constitutive model by comparing the calculation results obtained from the constitutive model to some experimental data in the literature. Table A1 in the Appendix list the material parameters used in the calculation process.

First, for assessing the description accuracy of the orthotropy, experimental data given in the study by Reilly and Burstein (1975) were referred. The data includes uniaxial tension and compression test data with various directional cortical bone extracted from the bovine femur. Figure 4 shows the comparison of the stress–strain curves obtained from the calculation results with those of the mechanical tests conducted by Reilly and Burstein. As only the elastic modulus and the failure point are present in those data, the slope of the elastic region is represented with dotted lines, then the failure point is plotted in Fig. 4. The calculation results were in good agreement with the experimental data for anisotropy and asymmetry in tension and compression. In addition, the difference in the failure time in each direction is proved. This suggests that the orthotropic damage evolution can be accurately described using the proposed constitutive model.

Second, we assessed the description accuracy of the strain-rate dependency of the elastic modulus and damage evolution by comparing the results with the experimental data obtained by Wood (1971) and McElhaney (1966). The
tests conducted by Wood were uniaxial tensile tests of human cranium for three different strain rates of 150, 10, and 0.01 /s. The tests conducted by McElhaney are uniaxial compressive tests of embalmed human femur for six different strain rates of 1500, 300, 1, 0.1, 0.01, and 0.001/s. Figures 5 and 6 show the comparisons of the stress–strain curves of the calculation results with those of the mechanical tests conducted by Wood and McElhaney, respectively. Both the calculation results generally agreed with the experimental data. This suggests that the proposed model can accurately describe the strain-rate dependency of the elastic modulus, yield stress, and damage evolution.

Third, by comparing with the experimental data obtained by Yamada (1970) for various loading conditions, we validate the competence in representing the differences in the mechanical properties depending on the loading condition. Figure 7 shows the comparisons of the stress–strain curves of the calculation results with those of the mechanical tests conducted by Yamada. It is verified that the proposed constitutive model could describe the differences in the elastic modulus and yield stress for the tensile, compressive, and torsional loadings. In addition, the model can represent the failure occurrence during torsional loading, rather relatively late.

Consequently, it was verified that the proposed constitutive model of cortical bone could describe the anisotropy, strain-rate dependency, and asymmetry in tension and compression with regard to the elastic modulus, yield stress, and damage evolution. Hence, the model is useful for fracture prediction.

4. Fracture pattern simulation

Generally, fracture patterns are believed to depend on the anisotropy and asymmetry in tension or compression in cortical bone. In this chapter, to verify the representative accuracy of the fracture patterns with the introduction of the proposed constitutive model, we performed FE analyses by applying the constitutive model to the user material subroutine option “UMAT” of LS-DYNA ver. 971 (LSTC, USA). In this subroutine, the stress components are...
calculated using the proposed constitutive model with strain increment passed from the main program in LS-DYNA. At the same time, time-dependent variables, such as isotropic hardening variable and damage tensor components, are calculated, and then the fracture is determined based on the damage tensor.

Figure 8 shows the model used for the analyses. For omitting the influence of the shape of the FE model and inspecting the own property of the constitutive model, we adopted a simple cylindrical model of the human femur. The inside part, cancellous bone, is modeled as an isotropic elastoplastic material (E = 40 MPa, ν = 0.45), while the outside part, cortical bone, is modeled using the proposed constitutive equation. The loads are applied by controlling the rigid parts fixed to both ends of the cylinder at constant rates. The elements judged as failure are eliminated via internal processing in LS-DYNA, and then the inside part is exposed. The diameters of cancellous bone part and cortical bone part are 20 mm and 26 mm, respectively, and the length of the diaphysis is 240 mm. However, for compressive loading, a shorter model with a diaphysis of 150 mm in length was used to prevent the cylinder from buckling as far as possible.

Furthermore, we performed simulations on the FE model under some loading conditions such as tension, compression, and torsion, and then observed fracture patterns.

