Optimal design of a hysteretically damped dynamic vibration absorber

Toshihiko ASAMI*, Yoshito MIZUKAWA* and Keisuke YAMADA**

* Department of Mechanical Engineering, University of Hyogo
2167 Shosha, Himeji, Hyogo, 671-2280, Japan
E-mail: asami@eng.u-hyogo.ac.jp

** Department of Mechanical Engineering, Kansai University
3-3-35 Yamate-cho, Suita, Osaka, 564-8680, Japan

Received: 7 October 2019; Revised: 9 December 2019; Accepted: 3 March 2020

Abstract

In the optimization of dynamic vibration absorbers (DVAs), it is generally assumed that the damping force changes in proportion to the velocity of the object; this damping is called viscous damping. However, many DVAs used in practical applications are made of polymeric rubber materials having both restorative and damping effects. This polymer material is considered to show a hysteretic damping force that is proportional to the displacement rather than the velocity of the object. Despite the widespread use of such hysteretically damped DVAs, there are very few studies on their optimal design, and the design formula of the well-known general viscously damped DVA is presently used for the design of this type of DVA. This article reports the optimal solution of this hysteretically damped DVA. For generality, it is assumed that the primary system also has structural damping that can be treated as hysteretic damping. Three optimization criteria, namely the H_∞ optimization, H_2 optimization, and stability maximization criteria, were adopted for the optimization of the DVA. For the H_∞ optimization and stability maximization criteria, exact algebraic solutions were successfully derived, and for the H_2 criterion, simultaneous equations with six unknowns and their numerical solutions were obtained.

Keywords : Vibration, Optimal design, Hysteretically damped dynamic vibration absorber, H_∞ optimization criterion, H_2 optimization criterion, Stability maximization criterion, Damped primary system

1. Introduction

A dynamic vibration absorber (DVA) is an auxiliary vibrating body that can be attached to an object, called the primary system, to suppress the vibrations of that object. Since Ormondroyd and Den Hartog first proposed an optimization criterion for a damped DVA in 1928, many researchers have tried to optimize the DVA. The DVA optimization criteria proposed so far can be classified into three categories (Asami et al., 2002): H_∞ optimization, H_2 optimization, and stability maximization. H_∞ optimization is the earliest proposed criterion (Ormondroyd and Den Hartog, 1928) and aims to minimize the maximum amplitude response (called the H_∞ norm). The second optimization criterion—the H_2 optimization criterion—was devised to minimize the kinetic energy over the entire frequency range of the primary system subjected to random excitation (Crandall and Mark, 1963). This can be achieved by minimizing the area under the curve of the square of the frequency response function for the primary system. Both criteria are design methods that focus on the steady-state response of a primary system subjected to forced excitation. On the other hand, in the stability maximization criterion proposed by Nishihara and Matsuhisa in 1997, the design goal of the DVA is to attenuate the free vibration response of the primary system in the shortest time.

In optimization studies, the DVA has been assumed to show linear viscous damping characteristics, which are the easiest to handle mathematically. Regarding the optimal design of this type of DVA attached to a damped primary system, an exact algebraic solution has been found for all criteria except for H_∞ optimization. However, many DVAs in practical use are composed of a combination of an auxiliary mass and a vibration isolator made of a rubber polymer material, rather

than a metal spring and a dashpot arranged in parallel. This type of polymer material does not show viscous damping, in which the damping force changes in proportion to the velocity of the object, but is instead said to display hysteretic damping, where the force varies in proportion to the displacement of the object. However, there has been very little research on the optimal design of this hysteretically damped DVA (Asami et al., 2000; Asami and Nishihara, 2003), and in practice, it is actually designed using the optimal design formula for the viscously damped DVA.

The aim of the present study was to derive a formula for the optimal design of a hysteretically damped DVA. For the sake of generality, the authors considered that the primary system also has hysteretic structural damping and optimized the DVA according to the above three criteria.

2. Analytical models and definition of dimensionless parameters

Figures 1 and 2 respectively show the analytical models of the hysteretically damped and general viscously damped systems considered in this study. In these figures, the m_1 – k_1 system represents the primary system to be controlled, and the m_2 – k_2 system represents the DVA. Additionally, Figs. 1(a) and 2(a) are force excitation systems, in which the external force acts directly on the primary mass, and Figs. 1(b) and 2(b) are motion excitation systems, in which a sinusoidal displacement is applied to the foundation. If there is no damping in the primary system ($\eta_1 = 0$ in Fig. 1, $c_1 = 0$ in Fig. 2), the DVA will have the same optimal parameters regardless of whether the system is a force or motion excitation system. However, when there is damping in the primary system, there is a difference between the optimal solutions of the two systems in the case of a viscously damped primary system but not in the case of a hysteretically damped system. This is a major difference between general viscously damped and hysteretically damped systems.

In this study, calculations were performed using the following dimensionless parameters:

$$\mu = \frac{m_2}{m_1}, \quad \nu = \frac{\omega_2}{\omega_1}, \quad \lambda = \frac{\omega}{\omega_1}, \quad \zeta_1 = \frac{c_1}{2m_1\omega_1}, \quad \zeta_2 = \frac{c_2}{2m_2\omega_2}, \quad (1)$$

where ω_1 and ω_2 are defined as

$$\omega_1 = \sqrt{k_1/m_1}, \quad \omega_2 = \sqrt{k_2/m_2}. \quad (2)$$

The symbol η shown in Fig. 1 is a dimensionless parameter called the loss factor. It is known that the damping ratio ζ defined as in Eq. (1) is related to the loss factor as $\eta = 2\zeta$ at an excitation frequency ratio of $\lambda = 1$.

Here, there is an important matter to note in the analytical model shown in Fig. 1. It has been demonstrated that if a hysteretically damped system is represented with a complex spring element as shown in Fig. 1, an unstable pole (characteristic root) appears in the system (Nagaike and Nagamatsu, 1985). Several methods have been proposed to find a convergent solution in this case by avoiding the divergence of free vibration. In this study, the authors adopted a method using the inverse Fourier transform to obtain a stable free vibration response in the hysteretically damped system.

Fig. 1 Analytical model of a hysteretically damped system subjected to (a) force, and (b) motion excitation

Fig. 2 Analytical model of a viscously damped system subjected to (a) force, and (b) motion excitation

3. Organization of this paper

As mentioned in the introduction, most DVAs in practical use are of the hysteresis damping type, whereas the equations used in their design are optimal solutions for the viscously damped DVA. Considering that many DVA designers only know the design formula for viscously damped DVA, the authors compare the difference between the optimal solutions of hysteretically and viscously damped DVAs throughout this paper.

DVAs are used in various vibration environments. The authors believe that H_∞ optimization is desirable in environments subject to sinusoidal excitation, and that H_2 optimization is desirable in environments subject to random excitation. Furthermore, if you want to suppress the free vibration of the primary system as quickly as possible, you should design the DVA according to the stability maximization criterion. In this paper, the authors will sequentially show the optimal design expressions or numerical solutions of the hysteretically and viscously damped DVAs according to these three types of optimization criteria. In addition, to confirm the vibration suppression effect of these DVAs, the frequency response and time history response of the vibratory systems are shown for typical mass ratios ($\mu = 0.05$ and 0.1).

4. Basic difference between hysteretic damping and general viscous damping

Before the optimization of the DVA is presented, it is necessary to clarify the difference between hysteretic damping and viscous damping. Figure 3 shows a hysteretically damped system and a general viscously damped system, each composed of a single mass. When the mass m of either of these systems is subjected to a sinusoidal displacement described by $x(t) = a \sin \omega t$, the force F_T is transmitted to the foundation of the system via a spring and a dashpot. If the loss factor η for the hysteretically damped system is 0.2 and the damping ratio ζ for the general viscously damped system is 0.1, the hysteresis curves describing the relationship between the transmitted force and input displacement for each system are as shown in Figs. 4(a) and (b). This phenomenon is caused by the phase difference between the sinusoidal force and displacement. The area enclosed by the hysteresis loop represents the magnitude of the dissipated energy. One of the features of hysteretic damping is that the shape of the loop does not change with the rate at which the operating point passes around the loop. In contrast, in a viscously damped system, as the velocity of the mass increases (i.e., the dimensionless frequency λ increases), the area of the ellipse and the damping force also increase. Conversely, when the mass velocity—and hence the frequency—approaches zero, the viscous damping force converges to zero, whereas the hysteretic damping force remains the same.

