Electroelastic field in a piezoelectric solid cylinder with D_{∞} symmetry subjected to transverse mechanical load

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Abstract

For the safe operation of devices comprised of bodies with D_{∞} symmetry, the electroelastic field in a solid cylinder subjected to a non-axisymmetric surface load of radial and circumferential stresses was investigated. The analytical technique constructed previously, the Fourier integral technique with respect to the axial coordinate, and the Fourier series technique with respect to the circumferential coordinate were employed. Then the three-dimensional distributions of the electroelastic field quantities, such as displacement, electric potential, electric field, strain, stress, and electric displacement, were determined. In addition, the resultant forces and moments on the axial section were formulated. Using numerical calculations assuming health monitoring or NDE techniques, it was found that the structures of the electroelastic field were dependent on the composition of the combined load even when the integrated effect of the combined load was constant and that the detailed three-dimensional analyses were absolutely significant. Then, the primary coupling behavior in the D_{∞} bodies was investigated for health monitoring, and it was found that the stress-strain relationship was substantially elastic but the developed electric displacement was affected by both of the piezoelectric and dielectric effects comparably and that the coupled analyses were also significant. Moreover, an elementary methodology to decompose the combined load into components was suggested and was successfully implemented for a concrete example.

Keywords: Electroelasticity, D_{∞} symmetry, Solid cylinder, Non-axisymmetry, Wood, Poly-L-lactic acid

1. Introduction

Recently, wood and poly-L-lactic acid (PLLA) have attracted considerable attention as promising materials that are carbon neutral and biodegradable. Both materials exhibit piezoelectricity (Fukada, 1955, 1991). The piezoelectricity of wood has been used in nondestructive evaluation (NDE) techniques (Galligan and Bertholf, 1963, Smetana and Kelso, 1971, Knuffel and Pizzi, 1986, Knuffel, 1988, Nakai et al., 1998, Nakai and Ando, 1998, Nakai et al., 2004, 2006). The piezoelectricity of PLLA has been used to develop human-machine interfaces (HMIs) for informational devices (Ando et al., 2012, 2013) and surgical instruments, such as microtweezers and catheters (Tajitsu et al., 2004, 2005, Tajitsu, 2006, 2008).

From a mesoscopic viewpoint, wood materials and films or fibers made of PLLA have a macro-symmetry of D_{∞} in common (Fukada, 1955, Tajitsu et al., 2004, 2005, Tajitsu, 2006, 2008), where the only electroelastic coupling is the coupling between the shear deformation in the plane parallel to the ∞ -fold rotation axis and the electric poling perpendicular to the sheared plane, as shown in Fig. 1. From this viewpoint, wood in NDE techniques and PLLA as a component of HMIs can be regarded as the D_{∞} bodies which receive a mechanical load input and produce an electrical signal output.

Ideally, the mechanical load in these applications is the intended input. However, during operation of these

applications, the D_∞ bodies are susceptible to unintended mechanical disturbances. Moreover, surgical instruments, such as microtweezers and catheters, are inevitably involved in mechanical disturbances owing to their surroundings such as narrow and intricate blood vessel. For the safe operation of these applications, the electroelastic response to a disturbance of the constituent D_∞ bodies needs to be evaluated. This need motivated the present authors to construct an analytical technique for the electroelastic field of the D_∞ bodies placed in a Cartesian coordinate system (Ishihara et al., 2015a). Using this technique, the electroelastic field in a semi-infinite body subjected to surface shear stresses (Ishihara et al., 2015b, 2015c) was investigated.

Moreover, because the wood structures in the NDE techniques have often cylindrical boundaries and the microtweezers and catheters comprised of PLLA fibers have cylindrical shapes in nature, an analysis of the electroelastic field under a cylindrical framework is also required. Along this requirement, the axisymmetric electroelastic field in the D_∞ cylinder subjected to an axisymmetric mechanical load was analyzed (Ishihara et al., 2016a). For the non-axisymmetric field, the problems of the cylinders subjected to several kinds of transverse electric fields were solved by the present authors assuming the actuator applications (Ishihara et al., 2016b, 2016c); however, an analysis was not performed for a body subjected to a mechanical load. For the microtweezer, catheter, and wooden material applications, the axisymmetric conditions are considered to be too ideal to describe the actual electroelastic field, and a non-axisymmetric condition is required.

Therefore, in this study, a non-axisymmetric electroelastic field created from a mechanical load is analyzed theoretically for a cylindrical body with D_∞ symmetry. First, the solid cylinder is non-axisymmetrically subjected to the normal and shear stresses which could possibly cut the cylinder. The previously constructed analytical technique is briefly summarized. In this technique, the electroelastic field quantities, such as displacement, electric field, strain, stress, and electric displacement, are expressed in terms of the elastic displacement potential function, the piezoelectric displacement potential function, and the electric potential function. The governing equations of these potential functions are shown. Then, the governing equations are solved using the Fourier integral technique with respect to the axial coordinate and the Fourier series technique with respect to the circumferential coordinate (Courant and Hilbert, 2004). The three-dimensional distributions of the electroelastic field quantities are formulated. By integrating the obtained distributions over the axial cross section, the resultant forces and moments on that section are formulated, and the overall electroelastic properties are determined. Moreover, numerical calculations are performed to elucidate the structure of the electroelastic field, while considering the detailed behaviors that would be overlooked under the overall formulation. Then, from a viewpoint of health monitoring, a method for detecting the mechanical disturbances is investigated.

Fig. 1 Electroelastic coupling in the D_∞ body.

2. Theoretical analysis

2.1 Analytical model

As shown in Fig. 2, the analytical model is a solid cylinder with a cylindrical coordinate system (r, θ, z) , where the z axis is parallel to the ∞ -fold rotation axis of D_∞ symmetry. A Cartesian coordinate system (x, y, z) is also indicated. The surface $r=R$ is subjected to a non-axisymmetric stress distribution as the unintended mechanical disturbance comprised of the radial stress $\sigma_0 f_\sigma(\theta, z)$ and the circumferential stress $\tau_0 f_\tau(\theta, z)$, where σ_0 and τ_0 denote the representative values of the respective stress components. The distribution function $f_\sigma(\theta, z)$ is symmetric and anti-symmetric for $y=0$ and $x=0$, respectively, the distribution function $f_\tau(\theta, z)$ is anti-symmetric and symmetric for $y=0$ and $x=0$, respectively, and both distribution functions are anti-symmetric for $z=0$. The surface $r=R$ is assumed to be free from the stress in the z -direction. These distributions model the transverse

mechanical load that deflects the cylinder in the zx -plane and possibly cut the cylinder at the cross section of $z=0$. The resultant forces in the x -, y -, and z -directions and the resultant moments around the x - and z -axes at the infinite ends of the cylinder are set to zero, whereas the resultant moments around y -axis at the infinite ends support the resultant moment due to the mechanical disturbance comprised of $\sigma_0 f_\sigma(\theta, z)$ and $\tau_0 f_\tau(\theta, z)$. The electric potential on the surface $r=R$ is set to zero, assuming that the radial electric displacement is measured on the surface through an electrode. On the other hand, the axial electric displacement at the infinite ends is set to zero, assuming that those ends are free from the surface electric charge.

