Thermoelectroelastic response of a piezoelectric cylinder with $D_\infty$ symmetry under axisymmetric mechanical and thermal loading

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Abstract
Carbon-neutral and biodegradable materials such as wood and poly-L-lactic acid play an important role in reducing environmental loads. Therefore, this study theoretically analyzed the thermoelectroelastic problem for a solid cylinder with $D_\infty$ symmetry subjected to a distributed torsional shear stress as a mechanical input and to a nonuniform temperature distribution as a thermal disturbance, in order to gain an elementary understanding of the thermal effects on the electroelastic field in cylindrical bodies with $D_\infty$ symmetry. The displacement components are expressed in terms of the elastic displacement potential function, the piezoelectric displacement potential function, and the thermoelastic displacement potential functions, and the electric field components are expressed in terms of the electric potential function. Subsequent to presenting the fundamental equations for thermoelectroelasticity, the governing equations for the above potential functions are derived based on the above fundamental equations. These governing equations are solved by a Fourier transform technique, and the theoretical solutions of the thermoelectroelastic field quantities are obtained. Furthermore, by performing numerical calculations, the distributions of the field quantities are illustrated graphically, and the structures of the thermoelectroelastic field are discussed qualitatively and quantitatively. Moreover, by quantitatively evaluating the ratio of the thermally disturbed electric displacement to the undisturbed electric displacement reflecting the mechanical input, the necessity of the thermoelectroelastic analyses, as treated in this paper, is clearly demonstrated.

Keywords: Thermoelectroelasticity, $D_\infty$ symmetry, Theoretical analysis, Cylindrical body, Wood, Poly-L-lactic acid

1. Introduction
Carbon neutrality and biodegradability have attracted considerable attention recently because of an increasing demand for a reduction in environmental loads. From the viewpoint of engineering production, wood and poly-L-lactic acid (PLLA) are two of the most promising candidates for achieving these aims. Wood, which was found to be piezoelectric by Fukada (1955), has been the subject of electroelastic problems aimed at developing nondestructive evaluation (NDE) techniques (Galligan and Bertholf, 1963, Smetana and Kelso, 1971, Knuffel and Pizzi, 1986, Knuffel, 1988, Nakai, et al., 1998, Nakai and Ando, 1998, Nakai, et al., 2004, 2006). On the other hand, PLLA is expected to be employed in human–machine interfaces (HMI)s for devices such as smart phones, tablet computers, and gaming devices (Ando, et al., 2012, 2013) and also in surgical instruments such as microwezeers and catheters (Tajitsu, et al., 2004, 2005, Tajitsu, 2006, 2008).

The development of NDE techniques for wood using piezoelectric effects and the mechanical design of devices
composed of PLLA require analytical solutions to the relevant electroelastic problems, because they are some of the few tools that can reveal the field quantities within such bodies. Motivated by these requirements, some researchers have been approaching the relevant electroelastic problems from the mesoscopic viewpoint in which both wooden materials and films or fibers made of PLLA have a macrosymmetry of $D_\infty$ in common (Fukada, 1955, Ando, et al., 2012, 2013, Tajitsu, et al., 2004, 2005, Tajitsu, 2006, 2008). $D_\infty$ symmetry is characterized by an $\infty$-fold rotation axis and a two-fold rotation axis perpendicular to it (Kim, 1999). The only electroelastic coupling in bodies with $D_\infty$ symmetry is the coupling between the shear deformation in the plane parallel to the $\infty$-fold rotation axis and the electric poling perpendicular to the sheared plane, as shown in Fig. 1. We derived the constitutive equations for such bodies and constructed a general solution technique for their electroelastic problems in a Cartesian coordinate system (Ishihara, et al., 2015). Furthermore, we extended the technique to treat problems in a cylindrical coordinate system (Ishihara, et al., 2016a) because cylindrical columns or defects are often found in wood engineering, and microwebers and catheters made of PLLA are naturally cylindrical. Using this extended technique, we analytically revealed the electroelastic fields in a solid cylinder subjected to a mechanical (Ishihara, et al., 2016b) or electrical (Ishihara, et al., 2016a, 2016c) load.

The theoretical analyses mentioned above (Ishihara, et al., 2015, 2016a, 2016b 2016c) were performed for isothermal cases because bodies with $D_\infty$ symmetry do not intrinsically exhibit pyroelectric effects (Ando, et al., 2012, 2013, Ishihara, et al., 2015). Meanwhile, as found in PLLA structures for HMIs and surgical instruments, $D_\infty$ bodies are inevitably exposed to a nonuniform temperature distribution due to heat flow from the surroundings. Such a temperature distribution gives rise to shear strains, especially in regions with relatively large temperature gradients, which in turn induce a change in electric poling through the coupling nature of $D_\infty$ symmetry, as shown in Fig. 1. It signifies that in engineering applications, such as NDE by piezoelectric effects and devices made of PLLA, the thermal environment affects the electrical output signals. Therefore, thermoelectroelastic analyses, which reveal the effects of the thermal environment on the electroelastic field, are absolutely essential in order to ensure the safe operation of these applications.

