Measurement method for axisymmetric 3D stress-tensor fields using background-oriented schlieren

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Abstract

Stress fields affect various characteristics of materials. However, measurement methods for three-dimensional stress fields tend to be complicated because stress tensors generally consist of six independent components. We therefore propose a simple method for measuring axisymmetric three-dimensional stress fields by measuring deflection angles of light rays using the background-oriented schlieren technique. Assuming axial symmetry of the stress field, we use the Lagrangian optics to derive a relationship between two-dimensional deflection angle and three-dimensional stress. Using independent ray polarizations, the method can reconstruct the respective refractive indices from the deflection angle. Equilibrium equations for the stress tensor can be written under the assumption of axial symmetry. By applying appropriate approximations, we formulate normal and shear stress-tensor components, which enables reconstruction of these components from refractive indices. Stress-tensor components can be experimentally generated by applying a load to a glass plate surface on the basis of the Hertz contact theory. The proposed method reconstructs the stress-tensor components from a refractive index field that is induced by the stress in the glass. The reconstructed axial-stress tensor σ_{zz} and shear-stress tensor τ_{rz} well agree with numeric calculations using structural analysis software. Other stress-tensor components, however, slightly deviate from those of the numeric calculations.

Keywords: Nondestructive evaluation, Stress measurement, Contact, Background-oriented schlieren, Solid mechanics

1. Background

Stress fields affect various characteristics of materials. One example is technology for intentionally applying stress to improve circuit performance in semiconductors (Takagi et al., 2001). In addition, it is known that surface residual stress provides a crack-resistance effect in some materials. This can be realized by various technologies such as chemical glass strengthening (Stefan et al., 2010), surface hardening of steel (Hayashi et al., 1969), and shot peening for metal components (Inoue et al., 1989). To assess the quality of these technologies and take measures to improve quality, it is necessary to obtain three-dimensional stress field information.

Internal stress in a material changes the refractive index of optical components such as lenses and prisms. The internal stress often deteriorates quality of optical components due to the refractive index change. For example, in manufacturing processing and use, injection molding conditions can cause residual stress (Zhao et al., 2009) (Young, 2005), and internal stress is generated by temperature distributions due to laser light absorption (Ohno et al., 2018). Therefore, a simple, low-cost, in situ stress measurement method is required for achieving appropriate conditions for processing and use.

In conventional methods, the three-dimensional stress field can be measured by analyzing the photoelastic effect based on birefringence phenomena (Aben et al., 2000, 2008) (Ramesh et al., 2016) or by analyzing X-ray diffraction caused by lattice deformation (The Society of Materials Science, 2002) (Webster et al., 1998). These methods have been studied for a long time and are powerful tools for experimental stress analysis.

Another method for analyzing three-dimensional stress field experimentally with simple setup is proposed here. Recent developments in open-source image processing libraries (OpenCV), in particular opticalflow algorithms (Farneback, 2000, 2001), have made it easier to use background-oriented schlieren (BOS) technique (Raffel, 2015) (Hayasaka et al., 2016) (Sueishi et al., 2017) (Ota et al., 2015) (Nicolas et al., 2016) (Toya et al., 2018, 2019). Refractive index field can be measured by BOS quantitatively with a relatively simple optical system. Therefore, we are developing a new stress measurement method using BOS technique. The proposed method is based on the phenomenon that the refractive index changes with density variation of the object due to stress. For transparent objects, this can be measured by a simple experimental system, because visible light can be used.

We also quantify the axisymmetric three-dimensional stress field that is experimentally generated by applying a load with a resin sphere to a sufficiently large glass plate on the basis of the Hertz contact theory (Hertz, 1896).

2. Method for measuring axisymmetric stress-tensor field

2.1 Refractive index reconstruction method with scalar potential using the BOS technique

A light ray passing through an inhomogeneous refractive index field is deflected according to the refractive index gradient. The deflection angle $\boldsymbol{\varepsilon}$ of the ray is defined as shown in Figure 1. When the refractive index near the origin exceeds the ambient refractive index, a light ray traveling from the source plane along the X -axis direction is deflected toward the origin when passing near the origin. The two-dimensional deflection angle $\boldsymbol{\varepsilon}$ is represented by projecting angles ε_Y and ε_Z onto the X - Y and X - Z planes, respectively. Cylindrical coordinates (r, θ, z) are defined as shown in Fig. 1.