Fig. 2 Analytical model.

2.2 Analytical technique

In this subsection, the analytical technique for determining the electroelastic field in the D_∞ bodies placed in the cylindrical coordinate system (Ishihara et al., 2016b) is described briefly.

The components of the displacement (u_r, u_θ, u_z) and electric field (E_r, E_θ, E_z) are expressed in Eqs (1) and (2) using the elastic displacement potential function φ_i ($i=1,2$), the piezoelectric displacement potential function $\mathcal{G}_i$ ($i=1,2$), and the electric potential function Φ .

$$u_r = \sum_{i=1}^2 \left(\frac{\partial \varphi_i}{\partial r} + \frac{1}{r} \frac{\partial \mathcal{G}_i}{\partial \theta} \right), u_\theta = \sum_{i=1}^2 \left(\frac{1}{r} \frac{\partial \varphi_i}{\partial \theta} - \frac{\partial \mathcal{G}_i}{\partial r} \right), u_z = \frac{\partial}{\partial z} \sum_{i=1}^2 k_i \varphi_i, \quad (1)$$

$$E_r = -\frac{\partial \Phi}{\partial r}, E_\theta = -\frac{1}{r} \frac{\partial \Phi}{\partial \theta}, E_z = -\frac{\partial \Phi}{\partial z}. \quad (2)$$

The definition of k_i ($i=1,2$) is described later. The components of the strain, ε_{kl} , are related to the displacement components by the displacement-strain relationship in Eq. (3).

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \varepsilon_{\theta\theta} = \frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta}, \varepsilon_{zz} = \frac{\partial u_z}{\partial z}, 2\varepsilon_{\theta z} = \frac{\partial u_\theta}{\partial z} + \frac{1}{r} \frac{\partial u_z}{\partial \theta}, 2\varepsilon_{rz} = \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}, 2\varepsilon_{r\theta} = \frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}. \quad (3)$$

The components of the stress and electric displacement, σ_{ij} and D_i , respectively, are related to the components of the strain and electric field by the constitutive equations shown in Eqs (4) and (5).

$$\begin{Bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{zz} \\ \sigma_{\theta z} \\ \sigma_{rz} \\ \sigma_{r\theta} \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ & c_{11} & c_{13} & 0 & 0 & 0 \\ & & c_{33} & 0 & 0 & 0 \\ & & & c_{44} & 0 & 0 \\ & & & & c_{44} & 0 \\ & & & & & (c_{11} - c_{12})/2 \end{bmatrix} \begin{Bmatrix} \varepsilon_{rr} \\ \varepsilon_{\theta\theta} \\ \varepsilon_{zz} \\ 2\varepsilon_{\theta z} \\ 2\varepsilon_{rz} \\ 2\varepsilon_{r\theta} \end{Bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ e_{14} & 0 & 0 \\ 0 & -e_{14} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} E_r \\ E_\theta \\ E_z \end{Bmatrix}, \quad (4)$$

$$\begin{Bmatrix} D_r \\ D_\theta \\ D_z \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & e_{14} & 0 & 0 \\ 0 & 0 & 0 & 0 & -e_{14} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \varepsilon_{rr} \\ \varepsilon_{\theta\theta} \\ \varepsilon_{zz} \\ 2\varepsilon_{\theta z} \\ 2\varepsilon_{zr} \\ 2\varepsilon_{r\theta} \end{Bmatrix} + \begin{bmatrix} \eta_{11} & 0 & 0 \\ & \eta_{11} & 0 \\ \text{sym.} & & \eta_{33} \end{bmatrix} \begin{Bmatrix} E_r \\ E_\theta \\ E_z \end{Bmatrix}. \quad (5)$$

The parameters c_{ij} , η_{kl} , and e_{kj} denote the elastic stiffness constant, dielectric constant, and piezoelectric constant, respectively. Equations (4) and (5) account for the coupling effect through the sole piezoelectric constant e_{14} , as shown in Fig. 1.

Moreover, the resultant forces in the x -, y -, and z -directions and the resultant moments around the x -, y -, and z -axes are defined in the cross section perpendicular to the z -axis, S_z , as follows.

$$(N_x, N_y, N_z) = \int_{S_z} (\sigma_{zx}, \sigma_{yz}, \sigma_{zz}) dS_z, \quad (M_x, M_y, M_z) = \int_{S_z} (y\sigma_{zz}, -x\sigma_{zz}, r\sigma_{\theta z}) dS_z, \quad (6)$$

where dS_z denotes a small elemental area of the cylindrical cross section S_z ; σ_{zx} and σ_{yz} denote the Cartesian components of the stress.

The governing equations for φ_i , ϑ_i , and Φ are given in Eq. (7).

$$\left\{ \begin{aligned} \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \mu_i \frac{\partial^2}{\partial z^2} \right) \varphi_i &= 0, \quad \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \nu_i \frac{\partial^2}{\partial z^2} \right) \vartheta_i = 0, \\ \frac{e_{14}}{c_{44}} \mu_3 \frac{\partial \Phi}{\partial z} &= \sum_{i=1}^2 \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \mu_3 \frac{\partial^2}{\partial z^2} \right) \vartheta_i \end{aligned} \right\}, \quad (7)$$

where μ_i and ν_i ($i=1,2$) are the roots of the quadratic equations, respectively, for μ and ν given by

$$c_{11}c_{44}\mu^2 - (c_{11}c_{33} - c_{13}^2 - 2c_{13}c_{44})\mu + c_{33}c_{44} = 0, \quad \nu^2 - [\mu_3(1 + k_{\text{couple}}) + \eta]\nu + \mu_3\eta = 0, \quad (8)$$

and

$$\mu_3 = \frac{2c_{44}}{c_{11} - c_{12}}, \quad \eta = \frac{\eta_{33}}{\eta_{11}}, \quad k_{\text{couple}} = \frac{e_{14}}{\sqrt{c_{44}\eta_{11}}}, \quad k_i = \frac{c_{11}\mu_i - c_{44}}{c_{13} + c_{44}} = \frac{(c_{13} + c_{44})\mu_i}{c_{33} - c_{44}\mu_i}. \quad (9)$$

It should be noted that the conditions in Eq. (8) are required for the *three* kinds of potentials φ_i , ϑ_i , and Φ to satisfy the *four* fundamental equations, namely, three equations for the equilibrium of stresses and the Gauss law.