The boundary conditions applied to the rigid parts were as follows:

- Tensile loading: constant displacement velocity 50 mm/s (one rigid part only)
- Compressive loading: constant displacement velocity 50 mm/s (one rigid part only)
- Torsional loading: constant angular velocity 10 rad/s (both rigid parts)

Figures 9(a) and (b) show distribution of the maximum principal stress and damage variable just before the fracture in tensile loading, respectively. Larger tensile stresses were observed around the middle region of the diaphysis, and then damage accumulations were also observed at the same area. Consequently, transverse fracture in the middle region of the diaphysis was observed as shown in Fig. 9(c). Figures 10(a) and (b) show distribution of the maximum shear stress and damage variable just before the fracture in compressive loading, respectively. In the early phase, a buckling occurred and some element failed due to tensile stress. However, thereafter, larger maximum shear stresses were observed at the counter side of the early failed elements as shown in Fig. 10(a). In addition, as shown in Fig. 10(b), damage variables were also accumulated around the early failed elements. As a result, the crack grew in the oblique direction, and finally the tendency of oblique fracture was observed (Fig. 10(c)). Figures 11(a) and (b) show distribution of the maximum shear stress and damage variable just before the fracture in torsional loading, respectively. Although the maximum shear stress was distributed around the whole diaphysis, the damage variables were accumulated locally. As a result, the crack developed spirally around the diaphyseal axis, i.e., spiral fracture occurred as shown in Figs. 11(c) and (d).
The cause of transverse fracture observed in tensile loading is considered as the tensile stress in longitudinal direction as shown in Fig. 9(a). On the other hand, the oblique fracture in compressive loading is caused by the shear stress as shown in Fig. 10(a) because the compressive strength in the longitudinal direction is stiffer than the shear strength. For torsional loading, the damage accumulation in circumferential direction by tensile stress due to torsion is assumed as the main cause of spiral fracture. As described above, these fracture patterns depend on the difference in the strength and damage evolution of each direction. Therefore the proposed constitutive model including the anisotropy and the asymmetry according to loading direction could reproduce the difference in some characteristic fracture patterns.

To confirm the expression of the strain rate dependency, an additional simulation was performed for tensile loading with ten times as high as the original velocity (500 mm/s). Figure 12 shows comparisons of stress–strain curves and damage variable–strain curves at the first failed elements in the diaphysis, respectively. Owing to the proposed constitutive model, it was found that the differences in the elastic moduli, yield stress, and failure stress depending on the loading rate were described (Fig. 12(a)). In addition, the difference in the damage evolution rate with strain rate was also observed as shown in Fig. 12(b). These properties formulated in the proposed model would be effective for predicting the fracture location and timing under various loading situations.

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This study has some limitations. The analyses performed in this study were only simple simulations on a cylindrical model; hence, we need to validate the constitutive model for reproducibility of fracture patterns and sites using more detailed femur model in the future. In addition, some uncertain factors exist with regard to the identification of material parameters in the proposed constitutive model. For example, we found that the behavior of deformation and fracture is sensitive to the Poisson’s ratio in some simulation results. We require further consideration from these perspectives.

5. Conclusion

In this study, we formulated a new constitutive model describing most of the mechanical properties of cortical bone to predict injuries more accurately. The validation of the model was performed by comparing the calculation results of the proposed model with the experimental data obtained in some literature. Hence, the model was verified to have a possibility to describe mechanical properties of cortical bone, as explained below.

- anisotropy of elastic modulus and yield stress with strain-rate dependency
- asymmetric stress–strain curves in tension and compression
- orthotropic damage evolution with strain-rate dependency

In addition, to apply the proposed model to general injury prediction, we performed fracture pattern simulations on a cylindrical FE model of the human femur using the proposed constitutive model. Hence, the tendency of some characteristic fracture patterns seen in actual cases, such as the transverse fracture in tensile loading, oblique fracture in compressive loading, and spiral fracture in torsional loading, were largely reproduced.

However, some uncertain factors exist with regard to the identification of the material parameters in the proposed constitutive model; hence, we need further study for better injury prediction.

As described above, the applicability of the proposed constitutive model was mostly verified. The model has the potential for more accurate injury prediction in the future.

Appendix: Identification of the material parameters

The constitutive model proposed in this study consists of the viscoelastic, viscoplastic, isotropic hardening, and damage evolution parts. Based on the experimental data, material parameters for each part were determined. In this appendix, the methods of identifying the material parameters are explained.