The two-dimensional deflection angle ε_Y and ε_Z of a light ray passing through a refractive index field generated by stress is able to be measured by the BOS technique. That is, the deflection angles of two independent polarizations are measured using a simple experimental system that uses camera images. In the following, linear polarizations along the Y - and Z -axes are respectively referred to as horizontal and vertical polarizations.

A three-dimensional refractive index field deflects light rays passing along X -direction from the source plane. The two-dimensional deflection angle distribution obtained using the BOS technique can be written using the scalar potential ϕ on the basis of the Lagrangian optics, and can be defined as

$$\boldsymbol{\varepsilon} \equiv \begin{pmatrix} 0 \\ \varepsilon_Y \\ \varepsilon_Z \end{pmatrix} = \nabla \phi, \quad (1)$$

under the assumption of an axial symmetric field (Ohno et al., 2019). The deviations in the refractive index is then reconstructed according to an algorithm shown in the Ohno et al., 2019 under the assumption of small deflection angles. The refractive index deviation occurs continuously in the path near the origin so the ray deflects continuously. In this algorithm, the ray deflection is integrated over the ray path along X -direction. However, the ray deflection occurs close to the Y - Z plane, which ensures that the position of the ray deflection is assumed in the Y - Z plane.

By reconstruction with the scalar potential ϕ , deviations in the refractive index in the Y - Z plane for horizontal polarization Δn_h and for vertical polarization Δn_v are precisely obtained based on an accurate ray-tracing solution, using the deflection angle distribution obtained using the BOS technique.

Fig. 1 Definitions of ray deflection angle ϵ , Cartesian coordinates (X, Y, Z) , and cylindrical coordinates (r, θ, z) . When the refractive index near the origin exceeds the ambient refractive index, the refractive index gradient deflects light rays passing near the origin toward the origin.

2.2 Basic stress-tensor and refractive-index equations

When the body forces are neglected, equilibrium equations of stresses in cylindrical coordinates are written as

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0, \quad (2)$$

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{2\tau_{r\theta}}{r} = 0, \quad (3)$$

$$\frac{\partial \tau_{rz}}{\partial r} + \frac{1}{r} \frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\tau_{rz}}{r} = 0, \quad (4)$$

where, σ_{rr} , $\sigma_{\theta\theta}$ and σ_{zz} are the vertical stress in r , θ and z direction respectively, and $\tau_{r\theta}$, $\tau_{\theta z}$ and τ_{rz} are the shear stress in r - θ , θ - z and r - z plane respectively. When the stress field is axially symmetric, the stress in the θ direction is neglected, and become $\tau_{r\theta} = \tau_{\theta z} = 0$, equilibrium equations (2), (3) and (4) are rewritten as

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0, \quad (5)$$

$$\frac{\partial \tau_{rz}}{\partial r} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\tau_{rz}}{r} = 0. \quad (6)$$

As a result, the state of axially symmetric stress can be described using four stress tensor components: radial stress σ_{rr} , hoop stress $\sigma_{\theta\theta}$, axial stress σ_{zz} , and shear stress τ_{rz} .

When the light ray deflection is assumed to occur in the Y - Z plane, deviation in the refractive index in the Y - Z plane for polarizations caused by a stress can be assumed using stress components as

$$\Delta n_h = K_1 \sigma_{rr} + K_2 (\sigma_{\theta\theta} + \sigma_{zz}), \quad (7)$$

$$\Delta n_v = K_2 (\sigma_{rr} + \sigma_{\theta\theta}) + K_1 \sigma_{zz}, \quad (8)$$

where K_1 and K_2 respectively represent optical-stress coefficients for polarizations that are parallel and perpendicular to the stress direction (Theocaris et al., 1979) (SCHOTT).