In this paper, therefore, we theoretically analyze the thermoelectroelastic field in a cylindrical body with $D_\infty$ symmetry. As a first step, we choose a solid cylinder axisymmetrically subjected to the combined mechanical and thermal loading in which a distributed torsional shear stress and a nonuniform temperature distribution are applied to its surface as a mechanical input and a thermal disturbance, respectively. Subsequent to presenting the fundamental equations for thermoelectroelasticity, the displacement components are expressed in terms of the elastic displacement potential function, the piezoelastic displacement potential function, and the thermoelastic displacement potential functions, and the electric field components are expressed in terms of the electric potential function. The governing equations for these potential functions are obtained by the fundamental equations mentioned above. The governing equations are solved by use of a Fourier transform technique, and the thermoelectroelastic field quantities are formulated. Moreover, numerical calculations are performed to elucidate the distributions of field quantities within the body and to investigate the effects of thermal disturbance on the electrical responses to the mechanical input.

2. Theoretical analysis

2.1 Analytical model

In operation of the NDE techniques for wood by its piezoelectric effects, the internal state of the object is inspected by means of electric signal in response to the intended mechanical input. In operation of the sensors made of PLLA, the...
mechanical input is detected by the resulting electric signal. In both these applications, the relation between the mechanical input and electrical output signal is one of the key subjects. Moreover, the relation is inevitably disturbed by the thermal environment, which must also be considered. Although the geometric or thermoelastic conditions of these applications are diverse, this paper addresses one of the most elementary models as a first attempt to formulate the thermoelastic field under the cylindrical system.

As shown in Fig. 2, the analytical model is a solid cylinder in the cylindrical coordinate system \((r, \theta, z)\), defined so that the \(z\) axis is parallel to the \(\infty\)-fold rotation axis of \(D_\infty\) symmetry. The surface \(r = R\) is subject to the shear stress distribution \(\tau_r f_r(z)\), the resultant effect of which serves as the torque around the \(z\) axis and is intended to model the mechanical input into the cylinder. Moreover, the surface is exposed to the temperature distribution \(T_0 f_T(z)\) as the thermal disturbance that affects the electrical responses to the mechanical input. \(\tau_0\) and \(T_0\) denote the characteristic values of the shear stress and temperature, respectively, and the distribution functions \(f_r(z)\) and \(f_T(z)\) are assumed to be symmetrical with respect to \(z = 0\). The electric potential on the surface \(r = R\) is set to zero. The radial and tangential displacements, axial stress, axial electric field, and temperature at the infinite ends of the cylinder are set to zero.

### 2.2 Fundamental equations

The constitutive equations for the \(D_\infty\) bodies are given as (Ishihara, et al., 2015)

\[
\begin{align*}
\left[ \begin{array}{c}
\sigma_{rr} \\
\sigma_{r\theta} \\
\sigma_{zr} \\
\sigma_{rz} \\
\sigma_{\theta\theta} \\
\sigma_{zz}
\end{array} \right] &= \left[ \begin{array}{cccccc}
c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\
c_{12} & c_{22} & c_{23} & 0 & 0 & 0 \\
c_{13} & c_{23} & c_{33} & 0 & 0 & 0 \\
c_{44} & & & & & \\
c_{11} & c_{12} & & & & \\
& & & & &
\end{array} \right] \left[ \begin{array}{c}
\varepsilon_{rr} \\
\varepsilon_{r\theta} \\
\varepsilon_{zr} \\
\varepsilon_{rz} \\
\varepsilon_{\theta\theta} \\
\varepsilon_{zz}
\end{array} \right] + \left[ \begin{array}{c}
\frac{E_r}{E_0} \\
\frac{E_{r\theta}}{E_0} \\
\frac{E_{zr}}{E_0} \\
\frac{E_{rz}}{E_0} \\
\frac{E_{\theta\theta}}{E_0} \\
\frac{E_{zz}}{E_0}
\end{array} \right],
\end{align*}
\]

\[
\begin{align*}
\left[ \begin{array}{c}
D_r \\
D_{r\theta} \\
D_z
\end{array} \right] &= \left[ \begin{array}{ccc}
e_{r14} & 0 & 0 \\
e_{r14} & 0 & 0 \\
e_{r14} & 0 & 0
\end{array} \right] + \left[ \begin{array}{ccc}
\eta_{11} & 0 & 0 \\
\eta_{11} & 0 & 0 \\
\eta_{33} & 0 & 0
\end{array} \right] \left[ \begin{array}{c}
E_r \\
E_{r\theta} \\
E_z
\end{array} \right] = 0,
\end{align*}
\]

where \((\varepsilon_{rr}, \varepsilon_{r\theta}, \varepsilon_{zr}, \varepsilon_{rz}, \varepsilon_{\theta\theta}, \varepsilon_{zz})\), \((\sigma_{rr}, \sigma_{r\theta}, \sigma_{zr}, \sigma_{rz}, \sigma_{\theta\theta}, \sigma_{zz})\), \((E_r, E_{r\theta}, E_z)\), and \((D_r, D_{r\theta}, D_z)\) denote the components of the strain, stress, electric field, and electric displacement, respectively; \(c_{ij}, \eta_{ij}, e_{ij}\), and \(\beta_i\) denote the elastic stiffness constant, dielectric constant, piezoelectric constant, and thermoelastic constant, respectively; and \(T\) denotes the temperature. Equations (1) and (2) account for the coupling, through the sole piezoelectric constant \(14\), between the shear deformation in the plane parallel to the \(\infty\)-fold rotation axis and the electric poling perpendicular to the sheared plane, as depicted in Fig. 1. The displacement–strain relations are given by