Figure 5 shows load–displacement data for a rubber bush used in automobile suspensions obtained in a previous study (Shibata et al., 1993). It can be seen from this figure that the shape of the hysteresis loop remains almost the same even as the excitation frequency is varied from 0 to 100 Hz. This result demonstrates that a vibration isolator made of a polymer material has characteristics resembling the hysteretic damping behavior shown in Fig. 4(a).

Special attention must be paid to the nonlinearity of the vibration-isolating rubber. The vibration-isolating rubber bush whose load–displacement curve is shown in Fig. 5 has a softening characteristic. In another experiment by the same research team, an isolator with a hardening spring characteristic was reported (Shibata et al., 1996). In the experiment shown in Fig. 5, an extremely small displacement was applied to the bush. These nonlinearities were not considered in this study.

5. H_∞ optimization of hysteretically damped dynamic vibration absorber

The evaluation index for H_∞ optimization is the H_∞ norm, which is defined as (Asami et al., 2002):

$$h_{\max} = \left| \frac{x_1}{f/k_1} \right|_{\max} \quad \text{or} \quad h_{\max} = \left| \frac{x_1}{x_0} \right|_{\max}. \quad (3)$$

Minimizing $h_{\max}$ is the goal of H_∞ optimization, and the minimum value of $h_{\max}$ is denoted by $h_{\min}$.

Fig. 3 Single-mass vibratory system: (a) Hysteretically damped system, and (b) Viscously damped system

5.1. H_∞ -optimal solution for undamped systems

Exact algebraic solutions have already been obtained for the case in which there is no damping in the primary system (Asami et al., 2000; Asami and Nishihara, 2003). First, for the hysteretically damped system,

$$\nu_{\text{opt}} = \frac{1}{1+\mu}, \quad \eta_{2\text{opt}} = \sqrt{\frac{\mu(3+\mu)}{2}}. \quad (4)$$

Next, for the general viscously damped system,

$$\nu_{\text{opt}} = \frac{2}{1+\mu} \sqrt{\frac{2}{3} \frac{16+23\mu+9\mu^2+2(2+\mu)\sqrt{4+3\mu}}{64+80\mu+27\mu^2}}, \quad \zeta_{2\text{opt}} = \frac{1}{4} \sqrt{\frac{8+9\mu-4\sqrt{4+3\mu}}{1+\mu}}. \quad (5)$$

Figure 6(a) shows the H_∞ -optimal solution for the undamped system calculated from Eqs. (4) and (5). The difference between the optimal tuning ratios ν_{opt} is very small, and the two lines are indistinguishable in Fig. 6(a). In addition, because the relationship $\zeta = \eta/2$ has been established by setting $\lambda = 1$, the value of $\eta_{2\text{opt}}/2$ is drawn with a dashed line in this figure; it can be seen that this dashed line is slightly larger than $\zeta_{2\text{opt}}$. Furthermore, the minimized resonance amplitude h_{min} reveals that the value for the hysteretically damped system (black line) is greater than that for the viscously damped system (gray line). In other words, if there is no damping in the primary system, the hysteretically damped DVA is inferior to the viscously damped DVA.

The frequency response functions for the system with the optimized hysteretically and viscously damped DVAs are shown in Figs. 7(a) and (b), respectively. Again, as the mass ratio μ increases, the resonance points are kept lower in the system with the viscously damped DVA. However, in a practical mass range ($\mu \leq 0.2$), the difference is negligibly small.

5.2. H_∞ -optimal solution for damped systems

Similarly, the exact solution of the H_∞ optimization for the hysteretically damped system was obtained in this study.

Fig. 4 Relationship between input displacement x and transmitted force F_T : (a: left) Hysteretically damped system, and (b: right) Viscously damped system

Fig. 5 Hysteretic load-displacement relationships for a rubber bush (natural rubber polymer) used in a car suspension (Shibata et al., 1993)

The procedure for deriving the solution for a single-mass DVA has been described in detail by Nishihara and Asami in 2002. The optimal parameters are given by

$$v_{\text{opt}} = \frac{1}{1+\mu} \sqrt{\frac{\text{Num}v}{\text{Den}v}}, \quad \eta_{2\text{opt}} = \sqrt{\frac{r_{\min}^2 + \eta_1^2}{(1+\mu)^2 v_{\text{opt}}^4}} - 1, \quad h_{\min} = \sqrt{\frac{1}{1-r_{\min}^2}}, \quad (6)$$

where

$$\left. \begin{aligned} \text{Num}v &= [2 + \mu - 2r_{\min}^2(1+\mu)][(1+\mu)^2 - 2r_{\min}^2(2+\mu+\mu^2)] \\ &\quad - 2[5 + 7\mu + 6\mu^2 + 2\mu^3 - 2r_{\min}^2(4 + 6\mu + 2\mu^2 + \mu^3)]\eta_1^2 + 4(2 + 3\mu)\eta_1^4 \\ \text{Den}v &= -[1 + \mu + 2r_{\min}^2(1-\mu)][2 + \mu - 2r_{\min}^2(1+\mu)] - 2[1 - 3\mu - 2\mu^2 - 2r_{\min}^2(2 - \mu^2)]\eta_1^2 + 4\eta_1^4 \end{aligned} \right\} \quad (7)$$

and

$$\left. \begin{aligned} r_{\min} &= \sqrt{-\frac{h_1}{3} + 2\sqrt{\frac{p}{3}}\cos(\theta + 2\pi/3)} \quad \text{or} \quad \sqrt{-\frac{h_1}{3} + 2\sqrt{\frac{p}{3}}\cos(\theta + 4\pi/3)} \\ \theta &= \frac{1}{3}\cos^{-1}\frac{3q/p}{2\sqrt{p/3}}, \quad p = \frac{1}{3}h_1^2 - h_2, \quad q = \frac{1}{3}h_1h_2 - \frac{2}{27}h_1^3 - h_3, \quad h_1 = \frac{g_1}{g_0}, \quad h_2 = \frac{g_2}{g_0}, \quad h_3 = \frac{g_3}{g_0} \\ g_0 &= 4(1-\mu^2)^2, \quad g_1 = -4(1+\mu)(3+3\mu^2+2\mu^3) + 4\eta_1^2(3-4\mu^2+\mu^4) \\ g_2 &= (2+\mu)(6+9\mu+12\mu^2+5\mu^3) - 4\eta_1^2(6+6\mu+\mu^2+5\mu^3+2\mu^4) + 4\eta_1^4(3-2\mu^2) \\ g_3 &= -(1+\mu)^2(2+\mu)^2 + 4\eta_1^2(3+6\mu+8\mu^2+5\mu^3+\mu^4) + 4\eta_1^4(3+3\mu-2\mu^2) + 4\eta_1^6. \end{aligned} \right\} \quad (8)$$

This solution was derived for the force excitation system shown in Fig. 1(a), and a different solution can be derived for the motion excitation system shown in Fig. 1(b). Both solutions were found to be numerically the same, so the latter solution is omitted here. However, as will be described later, the minimized resonance amplitude $h_{\min}$ is greater than that calculated from Eqs. (6)–(8) above.

Equation (8) was obtained by solving a cubic equation for r^2 , where $r^2 = 1 - 1/h_{\max}^2$. If this cubic equation is solved using Cardano's method, it falls into the casus irreducibilis where three real roots appear. One of the three real roots is the solution we want, but it contains the imaginary unit i . To avoid including the imaginary unit in the expression of the real solution, this equation was solved with Viète's method using the triple-angle formula for the trigonometric function. As a result, it was found that the desired solution was included in one of the two $r_{\min}$ expressions given in Eq. (8). Thus, there are two algebraic expressions describing the solution, but the first can sufficiently cover the normal application of DVAs.

Figure 8 shows the H_{∞} -optimal solutions at a range of appropriate mass ratios μ plotted against the loss factor η_1 of the primary system. This H_{∞} -optimal solution has a computational limitation; that is, the value of μ cannot be 1. Therefore, in Fig. 8, the solution is only plotted for values of μ of up to 0.99. As shown in this figure, when η_1 becomes zero, the values of v_{opt} and $\eta_{2\text{opt}}$ agree with the values calculated by Eq. (4). This plot also shows that as the primary damping η_1 increases, v_{opt} decreases monotonically and $\eta_{2\text{opt}}$ increases monotonically. This trend is completely opposite to that for the H_2 -optimal solution presented later.