A.1 Identification in viscoelastic part

First, identification of the parameters \( \lambda_0, \mu_0, \alpha_0, \beta_0, \gamma_0, \delta_0, \varepsilon_0, \zeta_0, \) and \( \eta_0 \) with respect to the stiffness tensor \( C_0 \) is noted. We defined the first, second, and third axes as the circumferential, radial, and longitudinal directions, respectively. Then, the structural tensors are described as \( A_{ij} = \delta_{i3}\delta_{j3}, \) and \( G_{ij} = \delta_{i3}\delta_{j1}. \)

The elastic relationships obtained from the uniaxial tension and compression tests in each axial direction and the shear tests in 1-2, 2-3, and 3-1 planes are represented below.

\[
\varepsilon_{11} = \frac{1}{E_1} \sigma_{11} - \frac{\nu_{12}}{E_2} \sigma_{22} - \frac{\nu_{31}}{E_3} \sigma_{33}
\]
(48)

\[
\varepsilon_{22} = -\frac{\nu_{12}}{E_1} \sigma_{11} + \frac{1}{E_2} \sigma_{22} - \frac{\nu_{23}}{E_3} \sigma_{33}
\]
(49)

\[
\varepsilon_{33} = -\frac{\nu_{31}}{E_1} \sigma_{11} - \frac{\nu_{23}}{E_2} \sigma_{22} + \frac{1}{E_3} \sigma_{33}
\]
(50)

\[
\varepsilon_{23} = \frac{1}{2G_{23}} \sigma_{23}
\]
(51)

\[
\varepsilon_{31} = \frac{1}{2G_{31}} \sigma_{31}
\]
(52)
\[ \varepsilon_{12} = \frac{1}{2G_{12}} \sigma_{12} \]  

(53)

The asymmetric elastic moduli in tension and compression are described as follows.

\[ E_1 = E_{1t} \left(1 + \bar{J}_3 \right)/2 + E_{1c} \left(1 - \bar{J}_3 \right)/2 \]  

(54)

\[ E_2 = E_{2t} \left(1 + \bar{J}_3 \right)/2 + E_{2c} \left(1 - \bar{J}_3 \right)/2 \]  

(55)

\[ E_3 = E_{3t} \left(1 + \bar{J}_3 \right)/2 + E_{3c} \left(1 - \bar{J}_3 \right)/2 \]  

(56)

The value of \( \bar{J}_3 \) is defined by the second invariant \( J_2 \) and the third invariant \( J_3 \) of the deviatoric stress tensor \( S_{ij} = \sigma_{ij} - \sigma_{kk} \delta_{ij}/3 \).

\[ \bar{J}_3 = \frac{3\sqrt{2}}{2} J_3 / (J_2)^{3/2} \]  

(57)

\[ J_2 = \frac{1}{2} S_{ij} S_{ij} \]  

(58)

\[ J_3 = \frac{1}{3} S_{ij} S_{jk} S_{ki} \]  

(59)

The value of \( \bar{J}_3 \) equals 1 in the uniaxial tension, -1 in the uniaxial compression, and 0 in pure torsion; hence, \( E_{1t} \), \( E_{2t} \), and \( E_{3t} \) correspond to the elastic moduli in tension against each axial direction \( E_{1c} \), \( E_{2c} \), and \( E_{3c} \) in compression, respectively.

The matrix description of the stiffness tensor in general orthotropic elastic material is given below, referring to the study by Nakahara et al. (2001).

\[
\mathbb{C}_0 = \begin{bmatrix}
c_{11} & c_{12} & c_{31} & 0 & 0 & 0 \\
c_{12} & c_{22} & c_{23} & 0 & 0 & 0 \\
c_{31} & c_{23} & c_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & c_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & c_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & c_{66}
\end{bmatrix}
\]  

(60)

\[
c_{11} = \frac{1}{E_3} - \frac{v_{23}^2}{E_2} = \frac{1}{E_3} - \frac{v_{31}^2}{E_2} = \frac{1}{E_3} - \frac{v_{12}^2}{E_1}, \quad c_{22} = \frac{1}{E_3} - \frac{v_{31}^2}{E_1} = \frac{1}{E_3} - \frac{v_{12}^2}{E_1}, \quad c_{33} = \frac{1}{E_3} - \frac{v_{23}^2}{E_1} \\
c_{12} = \frac{v_{31} \rho_{12}}{E_2} + \frac{v_{12} \rho_{23}}{E_1}, \quad c_{23} = \frac{v_{31} \rho_{23}}{E_2} + \frac{v_{23} \rho_{12}}{E_1}, \quad c_{31} = \frac{v_{12} \rho_{12}}{E_2} + \frac{v_{31} \rho_{23}}{E_1}
\]  