\[
\varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \quad \varepsilon_{r\theta} = \frac{u_r}{r}, \quad \varepsilon_{zr} = \frac{\partial u_z}{\partial z}, \quad \varepsilon_{rz} = \frac{\partial u_r}{\partial z}, \quad \varepsilon_{\theta\theta} = \frac{\partial u_\theta}{\partial r}, \quad \varepsilon_{zz} = \frac{\partial u_z}{\partial z},
\]

where \((u_r, u_\theta, u_z)\) denotes the components of the displacement. The equilibrium equations of stress and the Gauss law are given, respectively, by

\[
\frac{\partial\sigma_{rr}}{\partial r} + \frac{\partial\sigma_{r\theta}}{\partial \theta} + \frac{\partial\sigma_{zr}}{\partial z} = 0, \quad \frac{\partial\sigma_{r\theta}}{\partial r} + \frac{\partial\sigma_{\theta\theta}}{\partial \theta} + \frac{\partial\sigma_{rz}}{\partial z} = 0, \quad \frac{\partial\sigma_{zr}}{\partial r} + \frac{\partial\sigma_{rz}}{\partial \theta} + \frac{\partial\sigma_{zz}}{\partial z} = 0,
\]

\[
\frac{\partial D_r}{\partial r} + \frac{D_{r\theta}}{r} + \frac{\partial D_z}{\partial z} = 0.
\]
The temperature is governed by the steady heat conduction equation as
\[
\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \lambda \frac{\partial^2 T}{\partial z^2} = 0 \left( \lambda = \frac{\lambda_1}{\lambda_3} \right),
\] (6)
where \( \lambda_1 \) and \( \lambda_3 \) denote the thermal conductivities in the \( r \)- and \( z \)-directions, respectively.

### 2.3 Potential functions

The displacement potential functions \( \varphi \), \( \theta \), \( \Omega \), and \( \chi \) are introduced as
\[
u_r = \frac{\partial \varphi}{\partial r}, \quad \nu_\theta = -\frac{\partial \varphi}{\partial \theta}, \quad \nu_z = k \frac{\partial \varphi}{\partial z} + \frac{\partial \chi}{\partial z},
\] (7)
where \( k \) is an unknown coefficient at present. The electric field components are expressed by the electric potential function \( \Phi \) as
\[
E_r = -\frac{\partial \Phi}{\partial r}, \quad E_\theta = 0, \quad E_z = -\frac{\partial \Phi}{\partial z}.
\] (8)

Substituting Eqs (3), (7), and (8) into Eqs (1) and (2), and then the results into Eqs (4) and (5) gives the nonhomogeneous equations
\[
\begin{align*}
\begin{bmatrix}
 c_{11} \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) + c_{44} \frac{\partial^2}{\partial z^2} \\
 (c_{13} + c_{44}) \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right)
\end{bmatrix} \Omega + \begin{bmatrix}
 c_{13} + c_{44} \frac{\partial^2 \chi}{\partial z^2} \\
 c_{44} \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right)
\end{bmatrix} \chi = \beta_1 T,
\end{align*}
\] (9)
for the thermal part of the problem, and the homogeneous equations
\[
\begin{align*}
\begin{bmatrix}
 c_{11} \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) + kc_{13} + \left( 1 + k \right) c_{44} \frac{\partial^2}{\partial z^2} \\
 \left[ c_{13} + k c_{44} \right] \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right)
\end{bmatrix} \varphi = 0, \quad \beta_1 \neq 0,
\end{align*}
\] (10)
\[
\begin{align*}
\begin{bmatrix}
 c_{11} \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) + \left[ c_{13} + \left( 1 + k \right) c_{44} \right] \frac{\partial^2}{\partial z^2} \\
 c_{11} \frac{1}{r} \frac{\partial}{\partial r} + \left[ c_{44} \frac{\partial^2}{\partial z^2} + \left( 1 + k \right) c_{44} \frac{\partial}{\partial r} \right]
\end{bmatrix} \theta = 0,
\end{align*}
\] (11)
\[
\begin{align*}
\begin{bmatrix}
 c_{11} \frac{1}{r} \frac{\partial}{\partial r} - c_{12} \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + c_{44} \frac{\partial^2}{\partial z^2} \\
 c_{11} \frac{1}{r} \frac{\partial}{\partial r} + c_{44} \frac{\partial^2}{\partial z^2}
\end{bmatrix} \Phi = 0,
\end{align*}
\] (12)
\[
\begin{align*}
\begin{bmatrix}
 c_{11} \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial}{\partial r} \\
 \left[ c_{13} + (1 + k) c_{44} \right] \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + c_{44} \frac{\partial^2}{\partial z^2}
\end{bmatrix} \varphi = 0,
\end{align*}
\] (13)
for the isothermal counterpart. It should be noted that Eqs (9)–(11) are similar to the results for a transversely isotropic elastic body (Noda, et al., 1983, Elliott, 1949) and that Eqs (12) and (13) govern the unique coupling behavior, as depicted in Fig. 1, through \( e_{14} \).