2.3 Reconstruction of the stress field from the refractive index field

When a flat body and a spherical body are brought into contact at the origin of the cylindrical coordinates and a vertical load is applied to the flat body, the stress component σ_{zz} inside the flat body becomes large compared with other stress components when taking the z -axis as the load direction. Assuming the absolute value of the optical-stress coefficients as $|K_1| \ll |K_2|$ and assuming that σ_{zz} is larger than σ_{rr} and σ_{zz} is larger than $\sigma_{\theta\theta}$, axial stress σ_{zz} as represented by Eq. (7) can be approximated as

$$\sigma_{zz} \approx \frac{\Delta n_h}{K_2}. \quad (9)$$

Using Eq. (9), hoop stress $\sigma_{\theta\theta}$ represented by Eq. (8) can be written as

$$\sigma_{\theta\theta} = A - \sigma_{rr}, \quad (10)$$

$$A = \frac{1}{K_2} \Delta n_v - \frac{K_1}{K_2^2} \Delta n_h.$$

By transforming Eq. (6), shear stress τ_{rz} can be written as

$$\tau_{rz} = -\frac{1}{r} \frac{\partial}{\partial z} \int_0^\infty \sigma_{zz} r dr. \quad (11)$$

From Eqs. (5), (10), and (11), radial stress σ_{rr} can be written as

$$\sigma_{rr} = \frac{1}{r^2} \int_0^\infty r^2 \left(\frac{A}{r} - \frac{\partial \tau_{rz}}{\partial z} \right) dr. \quad (12)$$

Then, hoop stress $\sigma_{\theta\theta}$ can be derived from Eq. (10). As a result, all stress components can be reconstructed from the measured two-dimensional deviation distribution of the refractive indices of the two independent polarizations.

3. Stress-field model and numerical calculation

3.1 Stress-field model

Hertz contact theory is important and essential in discussions of contact stress, and is widely used for difficult-to-measure contact problems such as those for bearings and gears (Sadeghi et al., 2009) (Asakura Publishing, 2012) (Ohmsha, 2010). According to Hertz contact theory, stress on the z -axis in cylindrical coordinates near the contact area between two spherical elastic bodies is given as

$$\sigma_{zz}(r=0) = -\frac{a^2}{a^2 + z^2} \frac{3W}{2\pi a^2}, \quad (13)$$

$$\sigma_{rr}(r=0) = \sigma_{\theta\theta}(r=0) = -\left[(1 + \nu_1) \frac{z}{a} \left(-\frac{\pi}{2} + \frac{a}{z} + \sin^{-1} \frac{z}{\sqrt{a^2 + z^2}} \right) - \frac{a^2}{2(a^2 + z^2)} \right] \frac{3W}{2\pi a^2}, \quad (14)$$

$$a = \sqrt[3]{\frac{3WR}{2E}}, \quad \frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}, \quad \frac{2}{E} = \frac{1 - \nu_1^2}{E_1} + \frac{1 - \nu_2^2}{E_2},$$

where, E_1 and E_2 are Young's modulus values for the two bodies, ν_1 and ν_2 are the Poisson's ratio for the former and latter bodies, R_1 and R_2 are radii of the former and latter bodies, and W is the contact load. Stress near the contact area between a flat body and a spherical body is given when R_1 is infinite.

3.2 Structural analysis

Stress distributions within the glass plate were numerically calculated at positions other than on the z-axis using structural analysis software (ANSYS Ver. 19.0, ANSYS, Inc). The glass plate and the resin sphere are modeled as a two-dimensional axially symmetric model, which has a free-slip boundary condition at the contact surface and a square mesh for which glass plate resolution is set as 0.05 mm. The value of the mesh intrusion condition for the contact surface is adjusted so that calculated stress on the z-axis approximates the theoretical solution from Hertz contact theory. The Young's modulus values, Poisson's ratio, densities and radii of the glass and the resin set for this numerical calculation are shown in Table 2, and contact load is set to 40 N.

Figure 2 shows the results of this numerical calculation. Fig. 2(a) shows the numerically calculated results and theoretical results for stress components on the z-axis. The results provide analysis conditions that calculate stress components on the z-axis as approximately equal to the theoretical values. Fig. 2(b) shows the calculated results for the distribution of axial stress σ_{zz} using the obtained analysis conditions.