(61)

\[
A = 1 - 2v_{12}v_{23}v_{31} - \frac{E_1 v_{23}^2}{E_3} - \frac{E_2 v_{12}^2}{E_3} - \frac{E_3 v_{31}^2}{E_2}
\]

The parameters \( E_1, E_2, E_3, v_{23}, v_{31}, \) and \( v_{12} \) are elastic moduli and the Poisson’s ratio obtained from the uniaxial loading tests, respectively. The parameters \( G_{23}, G_{31}, \) and \( G_{12} \) are the shear moduli. The relationship between the coefficients of the tensor form (Eq. (13)) of the stiffness tensor and the components of the matrix form (Eq. (60)) is derived as follows.
\[ \lambda_0 = c_{22} - 2(c_{44} - c_{55} + c_{66}) \]
\[ \mu_0 = c_{44} - c_{55} + c_{66} \]
\[ \alpha_0 = -c_{22} + c_{23} + 2(c_{44} - c_{55} + c_{66}) \]
\[ \beta_0 = 2(c_{55} - c_{66}) \]
\[ \gamma_0 = c_{22} + c_{33} - 2c_{23} - 4c_{44} \]
\[ \delta_0 = -c_{22} + c_{12} + 2(c_{44} - c_{55} + c_{66}) \]
\[ \epsilon_0 = 2(-c_{44} + c_{55}) \]
\[ \zeta_0 = c_1 + c_2 - 2c_{12} - 4c_{56} \]
\[ \eta_0 = c_{22} - c_{12} - c_{23} + c_{31} - 2(c_{44} - c_{55} + c_{66}) \]

Hence, finally, the following equations are derived using \( E_1, E_2, E_3, v_{23}, v_{31}, v_{12}, G_{23}, G_{31}, \) and \( G_{12}. \)

\[ \lambda_0 = \frac{1}{E_1} - \frac{v_{31}^2}{E_3} - 2(G_{23} - G_{31} + G_{12}) \]
\[ \mu_0 = G_{23} - G_{31} + G_{12} \]
\[ \alpha_0 = -\frac{1}{E_1} \frac{v_{31}^2}{E_3} + \frac{v_{31}v_{12}}{E_3} + \frac{v_{23}}{E_2} + 2(G_{23} - G_{31} + G_{12}) \]
\[ \beta_0 = 2(G_{31} - G_{12}) \]
\[ \gamma_0 = \frac{1}{E_1} \frac{v_{23}^2}{E_3} + \frac{1}{E_2} - \frac{v_{12}^2}{E_1} - 2 \frac{v_{31}v_{12}}{E_3} + \frac{v_{23}}{E_2} - 4G_{23} \]
\[ \delta_0 = -\frac{1}{E_1} \frac{v_{23}^2}{E_3} + \frac{1}{E_2} - \frac{v_{12}^2}{E_1} + 2(G_{23} - G_{31} + G_{12}) \]
\[ \epsilon_0 = 2(-G_{23} + G_{31}) \]
\[ \zeta_0 = \frac{1}{E_1} \frac{v_{23}^2}{E_2} + \frac{1}{E_3} - \frac{v_{12}^2}{E_2} - 2 \frac{v_{23}v_{31}}{E_3} + \frac{v_{12}}{E_1} - 4G_{12} \]
\[ \eta_0 = \frac{1}{E_1} \frac{v_{23}^2}{E_3} - \frac{1}{E_2} \frac{v_{23}v_{31}}{E_2} + \frac{v_{12}}{E_1} - \frac{v_{31}v_{12}}{E_3} + \frac{v_{23}}{E_2} + \frac{v_{12}v_{23}}{E_1} + \frac{v_{31}}{E_3} - 2(G_{23} - G_{31} + G_{12}) \]

### A.2 Identification in the viscous equations

In this section, we describe the parameters correlated to the viscous equations, i.e., coefficients of the tensors \( C_i \) and \( G_i. \)

With regard to \( C_i, \) the modulus in the \( i^{th} \) spring component of the generalized Maxwell model is obtained in a manner similar to the determination of the elastic modulus given in the previous section. Hence, the parameters \( \lambda_i, \mu_i, \alpha_i, \beta_i, \gamma_i, \delta_i, \epsilon_i, \zeta_i, \) and \( \eta_i \) in the \( i^{th} \) stiffness tensor \( C_i \) are determined via the material parameters \( E_1^{C_i}, E_2^{C_i}, E_3^{C_i}, v_{23}^{C_i}, v_{31}^{C_i}, v_{12}^{C_i}, G_{23}^{C_i}, G_{31}^{C_i}, \) and \( G_{12}^{C_i} \) obtained from the experiments at sufficiently slow strain rates.