Because Eqs (10) and (11), both for \( \varphi \), must be identical, the relation
\[
\frac{k c_{11} + (1 + k) c_{44}}{c_{11}} = \frac{k c_{13}}{c_{13} + (1 + k) c_{44}} = \mu
\] (14)
must hold, which, by eliminating \( k \), leads to a quadratic equation for \( \mu \) given by...
\[ c_{11}c_{44} \mu^2 + (c_{11}c_{13} - c_{11}^2 - 2c_{11}c_{44}) \mu + c_{13}c_{44} = 0. \]  
(15)

For the two roots, \( \mu_1 \) and \( \mu_2 \), of Eq. (15), we denote the corresponding \( k \) as \( k_1 \) and \( k_2 \), respectively, and the corresponding \( \phi \) as \( \phi_1 \) and \( \phi_2 \), respectively. Then, from Eqs (10), (11), and (14), the governing equations for \( \phi_1 \) and \( \phi_2 \) are obtained as

\[ \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \mu \frac{\partial^2}{\partial z^2} \right) \phi_i = 0 \quad (i = 1, 2), \]
(16)

and the constants \( k_1 \) and \( k_2 \) as

\[ k_i = \frac{c_{11} \mu_i - c_{44}}{c_{11} + c_{44}} = \frac{(c_{11} + c_{44}) \mu_i}{c_{33} - c_{44} \mu_i} \quad (i = 1, 2). \]
(17)

The governing equation for \( \vartheta \) is obtained by eliminating \( \Phi \) in Eqs (12) and (13) as

\[ \left[ \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \mu \frac{\partial^2}{\partial z^2} \right)^2 + \mu (1 + k_{\text{couple}}^2) + \eta \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \nu_1 \frac{\partial^2}{\partial z^2} + \nu_2 \frac{\partial^2}{\partial z^2} \right) \right] \vartheta = 0, \]
(18)

where

\[ \mu_1 = \frac{2c_{44}}{c_{11} - c_{12}}, \quad \eta = \frac{\eta_{13}}{\eta_{11}}, \quad k_{\text{couple}} = \frac{e_{14}}{\sqrt{c_{44} \eta_{11}}}. \]
(19)

Equation (18) can then be rewritten as

\[ \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \nu_1 \frac{\partial^2}{\partial z^2} \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \nu_2 \frac{\partial^2}{\partial z^2} \right) \right) \vartheta = 0, \]
(20)

where \( \nu_1 \) and \( \nu_2 \) are the two roots of a quadratic equation with respect to \( \nu \) as follows:

\[ \nu^2 - \left[ \mu_i (1 + k_{\text{couple}}^2) + \eta \right] \nu + \mu_i \eta = 0. \]
(21)

A general solution to Eq. (20) is obtained as

\[ \vartheta = \vartheta_1 + \vartheta_2, \]
(22)

where the governing equations for \( \vartheta_1 \) and \( \vartheta_2 \) are given by

\[ \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \nu_i \frac{\partial^2}{\partial z^2} \right) \vartheta_i = 0 \quad (i = 1, 2). \]
(23)

Furthermore, \( \Phi \) is obtained by substituting \( \vartheta \) into Eq. (12).

### 2.4 Analysis

The boundary conditions are described for the temperature field as

\[ T = T_0 f_r(z) \quad \text{at} \quad r = R, \quad T \to 0 \quad \text{for} \quad |z| \to \infty \]
(24)
and for the resulting thermoelectroelastic field as

$$\sigma_{rr} = 0, \sigma_{r\theta} = r_0 f_r(u), \sigma_{r\phi} = 0, \Phi = 0 \text{ at } r = R; \quad u_r \to 0, u_\theta \to 0, \sigma_{zz} \to 0, E_z \to 0 \text{ for } |r| \to \infty.$$  \hfill (25)

As for the thermal part of the problem, by applying the Fourier cosine transform with respect to \( z \) and its inversion (Sneddon, 1972) to Eq. (6), and considering Eq. (24) and the finiteness of the temperature at \( r = 0 \), the solution of temperature is obtained as

$$T = T_0 \int_0^\infty f^*_T(\alpha) \left[ \frac{I_0(\sqrt{\lambda} r^2)}{I_0(\sqrt{\kappa} r^2)} \cos(\alpha z) \right] d\alpha,$$  \hfill (26)

where \( f^*_T(\alpha) \) denotes the Fourier cosine transform of distribution function \( f_T(z) \), defined by

$$f^*_T(\alpha) = \frac{2}{\pi} \int_0^\infty f_T(z) \cos(\alpha z) dz.$$

By examining the structures of Eqs (9) and (26), the particular solution to Eq. (9) has the form

$$\alpha^2 \begin{bmatrix} E_{i1}(\alpha) \\ E_{i2}(\alpha) \end{bmatrix} = \frac{\beta_i T_0}{e_{44}} f^*_T(\alpha) \begin{bmatrix} E^*_i \\ E^*_j \end{bmatrix},$$

where

$$\begin{bmatrix} E^*_i \\ E^*_j \end{bmatrix} = \frac{e_{44}}{\beta_i c_1 c_{44} \alpha^2 - c_1 c_{44} - 2 c_1 c_{44} \alpha^2 + c_{33} c_{44}} \begin{bmatrix} \beta_i(c_{44} - c_{33}) + \beta_j(c_{44} + c_{33}) \\ - \beta_i(c_{44} + c_{33}) + \beta_j(c_{44} - c_{33}) \end{bmatrix}.$$  \hfill (30)