Fig. 2 Numerical calculation of the stress tensor. Structural analysis conditions are obtained by comparing numerical calculation results and Hertzian contact stress on the z-axis so that the calculated stress on the z-axis approximates the theoretical solution. Stress distribution at positions other than on the z-axis are numerically calculated using structural analysis software.

4. Reconstructed results with measured data

4.1 Setup of BOS measurement

Figure 3(a) illustrates a BOS measurement and Fig. 3(b) shows the optical system used for BOS measurements in the X-Y plane. This experiment measures a stress field near a contact area between a bottom-supported glass plate and a resin sphere, with the sphere applying a vertical load onto the glass. The glass plate has standard refractive index n_0 and is placed in ambient air of refractive index n_{Air} , which is assumed to be 1. The origin O in the system is the center of the glass surface that is the contact area. To measure the deflection angle of rays passing through the glass in the BOS measurement, we used photographs of a dot-pattern background that is observed through the glass. When the refractive index of the glass is uniform, a ray from point P on the background goes straight through the glass and is observed at point Q on the camera's image plane. In contrast, the glass density locally increases near the contact area when a load is applied, increasing the refractive index near the contact area. This deflects rays passing through the glass near the contact area, so a ray from point P on the background is observed at point Q' on the image plane. Point P is thus observed as if it had moved to point P'. Displacement of background dot PP' is measured by comparing photographs from before and after load application, and dot displacement on image plane QQ' is measured using the pixel pitch of the image sensor.

(a) A BOS measurement.

(b) Optical system (omitting refraction of light rays on the glass surface).

Fig. 3 BOS measurement setup, with (a) a schematic illustration of BOS measurements and (b) the optical system in the X-Y plane.

In Fig. 3(b), the background on which dot patterns are printed is set on the plane at $X = X_D$, and the camera image plane is set on the plane at $X = X_i$. A light ray deflected by the change in refractive index is represented as a solid red line, while a light path with no refractive index change is represented as a solid blue line. The dashed red line is the path extended to point P' , where point P is observed from the camera after the refractive index changed. The distance through the glass along the X -axis is d , the distance from the origin O to the background is L_D , the distance from origin O to the lens is L_A , and the distance from the lens to the image plane is L_i . Table 1 summarizes parameter values for the optical system. Cartesian coordinates for points are represented like (X_i, q_Y, q_Z) .

Reconstruction of the refractive index field with scalar potential ϕ assumes that a light ray deflects at a point on the Y - Z plane. Representing the point where light ray deflection occurs as H , the position of a pattern dot projected on the image plane along the Y -axis can be written using the coordinate of point H and a magnification ratio M_1 of the optical lens system, giving

$$q'_Y = -M_1 h_Y, \quad (15)$$

where the magnification ratio can be written with L_A , L_i , and d , from geometrical optics considerations as

$$M_1 = \frac{L_i}{L_A + (1/n_0 - 1)d/2}. \quad (16)$$

Dot displacement along the Y -axis on the image plane due to light ray deflection can be written using magnification ratio M_2 of the optical lens system, deflection angle along the Y -axis ε_Y , L_D , d , and standard value n_0 as

$$q'_Y - q_Y = M_2 \varepsilon_Y n_0 \left(L_D + \left(\frac{1}{n_0} - 1 \right) \frac{d}{2} \right), \quad (17)$$

where the magnification ratio can be written using L_D , L_A , L_i , and d from geometrical optics considerations as

$$M_2 = \frac{L_i}{L_D + L_A + (1/n_0 - 1)d}. \quad (18)$$

Similar to Eq. (17), dot displacement along the Z -axis on the image plane can be written as

$$q'_Z - q_Z = M_2 \varepsilon_Z n_0 \left(L_D + \left(\frac{1}{n_0} - 1 \right) \frac{d}{2} \right). \quad (19)$$

Fig. 4 Perspective view of the experimental setup. A resin sphere (POM) is connected to a compression machine and contacts the center of a glass plate (BK7) at a compression load W of 40 N.