2.3 Analysis

The boundary conditions are described in Eq. (10).

$$\left\{ \begin{aligned} \sigma_{rr} &= \sigma_0 f_\sigma(\theta, z), \quad \sigma_{r\theta} = -\tau_0 f_\tau(\theta, z), \quad \sigma_{zr} = 0, \quad \Phi = 0 \text{ at } r = R; \\ N_x &\rightarrow 0, \quad N_y \rightarrow 0, \quad N_z \rightarrow 0, \quad M_x \rightarrow 0, \quad M_y \rightarrow \mp \frac{M_{\text{disturbance}}}{2}, \quad M_z \rightarrow 0, \quad D_z \rightarrow 0 \text{ for } |z| \rightarrow \infty \end{aligned} \right\}. \quad (10)$$

In Eq. (10), $M_{\text{disturbance}}$ denotes the resultant moment around y -axis due to the mechanical disturbance comprised of $\sigma_0 f_\sigma(\theta, z)$ and $\tau_0 f_\tau(\theta, z)$, and is defined by

$$M_{\text{disturbance}} = \int_{-\infty}^{\infty} z f_{\text{disturbance}}(z) dz, \quad (11)$$

where $f_{\text{disturbance}}(z)$ denotes the resultant force in x -direction per unit length of the cylinder, and is defined by

$$f_{\text{disturbance}}(z) = \int_{-\pi}^{\pi} \cos \theta \cdot \sigma_0 f_{\sigma}(\theta, z) R d\theta + \int_{-\pi}^{\pi} \sin \theta \cdot \tau_0 f_{\tau}(\theta, z) R d\theta. \quad (12)$$

By applying the Fourier integral technique with respect to z and the Fourier series technique with respect to θ (Courant and Hilbert, 2004) and considering the symmetry and finiteness of the electroelastic field, the solutions to the first and second equations of Eq. (7) are obtained in Eq. (13).

$$\left. \begin{aligned} \varphi_i &= \sum_{n=1}^{\infty} \left[\int_0^{\infty} A_{2n-1,i}(\alpha) I_{2n-1}(\sqrt{\mu_i} \alpha r) \sin(\alpha z) d\alpha \right] \cos(2n-1)\theta, \\ g_i &= \sum_{n=1}^{\infty} \left[\int_0^{\infty} C_{2n-1,i}(\alpha) I_{2n-1}(\sqrt{\nu_i} \alpha r) \sin(\alpha z) d\alpha \right] \sin(2n-1)\theta \end{aligned} \right\}. \quad (13)$$

$I_n(\cdot)$ denotes the modified Bessel function of the first kind of order n ; $A_{2n-1,i}(\alpha)$ and $C_{2n-1,i}(\alpha)$ ($i=1,2$) are constants to be determined by Eq. (10). Furthermore, by substituting Eq. (13) into the third equation of Eq. (7) and integrating the result with respect to z , the solution of the electric potential function is obtained in Eq. (14).

$$\Phi = -\frac{c_{44}}{e_{14}\mu_3} \sum_{i=1}^2 \sum_{n=1}^{\infty} \left[\int_0^{\infty} (\nu_i - \mu_3) \alpha C_{2n-1,i}(\alpha) I_{2n-1}(\sqrt{\nu_i} \alpha r) \cos(\alpha z) d\alpha \right] \sin(2n-1)\theta. \quad (14)$$

By substituting Eqs (13) and (14) into Eqs (1)–(3), the electroelastic field quantities are obtained in Eqs (15)–(17).

$$\left. \begin{aligned} u_r &= \sum_{i=1}^2 \sum_{n=1}^{\infty} \left\langle \int_0^{\infty} \left\{ \sqrt{\mu_i} \alpha A_{2n-1,i}(\alpha) \frac{1}{2} [I_{2n-2}(\sqrt{\mu_i} \alpha r) + I_{2n}(\sqrt{\mu_i} \alpha r)] \right. \right. \\ &\quad \left. \left. + \sqrt{\nu_i} \alpha C_{2n-1,i}(\alpha) \frac{1}{2} [I_{2n-2}(\sqrt{\nu_i} \alpha r) - I_{2n}(\sqrt{\nu_i} \alpha r)] \right\} \sin(\alpha z) d\alpha \right\rangle \cos(2n-1)\theta, \\ u_{\theta} &= -\sum_{i=1}^2 \sum_{n=1}^{\infty} \left\langle \int_0^{\infty} \left\{ \sqrt{\mu_i} \alpha A_{2n-1,i}(\alpha) \frac{1}{2} [I_{2n-2}(\sqrt{\mu_i} \alpha r) - I_{2n}(\sqrt{\mu_i} \alpha r)] \right. \right. \\ &\quad \left. \left. + \sqrt{\nu_i} \alpha C_{2n-1,i}(\alpha) \frac{1}{2} [I_{2n-2}(\sqrt{\nu_i} \alpha r) + I_{2n}(\sqrt{\nu_i} \alpha r)] \right\} \sin(\alpha z) d\alpha \right\rangle \sin(2n-1)\theta, \\ u_z &= \sum_{i=1}^2 \sum_{n=1}^{\infty} \left[\int_0^{\infty} k_i \alpha A_{2n-1,i}(\alpha) I_{2n-1}(\sqrt{\mu_i} \alpha r) \cos(\alpha z) d\alpha \right] \cos(2n-1)\theta \end{aligned} \right\}, \quad (15)$$

$$\left. \begin{aligned} E_r &= \frac{c_{44}}{e_{14}\mu_3} \sum_{i=1}^2 \sum_{n=1}^{\infty} \left\{ \int_0^{\infty} (\nu_i - \mu_3) \sqrt{\nu_i} \alpha^2 C_{2n-1,i}(\alpha) \frac{1}{2} [I_{2n-2}(\sqrt{\nu_i} \alpha r) + I_{2n}(\sqrt{\nu_i} \alpha r)] \cos(\alpha z) d\alpha \right\} \sin(2n-1)\theta, \\ E_{\theta} &= \frac{c_{44}}{e_{14}\mu_3} \sum_{i=1}^2 \sum_{n=1}^{\infty} \left\{ \int_0^{\infty} (\nu_i - \mu_3) \sqrt{\nu_i} \alpha^2 C_{2n-1,i}(\alpha) \frac{1}{2} [I_{2n-2}(\sqrt{\nu_i} \alpha r) - I_{2n}(\sqrt{\nu_i} \alpha r)] \cos(\alpha z) d\alpha \right\} \cos(2n-1)\theta, \\ E_z &= -\frac{c_{44}}{e_{14}\mu_3} \sum_{i=1}^2 \sum_{n=1}^{\infty} \left[\int_0^{\infty} (\nu_i - \mu_3) \alpha^2 C_{2n-1,i}(\alpha) I_{2n-1}(\sqrt{\nu_i} \alpha r) \sin(\alpha z) d\alpha \right] \sin(2n-1)\theta \end{aligned} \right\}, \quad (16)$$