As for the isothermal counterpart, by applying the Fourier cosine transform with respect to \( z \) and its inversion (Sneddon, 1972) to Eqs (16) and (23), and considering the finiteness of the field at \( r = 0 \) and \( |z| \to \infty \), the general solutions to Eqs (16) and (23) are obtained as

$$\varphi_i = \int_0^\infty A_i(\alpha) I_0(\sqrt{\mu_i} r^2) \cos(\alpha z) d\alpha, \quad \varphi_j = \int_0^\infty C_i(\alpha) I_0(\sqrt{\nu_i} r^2) \cos(\alpha z) d\alpha \quad (i = 1, 2),$$

where \( A_i(\alpha) \) and \( C_i(\alpha) \) \( (i = 1, 2) \) are constants to be determined by Eq. (25). Furthermore, by substituting Eq. (31) into Eq. (22) and then into Eq. (12), and integrating the result with respect to \( z \), the general solution of the electric potential function is obtained as

$$\Phi = \frac{e_{44}}{c_{44} \mu_i} \sum_{i=1}^2 \int_0^\infty (\nu_i - \mu_i) \alpha C_i(\alpha) I_0(\sqrt{\nu_i} r^2) \sin(\alpha z) d\alpha.$$  \hfill (32)

By substituting Eqs (28), (31), and (32) into Eqs (3), (7), and (8), the thermoelectroelastic field quantities are obtained as
Equation (36) is then solved as

\[ u_i = \sum_{i=1}^{n} u_i, \]

\[ u_j = \sum_{j=1}^{m} u_j, \]

\[ u_k = \sum_{k=1}^{l} u_k, \]

\[ E_i = \frac{2\pi}{c_{44}} \sum_{i=1}^{n} \int_{0}^{\infty} \left( u_i - \mu_i \right) I_i \left( \sqrt{u_i} \cos(\alpha) \right) \sin(\alpha) d\alpha, \]

\[ E_j = \frac{2\pi}{c_{44}} \sum_{j=1}^{m} \int_{0}^{\infty} \left( u_j - \mu_j \right) I_j \left( \sqrt{u_j} \cos(\alpha) \right) \sin(\alpha) d\alpha, \]

\[ E_k = \frac{2\pi}{c_{44}} \sum_{k=1}^{l} \int_{0}^{\infty} \left( u_k - \mu_k \right) I_k \left( \sqrt{u_k} \cos(\alpha) \right) \sin(\alpha) d\alpha, \]

The components of the stress and electric displacement can be obtained by substituting Eqs (26), (34), and (35) into Eqs (1) and (2), but these results are omitted here for brevity. By substituting the thus-obtained stress components and Eq. (32) into Eq. (25), a set of simultaneous equations for \( A_i(\alpha) \) and \( C_i(\alpha) \) \((i = 1, 2)\) is obtained as

\[ \sum_{i=1}^{2} \alpha^2 A_i(\alpha) \left[ (1 + k_i) I_i \left( \sqrt{\mu_i} \right) - \mu_i \frac{I_0 \left( \sqrt{\mu_i} \right) - I_1 \left( \sqrt{\mu_i} \right)}{\mu_i} \right] = \frac{\beta T_n}{c_{44}} f_i^* (\alpha) - \mu_i \frac{I_0 \left( \sqrt{\mu_i} \right) - I_1 \left( \sqrt{\mu_i} \right)}{\mu_i} \left( E_{1i}^* + E_{2i}^* \right), \]

\[ \sum_{i=1}^{2} \alpha^2 C_i(\alpha) \left[ (1 + k_i) I_i \left( \sqrt{\mu_i} \right) \right] = -\frac{\beta T_n}{c_{44}} f_i^* (\alpha) \sqrt{\mu_i} \frac{I_0 \left( \sqrt{\mu_i} \right) - I_1 \left( \sqrt{\mu_i} \right)}{I_0 \left( \sqrt{\lambda_i} \right)} \left( E_{1i}^* + E_{2i}^* \right), \]

where \( f_i^* (\alpha) \) denotes the Fourier cosine transform of distribution function \( f_i(z) \), defined by