Although the proposed constitutive model could describe the orthotropic property in the viscoelastic part, the number of the parameters is enormous and the anisotropy could be sufficiently described by the stiffness tensor \( C_0. \) Therefore, in this study, the relationships between the coefficients of \( C_i \) are assumed as follows.

\[ E_1^{C_i} = E_2^{C_i} = E_3^{C_i} = E_1^{C_0}, v_{23}^{C_i} = v_{31}^{C_i} = v_{12}^{C_i} = v_C, G_{23}^{C_i} = G_{31}^{C_i} = G_{12}^{C_i} = \frac{E_1^{C_0}}{2(1 - v_C)} \ (i = 1, \ldots, n) \]

The values \( \lambda_{gi}, \mu_{gi}, \alpha_{gi}, \beta_{gi}, \gamma_{gi}, \delta_{gi}, \epsilon_{gi}, \zeta_{gi}, \) and \( \eta_{gi}, \) i.e., parameters of viscous coefficient tensor \( G_i \) in the \( i^{th} \) dashpot component of the generalized Maxwell model are similarly determined using the values of \( E_1^{G_i}, E_2^{G_i}, E_3^{G_i}, \)
\( v_{12}^{G_i}, v_{23}^{G_i}, v_{31}^{G_i}, G_{12}^{G_i}, G_{23}^{G_i}, \) and \( c_{31}^{G_i} \), and the isotropy is also assumed.

\[
E_1^{G_i} = E_2^{G_i} = E_3^{G_i} = E_{G_i}, \quad \gamma_{12}^{G_i} = \gamma_{23}^{G_i} = \gamma_{31}^{G_i} = \gamma_{G_i}, \quad G_{12}^{G_i} = G_{23}^{G_i} = \sigma_{G_i} = \frac{E_{G_i}}{2(1 - \nu_{G_i})} \quad (i = 1, \ldots, n)
\]

(A.3) Identification in the viscoplastic part

In the viscoplastic part, we adopted the Tsai–Wu type yield function for describing the orthotropic yield stress and asymmetry in tension and compression.

\[
g(\sigma, D, A_0, G_0) = \sigma_y \left( g_1 \sigma_{11} + g_2 \sigma_{22} + g_3 \sigma_{33} + g_{12} \sigma_{12} + g_{23} \sigma_{23} + g_{31} \sigma_{31} + F_{11} \sigma_{11} + F_{22} \sigma_{22} + F_{33} \sigma_{33} + 2F_{23} \sigma_{23} + 2F_{31} \sigma_{31} + 2F_{12} \sigma_{12} + F_{44} \sigma_{44} + F_{55} \sigma_{55} + F_{66} \sigma_{66} \right)^{1/2}
\]

(82)

Equation (82) can be rewritten by using each stress component, as given below.

\[
g = \sigma_y \left( F_{11} \sigma_{11} + F_{22} \sigma_{22} + F_{33} \sigma_{33} + F_{12} \sigma_{12} + F_{23} \sigma_{23} + F_{31} \sigma_{31} + 2F_{23} \sigma_{23} + 2F_{31} \sigma_{31} + 2F_{12} \sigma_{12} \right)^{1/2}
\]

(83)

In Eq. (83), suffixes 1, 2, and 3 represent the circumferential, radial, and longitudinal directions, respectively. In addition, the coefficients are defined as follows.

\[
\begin{align*}
F_1 &= \frac{1}{\sigma^{+}_c} - \frac{1}{\sigma^-_c} & F_2 &= \frac{1}{\sigma^{+}_r} - \frac{1}{\sigma^-_r} & F_3 &= \frac{1}{\sigma^{+}_l} - \frac{1}{\sigma^-_l} \\
F_{11} &= \frac{1}{\sigma^{+}_{11}} - \frac{1}{\sigma^{-}_{11}} & F_{22} &= \frac{1}{\sigma^{+}_{22}} - \frac{1}{\sigma^{-}_{22}} & F_{33} &= \frac{1}{\sigma^{+}_{33}} - \frac{1}{\sigma^{-}_{33}} & F_{44} &= \frac{1}{\tau^{+}_{23}} - \frac{1}{\tau^{-}_{23}} & F_{55} &= \frac{1}{\tau^{+}_{31}} - \frac{1}{\tau^{-}_{31}} & F_{66} &= \frac{1}{\tau^{+}_{12}} - \frac{1}{\tau^{-}_{12}}
\end{align*}
\]