$$\begin{aligned}
\varepsilon_{rr} &= \sum_{i=1}^2 \sum_{n=1}^{\infty} \left\langle \int_0^{\infty} \left\{ \mu_i \alpha^2 A_{2n-1,i}(\alpha) \frac{1}{4} [I_{2n-3}(\sqrt{\mu_i} \alpha r) + 2I_{2n-1}(\sqrt{\mu_i} \alpha r) + I_{2n+1}(\sqrt{\mu_i} \alpha r)] \right. \right. \\
&\quad \left. \left. + \nu_i \alpha^2 C_{2n-1,i}(\alpha) \frac{1}{4} [I_{2n-3}(\sqrt{\nu_i} \alpha r) - I_{2n+1}(\sqrt{\nu_i} \alpha r)] \right\} \sin(\alpha z) d\alpha \right\rangle \cos(2n-1)\theta, \\
\varepsilon_{\theta\theta} &= -\sum_{i=1}^2 \sum_{n=1}^{\infty} \left\langle \int_0^{\infty} \left\{ \mu_i \alpha^2 A_{2n-1,i}(\alpha) \frac{1}{4} [I_{2n-3}(\sqrt{\mu_i} \alpha r) - 2I_{2n-1}(\sqrt{\mu_i} \alpha r) + I_{2n+1}(\sqrt{\mu_i} \alpha r)] \right. \right. \\
&\quad \left. \left. + \nu_i \alpha^2 C_{2n-1,i}(\alpha) \frac{1}{4} [I_{2n-3}(\sqrt{\nu_i} \alpha r) - I_{2n+1}(\sqrt{\nu_i} \alpha r)] \right\} \sin(\alpha z) d\alpha \right\rangle \cos(2n-1)\theta, \\
\varepsilon_{zz} &= \sum_{i=1}^2 \sum_{n=1}^{\infty} \left[\int_0^{\infty} (-k_i \alpha^2) A_{2n-1,i}(\alpha) I_{2n-1}(\sqrt{\mu_i} \alpha r) \sin(\alpha z) d\alpha \right] \cos(2n-1)\theta, \\
2\varepsilon_{\theta z} &= -\sum_{i=1}^2 \sum_{n=1}^{\infty} \left\langle \int_0^{\infty} \left\{ (1+k_i) \sqrt{\mu_i} \alpha^2 A_{2n-1,i}(\alpha) \frac{1}{2} [I_{2n-2}(\sqrt{\mu_i} \alpha r) - I_{2n}(\sqrt{\mu_i} \alpha r)] \right. \right. \\
&\quad \left. \left. + \sqrt{\nu_i} \alpha^2 C_{2n-1,i}(\alpha) \frac{1}{2} [I_{2n-2}(\sqrt{\nu_i} \alpha r) + I_{2n}(\sqrt{\nu_i} \alpha r)] \right\} \cos(\alpha z) d\alpha \right\rangle \sin(2n-1)\theta, \\
2\varepsilon_{rz} &= \sum_{i=1}^2 \sum_{n=1}^{\infty} \left\langle \int_0^{\infty} \left\{ (1+k_i) \sqrt{\mu_i} \alpha^2 A_{2n-1,i}(\alpha) \frac{1}{2} [I_{2n-2}(\sqrt{\mu_i} \alpha r) + I_{2n}(\sqrt{\mu_i} \alpha r)] \right. \right. \\
&\quad \left. \left. + \sqrt{\nu_i} \alpha^2 C_{2n-1,i}(\alpha) \frac{1}{2} [I_{2n-2}(\sqrt{\nu_i} \alpha r) - I_{2n}(\sqrt{\nu_i} \alpha r)] \right\} \cos(\alpha z) d\alpha \right\rangle \cos(2n-1)\theta, \\
2\varepsilon_{r\theta} &= -2 \sum_{i=1}^2 \sum_{n=1}^{\infty} \left\langle \int_0^{\infty} \left\{ \mu_i \alpha^2 A_{2n-1,i}(\alpha) \frac{1}{4} [I_{2n-3}(\sqrt{\mu_i} \alpha r) - I_{2n+1}(\sqrt{\mu_i} \alpha r)] \right. \right. \\
&\quad \left. \left. + \nu_i \alpha^2 C_{2n-1,i}(\alpha) \frac{1}{4} [I_{2n-3}(\sqrt{\nu_i} \alpha r) + I_{2n+1}(\sqrt{\nu_i} \alpha r)] \right\} \sin(\alpha z) d\alpha \right\rangle \sin(2n-1)\theta
\end{aligned} \quad (17)$$

The components of the stress and electric displacement can be obtained by substituting Eqs (16) and (17) into Eqs (4) and (5); however, these results are omitted for brevity. By substituting the obtained stress components and Eq. (12) into the first-fourth conditions in Eq. (10), a set of simultaneous equations for $A_{2n-1,i}(\alpha)$ and $C_{2n-1,i}(\alpha)$ ($i=1,2$) is obtained in Eq. (18).

$$\begin{aligned}
&\sum_{i=1}^2 \left\langle \alpha^2 A_{2n-1,i}(\alpha) \left\{ [\mu_3(1+k_i) - \mu_i] I_{2n-1}(\sqrt{\mu_i} \alpha R) + 2\mu_i \frac{1}{4} [I_{2n-3}(\sqrt{\mu_i} \alpha R) + I_{2n+1}(\sqrt{\mu_i} \alpha R)] \right\} \right. \\
&\quad \left. + 2\nu_i \alpha^2 C_{2n-1,i}(\alpha) \frac{1}{4} [I_{2n-3}(\sqrt{\nu_i} \alpha R) - I_{2n+1}(\sqrt{\nu_i} \alpha R)] \right\rangle = \frac{\mu_3}{c_{44}} \sigma_0 f_{\sigma,2n-1}^*(\alpha), \\
&\sum_{i=1}^2 \left\langle -2\mu_i \alpha^2 A_{2n-1,i}(\alpha) \frac{1}{4} [I_{2n-3}(\sqrt{\mu_i} \alpha R) - I_{2n+1}(\sqrt{\mu_i} \alpha R)] \right. \\
&\quad \left. - 2\nu_i \alpha^2 C_{2n-1,i}(\alpha) \frac{1}{4} [I_{2n-3}(\sqrt{\nu_i} \alpha R) + I_{2n+1}(\sqrt{\nu_i} \alpha R)] \right\rangle = -\frac{\mu_3}{c_{44}} \tau_0 f_{\tau,2n-1}^*(\alpha), \\
&\sum_{i=1}^2 \left\langle \left\{ (1+k_i) \sqrt{\mu_i} \alpha^2 A_{2n-1,i}(\alpha) \frac{1}{2} [I_{2n-2}(\sqrt{\mu_i} \alpha R) + I_{2n}(\sqrt{\mu_i} \alpha R)] \right. \right. \\
&\quad \left. \left. + \frac{1}{\mu_3} \nu_i \sqrt{\nu_i} \alpha^2 C_{2n-1,i}(\alpha) \frac{1}{2} [I_{2n-2}(\sqrt{\nu_i} \alpha R) - I_{2n}(\sqrt{\nu_i} \alpha R)] \right\} \right\rangle = 0, \\
&\sum_{i=1}^2 (\mu_3 - \nu_i) \alpha^2 C_{2n-1,i}(\alpha) I_{2n-1}(\sqrt{\nu_i} \alpha R) = 0
\end{aligned} \quad (18)$$

$f_{\sigma,2n-1}^*(\alpha)$ and $f_{\tau,2n-1}^*(\alpha)$ denote the coefficients in the Fourier expressions of the distribution functions and are shown in Eqs (19) and (20).