\[ f_i^* (\alpha) = \frac{2}{\pi} \int_{0}^{\infty} f_i(z) \cos(\alpha) dz. \]
\[ \alpha^2 \left[ A_i(\alpha) \right] = \frac{B T \tau \alpha^{1+\epsilon}(\alpha)}{c_{44}} \left[ \alpha^2 \left[ A_i(\alpha) \right] \right] \]
\[ \alpha^2 \left[ C_i(\alpha) \right] = \frac{\tau_0 \alpha^{1+\epsilon}(\alpha)}{c_{44}} \left[ \alpha^2 \left[ C_i(\alpha) \right] \right] \]
\]
\[ \Delta_s(\alpha) = (1 + k_1) \left\{ \left[ \frac{\mu_z I_0}{\sqrt{\mu_z \alpha R}} \right] - I_2 \left[ \frac{\sqrt{\mu_z \alpha R}}{I_0} \right] \right\} \]
\[ \Delta_c(\alpha) = (\mu_4 - \mu_5) I_0 \left[ \frac{\sqrt{\mu_z \alpha R}}{I_0} \right] - I_2 \left[ \frac{\sqrt{\mu_z \alpha R}}{I_0} \right] k_4 \]
\[ \Delta_s(\alpha) = (1 + k_4) \left\{ \left[ \frac{\mu_z I_0}{\sqrt{\mu_z \alpha R}} \right] - I_2 \left[ \frac{\sqrt{\mu_z \alpha R}}{I_0} \right] \right\} \]
\[ \Delta_c(\alpha) = (\mu_4 - \mu_5) I_0 \left[ \frac{\sqrt{\mu_z \alpha R}}{I_0} \right] - I_2 \left[ \frac{\sqrt{\mu_z \alpha R}}{I_0} \right] k_4 \]
\[ A_i'(\alpha) = (1 + k_4) \left\{ \left[ \frac{\mu_z I_0}{\sqrt{\mu_z \alpha R}} \right] - I_2 \left[ \frac{\sqrt{\mu_z \alpha R}}{I_0} \right] \right\} \]
\[ C_i'(\alpha) = (\mu_4 - \mu_5) I_0 \left[ \frac{\sqrt{\mu_z \alpha R}}{I_0} \right] - I_2 \left[ \frac{\sqrt{\mu_z \alpha R}}{I_0} \right] k_4 \]

\( A_i'(\alpha) \) and \( C_i'(\alpha) \) are obtained by interchanging the subscripts "1" and "2" of the \( k \), \( \mu \), and \( \nu \) terms in \( -A_i'(\alpha) \) and \( -C_i'(\alpha) \), respectively. The thermoelectroelastic field quantities can then be formulated by substituting Eqs (38) and (39) into Eqs (32)–(35), which is the prime achievement in this paper. The great significance of the thus-obtained formulas is that they can provide the field quantities within the body explicitly, which are almost impossible to obtain experimentally or numerically.

Care should be taken to the following indeterminacy in the formula for \( u_\theta \) described by the second of Eq. (33) and Eqs (38) and (39). It does not satisfy \( \lim_{\theta \to \infty} u_\theta = 0 \) but does provide the difference \( u_\theta(r, z') - u_\theta(r, z) \) \( (z' \neq z) \) or the corresponding infinitesimal, \( 2\varepsilon_\theta \), defined by the fourth of Eq. (3). Therefore, \( u_\theta \) at any points can be calculated by integrating that difference under the boundary condition \( u_\theta(r, \pm \alpha) = 0 \). In spite of that, it should be noted that \( u_\theta \) in the principal part, i.e., \( O(z/R) \sim 1 \), tends to infinity because the cylinder is subjected to a torque over the entire region. The method to overcome this difficulty is presented in Subsection 3.2.

3. Numerical calculations

3.1 Numerical specifications

The distribution functions for the shear stress and temperature, \( f_i(z) \) and \( f_i(z) \), are assumed to have Gaussian distributions with respect to \( z \) with effective widths \( \delta_i \) and \( \delta_i \), respectively, as follows:

\[ f_i(z) = \exp \left( -\frac{z^2}{\delta_i^2} \right) \]
\[ f_i(z) = \exp \left( -\frac{z^2}{\delta_i^2} \right) \]

To illustrate the numerical results, the following nondimensional quantities are introduced:

\[ \left( \bar{r}, \bar{z}, \bar{\delta}_i, \bar{\delta}_i \right) = \left( \frac{r, z}{R}, \frac{\delta_i}{c_{44}} \right) \]
\[ \left( \bar{u}_r, \bar{u}_\theta, \bar{u}_z \right) = \left( \frac{u_r, u_\theta, u_z}{c_{44}} \right) \]
\[ \left( \bar{\sigma}_r, \bar{\sigma}_\theta, \bar{\sigma}_z \right) = \left( \frac{\sigma_r, \sigma_\theta, \sigma_z}{c_{44}} \right) \]

PLLA is chosen as the body with \( D_\alpha \) symmetry. Referring to the literature (Ishihara, et al., 2015, 2016b, Weathermax [DOI: 10.1299/mej.16-00609] © 2017 The Japan Society of Mechanical Engineers
and Stamm, 1946, Glass and Zelinka, 2010), the nondimensional material constants are assumed to be \( \hat{c}_{11} = 1.1190 \), \( \hat{c}_{12} = 0.39659 \), \( \hat{c}_{13} = 0.63578 \), \( \hat{c}_{15} = 16.533 \), \( \eta = 1.3369 \), \( k_{\text{couple}} = -10^{-3} \), \( \lambda = 1.8 \), and \( \beta_1 = 1.9712 \). It should be noted that \( k_{\text{couple}} \) —and therefore, \( e_{14} \) from Eq. (19)—is negative. The effective widths of shear stress and temperature are taken to be \( \hat{\delta}_\tau = 1 \) and \( \hat{\delta}_T = 1 \), respectively. The signs for nondimensional quantities, \( \hat{\ldots} \), are omitted hereafter.