(84)

The superscripts “+” and “−” represent the tension and compression, respectively. The coefficients are derived as the initial yield stresses in tension, compression, or shear tests according to each axial direction under the condition of a slowest possible strain rate. The parameter \( F_i \) describes the asymmetry in tension and compression, while \( F_{ij} (i \neq j) \) represents the orthotropic yield stress.

Moreover, although \( F_{ij} (i \neq j) \) should be identified from the equibiaxial test, the test is so difficult to conduct that the data for bone does not exist. However, \( F_{ij} (i \neq j) \) contributes to the lateral viscoplastic strain rate in the uniaxial loading condition. Hence, assuming that the lateral strain shrinks during the tensile loading and expands during the compressive loading, we imposed the following restrictions on the identification of \( F_{ij} (i \neq j) \).

\[
\begin{align*}
F_{ij} &< -\frac{F_i}{2\sigma^{+}_{ii}} \\
F_{ij} &< -\frac{F_i}{2\sigma^{+}_{jj}}
\end{align*} \quad (i \neq j)
\]

(85)

In Eq. (85), \( \sigma^{+}_{ii} \) and \( \sigma^{+}_{jj} \) are defined as the maximum values of the ii and jj components in the reachable stress state after yielding, respectively.

In addition, we defined \( \sigma_y \), as given in the following equation.

\[
\sigma_y = \left[ \frac{1}{6} \left( \frac{1}{\sigma^{+}_c} \right)^2 + \left( \frac{1}{\sigma^{+}_r} \right)^2 + \left( \frac{1}{\sigma^{+}_l} \right)^2 + \left( \frac{1}{\sigma^{-}_r} \right)^2 + \left( \frac{1}{\sigma^{-}_l} \right)^2 \right]^{1/2}
\]

(86)

The parameters \( K \) and \( N_i \) in Eqs. (30) and (31) are identified as describing the differences in the initial yield stress in the stress–viscoplastic strain curves under various strain rates.
A.4 Identification in the isotropic hardening part

The isotropic hardening variable $r$ is introduced in the proposed constitutive model. The parameters in the evolution equation for the isotropic hardening variable $\dot{r}$ and conjugate force $R$ are identified by reproducing the stress–viscoplastic curves. First, paying attention to the low value of hardening in the second term of Eq. (31) in the case of high strain rates, the parameters $b$ and $R^\infty$ are determined at first by matching the hardening curve under the fastest strain rate. Next, assuming that the parameters $C$ and $b_0$ describe the effect of viscosity, the parameters are identified by reproducing the hardening curves in each strain rate, i.e., estimated from the difference in the hardening curves between the fastest strain rate and the slowest rate.

A.5 Identification in the damage evolution

As the influence of the damage variables appears in the latter half of the viscoplastic phase, the parameters in the damage evolution equation are identified using the stress–strain curves in this phase. The parameters $N_2$ and $m$ dominate the differences in the fracture time in the stress–viscoplastic strain curves under various strain rates given in Eqs. (45)-(47). Moreover, the parameters $a_L$, $b_L$, and $c_L$ describe the damage cumulative rates in the longitudinal axial direction without considering the strain rate. Similarly, $a_C$, $b_C$, and $c_C$ correspond to those in the circumferential direction and $a_R$, $b_R$, and $c_R$ correspond to those in the radial direction.

Hence, for the identification of the damage evolution equation, $N_2$ and $m$ are identified by describing the differences in the fracture time in the experiments under the fastest and slowest strain rates by assigning temporary values to $a_L$, $b_L$, $c_L$, $a_C$, $b_C$, $c_C$, $a_R$, $b_R$, and $c_R$ first. Then, $c_L$, $c_C$, and $c_R$ are adjusted by fitting to the fracture time in the experiment under the slowest strain rate. Finally, $a_L$ and $b_L$ are estimated by describing the fracture time in the tensile and compressive loadings in the longitudinal axial direction, and similarly $a_C$, $b_C$, $a_R$, and $b_R$ are identified for the transverse direction.

<table>
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<td>0.22</td>
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