$$f_{\sigma}(\theta, z) = \sum_{n=1}^{\infty} \left[\int_0^{\infty} f_{\sigma,2n-1}^*(\alpha) \sin(\alpha z) d\alpha \right] \cos(2n-1)\theta, \quad f_{\tau}(\theta, z) = \sum_{n=1}^{\infty} \left[\int_0^{\infty} f_{\tau,2n-1}^*(\alpha) \sin(\alpha z) d\alpha \right] \sin(2n-1)\theta, \quad (19)$$

$$\left. \begin{aligned} f_{\sigma,2n-1}^*(\alpha) &= \frac{2}{\pi} \int_0^\infty \left[\frac{1}{\pi} \int_{-\pi}^\pi f_\sigma(\theta, z) \cos(2n-1)\theta d\theta \right] \sin(\alpha z) dz, \\ f_{\tau,2n-1}^*(\alpha) &= \frac{2}{\pi} \int_0^\infty \left[\frac{1}{\pi} \int_{-\pi}^\pi f_\tau(\theta, z) \sin(2n-1)\theta d\theta \right] \sin(\alpha z) dz \end{aligned} \right\}. \quad (20)$$

Equation (18) is then solved as

$$\alpha^2 \begin{Bmatrix} A_{2n-1,i}^*(\alpha) \\ C_{2n-1,i}^*(\alpha) \end{Bmatrix} = \frac{1}{c_{44}} \cdot \frac{1}{\Delta_{2n-1}(\alpha)} \begin{bmatrix} A_{2n-1,i}^{*,rr}(\alpha) & A_{2n-1,i}^{*,r\theta}(\alpha) \\ C_{2n-1,i}^{*,rr}(\alpha) & C_{2n-1,i}^{*,r\theta}(\alpha) \end{bmatrix} \begin{Bmatrix} \mu_3 \sigma_0 f_{\sigma,2n-1}^*(\alpha) \\ -\mu_3 \tau_0 f_{\tau,2n-1}^*(\alpha) \end{Bmatrix} \quad (i=1,2), \quad (21)$$

where

$$\left. \begin{aligned} \Delta_{2n-1}(\alpha) &\equiv \begin{vmatrix} a_{2n-1,1}^{rr} & a_{2n-1,2}^{rr} & c_{2n-1,1}^{rr} & c_{2n-1,2}^{rr} \\ a_{2n-1,1}^{r\theta} & a_{2n-1,2}^{r\theta} & c_{2n-1,1}^{r\theta} & c_{2n-1,2}^{r\theta} \\ a_{2n-1,1}^{zr} & a_{2n-1,2}^{zr} & c_{2n-1,1}^{zr} & c_{2n-1,2}^{zr} \\ 0 & 0 & c_{2n-1,1}^\Phi & c_{2n-1,2}^\Phi \end{vmatrix}, \\ A_{2n-1,1}^{*,l}(\alpha) &\equiv \begin{vmatrix} \delta_{(rr,l)} & a_{2n-1,2}^{rr} & c_{2n-1,1}^{rr} & c_{2n-1,2}^{rr} \\ \delta_{(r\theta,l)} & a_{2n-1,2}^{r\theta} & c_{2n-1,1}^{r\theta} & c_{2n-1,2}^{r\theta} \\ 0 & a_{2n-1,2}^{zr} & c_{2n-1,1}^{zr} & c_{2n-1,2}^{zr} \\ 0 & 0 & c_{2n-1,1}^\Phi & c_{2n-1,2}^\Phi \end{vmatrix}, \quad C_{2n-1,1}^{*,l}(\alpha) \equiv \begin{vmatrix} a_{2n-1,1}^{rr} & a_{2n-1,2}^{rr} & \delta_{(rr,l)} & c_{2n-1,2}^{rr} \\ a_{2n-1,1}^{r\theta} & a_{2n-1,2}^{r\theta} & \delta_{(r\theta,l)} & c_{2n-1,2}^{r\theta} \\ a_{2n-1,1}^{zr} & a_{2n-1,2}^{zr} & 0 & c_{2n-1,2}^{zr} \\ 0 & 0 & 0 & c_{2n-1,2}^\Phi \end{vmatrix}, \quad \delta_{(l',l)} \equiv \begin{cases} 1 & l' = l, \\ 0 & l' \neq l \end{cases}, \\ a_{2n-1,i}^{rr} &\equiv [\mu_3(1+k_i) - \mu_i] I_{2n-1}(\sqrt{\mu_i} \alpha R) + \mu_i \frac{1}{2} [I_{2n-3}(\sqrt{\mu_i} \alpha R) + I_{2n+1}(\sqrt{\mu_i} \alpha R)], \\ a_{2n-1,i}^{r\theta} &\equiv -\mu_i \frac{1}{2} [I_{2n-3}(\sqrt{\mu_i} \alpha R) - I_{2n+1}(\sqrt{\mu_i} \alpha R)], \quad a_{2n-1,i}^{zr} \equiv (1+k_i) \sqrt{\mu_i} \frac{1}{2} [I_{2n-2}(\sqrt{\mu_i} \alpha R) + I_{2n}(\sqrt{\mu_i} \alpha R)], \\ c_{2n-1,i}^{rr} &\equiv \nu_i \frac{1}{2} [I_{2n-3}(\sqrt{\nu_i} \alpha R) - I_{2n+1}(\sqrt{\nu_i} \alpha R)], \quad c_{2n-1,i}^{r\theta} \equiv -\nu_i \frac{1}{2} [I_{2n-3}(\sqrt{\nu_i} \alpha R) + I_{2n+1}(\sqrt{\nu_i} \alpha R)], \\ c_{2n-1,i}^{zr} &\equiv \frac{1}{\mu_3} \nu_i \sqrt{\nu_i} \frac{1}{2} [I_{2n-2}(\sqrt{\nu_i} \alpha R) - I_{2n}(\sqrt{\nu_i} \alpha R)], \quad c_{2n-1,i}^\Phi \equiv (\mu_3 - \nu_i) I_{2n-1}(\sqrt{\nu_i} \alpha R) \end{aligned} \right\}, \quad (22)$$

$A_{2n-1,2}^{*,l}(\alpha)$ and $C_{2n-1,2}^{*,l}(\alpha)$ are obtained by interchanging the subscripts "1" and "2" of the k , μ , and ν terms in $A_{2n-1,1}^{*,l}(\alpha)$ and $C_{2n-1,1}^{*,l}(\alpha)$, respectively. The electroelastic field quantities can then be formulated by substituting Eqs (21) and (22) into Eqs (14)–(17) and then into Eqs (4) and (5). This is the prime achievement of this study. The great significance of the obtained formulas is that they can provide the field quantities within the body explicitly. These are almost impossible to obtain experimentally or numerically.