Figure 4 shows a perspective view of the experimental setup, in which a glass plate (BK7: S-BSL 7) is placed on the Z -axis positioning stage, and a resin sphere (POM: polyacetal, standard-grade, MISUMI) contacts the center of the glass. A compression load W of 40 N is introduced onto the glass with the resin sphere connected to a compression machine (AG-300kNX, Shimadzu). LED lighting (LLBK1, Aitec) with mean wavelength of 550 nm illuminates the dot pattern background. The glass plate had length d of 48.9 mm, width 50 mm, and thickness 8 mm. The standard refractive index n_0 is 1.51825 for the mean wavelength of the LED lighting of 550 nm at ambient temperature 23.5°C. The camera (DXC-RX10M2, Sony) has sensor pixel pitch 2.41 μm and sensor size 13.2×8.8 mm. We attached a linearly polarized light filter (517628, Kenko) to the camera to capture images of polarized rays. The experiment was carried out at room temperature.

Figure 5 shows the optical system of the lens. Since the lens focus is on the background in BOS measurements, the contact area that is the measurement target is blurred. The diameter of the entrance pupil of the optical system (d_i) is calculated as f/F , where F is the F-number of the optical system and f is the focal length of the lens. The diameter of blur circle d_b is calculated from the thin lens formula as the following equations, using converted distances L'_b and L'_A to consider the optical distance of the glass.

$$\frac{1}{f} = \frac{1}{L_i} + \frac{1}{L'_D + L'_A} = \frac{1}{L_i} + \frac{1}{L_D + L_A + \left(\frac{1}{n_0} - 1 \right) d}, \quad (20)$$

$$\frac{1}{f} = \frac{1}{L_b} + \frac{1}{L'_A} = \frac{1}{L_b} + \frac{1}{L_A + \left(\frac{1}{n_0} - 1\right)\frac{d}{2}}, \quad (21)$$

$$\frac{d_b}{d_l} = \frac{L_b - L_i}{L_b}. \quad (22)$$

Fig. 5 Optical system for focusing. Since the lens focuses on the background in BOS measurements, focus at the contact area (the glass center) that is the measurement target is blurred.

Table 1 lists each of the values mentioned above, and Table 2 summarizes physical properties of the glass (OHARA) and resin sphere (MISUMI) at the ambient temperature. Young's modulus and Poisson's ratio for the resin material are unpublished, so we used values for a comparable resin with similar mechanical properties from another manufacturer (Polyplastics, Toray Plastics Precision). Lateral surface deformation of the glass under load is assumed to be small, so we neglected any resulting path changes due to refraction. Similarly, polarization rotation due to the photoelastic effect in the glass under load is assumed to be small, we neglected any resulting image changes due to polarization rotation. The maximum Hertz pressure at a load W of 40 N is 55 MPa, which is sufficiently smaller than the compressive strength of the resin at 5% deformation of 103 MPa.

Table 1. Experiment values

Glass length, d [mm]	48.9
L_D [mm]	64.5
$L_A + L_i$ [mm]	348
Focal length, f [mm]	73.3
Camera F-number, F [-]	16
Magnification ratio, M_1 [-]	0.195
Magnification ratio, M_2 [-]	0.170

Table 2. Physical properties of the glass and resin materials

Glass: Young's modulus, E_1 [MPa]	8.00×10^4
Glass: Poisson's ratio, ν_1 [-]	0.205
Glass: density, ρ_1 [kg/m ³]	2520
Glass: radius, R_1 [mm]	∞
Glass: refractive index, n_0 [-]	1.51825
Glass: optical-stress coefficient, K_1 [/MPa]	-0.25×10^{-6}
Glass: optical-stress coefficient, K_2 [/MPa]	-2.7×10^{-6}
Resin: Young's modulus, E_2 [MPa]	2.70×10^3
Resin: Poisson's ratio, ν_2 [-]	0.35
Resin: density, ρ_2 [kg/m ³]	1410
Resin: radius, R_2 [mm]	20.0
Resin: compressive strength at 5% deformation [MPa]	103