### 3.2 Field due to surface shear

In this subsection, the field due to only the surface shear, i.e., the case of \( T_\theta = 0 \), is investigated. This condition is intended to model the ideal environment excluding the thermal disturbance in applications such as the NDE techniques for wood by its piezoelectric effects and the sensors made of PLLA.

Figure 3 shows the distribution of the displacement vectors projected on plane \( z = 1 \), namely \( (u_r, u_\theta) \). The vectors are depicted on a scale of 1:50, and the relative tangential displacement defined by

\[
u'_\theta = u_\theta - r \cdot (u_\theta)_{r=0, z=0}
\]

denotes the tangential displacement relative to the radial line that connects the origin \( (r,z) = (0,0) \) and the tangentially displaced point of \( (r,z) = (1,0) \). It should be noted that \( u'_\theta \) is introduced to overcome the difficulty that \( u_\theta \) at the region under discussion tends to infinity because the cylinder is subjected to a torque over the entire region when its infinite ends are fixed, as stated in Subsection 2.1. From Fig. 3 and also from \( u_r, u_z = 0 \) for \( T_\theta = 0 \) as found from Eqs (29), (33), and (38), it is found that the cylinder experiences only the tangential displacement. Figure 4 shows that the radial distribution of torsional shear strain \( 2\varepsilon_{rb} \) is almost linear for a relatively large \( z \) and tends to exhibit nonlinearity with decreasing \( z \). This tendency is supported by the relation between the (nondimensional) torque \( M_z \) and surface stress \( (\sigma_{rb})_{r=1} \) as

\[
\frac{dM_z}{dz} = -1 \times 2\pi \cdot 1 \cdot (\sigma_{rb})_{r=1},
\]

which can be derived by integrating the second of Eq. (4) from \( r = 0 \) to \( r = 1 \) with \( r \times 2\pi dr \) multiplied (or by considering the equilibrium of moments around \( z \)-axis acting on the free body with infinitesimal axial length). Equation (43) implies that, if \( (\sigma_{rb})_{r=1} \neq 0 \) as found in the principal part \( O(z) \sim 1 \) of Eq. (40), the torque \( M_z \) changes with \( z \) and, therefore, the mechanical circumstances differ from those of the Saint-Venant's torsion problem for an axisymmetric shaft with a uniform cross section (Timoshenko, 1955).

Then, the distribution of the resulting electric displacement through the coupling effect, as depicted in Fig. 1, is shown in Fig. 5, where the vectors projected on planes \( z = 1 \) and \( r = 0.5 \) are shown in Fig. 5 (a) and (b), respectively, and the vectors are depicted on a scale of 12:1. From Fig. 5 (a), it is found that the vectors \( (D_r, D_\theta) \) projected on plane \( z = 1 \) are distributed radially. This property is also verified as follows. From the second of Eq. (2) together with Eqs (19) and (41), and \( e_{14} < 0 \), the nondimensional relation is obtained as
\( D_\theta = k_{\text{couple}} \frac{1}{2} \varepsilon_\varphi + E_\theta. \)  

Meanwhile, the strain \( 2\varepsilon_\varphi \) is found to be absent by considering the fifth of Eq. (3) and \( u_r = u_z = 0 \), as mentioned above. Moreover, as seen in the second of Eq. (8), \( E_\theta \) is also absent, which originates from the axisymmetry of the electroelastic field. These two characteristics make \( D_\theta \) absent in Eq. (44), resulting in the axial distribution in Fig. 5(a). Furthermore, the electric displacement vectors projected on plane \( r = 0.5 \), \( (D_r,D_\theta) \), are distributed horizontally as found in Fig. 5(b). These streamlines of electric displacement are greatly significant because they can provide the information on the internal structure of the sound material under discussion. For the NDE techniques for wood and the sensors made of PLLA mentioned above, this information is employed as a reference to detect unexpectedly-generated defects by observing the changes in the streamlines, and serves for appropriate operation of these applications.

### 3.3 Field due to surface shear and temperature

In this subsection, the field due to both the shear and temperature on the surface is investigated. The characteristic temperature is assumed to be \( T_0 = 1 \). This condition is intended to model the realistic environment where the operation of the engineering applications mentioned in Subsection 3.2 is inevitably disturbed by a nonuniform temperature distribution.