Furthermore, by transforming the obtained cylindrical stress components into the Cartesian components and substituting the results into Eq. (6), the resultant forces and moments are formulated in Eq. (23).

$$\left. \begin{aligned} N_x &= c_{44} \sum_{i=1}^2 \left\{ \int_0^\infty \pi \alpha R \left[(1+k_i) A_{1,i}(\alpha) I_1(\sqrt{\mu_i} \alpha R) + \frac{1}{\mu_3} \nu_i C_{1,i}(\alpha) I_1(\sqrt{\nu_i} \alpha R) \right] \cos(\alpha z) d\alpha \right\}, \\ M_y &= c_{44} \sum_{i=1}^2 \left\{ \int_0^\infty \pi (1+k_i) A_{1,i}(\alpha) \left[\sqrt{\mu_i} \alpha R^2 I_0(\sqrt{\mu_i} \alpha R) - 2 R I_1(\sqrt{\mu_i} \alpha R) \right] \sin(\alpha z) d\alpha \right\}, \\ N_y &= 0, \quad N_z = 0, \quad M_x = 0, \quad M_z = 0 \end{aligned} \right\}. \quad (23)$$

From Eq. (23), it is found that the sixth, seventh, eighth, and tenth conditions in Eq. (10) are satisfied. While the third–sixth equations of Eq. (23) are expected by the symmetry shown in Fig. 2, the first and second equations of Eq. (23) only depend on $n=1$ with respect to the circumferential periodicity derived from Eqs (13) and (14). Moreover,

when considering the first–third equations of Eq. (18), the formulas for N_x and M_y are simplified to Eq. (24).

$$N_x = \pi R \int_0^\infty \left[\sigma_0 \frac{f_{\sigma,1}^*(\alpha)}{\alpha} + \tau_0 \frac{f_{\tau,1}^*(\alpha)}{\alpha} \right] \cos(\alpha z) d\alpha, \quad M_y = -\pi R \int_0^\infty \left[\sigma_0 \frac{1}{\alpha^2} f_{\sigma,1}^*(\alpha) + \tau_0 \frac{1}{\alpha^2} f_{\tau,1}^*(\alpha) \right] \sin(\alpha z) d\alpha. \quad (24)$$

Subsequently, the non-zero resultant force and moment, N_x and M_y , are referred to as the shear force and bending moment, respectively. Meanwhile, by substituting Eq. (19) into Eq. (12) and integrating the result with respect to θ , the resultant force in x -direction per unit width of the cylinder is formulated in Eq. (25).

$$f_{\text{disturbance}}(z) = \pi R \int_0^\infty [\sigma_0 f_{\sigma,1}^*(\alpha) + \tau_0 f_{\tau,1}^*(\alpha)] \sin(\alpha z) d\alpha. \quad (25)$$

From Eqs (24) and (25), the obtained solutions are found to satisfy the relations:

$$\frac{dN_x}{dz} = -f_x(z), \quad \frac{dM_y}{dz} = -N_x, \quad (26)$$

which implies the equilibrium of forces in x -direction and that of moments around y -axis for an arbitrary z . Furthermore, by multiplying the first of Eq. (26) by z , adding the result to the second of Eq. (26), and integrating the result with respect to $|z| < \infty$, the obtained solutions are found to satisfy the relation which describes the equilibrium of the moments around y -axis for the entire cylinder in Eq. (27).

$$(zN_x + M_y)_{z \rightarrow +\infty} - (zN_x + M_y)_{z \rightarrow -\infty} \rightarrow -M_{\text{disturbance}}. \quad (27)$$

On the other hand, by considering the equilibrium of forces in x -direction for the entire cylinder and the anti-symmetry of the resultant force per unit width $f_{\text{disturbance}}(z)$ with respect to $z = 0$, the supporting forces at the infinite ends can be set to zero as in Eq. (28).

$$(N_x)_{z \rightarrow \pm\infty} \rightarrow 0. \quad (28)$$

By considering Eqs (27) and (28) and the anti-symmetry of $(zN_x + M_y)$ with respect to $z = 0$, the ninth of Eq. (10) is found to be satisfied if the infinitesimal order in Eq. (28) is assumed to be higher than the infinite order of z . Whether the ninth of Eq. (10) is achieved depends on the infinitesimal order in Eq. (28), which in turn depends on the behavior of the function $f_{\text{disturbance}}(z)$ and therefore on the distribution functions $f_\sigma(\theta, z)$ and $f_\tau(\theta, z)$. This respect is illustrated in the following subsection.

2.4 Overall electroelastic circumstances

As found in Eq. (23) or (24), the shear force N_x and the bending moment M_y only depend on $n=1$ with respect to the circumferential periodicity derived from Eqs (13) and (14). Therefore, the formulas for the surface stress distributions are chosen to include only the component of $n=1$ in Eq. (19) and are shown in Eq. (29).

$$f_\sigma(\theta, z) = \cos \theta \cdot \frac{R}{\delta} \cdot \frac{z}{\delta} \exp\left(-\frac{z^2}{\delta^2}\right), \quad f_\tau(\theta, z) = \sin \theta \cdot \frac{R}{\delta} \cdot \frac{z}{\delta} \exp\left(-\frac{z^2}{\delta^2}\right), \quad (29)$$

The parameter δ denotes the effective width of the application. By substituting Eq. (29) into Eq. (20) and performing the integrations with respect to θ and z (Erdélyi et al., 1954), the coefficients in the Fourier expressions are calculated as

$$f_{\sigma,2n-1}^*(\alpha) = \begin{cases} \frac{R\alpha\delta}{2\sqrt{\pi}} \exp\left(-\frac{\alpha^2\delta^2}{4}\right) : n=1, \\ 0 : n>1 \end{cases}, f_{\tau,2n-1}^*(\alpha) = \begin{cases} \frac{R\alpha\delta}{2\sqrt{\pi}} \exp\left(-\frac{\alpha^2\delta^2}{4}\right) : n=1, \\ 0 : n>1 \end{cases}. \quad (30)$$

In that case, by substituting Eq. (30) into Eq. (24) and performing the integrations with respect to α (Erdélyi et al., 1954), the explicit formulas for the shear force and bending moment are obtained with Eq. (31).

$$N_x = \tau_{\text{avg}} \cdot A \cdot \exp\left(-\frac{z^2}{\delta^2}\right), M_y = -\sigma_{\text{max}} \cdot Z \cdot \text{erf}\left(\frac{z}{\delta}\right), \quad (31)$$

where $\text{erf}(\bullet)$ denotes the Gauss error function defined by $\text{erf}(x) \equiv \frac{2}{\sqrt{\pi}} \cdot \int_0^x \exp(-\xi^2) d\xi$ and

$$A \equiv \pi R^2, Z \equiv \frac{\pi R^3}{4}, \tau_{\text{avg}} \equiv \frac{\sigma_0 + \tau_0}{2}, \sigma_{\text{max}} \equiv (\sigma_0 + \tau_0) \frac{\sqrt{\pi} \delta}{R}. \quad (32)$$

In Eqs (31) and (32), A and Z denote the cross-sectional area and sectional modulus of the cylinder, respectively; the parameter τ_{avg} is interpreted as the average shear stress acting on the plane $z=0$; and σ_{max} is interpreted as the maximum bending stress acting on the plane $z \rightarrow \infty$. It should be noted that the result in Eq. (31) illustrates the fifth and ninth of Eq. (10). The variations of the shear force and bending moment with the axial location, given by Eq. (31), are shown in Fig. 3. The effect of the shear force is predominant over that of the bending moment in the principal part of the application, $O(z/\delta) \ll 1$, and vice versa for $z/\delta \rightarrow \infty$.

Fig. 3 Variations of the shear force and the bending moment with the axial position.