4.2 Data processing

Figure 6 shows a close-up around the contact area in a captured image, and Fig. 7 shows displacement of the dots contoured with a color scale for the area indicated by the dashed red square in Fig. 6. Dot displacements were obtained using OpenCV with the help of the Farneback opticalflow algorithm. Displacements are mean values obtained from captured images, specifically five continuous images taken before applying load and another five continuous images taken after applying load. Dots near the contact area move away from the contact area, due to the increased refractive index field around the contact area. Absolute displacement values are symmetrical about the Z-axis, meaning the refractive index field is axially symmetric about the z-axis in cylindrical coordinates. There was random displacement in the Y-axis direction and displacement of about two pixels in the Z-axis direction at the glass surface. These noises are likely due to Z-axis displacement of the positioning stage caused by the load on the glass. Blurring of the glass surface is due to the above-described focus blurring.

Fig. 6 The background dot pattern as observed through the glass. The analysis area includes the contact area between the glass and the resin sphere.

Fig. 7 Background dot displacement. 1 image pixel is about 12.4 μm . Dots near the contact area move away due to the increased refractive index field generated there.

Figure 8 shows deflection angles due to horizontal polarization (ε_{Y_h} , ε_{Z_h}) and vertical polarization (ε_{Y_v} , ε_{Z_v}) as calculated using Eqs. (17) and (19).

Fig. 8 Deflection angles of horizontal and vertical polarization calculated by applying the experimentally obtained dot displacement to Eqs. (17) and (19) using geometric relationships.

It is necessary to perform data processing on the obtained deflection angles to reconstruct refractive indices using the scalar potential ϕ . First, absolute values of the deflection angles are averaged over the symmetric Z-axis. Next, data processing of deflection angles at the glass surface is examined. Since the fast Fourier transformation is used in the refractive index reconstruction process, deflection angles at the $Z = 0$ glass surface must be spatially differentiated. However, deflection angles are discontinuous at the glass surface, so a virtual deflection angle field that is continuously connected at the surface is supplemented in a $Z < 0$ area (i.e., outside the glass). One candidate virtual deflection angle that is continuous and differentiable at the $Z = 0$ surface can be written as

$$\varepsilon_Y(Y, Z) = (2\varepsilon_Y(Y, 0) - \varepsilon_Y(Y, -Z))\exp\left[-\left(\frac{Z}{R_v}\right)^2\right], \quad (23)$$

$$\varepsilon_Z(Y, Z) = (2\varepsilon_Z(Y, 0) - \varepsilon_Z(Y, -Z))\exp\left[-\left(\frac{Z}{R_v}\right)^2\right], \quad (24)$$

where $Z < 0$ and R_v is an arbitrary constant smaller than the typical analysis area size. Equations (23) and (24) thus represent the entire analysis area, including the virtual field. Note that deflection angles at $Z > 0$ remain the same, so this supplementary process does not affect reconstruction there. Figure 9 shows processed deflection angle distributions for horizontal polarization ($\varepsilon_{Y,h}$, $\varepsilon_{Z,h}$) and vertical polarization ($\varepsilon_{Y,v}$, $\varepsilon_{Z,v}$).

Fig. 9 Processed deflection angles of horizontal and vertical polarization for reconstructing refractive indices using the scalar potential ϕ .

4.3 Reconstruction of the refractive index deviation

Figure 10 shows the refractive index deviation due to an applied load from the experimental result using the scalar potential and calculation result using Eqs. (7) and (8) for numerically analyzed stress. Note that the effect of the continuous deflection of ray caused by stress distribution along the X -direction is taken into account by algorithm using the scalar potential cited in Section 2.1. The experimentally obtained refractive index agrees well with the calculated results for both independent polarizations about the refractive index field in the r - z plane as shown in Fig. 10(a) and on the z -axis as shown in Fig. 10(b).

(a) Refractive index field in the r - z plane for horizontal polarization and vertical polarization.

(b) Refractive index on the z -axis for horizontal polarization and vertical polarization.

Fig. 10 Refractive index deviation due to an applied load from the experimental and calculated results for horizontal polarization Δn_h and vertical polarization Δn_v . The experimental results agree well with the calculated results for both independent polarizations.