Figure 9 shows the resonance amplitude $h_{\min}$ of the primary system to which the H_{∞} -optimal DVA is attached. Figures 9(a) and (b) respectively show the $h_{\min}$ values for the force excitation system, calculated using Eq. (8), and the

Fig. 6 Optimal solutions for hysteretically and viscously damped DVAs attached to an undamped primary system: (a: left) H_{∞} -optimal solutions, and (b: right) H_2 -optimal solutions

Fig. 7 Frequency response functions for an undamped primary system with an optimally tuned and damped DVA: (a: left) Hysteretically damped system, and (b: right) Viscously damped system

motion excitation system, the expression for which is omitted here. The major difference between these $h_{\min}$ solutions is that the former can take a value of 1 or less whereas the latter cannot be less than 1. The reason for this can be understood from the form of the frequency response function, which will be shown later. It can be seen from this figure that the resonance amplitude is lower when a larger DVA is installed. The curve of $\mu = 0$ shown as a gray line in this figure represents the resonance amplitude of the primary system when no DVA is attached. Because the difference between this curve and the curve for $\mu = 0.05$, shown as a black line, is large, it can be said that a considerable vibration suppression effect can be expected even if a small DVA is attached.

Fig. 8 H_{∞} -optimal solutions for a hysteretically damped system: (a: left) H_{∞} -optimal tuning ratio v_{opt} , and (b: right) H_{∞} -optimal loss factor $\eta_{2\text{opt}}$

Fig. 9 Minimized amplitudes for hysteretically damped systems: (a: left) Force excitation system, and (b: right) Motion excitation system

Figures 10 and 11 show the frequency response functions for the force excitation system in Fig. 1(a) and the motion excitation system in Fig. 1(b), respectively. In Fig. 10, the displacement response for the primary system on the vertical axis is nondimensionalized by dividing it by the static displacement $|f/k_1|$ of the primary mass. It should be noted that this static spring constant k_1 does not include the loss factor η_1 for the primary system. In the hysteretically damped system, a damping force acts on the system even when a static force is applied, as shown in Fig. 4(a). With the vertical axis nondimensionalized in this way, the dimensionless displacement will not be 1 even in the stationary state ($\lambda = 0$) and can be less than 1. Therefore, as shown in Fig. 10, the phenomenon of the dimensionless resonance amplitude becoming smaller than 1 occurs. In contrast, in the motion excitation system, the value of the transmissibility $|x_1/x_0|$ is always 1 at rest, so the resonance amplitude is always greater than 1, as shown in Fig. 11.

6. H_2 optimization of hysteretically damped dynamic vibration absorber

The evaluation index of the H_2 optimization criterion is the H_2 norm, which is defined as (Asami et al., 2002)

$$I_a = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \frac{x_1}{f/k_1} \right|^2 d\lambda \quad \text{or} \quad I_a = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \frac{x_1}{x_0} \right|^2 d\lambda. \quad (9)$$

Minimizing I_a is the goal of H_2 optimization, and this minimum value is written as I_{amin} .

6.1. H_2 -optimal solution for general viscously damped dynamic vibration absorber

Before going into the hysteretically damped system here, we will outline the H_2 optimization of the DVA for the conventional general viscously damped system. Actually, H_2 optimization of the hysteretically damped DVA cannot be performed by simply following the conventional method; thus, this paper proposes a modification to the conventional H_2 optimization procedure. First, the calculation result for the evaluation index I_a defined by Eq. (9) for the force excitation system shown in Fig. 2(a) is given by

$$I_a = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \frac{x_1}{f/k_1} \right|^2 d\lambda = \frac{1}{2\pi} \int_{-\infty}^{\infty} f_n(\lambda) d\lambda; \quad f_n(\lambda) = \frac{a_0\lambda^6 + a_1\lambda^4 + a_2\lambda^2 + a_3}{(\lambda^4 - b_2\lambda^2 + b_4)^2 + (-b_1\lambda^3 + b_3\lambda)^2} \quad (10)$$

$$\left. \begin{aligned} a_0 &= 0, \quad a_1 = 1, \quad a_2 = -2(1 - 2\zeta_2^2)v^2, \quad a_3 = v^4 \\ b_1 &= 2[\zeta_1 + \zeta_2(1 + \mu)v], \quad b_2 = 1 + 4\zeta_1\zeta_2v + (1 + \mu)v^2, \quad b_3 = 2v(\zeta_1v + \zeta_2), \quad b_4 = v^2. \end{aligned} \right\} \quad (11)$$

Because the system parameters defined in Eq. (1) are non-negative, all the parameters b_1 – b_4 in Eq. (11) are positive.

In the conventional method, the denominator of the function f_n in Eq. (10) is factorized as follows:

$$\text{Den}[f_n] = (\lambda^4 - ib_1\lambda^3 - b_2\lambda^2 + ib_3\lambda + b_4)(\lambda^4 + ib_1\lambda^3 - b_2\lambda^2 - ib_3\lambda + b_4). \quad (12)$$

Furthermore, Eq. (12) was formally factorized into the following linear expression:

$$\text{Den}[f_n] = (\lambda - i\lambda_1)(\lambda - i\lambda_2)(\lambda - i\lambda_3)(\lambda - i\lambda_4)(\lambda + i\lambda_1)(\lambda + i\lambda_2)(\lambda + i\lambda_3)(\lambda + i\lambda_4), \quad (13)$$

where the first four factors correspond to the first factor in Eq. (12). Applying the residue theorem (Kreyszig, 1999) to the function f_n after factorizing the denominator as Eq. (13) and executing the improper integral of Eq. (10) yields the evaluation index I_a in the H_2 optimization as

$$I_a = \frac{(b_2b_3 - b_1b_4)a_0 + b_3a_1 + b_1a_2 + (b_1b_2 - b_3)a_3/b_4}{2[b_1(b_2b_3 - b_1b_4) - b_3^2]}. \quad (14)$$

Fig. 10 Frequency responses for a hysteretic system subjected to force excitation: (a: left) $\mu = 0.05$, and (b: right) $\mu = 0.1$

Fig. 11 Frequency responses for a hysteretic system subjected to motion excitation: (a: left) $\mu = 0.05$, and (b: right) $\mu = 0.1$

This method is correct in the end, but it misleads many people to believe the pole (or characteristic root) of the function f_n is represented by a pure imaginary number. In practice, the constants λ_1 – λ_4 in Eq. (13) are all complex numbers. To clarify this point, the authors recommend using the following equation (Asami and Mizukawa, 2019) instead of Eq. (13):

$$\text{Den}[f_n] = \left. \begin{aligned} &(\lambda + \alpha_1 - i\beta_1)(\lambda - \alpha_1 - i\beta_1)(\lambda + \alpha_2 - i\beta_2)(\lambda - \alpha_2 - i\beta_2) \\ &\times (\lambda + \alpha_1 + i\beta_1)(\lambda - \alpha_1 + i\beta_1)(\lambda + \alpha_2 + i\beta_2)(\lambda - \alpha_2 + i\beta_2). \end{aligned} \right\} \quad (15)$$

With this equation, the generality is not lost even if α_1 , α_2 , β_1 , and β_2 are all positive real numbers. Expanding Eq. (15) and comparing the coefficients with Eq. (12) yields the following relational expressions with system parameters b_i ($i = 1$ –4):

$$\left. \begin{aligned} b_1 &= 2(\beta_1 + \beta_2), \quad b_2 = \alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2 + 4\beta_1\beta_2 \\ b_3 &= 2\beta_2(\alpha_1^2 + \beta_1^2) + 2\beta_1(\alpha_2^2 + \beta_2^2), \quad b_4 = (\alpha_1^2 + \beta_1^2)(\alpha_2^2 + \beta_2^2). \end{aligned} \right\} \quad (16)$$