3. Numerical calculations

3.1 Numerical specifications

To illustrate the numerical results, the following nondimensional quantities are introduced.

$$\left. \begin{aligned} (\hat{r}, \hat{z}, \hat{\delta}) &\equiv \frac{(r, z, \delta)}{R}, (\hat{c}_{11}, \hat{c}_{12}, \hat{c}_{13}, \hat{c}_{33}) \equiv \frac{(c_{11}, c_{12}, c_{13}, c_{33})}{c_{44}}, \\ \hat{\varepsilon}_{\theta z} &\equiv \frac{\varepsilon_{\theta z}}{\sigma_{\text{ref}}/c_{44}}, \hat{E}_r \equiv \frac{E_r}{\sigma_{\text{ref}}/|e_{14}|}, (\hat{\sigma}_{\theta z}, \hat{\sigma}_{zx}, \hat{\sigma}_{yz}) \equiv \frac{(\sigma_{\theta z}, \sigma_{zx}, \sigma_{yz})}{\sigma_{\text{ref}}}, (\hat{D}_r, \hat{D}_z) \equiv \frac{(D_r, D_z)}{\sigma_{\text{ref}}\eta_{11}/|e_{14}|} \end{aligned} \right\}. \quad (33)$$

The parameter σ_{ref} denotes a reference value of stress described later. PLLA is chosen as the body with D_∞ symmetry. Based on previous research (Ishihara et al., 2016a), the nondimensional material constants are assumed to be $\hat{c}_{11} = 1.1190$, $\hat{c}_{12} = 0.39659$, $\hat{c}_{13} = 0.63578$, $\hat{c}_{33} = 16.533$, $\eta = 1.3369$, and $k_{\text{couple}} = -10^{-1}$. The parameter k_{couple} and,

therefore, e_{14} from Eq. (9)—are negative. The effective width of the application is taken to be $\hat{\delta}=1$.

3.2 Structure of the electroelastic field

The structure of the electroelastic field is investigated. When considering Eqs (31) and (32), the field appears to be governed by the *sum* of the representative values of the radial and circumferential stresses, $\sigma_0 + \tau_0$, regardless of their composition. The shear force and bending moment given by Eqs (31) and (32), however, describe only the overall behaviors, i.e., the quantities integrated over the cylindrical cross section S_z . Therefore, the detailed behaviors that would be overlooked under the overall consideration are considered. Thus, the condition is introduced for a constant σ_{ref} in Eq. (34).

$$\sigma_0 = 2\sigma_{\text{ref}}\chi, \tau_0 = 2\sigma_{\text{ref}}(1-\chi). \quad (34)$$

Using Eq. (32), this leads to Eq. (35).

$$\tau_{\text{avg}} = \sigma_{\text{ref}}. \quad (35)$$

From Eqs (34) and (35), the average shear stress on $z=0$, τ_{avg} , is constant, and the nondimensional parameter χ describes the composition of σ_0 and τ_0 . Specifically, the condition $\chi=1$ signifies that the surface of the cylinder is only subjected to radial stress, and the condition $\chi=0$ signifies that the surface is only subjected to circumferential stress.

Figure 4 shows the distributions of the shear stress vectors $(\hat{\sigma}_{zx}, \hat{\sigma}_{yz})$ in the cylindrical cross section at $\hat{z}=0$, where the shear force N_x is found to attain the maximum magnitude from Eq. (31) and could potentially cut the cylinder. In Fig. 4, the directions and relative vector magnitudes are indicated by the direction and length of the arrows, respectively. The magnitude $\sqrt{\hat{\sigma}_{zx}^2 + \hat{\sigma}_{yz}^2}$ is indicated by contours, and the x - and y -axes are oriented for comparison with the right panels of Figs 2 (a) and (b). From Fig. 4, the shear stress vectors are mainly directed in the x -direction for $\chi=1$ and $\chi=0$; however, the circumferential component is more remarkable for $\chi=0$ than for $\chi=1$. Moreover, the maximum stress occurs at the core $\hat{r}=0$ and at the outer surface $(\hat{r}, \theta)=(1, \pi/2)$ for $\chi=1$ and $\chi=0$, respectively, and their magnitudes differ. Thus, the stress distribution profile is dependent on the composition parameter χ even when the average shear stress τ_{avg} is constant. This demonstrates the necessity for the detailed three-dimensional analysis of the electroelastic field performed in this study.

Fig. 4 Distributions of the shear stress vectors in the cylindrical cross section ($\hat{z}=0$).

From a viewpoint of health monitoring, the unintended mechanical disturbances must be detected. For the D_∞ bodies, the disturbances can be detected by the surface electric displacement (or electric poling) that develops through the coupling effect, as shown in Fig. 1. Using an elementary method, the magnitude of the mechanical disturbance τ_{avg} , i.e., the average shear stress on $z=0$, causes the torsional stress to be

$$(\sigma_{\theta z})_{(r,\theta)=(R,\pi/2)} = -(4/3) \cdot \tau_{\text{avg}} \quad (36)$$

on the neutral axis $x=0$ (Timoshenko, 1955), and the electric displacement on the outer surface $(r,\theta)=(R,\pi/2)$ through the coupling effect, as shown in Fig. 1, is roughly estimated to be

$$(D_r)_{(r,\theta)=(R,\pi/2)} = e_{14} \times [-(4/3) \cdot \tau_{\text{avg}} / c_{44}] \quad (37)$$

by multiplying the roughly-estimated strain $-(4/3) \cdot \tau_{\text{avg}} / c_{44}$ by the piezoelectric "e-constant" e_{14} . However, as shown in Fig. 4, the torsional stress $(\hat{\sigma}_{\theta z})_{(\hat{r},\theta)=(1,\pi/2)}$, depicted by the arrows at $(\hat{x},\hat{y})=(0,1)$, does depend on the composition parameter χ even if the average shear stress τ_{avg} is constant. Therefore, the elementary evaluation described by Eq. (36) is no longer valid. This again demonstrates the necessity for a detailed three-dimensional analysis.

Meanwhile, from Eqs (4) and (5), the coupling relation under evaluation is shown in Eq. (38).

$$\sigma_{\theta z} = c_{44} \cdot 2\varepsilon_{\theta z} - e_{14} E_r, \quad D_r = e_{14} \cdot 2\varepsilon_{\theta z} + \eta_{11} E_r. \quad (38)$$

Thus, the shear strain is determined by the shear stress and the radial electric field. In addition, the radial electric displacement is determined by the shear strain and the radial electric field. Therefore, Eq. (37) is not valid, even if Eq. (36) were to be acceptable. Consequently, the coupled electroelastic analysis in this study is essential. Thus, the coupling relation described by Eq. (38) is investigated quantitatively as follows. Equation (38) is nondimensionalized according to Eq. (33) and $e_{14} < 0$ and given as Eq. (39).

$$\hat{\sigma}_{\theta z} = 2\hat{\varepsilon}_{\theta z} + \hat{E}_r, \quad \hat{D}_r = -k_{\text{couple}}^2 \cdot 2\hat{\varepsilon}_{\theta z} + \hat{E}_r. \quad (39)$$

Figures 5 and 6 show the distributions of the field quantities participating in the first and second equations of Eq. (39), respectively, along the surface line $(\hat{r},\theta)=(1,\pi/2)$, where these quantities obtain their maximum values with respect to θ as found from Eqs (4), (5), (16), and (17). From Fig. 5 and the first equation of Eq. (39), the magnitude of $\hat{E}_r$ is negligible ($O(10^{-3})$, at most) compared with those of $\hat{\sigma}_{\theta z}$ and $2\hat{\varepsilon}_{\theta z}$, which are depicted by the overlapping curves in Fig. 5, and the shear strain is determined only by the shear stress. Therefore, the elementary evaluation of the strain, i.e., dividing the stress by the elastic constant, is substantially acceptable. From Fig. 6, however, the quantities in the second equation of Eq. (39) are comparable in magnitude. Therefore, the elementary evaluation of multiplying the strain by the piezoelectric e-constant, as shown in Eq. (37), is no longer valid. This clearly demonstrates the necessity for the coupled electroelastic analysis shown in this study.