5. Results

Figure 11 shows the experimental results for stress distribution obtained by applying the experimental refractive index deviation to Eqs. (9), (10), (11), and (12) and from numerically calculated results for the stress field. The reconstructed axial stress σ_{zz} and shear stress τ_{rz} well agree with numeric calculations from the structural analysis software. However, radial stress σ_{rr} and hoop stress $\sigma_{\theta\theta}$ slightly deviated from those calculations. The difference between experimental and calculated results for axial stress σ_{zz} was about 1.6 times at the origin O.

(a) Experimental stress-tensor results.

(b) Calculated stress-tensor results.

Fig. 11 Results of stress-tensor field reconstruction. The reconstructed axial stress σ_{zz} and shear stress τ_{rz} well agree with numeric calculations from the structural analysis software. However, radial stress σ_{rr} and hoop stress $\sigma_{\theta\theta}$ slightly deviated from those calculations.

6. Discussion

In Fig. 11, experimental stress near the glass surface are slightly deviated from calculated result. Similarly, In Fig. 10, experimental refractive index deviation used to calculate the stress also has slight differences between experiment and calculation near the glass surface. One of the causes of these errors is considered to be an experimental factor such as alignment errors of the images of two polarization, due to the positioning error of the camera, image blurring and diffraction limit of light etc.

In addition, in Fig. 11, the experimental results showed noise near the glass surface, especially for radial stress σ_{rr} and hoop stress $\sigma_{\theta\theta}$. To consider the cause of this noise, we measured the stress shown in Figure 12 with the F-number setting of the camera changed to $F = 11$, which increased the blur diameter d_b ($d_b = 0.345$ mm for $F = 16$, and 0.501 mm for $F = 11$), thereby smoothing the measured deflection angle field and reducing noise. This reduced noise inside the glass as intended, but, as Fig. 12 shows, noise near the surface was insufficiently decreased. This noise might be mainly generated when measured discrete refractive index fields are partially differentiated by Eqs. (11) and (12), as discontinuity of the measured deflection angle influences noise inside the glass.

However, the glass image near the surface might be influenced by total internal reflection of light at the surface and minute displacement of the glass. These effects caused discontinuities in the experimental deflection angles, because calculation error for dot displacements based on opticalflow processing become larger. There is thus need for improving BOS measurement technology through improved imaging devices and image processing technologies for reducing measurement noise.

Even so, the proposed method is advantageous in terms of adaptability to various light wavelengths, a simple measurement system and simple equipment applicable to on-site measurements. It is expected to become a cost-effective new measurement method for experimental stress analysis.

Fig. 12 Stress as measured with different F-number settings of the camera. Decreasing the F-number increases the blur diameter d_b and blur near the contact area. Noise inside the glass decreases, but noise near the glass surface is insufficiently reduced at $F = 11$. Note that vertical axis in this figure indicates the 0 to 0.5 mm portion of the vertical axis in Fig. 11.

7. Conclusion

We proposed a measurement method for axisymmetric three-dimensional distributions of stress tensors using the BOS technique. Assuming axial symmetry of the stress field, we used the Lagrangian optics to derive a relationship between two-dimensional deflection angle and three-dimensional stress. Using independent ray polarizations, a method was derived for reconstructing the respective refractive indices from the deflection angle. The stress-tensor equilibrium equations presented as Eqs. (5) and (6) can be written under the assumption of axial symmetry. Applying appropriate approximations represented by Eq. (9), we formulated normal and shear stress-tensor components for reconstruction from the refractive index fields presented as Eqs. (10), (11), and (12). We experimentally generated the axisymmetric refractive index field by applying to a glass plate surface a load that can be described by Hertz contact theory. Stress-tensor components were reconstructed from the refractive index field using the proposed method, as shown in Fig. 11. The reconstructed axial stress-tensor σ_{zz} and shear stress τ_{rz} well agree with numeric calculations from structural analysis software. Other stress-tensor components, however, slightly deviated from those numeric calculations. We believe that the proposed method is a promising method for measuring stress and is applicable to wide range of applications.

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