Substituting Eq. (15) for the denominator of the function f_n in Eq. (10) and performing integration using the residue theorem yields the following result:

$$\left. \begin{aligned} I_a &= I_{a\text{Num}}/I_{a\text{Den}} \\ I_{a\text{Num}} &= a_3[\alpha_1^2\beta_1 + \alpha_2^2\beta_2 + (\beta_1 + \beta_2)(\beta_1^2 + 3\beta_1\beta_2 + \beta_2^2)] + (\alpha_1^2 + \beta_1^2)(\alpha_2^2 + \beta_2^2)\{a_1\alpha_1^2\beta_2 + a_0\alpha_1^4\beta_2 + a_2(\beta_1 + \beta_2) \\ &\quad + a_1\beta_1[\alpha_2^2 + \beta_2(\beta_1 + \beta_2)] + a_0\beta_1[\alpha_2^4 + 2\alpha_2^2\beta_2(2\beta_1 + \beta_2)] + 2\alpha_1^2\beta_2(\beta_1 + 2\beta_2) + \beta_2(\beta_1 + \beta_2)(\beta_1^2 + 3\beta_1\beta_2 + \beta_2^2)\} \\ I_{a\text{Den}} &= 4\beta_1(\alpha_1^2 + \beta_1^2)\beta_2(\alpha_2^2 + \beta_2^2)\{\alpha_1^4 - 2\alpha_1^2[\alpha_2^2 - (\beta_1 + \beta_2)^2] + [\alpha_2^2 + (\beta_1 + \beta_2)^2]^2\}. \end{aligned} \right\} \quad (17)$$

Applying the relational expressions shown in Eq. (16) to the parameters α_1 , α_2 , β_1 , and β_2 of Eq. (17) yields the same expression as given in Eq. (14). As will be shown later, the parameters are all real numbers, and it can be proven that λ_1 – λ_4 in Eq. (13) are all complex numbers. Thus, in the case of a general viscously damped system, the improper integral of Eq. (10) can be expressed by system parameters as described by Eq. (14). Next, by solving the simultaneous equations obtained by the partial differentiation of this equation with respect to ν and ζ_2 , the optimal tuning ratio ν_{opt} and the optimal damping ratio $\zeta_{2\text{opt}}$ can be derived as follows:

$$\left. \begin{aligned} \nu_{\text{opt}}^2 &= -\frac{g_1}{4g_0} + \frac{1}{2} \sqrt{\frac{g_2}{3g_0}} - \frac{1}{2} \sqrt{\frac{3g_1^2}{4g_0^2} - \frac{2g_2}{g_0} - \frac{q_2}{3g_0} - \left(\frac{g_1^3}{4g_0^3} - \frac{g_1g_2}{g_0^2} + \frac{2g_3}{g_0}\right) \sqrt{\frac{3g_0}{q_2}}} \\ \zeta_{2\text{opt}} &= \frac{\zeta_1}{2\nu_{\text{opt}}[2 + \mu - 2(1 + \mu)^2\nu_{\text{opt}}^2]} \left[-1 + 2\mu\nu_{\text{opt}}^2 + (1 + \mu)^2\nu_{\text{opt}}^4 \right. \\ &\quad \left. + (1 - \nu_{\text{opt}}^2 - \mu\nu_{\text{opt}}^2) \sqrt{1 + 2(1 - \mu)\nu_{\text{opt}}^2 + (1 + \mu)^2\nu_{\text{opt}}^4} \right] \\ p_0 &= g_2^2 - 3g_1g_3 + 12g_0g_4, \quad p_1 = 2g_2^3 - 9g_1g_2g_3 + 27g_1^2g_4 + 27g_0g_3^2 - 72g_0g_2g_4 \\ q_1 &= p_1 + \sqrt{p_1^2 - 4p_0^3}, \quad q_2 = 3g_1^2/(4g_0) - 2g_2 + p_0(q_1/2)^{-1/3} + (q_1/2)^{1/3} \\ g_0 &= 4(1 - \zeta_1^2 + \mu)(1 + \mu)^5, \quad g_1 = 4\mu(1 + \mu)^2[4\zeta_1^4 - 5\zeta_1^2(1 + \mu) + (1 + \mu)^2] \\ g_2 &= -(1 + \mu)^2(8 - 8\zeta_1^2 + 12\mu + 3\mu^2), \quad g_3 = -2(2 - 6\zeta_1^2 + \mu)\mu(1 + \mu), \quad g_4 = -4\zeta_1^2(1 + \mu) + (2 + \mu)^2. \end{aligned} \right\} \quad (18)$$

When there is no damping in the primary system, the solution given in Eq. (18) reduces to the following simple solution:

$$\nu_{\text{opt}} = \frac{1}{1 + \mu} \sqrt{\frac{2 + \mu}{2}}, \quad \zeta_{2\text{opt}} = \sqrt{\frac{\mu(4 + 3\mu)}{8(1 + \mu)(2 + \mu)}}. \quad (19)$$

One of the authors has previously proposed a solution for a viscously damped DVA based on the H_2 criterion derived using Viète's method; here, it is derived using Cardano's method. These two solutions were confirmed to return the same value. However, the Cardano's method solution given in Eq. (18) requires a complex number calculation. Equation (18) is an exact solution for the force excitation system shown in Fig. 2(a), but because the algebraic solution was not obtained for the motion excitation system in Fig. 2(b), the original equations were instead solved numerically. The results are shown in Fig. 12. Figure 12(a) shows the optimal tuning ratio ν_{opt} , and Fig. 12(b) shows the optimal damping ratio $\zeta_{2\text{opt}}$; in this figure, the solid and dashed lines represent the solutions for the motion and force excitation systems, respectively. In the force excitation system, the optimal values decrease monotonically as the primary damping ζ_1 increases. The solution for the motion excitation system is almost the same as that for the force excitation system up to $\zeta_1 = 0.1$.

Figures 13(a) and (b) represent the trajectory of the characteristic root of a vibratory system with a viscously damped DVA optimized by the H_2 criterion for the force and motion excitation systems, respectively. As shown in the figure, both the real parts α_1 and α_2 and imaginary parts β_1 and β_2 of the characteristic roots take non-zero values. This confirms that the characteristic roots are indeed complex numbers. There are eight characteristic roots from the first to the fourth quadrant, but this figure shows only two roots in the first quadrant. The remaining characteristic roots can be obtained by reflecting the points shown in Fig. 13 about the real and imaginary axes. This figure shows the root locus when the primary damping ratio ζ_1 is varied from 0 to 0.6 and the mass ratio μ is varied from 0.01 to 1.0. In the force excitation system (Fig. 13(a)), the loci of the two characteristic roots never intersect, but in the motion excitation system (Fig. 13(b)), they show more complex behavior. In particular, the root loci of $\mu = 0.07$ and 0.075 are completely different from each other despite the mass ratios being very close.

6.2. H_2 -optimal solution for hysteretically damped dynamic vibration absorber

The function f_n obtained by squaring the compliance transfer function for the hysteretically damped system for the force excitation system in Fig. 1(a) is expressed as follows:

$$\left. \begin{aligned} f_n(\lambda) &= \frac{a_0\lambda^6 + a_1\lambda^4 + a_2\lambda^2 + a_3}{(\lambda^4 - b_2\lambda^2 + b_4)^2 + (-b_1\lambda^2 + b_3)^2} \\ a_0 &= 0, \quad a_1 = 1, \quad a_2 = -2\nu^2, \quad a_3 = (1 + \eta_2^2)\nu^4 \\ b_1 &= \eta_1 + \eta_2(1 + \mu)\nu^2, \quad b_2 = 1 + (1 + \mu)\nu^2, \quad b_3 = (\eta_1 + \eta_2)\nu^2, \quad b_4 = (1 - \eta_1\eta_2)\nu^2. \end{aligned} \right\} \quad (20)$$

The second term of the denominator of the function f_n is different from that for the viscously damped system. The denominator of the function f_n can be factored as follows:

$$\text{Den}[f_n] = [\lambda^4 - (b_2 + ib_1)\lambda^2 + b_4 + ib_3][\lambda^4 - (b_2 - ib_1)\lambda^2 + b_4 - ib_3]. \quad (21)$$

In this case, it is impossible to further factorize Eq. (21) into a linear form as was done with Eq. (13). However, it can be factorized as follows:

$$\left. \begin{aligned} \text{Den}[f_n] &= (\lambda - \alpha_1 - i\beta_1)(\lambda + \alpha_1 + i\beta_1)(\lambda - \alpha_2 - i\beta_2)(\lambda + \alpha_2 + i\beta_2) \\ &\quad \times (\lambda - \alpha_1 + i\beta_1)(\lambda + \alpha_1 - i\beta_1)(\lambda - \alpha_2 + i\beta_2)(\lambda + \alpha_2 - i\beta_2), \end{aligned} \right\} \quad (22)$$

where the first four factors in Eq. (22) correspond to the first factor in Eq. (21). Equations (22) and (15) are exactly the same except that the order of the eight factors is different. Therefore, it is possible to obtain the improper integral of Eq. (9)