(a) Radially loaded case, $\chi = 1$.

(b) Circumferentially loaded case, $\chi = 0$.

Fig. 5 Distributions of the field quantities participating in the inverse-piezoelectric equation $((\hat{r},\theta)=(1,\pi/2))$.

Fig. 6 Distributions of the field quantities participating in the direct-piezoelectric equation $((\hat{r}, \theta) = (1, \pi/2))$.

So far, the electric displacement has been regarded as the output signal used to detect the unintended mechanical disturbances modelled by the normal and shear stresses. Meanwhile, the electric displacement is governed by the Gauss law and the equilibrium equations of the stresses combined with the electroelastic coupling constitutive equations over the analytical domain (Ishihara et al., 2016b). In other words, the electric displacement reflects the internal structure of the domain. The NDE techniques using the piezoelectricity of the D_∞ bodies are based on this criterion. The surface electric displacement shown in Fig. 6 would exhibit different distributions if the domain had any defects. This difference would detect the internal defects. For detection, the reference information, i.e., a detailed profile of the electric displacement for the defect-free domain, is required at first. From such a viewpoint, the distributions of the electric displacement in an internal domain $(\theta = \pi/2)$ are shown in Fig. 7, where the circumferential component D_θ disappears as found in Eqs (5), (16), and (17), and the vectors are depicted with a scale of 1 to 10. From Fig. 7, the electric displacement vectors are mainly directed in the r -direction for $\chi = 1$ and $\chi = 0$; however, the axial component is more remarkable for $\chi = 0$ than for $\chi = 1$. The difference in the profile for χ occurs because the electric displacement is governed by a series of electroelastic coupling equations (Ishihara et al., 2016b) and, as shown in Figs 4 and 5, the profile of the stress as a coupling partner is dependent on χ . The most significant aspect found in Fig. 7 is the reference information without any defects for the NDE setup is dependent on the composition even when the average shear stress τ_{avg} is constant. Moreover, if any defects were introduced into the domain, the disturbance of the distribution of the electric displacement would be dependent on the composition parameter χ . Therefore, for the NDE techniques, the detailed three-dimensional and coupled analysis of the electroelastic field performed in this study is absolutely essential.

Fig. 7 Distribution of the electric displacement vectors in the circumferential cross section $(\theta = \pi/2)$.

3.3 Decomposition of the combined load

As shown previously, the electroelastic field is dependent on the composition parameter χ even when the average shear stress τ_{avg} is constant. From a viewpoint of health monitoring, τ_{avg} as the total amount and χ as the

composition, or equivalently both of σ_0 and τ_0 , should be detected. Specifically, the combined load should be decomposed into the respective components. The decomposition methodology is discussed below.

From the first equations of Eqs (5) and (16), the fourth equation of Eq. (17), and Eq. (21), the radial electric displacement D_r is linear with respect to σ_0 and τ_0 . Thus, the radial electric displacement D_r along the surface line $(r, \theta) = (R, \pi/2)$, which is regarded as a candidate for the output signal and shown in Fig. 6, is described as a function of the axial coordinate in Eq. (40).

$$(D_r)_{(r, \theta) = (R, \pi/2)} = \sigma_0 F_\sigma(z) + \tau_0 F_\tau(z). \quad (40)$$

Therefore, if the magnitudes of the electric displacement D_1 and D_2 at different positions z_1 and z_2 , respectively, are measured, a set of simultaneous equations is obtained as

$$D_1 = \sigma_0 F_\sigma(z_1) + \tau_0 F_\tau(z_1), \quad D_2 = \sigma_0 F_\sigma(z_2) + \tau_0 F_\tau(z_2). \quad (41)$$

Equation (41) is solved as Eq. (42).

$$\begin{Bmatrix} \sigma_0 \\ \tau_0 \end{Bmatrix} = \frac{1}{F_\sigma(z_1)F_\tau(z_2) - F_\tau(z_1)F_\sigma(z_2)} \begin{bmatrix} F_\tau(z_2) & -F_\tau(z_1) \\ -F_\sigma(z_2) & F_\sigma(z_1) \end{bmatrix} \begin{Bmatrix} D_1 \\ D_2 \end{Bmatrix}. \quad (42)$$

The decomposition described in Eq. (42) is graphically illustrated in Fig. 8, where each quantity is nondimensionalized as defined by Eq. (33) and given as Eq. (43).

$$(\hat{z}_1, \hat{z}_2) \equiv \frac{(z_1, z_2)}{R}, \quad (\hat{\sigma}_0, \hat{\tau}_0) \equiv \frac{(\sigma_0, \tau_0)}{\sigma_{\text{ref}}}, \quad (\hat{D}_1, \hat{D}_2) \equiv \frac{(D_1, D_2)}{\sigma_{\text{ref}} \eta_{11} / |e_{14}|}. \quad (43)$$

The blue and red contours denote the values of $\hat{D}_1$ and $\hat{D}_2$, respectively. For example, if the output signal is detected as $(\hat{D}_1, \hat{D}_2) \equiv (0.004, 0.001)$, the values of $\hat{\sigma}_0$ and $\hat{\tau}_0$ are the coordinates of the intersection of the corresponding contours in Fig. 8. This leads to $(\hat{\sigma}_0, \hat{\tau}_0) \equiv (1.538, 0.785)$. Although these procedures seem elementary and therefore require further refinements, they suggest the essential idea to decompose a combined load into its components.

Fig. 8 Decomposition of the combined load $(\hat{z}_1 = 1, \hat{z}_2 = 1.5)$.

4. Conclusion

For the safe operation of devices comprised of a body with D_∞ symmetry, the electroelastic field in a cylinder subjected to non-axisymmetrical normal and shear stresses that could possibly cut the cylinder was investigated. Using a previously constructed analytical technique and the Fourier integral and series techniques, the three-dimensional

distributions of the electroelastic field quantities and the resultant forces and moments on the axial section were formulated explicitly. Then, using numerical calculations assuming health monitoring or NDE techniques, it was found that the structures of the electroelastic field were dependent on the composition of the combined load even when the integrated effect of the combined load was constant. Next, the primary coupling behavior in the D_∞ bodies was investigated from a viewpoint of health monitoring, and the stress-strain relation was seen to be substantially elastic; however, the developed electric displacement was affected by both of the piezoelectric and dielectric effects comparably. These findings clearly demonstrate the necessity of the three-dimensional and coupled electroelastic analysis performed in this study. Moreover, an elementary but essential methodology to decompose the combined load into the components was suggested, and it was successfully implemented for a concrete example.

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