Figures 6 and 7 show the distributions of temperature and displacement vectors projected on an arbitrary plane of constant \( \theta \). As observed in Fig. 7, the portion subjected to a relatively high temperature shown in Fig. 6 relatively expands in the radial direction. Then, from the fifth of Eq. (3), such a nonuniform distribution of displacement is found to give rise to the shear strain \( 2\varepsilon_\varphi \), which is quite unlike the isothermal case treated in Subsection 3.2. The distribution of \( 2\varepsilon_\varphi \) on the same plane is shown in Fig. 8. The distribution in Fig. 8 actually coincides with the distributions of the shear stress \( \sigma_\varphi \) and the resulting electric displacement \( D_\theta \) for the following reasons. From the fifth of Eq. (1) along with Eq. (41) and \( e_{14} < 0 \), the nondimensional relation is obtained as follows:

\[ \sigma_\varphi = 2\varepsilon_\varphi - E_\theta. \]  

Equations (44) and (45) are valid regardless of whether the field is axisymmetric. On the other hand, \( E_\theta = 0 \), shown in the second of Eq. (8), originates from the axisymmetry of the field, as stated in Subsection 3.2. Consequently, from Eqs (44) and (45), it is found that the axisymmetry of the field makes \( 2\varepsilon_\varphi \), \( \sigma_\varphi \), and \( D_\theta \) proportional to one another as

\[ \sigma_\varphi = 2\varepsilon_\varphi = D_\theta/k_{\text{couple}}^2. \]  

Therefore, Fig. 8 also shows the distributions of the shear stress \( \sigma_\varphi \) and the resulting electric displacement \( D_\theta \). Furthermore, because the surface temperature gives rise to the shear strain \( 2\varepsilon_\varphi \) as stated in relation to Figs 6 and 7,
Eq. (46) signifies that the thermal load does affect the electric response.

$$D(z) = \sqrt{D_r^2 + D_\theta^2 + D_z^2}$$

**Fig. 6** Temperature distribution on $rz$-plane ($T_0 = 1$).

**Fig. 7** Displacement vectors on $rz$-plane ($T_0 = 1$).

**Fig. 8** Distribution of shear strain in $rz$-plane ($T_0 = 1$).

### 3.4 Effect of thermal load on electric response

In operations of the NDE techniques for wood by its piezoelectric effects or the sensors made of PLLA for instance, the electric signals resulting from mechanical inputs are detected as stated in Subsection 3.2. In these applications, the bodies under discussion are generally exposed to a nonuniform temperature distribution due to heat flow from the surroundings, which in turn induces the electric response, as described in Subsection 3.3. In other words, the thermal load may disturb the detection of electric signals in response to the mechanical inputs. From this standpoint, the effect of the thermal load on the electric response is investigated.

Figure 9 shows the distributions of the electric displacement vectors projected on plane $z = 1$, which are depicted on a scale of 12:1. By comparing Fig. 5 (a), Fig. 9 (a), and Fig. 9 (b), it is found that, as the characteristic temperature $T_0$ increases, the magnitude of the tangential electric displacement increases and the profile of distribution changes qualitatively, especially in the interior of the cylinder. In order to investigate the quantitative changes in the electric displacement, the variations of the magnitude $D(z) = \sqrt{D_r^2 + D_\theta^2 + D_z^2}$ for several interior points with the characteristic temperature are presented in Fig. 10. In Fig. 10, the left axis denotes the magnitude $D(z)$, and the right axis denotes the thermal error $R$, which is defined as the percentile ratio of the thermally disturbed magnitude to the undisturbed magnitude, formulated as

$$R = \frac{\left[D(z) - D(z)|_{T_0=0}\right]}{D(z)|_{T_0=0}} \times 100\%.$$  (47)

For the value $\beta_i = 33.052 \times 10^4$ [Pa/K] used to calculate numerical parameters given in Subsection 3.1, the dimensional characteristic values of $T_0 = 3$ [K] and $\tau_0 = 10^4$ [Pa] [k gf/cm$^2$], both of which are possible in reality, are found to correspond to the nondimensional characteristic value of $\tilde{T}_0 \approx 1$ from the fourth of Eq. (41). In the case of $\tilde{T}_0 = 1$, which is briefly described as $T_0 = 1$ in Fig. 10, the thermal error of 1.45% is observed for $(r,z) = (0.8, 1)$. The dimensional value of $T_0 = 12$ [K], which is also possible, corresponds to the nondimensional value of $T_0 \approx 4$, for which the thermal error $R$ is found to reach around 20%. The engineering applications mentioned above are inevitably exposed to a nonuniform temperature distribution due to heat flow from the surroundings: the bodies inspected by the NDE techniques are brought into contact with some interfaces, and the microtweezers and catheters made of PLLA are brought into contact with heat sources such as organs, body fluids, and peripheral devices. Therefore, the estimated errors as shown in Fig. 10 clearly demonstrate the necessity of the thermoelectroelastic analyses, as treated in this paper, and provide essential guidelines for safe design or operation.
4. Conclusion

In order to gain an elementary understanding of the effects of the thermal environment on the electroelastic field in cylindrical bodies with $D_{\infty}$ symmetry, the thermoelectroelastic field was theoretically analyzed by addressing a solid cylinder subjected to both a distributed torsional shear stress and a nonuniform temperature distribution on its surface. Following the introduction of the displacement potential functions and electric potential function to describe the field quantities, the governing equations for these potential functions were derived based on the fundamental equations of thermoelectroelasticity. These governing equations were solved by applying a Fourier transform technique, and the field quantities were formulated. Moreover, by performing numerical calculations, the distributions of field quantities, which are nearly impossible to measure experimentally, were revealed, and the necessity of the thermoelectroelastic analyses was clearly demonstrated.

References

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