Fig. 12 H_2 -optimal solutions for a viscously damped DVA attached to damped primary system: (a: left) H_2 -optimal tuning ratio ν_{opt} , and (b: right) H_2 -optimal damping ratio $\zeta_{2\text{opt}}$

Fig. 13 Characteristic root loci of viscously damped systems with an H_2 -optimal DVA:
(a: left) Force excitation system, and (b: right) Motion excitation system

in exactly the same way as in the viscous damping case, and the integration result is expressed by Eq. (17). However, the relationship between the characteristic root coordinates α_1 , α_2 , β_1 , and β_2 and the system parameters b_1 – b_4 are different from Eq. (16), and the result is expressed as

$$\left. \begin{aligned} b_1 &= 2(\alpha_1\beta_1 + \alpha_2\beta_2), & b_2 &= \alpha_1^2 + \alpha_2^2 - \beta_1^2 - \beta_2^2 \\ b_3 &= 2\alpha_2\beta_2(\alpha_1^2 - \beta_1^2) + 2\alpha_1\beta_1(\alpha_2^2 - \beta_2^2), & b_4 &= (\alpha_1^2 - \beta_1^2)(\alpha_2^2 - \beta_2^2) - 4\alpha_1\alpha_2\beta_1\beta_2. \end{aligned} \right\} \quad (23)$$

For hysteretically damped systems, as in the case of viscously damped systems, Eq. (17) cannot be replaced with Eq. (14), which is expressed using the system parameters. Therefore, a method of obtaining the optimal values directly from Eq. (17) is proposed here. First, because the four expressions for b_i ($i = 1$ – 4) in Eq. (20) and the corresponding four expressions in Eq. (23) are equivalent, the following simultaneous equations with four variables can be obtained:

$$\left. \begin{aligned} f_1 &= \eta_1 + \eta_2(1 + \mu)v^2 - 2(\alpha_1\beta_1 + \alpha_2\beta_2) = 0, & f_2 &= 1 + (1 + \mu)v^2 - \alpha_1^2 - \alpha_2^2 + \beta_1^2 + \beta_2^2 = 0 \\ f_3 &= (\eta_1 + \eta_2)v^2 - 2\alpha_2\beta_2(\alpha_1^2 - \beta_1^2) - 2\alpha_1\beta_1(\alpha_2^2 - \beta_2^2) = 0, & f_4 &= (1 - \eta_1\eta_2)v^2 - (\alpha_1^2 - \beta_1^2)(\alpha_2^2 - \beta_2^2) + 4\alpha_1\alpha_2\beta_1\beta_2 = 0. \end{aligned} \right\} \quad (24)$$

This is not the only expression needed for optimization. Next, to express Eq. (17) in a compact form, the system parameters are summarized in terms of a_i ($i = 0$ – 3), and the variables h_i ($i = 0$ – 3) are assigned as coefficients of a_i . This gives

$$I_a = a_0h_0 + a_1h_1 + a_2h_2 + a_3h_3. \quad (25)$$

If this equation is partially differentiated with respect to the two parameters v and η_2 , respectively, and set to zero, the following additional equations are obtained:

$$\left. \begin{aligned} f_5 &= \frac{\partial I_a}{\partial v} = \frac{\partial a_0}{\partial v}h_0 + \frac{\partial a_1}{\partial v}h_1 + \frac{\partial a_2}{\partial v}h_2 + \frac{\partial a_3}{\partial v}h_3 + a_0\frac{\partial h_0}{\partial v} + a_1\frac{\partial h_1}{\partial v} + a_2\frac{\partial h_2}{\partial v} + a_3\frac{\partial h_3}{\partial v} = 0 \\ f_6 &= \frac{\partial I_a}{\partial \eta_2} = \frac{\partial a_0}{\partial \eta_2}h_0 + \frac{\partial a_1}{\partial \eta_2}h_1 + \frac{\partial a_2}{\partial \eta_2}h_2 + \frac{\partial a_3}{\partial \eta_2}h_3 + a_0\frac{\partial h_0}{\partial \eta_2} + a_1\frac{\partial h_1}{\partial \eta_2} + a_2\frac{\partial h_2}{\partial \eta_2} + a_3\frac{\partial h_3}{\partial \eta_2} = 0. \end{aligned} \right\} \quad (26)$$

The function f_5 is computed as the sum of terms containing eight partial derivatives, the first four of which are easily computed. The partial derivative $\partial h_0/\partial v$ included in the fifth term is difficult to compute, so it is transformed as follows using the inverse function theorem:

$$\frac{\partial h_0}{\partial v} = \begin{bmatrix} \frac{\partial h_0}{\partial \alpha_1} \\ \frac{\partial h_0}{\partial \alpha_2} \\ \frac{\partial h_0}{\partial \beta_1} \\ \frac{\partial h_0}{\partial \beta_2} \end{bmatrix}^T \begin{bmatrix} \frac{\partial \alpha_1}{\partial b_1} & \frac{\partial \alpha_1}{\partial b_2} & \frac{\partial \alpha_1}{\partial b_3} & \frac{\partial \alpha_1}{\partial b_4} \\ \frac{\partial \alpha_2}{\partial b_1} & \frac{\partial \alpha_2}{\partial b_2} & \frac{\partial \alpha_2}{\partial b_3} & \frac{\partial \alpha_2}{\partial b_4} \\ \frac{\partial \beta_1}{\partial b_1} & \frac{\partial \beta_1}{\partial b_2} & \frac{\partial \beta_1}{\partial b_3} & \frac{\partial \beta_1}{\partial b_4} \\ \frac{\partial \beta_2}{\partial b_1} & \frac{\partial \beta_2}{\partial b_2} & \frac{\partial \beta_2}{\partial b_3} & \frac{\partial \beta_2}{\partial b_4} \end{bmatrix} \begin{bmatrix} \frac{\partial b_1}{\partial v} \\ \frac{\partial b_2}{\partial v} \\ \frac{\partial b_3}{\partial v} \\ \frac{\partial b_4}{\partial v} \end{bmatrix} = \begin{bmatrix} \frac{\partial h_0}{\partial \alpha_1} \\ \frac{\partial h_0}{\partial \alpha_2} \\ \frac{\partial h_0}{\partial \beta_1} \\ \frac{\partial h_0}{\partial \beta_2} \end{bmatrix}^T \begin{bmatrix} \frac{\partial b_1}{\partial \alpha_1} & \frac{\partial b_1}{\partial \alpha_2} & \frac{\partial b_1}{\partial \beta_1} & \frac{\partial b_1}{\partial \beta_2} \\ \frac{\partial b_2}{\partial \alpha_1} & \frac{\partial b_2}{\partial \alpha_2} & \frac{\partial b_2}{\partial \beta_1} & \frac{\partial b_2}{\partial \beta_2} \\ \frac{\partial b_3}{\partial \alpha_1} & \frac{\partial b_3}{\partial \alpha_2} & \frac{\partial b_3}{\partial \beta_1} & \frac{\partial b_3}{\partial \beta_2} \\ \frac{\partial b_4}{\partial \alpha_1} & \frac{\partial b_4}{\partial \alpha_2} & \frac{\partial b_4}{\partial \beta_1} & \frac{\partial b_4}{\partial \beta_2} \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial b_1}{\partial v} \\ \frac{\partial b_2}{\partial v} \\ \frac{\partial b_3}{\partial v} \\ \frac{\partial b_4}{\partial v} \end{bmatrix}. \quad (27)$$

All subsequent derivatives of the function f_5 can be calculated in the same way. The function f_6 can also be calculated in the same way, and as a result, Eq. (26) can be rewritten as

$$\left. \begin{aligned} f_5 &= 2\alpha_1^9\alpha_2^7\beta_1^3v + 4\alpha_1^8\alpha_2^8\beta_1^3v + \ll 14377 \text{ terms} \gg - \alpha_1\beta_1^4\beta_2^{12}\eta_2^3\mu v^5 = 0 \\ f_6 &= 2\alpha_1^{10}\alpha_2^7\beta_1^2v^2 + 2\alpha_1^9\alpha_2^8\beta_1^2v^2 + \ll 7455 \text{ terms} \gg + \alpha_1\beta_1^4\beta_2^{12}\eta_2^2\mu v^6 = 0. \end{aligned} \right\} \quad (28)$$

The H_2 -optimal solution is obtained by solving the simultaneous equations with six variables given in Eqs. (28) and (24) using the Newton–Raphson method. The solution for a loss factor of $\eta_1 = 0$ is shown in Fig. 6(b). This figure also shows the optimal solution for the viscously damped DVA represented by Eq. (19). Comparing these two solutions demonstrates that the optimal tuning ratio ν_{opt} for the hysteretically damped DVA is slightly larger than that for the viscously damped DVA and half of the optimal loss factor $\eta_{2\text{opt}}/2$ is also larger than the optimal damping ratio $\zeta_{2\text{opt}}$. Regarding the minimized evaluation index $I_{a\text{min}}$, the viscously damped DVA can achieve a smaller value than the hysteretically damped DVA, as in the case of the H_∞ criterion.

For the case where hysteretic damping exists in the primary system, the same calculation was performed for the motion excitation system shown in Fig. 1(b). As a result, the same H_2 -optimal solution was obtained for both the force and motion excitation systems, and the results are shown in Fig. 14. A comparison of Fig. 14 with Fig. 8 for the H_∞ -optimal solution reveals that the changes in the two optimal solutions as the primary damping increases are completely opposite. In comparison with the H_∞ -optimal solution, the H_2 -optimal solution is initially designed to have a larger ν_{opt} and a smaller $\eta_{2\text{opt}}$, and as the primary system damping increases, the difference between the two solutions increases.

Figures 15(a) and (b) show the minimum value of the evaluation index I_a in the H_2 optimization for the force and motion excitation systems, respectively. Again, the gray curve for $\mu = 0$ shows the value of the evaluation index for the primary system with no DVA attached. A comparison of Figs. 15 and 16, which respectively show the H_2 optimization results for the hysteretically and viscously damped systems, demonstrates that there is almost no difference between these two types of systems up to a loss factor of $\eta_1 = 0.3$ (damping ratio of $\zeta_1 = 0.15$). However, when the loss factor exceeds 0.3 (damping ratio exceeds 0.15), the evaluation index is smaller in the hysteretically damped system than in the viscously damped system. In the motion excitation system, this is because the damping force increases in proportion to the frequency in a viscously damped system, causing the viscously damped system to have a higher vibration transmissibility at high frequencies. In the force excitation system, the lower evaluation index for the hysteretically damped system is a result of the frequency response function starting from less than 1 at $\lambda = 0$, as will be described later.

Fig. 14 H_2 -optimal solutions for a hysteretically damped system: (a: left) H_2 -optimal tuning ratio ν_{opt} , and (b: right) H_2 -optimal solutions for a hysteretically damped system

Fig. 15 Minimized H_2 norm I_a for a hysteretically damped system: (a: left) Force excitation system, and (b: right) Motion excitation system

Figures 17 and 18 show the frequency responses of force and motion excitation systems, respectively, with a DVA optimally designed according to the H_2 criterion. Figures 17(a) and 18(a) show the response for a DVA with a mass ratio of $\mu = 0.05$, and Figs. 17(b) and 18(b) show the case for a mass ratio of $\mu = 0.1$. A comparison of Figs. 17 and 18 with Figs. 10 and 11 reveals that the first and second resonance points are respectively higher and lower in the H_2 optimization than in the H_∞ optimization. In the range of frequencies exceeding the second resonance point, the responses of the systems with the DVAs designed by the H_2 criterion are lower than for those designed by the H_∞ criterion. As a result, the performance index I_a in the system designed by the H_2 criterion is smaller over the entire frequency range. An interesting phenomenon is observed when Fig. 17 is compared with Fig. 19; that is, for the viscously damped system shown in Fig. 19, the anti-resonance point generated between the first and second resonance points moves toward lower frequencies as the primary system damping increases, whereas this point moves toward higher frequencies for the hysteretically damped system shown in Fig. 17. This phenomenon appears to occur because the damping force does not change with increasing frequency in the hysteretically damped system.

7. Stability maximization design of hysteretically damped dynamic vibration absorber

In stability maximization, the DVA is designed to maximize the following evaluation index (Nishihara and Matsuhisa, 1997):

$$\Lambda = -\max(\text{Re}[s_i]). \quad (29)$$

The value of Λ represents the distance between the rightmost characteristic root and the imaginary axis (Fig. 20). The free vibration of the system with DVAs designed by this criterion is attenuated most rapidly of the three criteria (Fig. 21). This is because the vibration does not return from the DVA to the primary system.

7.1. Procedure for deriving the optimal solution for the hysteretically damped system

The optimal design of the hysteretically damped DVA attached to the hysteretically damped primary system shown in Fig. 1(a) is presented here. The compliance transfer function for the system is expressed using complex numbers as

$$\frac{x_1}{f/k_1} = [(1 + i\eta_2)v^2 - \lambda^2] / \{\lambda^4 - [1 + i\eta_1 + (1 + i\eta_2)(1 + \mu)v^2] \lambda^2 - (\eta_1 - i)(\eta_2 - i)v^2\}. \quad (30)$$

The characteristic equation is obtained by setting the denominator of this equation to zero. If the substitution $s = i\lambda$ is performed, Eq. (30) becomes

$$s^4 + [1 + i\eta_1 + (1 + i\eta_2)(1 + \mu)v^2] s^2 - (\eta_1 - i)(\eta_2 - i)v^2 = 0. \quad (31)$$

Because the coefficients of the operator s are complex numbers, they can be expressed as follows:

$$s^4 + (a_{2r} + i a_{2i}) s^2 + a_{4r} + i a_{4i} = 0, \quad (32)$$

where

$$a_{2r} = 1 + (1 + \mu)v^2, \quad a_{2i} = \eta_1 + \eta_2(1 + \mu)v^2, \quad a_{4r} = (1 - \eta_1\eta_2)v^2, \quad a_{4i} = (\eta_1 + \eta_2)v^2. \quad (33)$$

Fig. 16 Minimized H_2 norm I_a for a viscously damped system: (a: left) Force excitation system, and (b: right) Motion excitation system

Fig. 17 Frequency response functions for a hysteretically damped system subjected to force excitation: (a: left) $\mu = 0.05$, and (b: right) $\mu = 0.1$

Fig. 18 Frequency response functions for a hysteretically damped system subjected to motion excitation: (a: left) $\mu = 0.05$, and (b: right) $\mu = 0.1$

Fig. 19 Frequency response functions of a viscously damped system subjected to force excitation: (a: left) $\mu = 0.05$, and (b: right) $\mu = 0.1$

The characteristic equation (Eq. (32)) is a quadratic equation with respect to s^2 , and its solution is given by

$$s^2 = \frac{-(a_{2r} + i a_{2i}) \pm \sqrt{(a_{2r} + i a_{2i})^2 - 4(a_{4r} + i a_{4i})}}{2} \quad (34)$$

Previous studies (Nishihara and Matsuhisa, 1997; Asami et al., 2018) have demonstrated that when the stability takes a maximum value, the characteristic equation has a double root. Therefore, the expression under the square root of Eq. (34) must be zero, which gives

$$a_{2r}^2 - a_{2i}^2 - 4a_{4r} = 0, \quad a_{2r}a_{2i} - 2a_{4i} = 0. \quad (35)$$

Substituting Eq. (33) into Eq. (35) yields

$$\left. \begin{aligned} (1 - \eta_2^2)(1 + \mu)^2 v^4 - 2(1 - \eta_1 \eta_2)(1 - \mu)v^2 + 1 - \eta_1^2 &= 0 \\ v_2(1 + \mu)^2 v^4 - (\eta_1 + \eta_2)(1 - \mu)v^2 + \eta_1 &= 0. \end{aligned} \right\} \quad (36)$$

The parameters μ and η_1 in Eq. (36) can be any positive real numbers. For any set of given values of μ and η_1 , the simultaneous equations in Eq. (36) have a positive double root (Asami et al., 2019)

$$v_{\text{opt}} = \frac{\sqrt{1 - \mu - 2\eta_1 \sqrt{\mu}}}{1 + \mu}, \quad \eta_{2\text{opt}} = \frac{\eta_1(1 - \mu) + 2\sqrt{\mu}}{1 - \mu - 2\eta_1 \sqrt{\mu}}. \quad (37)$$

Because this double root represents the solution for the optimal tuning conditions and the loss factors, the subscript “opt” is attached to the variables. When the characteristic equation takes a double root, the square of the characteristic root is the following single value:

$$s^2 = -\frac{a_{2r} + i a_{2i}}{2} \equiv -(a_0 + i b_0) = -\frac{1}{1 + \mu} [1 - \eta_1 \sqrt{\mu} + i(\eta_1 + \sqrt{\mu})]. \quad (38)$$

This squared characteristic root is located in the third quadrant of the complex plane. Therefore, the argument is in the range of 180° to 270° or 540° to 630° .

Next, taking the square root of Eq. (38) yields

$$s_{1,2} = \pm i \sqrt{a_0 + i b_0}. \quad (39)$$

This time, the arguments of the characteristic roots s_1 and s_2 are in the ranges of 90° to 135° and 270° to 315° , respectively. Therefore, s_1 and s_2 are respectively located the second and fourth quadrants of the complex plane.

Because the square root of a complex number is also a complex number, the characteristic roots can be expressed as

$$s_{1,2} = A_0 + i B_0. \quad (40)$$

Equations (39) and (40) are set equal to each other, then

$$-a_0 - i b_0 = A_0^2 - B_0^2 + 2i A_0 B_0. \quad (41)$$

Fig. 20 Characteristic roots of a three-degree-of-freedom vibratory system

Fig. 21 Free vibration response of an undamped primary system with DVAs designed by different optimization criteria

Because $A_0^2 \geq 0$ and $B_0^2 \geq 0$, the solution of Eq. (41) can be expressed as

$$A_0^2 = \frac{-a_0 + \sqrt{a_0^2 + b_0^2}}{2}, \quad B_0^2 = \frac{a_0 + \sqrt{a_0^2 + b_0^2}}{2}. \quad (42)$$

Finally, because s_1 is located in the second quadrant and s_2 in the fourth quadrant of the complex plane, their coordinates are given by

$$s_1 = -x_1 + i y_1, \quad s_2 = x_1 - i y_1, \quad (43)$$

where

$$x_1 = \sqrt{\left[-1 + \eta_1 \sqrt{\mu} + \sqrt{(1+\mu)(1+\eta_1^2)}\right]/[2(1+\mu)]}, \quad y_1 = \sqrt{\left[1 - \eta_1 \sqrt{\mu} + \sqrt{(1+\mu)(1+\eta_1^2)}\right]/[2(1+\mu)]}. \quad (44)$$

7.2. Optimal solution for the viscously damped system

The solution based on the stability criterion for the general viscously damped system is derived as follows (Nishihara and Matsuhisa, 1997):

$$\nu_{\text{opt}} = \frac{1}{1+\mu} \left(1 - \zeta_1 \sqrt{\frac{\mu}{1+\mu-\zeta_1^2}}\right), \quad \zeta_{2\text{opt}} = \frac{\zeta_1 + \sqrt{\mu(1+\mu-\zeta_1^2)}}{1+\mu}. \quad (45)$$

It has been confirmed that this optimal solution holds in the regions where the characteristic equation has complex conjugate roots and where it has two different double real roots.

In general viscously damped systems, it is known that when the damping ratio for the system increases, the characteristic equation takes two different real roots. First, the coordinates of the characteristic roots when taking the conjugate complex roots are expressed as follows:

$$s_{1,2} = x_1 \pm i y_1, \quad (46)$$

where

$$\left. \begin{aligned} x_1 &= -\frac{\zeta_1}{1+\mu} - \frac{1+\mu-2\zeta_1^2}{2(1+\mu)} \sqrt{\frac{\mu}{1+\mu-\zeta_1^2}} \\ y_1^2 &= \frac{1+\zeta_1^2}{1+\mu} - \frac{2\zeta_1^2}{(1+\mu)^2} - \frac{\mu}{4(1+\mu-\zeta_1^2)} - \frac{2\zeta_1 \sqrt{\mu(1+\mu-\zeta_1^2)}}{(1+\mu)^2} \end{aligned} \right\} \quad (47)$$

It should be noted that in Eq. (47), the expression of the coordinate y_1 represents its square.

Next, when the characteristic equation has two different double real roots, the roots are expressed as follows:

$$\left. \begin{aligned} s_1 = x_1 &= -\frac{\zeta_1}{1+\mu} - \frac{1+\mu-2\zeta_1^2}{2(1+\mu)} \sqrt{\frac{\mu}{1+\mu-\zeta_1^2}} \\ s_2 = x_2 &= s_1 + \frac{1}{1+\mu} \left[2\zeta_1 \sqrt{\mu(1+\mu-\zeta_1^2)} - \frac{(1+\mu)^2(4-\mu)/4 - \zeta_1^2(1+\mu)(2-\mu) + \zeta_1^4(1-\mu)}{1+\mu-\zeta_1^2} \right]^{1/2} \end{aligned} \right\} \quad (48)$$

Here, the stability of the system is determined by s_2 . For $\zeta_1 = 1$, this yields $s_1 = -0.5$ and $s_2 = \Lambda_{\text{max}} = 0$ except $\mu = 0$.

7.3. Solution visualization for undamped and damped primary systems

Figure 22 shows the stability-maximized solution of hysteretically and viscously damped DVAs optimized for undamped primary systems. A comparison of this figure with the H_∞ - and H_2 -optimal solutions shown in Figs. 6(a) and (b) reveals that the optimal tuning ratio ν_{opt} of the hysteretically damped DVA is clearly different from that of the viscously damped DVA, and that the optimal loss factor $\eta_{2\text{opt}}$ and damping ratio $\zeta_{2\text{opt}}$ of DVAs designed by the stability criterion are quite large. Again, the viscously damped DVA was found to achieve a better performance than the hysteretically damped DVA.

Figures 23(a) and (b) respectively show the optimal tuning ratio ν_{opt} and the optimal loss factor $\eta_{2\text{opt}}$ given by Eq. (37) plotted against the primary system damping η_1 . The optimal tuning ratio ν_{opt} and the optimal loss factor $\eta_{2\text{opt}}$ monotonically decrease and increase, respectively, with increasing primary system damping. This trend is the same as in Figs. 8(a) and (b) for the H_∞ -optimal solution. However, the optimal values themselves are quite dissimilar.

7.4. Characteristic root locus and stability

The root locus of the characteristic equation of the hysteretically damped system with the DVA designed by stability maximization can be calculated from Eqs. (43) and (44), and the result is shown in Fig. 24(a). Similarly, Fig. 24(b) shows the root locus for the viscously damped system calculated from Eqs. (46) and (47). The major difference between these loci is that the characteristic roots of the hysteretically damped system do not form complex conjugates; that is, some roots exist with their real part in the positive domain, and there is no real number solution. This means that the general solution for the hysteretically damped system cannot be expressed as a real number, because the original equation of motion is expressed in terms of complex numbers. The characteristic root that appears in the fourth quadrant makes the vibratory system unstable, but because this root does not appear as a real phenomenon, it will be ignored in this study.

Fig. 22 Stability-maximized solutions for damped DVAs attached to an undamped primary system

Fig. 23 Stability-maximized solutions for a hysteretically damped DVA attached to a damped primary system: (a: left) Stability-maximized tuning ratio v_{opt} , and (b: right) Stability-maximized loss factor η_{2opt}

Fig. 24 Trajectory of characteristic roots when the stability index is maximized (overall view of the roots): (a: left) Hysteretically damped system, and (b: right) Viscously damped system

Figure 25 represents only those roots shown in Fig. 24 that lie in the second quadrant. A comparison of Figs. 25(a) and (b) demonstrates that as the primary system damping increases, the natural angular frequency (value on the imaginary axis in the figure) of the system decreases in the viscously damped system, whereas it increases in the hysteretically damped system. Therefore, the overdamping situation that appears in the viscously damped system does not occur in the hysteretically damped system regardless of how large the system damping is.

The distance between the characteristic root and the imaginary axis ($x = 0$) represents the maximized stability. The maximized stability $\Lambda_{\max}$ is shown in Fig. 26. The stability of the hysteretic vibration system increases with both the loss factor η_1 of the primary system and the mass ratio μ of the DVA. In contrast, if the primary damping ratio ζ_1 exceeds 0.5 in the viscously damped system, the system stability decreases with increasing mass ratio μ . This can also be seen from the fact that the curve for $\zeta_1 = 0.5$ is almost vertical in Fig. 25(b). That is, if ζ_1 exceeds 0.5, the horizontal distance between the characteristic roots and the imaginary axis decreases as μ increases. In the viscously damped system, the characteristic root changes from a complex value to a real value when the system damping reaches a critical damping ratio (approximately 0.5–1.0). At this time, the coordinates of the characteristic root change suddenly from those described by Eq. (47) to those given in Eq. (48). This is why the slope of the curve in Fig. 26(b) changes discontinuously.

7.5. Frequency and free vibration responses of the primary system

Figure 27 shows the compliance transfer function for a system with a DVA optimally designed according to the stability criterion. If the DVA is optimally designed according to the stability criterion, the damping of the DVA is quite large, which causes the system to exhibit only one sharp resonance point, as shown in this figure.

In hysteretically damped systems, the general solution cannot be expressed using real numbers, so the free vibration response of the system subjected to arbitrary initial conditions cannot be calculated. However, the free vibration response for a system given an initial displacement can be calculated using the inverse Fourier transform of the frequency response

Fig. 25 Trajectory of characteristic roots when the stability index is maximized (roots in the second quadrant): (a: left) Hysteretically damped system, and (b: right) Viscously damped system

Fig. 26 Maximized stability index for a system with an optimally tuned and damped DVA: (a: left) Hysteretically damped system, and (b: right) Viscously damped system

function (Nagaike and Nagamatsu, 1985; Shimizu et al., 1998). An example of the calculation is shown in Fig. 28. Figures 28(a), (b), and (c) show the free vibration responses of the vibration system with the DVA optimally designed according to the H_∞ , H_2 , and stability criteria, respectively. The red line in each of these plots is the free vibration response of the hysteretically damped vibration system, and the blue line is the response of the viscously damped vibration system calculated under the same conditions (with the primary damping ratio ζ_1 set to half of the loss factor η_1). As shown in this figure, there is almost no difference between the two vibratory systems at this damping magnitude ($\zeta_1 = 0.1$) and DVA mass ratio ($\mu = 0.1$). The primary difference between the two vibratory systems is that the viscously damped system shows a delayed response due to its slightly longer vibration period. Among these three optimization criteria, the stability criterion causes the free vibration response of the system to converge the most rapidly.

The free vibration response of the hysteretically damped system given an initial velocity could not be calculated. Instead, the calculation of the free vibration response was carried out in a viscously damped system. The result is shown in Fig. 29. Together with Fig. 21, this figure indicates that the free vibration response of the system converges more quickly as the primary damping ratio ζ_1 increases from 0 to 0.1. In particular, in a vibration system designed using the stability criterion, the vibration converges most rapidly among the three criteria because no swing-back occurs. By analogy with Fig. 28, it is considered that this is almost the same in the hysteretically damped vibratory system.

8. Conclusion

The results presented in this paper indicate that the damping performance of the hysteretically damped DVA is inferior to that of the viscously damped DVA when the DVAs are designed according to the H_∞ , H_2 , and stability criteria. However, the difference in performance is so small that it is negligible for normal use when the mass ratio μ between the DVA and the primary system is 0.2 or less. The following conclusions were obtained regarding the optimal design of the hysteretically damped DVA.

- (1) The optimal tuning ratio ν_{opt} is almost the same as that for the general viscously damped DVA. However, if the DVA is designed based on the stability criterion, the tuning ratio is considerably smaller than that for the viscously damped DVA.
- (2) The loss factor η is generally assumed to be twice the damping ratio ζ . However, the optimal loss factor $\eta_{2\text{opt}}$ for the hysteretically damped DVA is slightly larger than twice the optimal damping ratio $\zeta_{2\text{opt}}$.
- (3) For the H_∞ optimization, an exact algebraic solution for the optimal DVA with a damped primary system was obtained.
- (4) For the H_2 optimization, the optimal tuning ratio and loss factor for the DVA and the coordinates of the characteristic roots can be obtained by numerically solving simultaneous algebraic equations with six unknowns.
- (5) For the stability criterion, a simple exact algebraic solution was successfully derived for the design parameters of the DVA attached to a hysteretically damped primary system.
- (6) In the hysteretically damped system, the general solution is not expressed in terms of complex conjugates as it is in the viscously damped system.
- (7) If the spring constant of the hysteretically damped system is expressed as a complex number, an unstable solution will be derived. However, the solution does not appear in the actual physical behavior of the system.

Fig. 27 Frequency response functions for a hysteretically damped system subjected to force excitation:
(a: left) $\mu = 0.05$, and (b: right) $\mu = 0.1$

Fig. 28 Free vibration responses for viscously and hysteretically damped systems optimized using different criteria: (a: top left) H_∞ -optimal system, (b: top right) H_2 -optimal system, and (c: bottom) Stability-maximized system

Fig. 29 Free vibration responses of a viscously damped system with DVAs designed with different optimization criteria: (a: left) Primary damping $\zeta_1 = 0.05$, and (b: right) Primary damping $\zeta_1 = 0.1$

Acknowledgment

We express our gratitude for being allowed to pursue this research, which was supported by a research grant (19K04276) from the Ministry of Education, Culture, Sports, Science and Technology of Japan.

References

- Asami, T., Nishihara, O., and Watanabe, S., Exact Algebraic Optimization of a Dynamic Vibration Absorber for Minimization of Maximum Amplitude Response (2nd Report, Hysteretic Damped Absorber), Transactions of the JSME, Ser. C, Vol.66, No.644 (2000), pp.1186-1193, (in Japanese).
- Asami, T., Nishihara, O., and Baz, A. M., Analytical Solutions to H_∞ and H_2 Optimization of Dynamic Vibration Absorbers Attached to Damped Linear Systems, ASME Journal of Vibration and Acoustics, Vol.124, No.2 (2002), pp.284-295.

- Asami, T., and Nishihara, O., Closed-Form Solution to H_∞ Optimization of Dynamic Vibration Absorbers (Application to Different Transfer Functions and Damping Systems), ASME Journal of Vibration and Acoustics, Vol.125, No.3 (2003), pp.398-405.
- Asami, T., Mizukawa, Y., and Ise, T., Optimal Design of Double-Mass Dynamic Vibration Absorbers Minimizing the Mobility Transfer Function, ASME Journal of Vibration and Acoustics, Vol.140, No.6 (2018), p.061012.
- Asami, T., and Mizukawa, Y., Optimal Design of a Hysteretically Damped Dynamic Vibration Absorber Based on H_∞ and H_2 Criteria, JSME Dynamics and Design Conference 2019, No.19-13 (2019), (in Japanese).
- Asami, T., Mizukawa, Y., and Yamada, K., Optimal Design of a Hysteretically Damped Dynamic Vibration Absorber Based on Stability Maximization Criterion, JSME Mechanical Engineering Congress 2019 (MECJ-19), No.19-1 (2019), (in Japanese).
- Crandall, S. H., and Mark, W. D., Random Vibration in Mechanical Systems (1963), p.71, Academic Press.
- Kreyszig, E., Advanced Engineering Mathematics, 8th ed. (1999), p.784, John Wiley & Sons, Inc., New York.
- Nagaike, M., and Nagamatsu, A., Study on the Mode Analysis (3rd Report, Fundamental Investigation for a Single-Degree-of-Freedom System), Transactions of the JSME, Ser. C, Vol.51, No.464 (1985), pp.710-718, (in Japanese).
- Nishihara, O., and Matsuhisa, H., Design and Tuning of Vibration Control Devices via Stability Criterion, Dynamics and Design Conference '97, No.97-10-1 (1997), pp.165-168, (in Japanese).
- Nishihara, O., and Asami, T., Closed-Form Solutions to the Exact Optimizations of Vibration Absorbers (Minimizations of the Maximum Amplitude Magnification Factors), ASME Journal of Vibration and Acoustics, Vol.124, No.4 (2002), pp.576-582.
- Ormondroyd, J., and Den Hartog, J. P., The Theory of the Dynamic Vibration Absorber, ASME Journal of Applied Mechanics, Vol.50, No.7 (1928), pp.9-22.
- Shibata, K., Misaji, K., and Kato, H., Vibration Characteristics of Rubber (Nonlinear Vibration Characteristics Depending on Frequency and Amplitude of Displacement), Transactions of the JSME, Ser. C, Vol.59, No.564 (1993), pp.2408-2414, (in Japanese).
- Shibata, K., Ichinose, H., Sakurai, H., and Osawa, K., Study on Nonlinear Vibration of Rubber Bearing for Base Isolation System With Hysteresis Restoring Force Characteristics of Hardening Type, Journal of Structural and Construction Engineering, Vol.61, No.490 (1996), pp.119-127, (in Japanese).
- Shimizu, N., et al., Damping Technology of Vibration (1998), Edited by JSME, p.34, (in Japanese), Yokendo Ltd., Tokyo.