Formulation of plastic strain distribution derived from long-distance travel of temperature distribution based on residual stress required for elastic shakedown behavior

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Abstract
In an experimental study that simulated a fast breeder reactor (FBR) vessel near the coolant surface, it was reported that the long distance travel of temperature distribution causes a new type of thermal ratcheting, even in the absence of primary stress. When the distance of temperature travel is moderate, the accumulation of the plastic strain due to this mechanism is finally saturated. Through the large number of Finite Element Analysis, we have found the strong relationship between hoop-membrane distributions of accumulated plastic strain and residual stress in this saturated case. Focusing on this relationship, we have aimed to predict the saturated distribution of the plastic strain based on the residual stress distribution that is required for the elastic shakedown behavior. In this paper, based on classical shell theory, we formulated the plastic strain distribution that brings uniform hoop-membrane stress in the given region. And then, we compared the formulated strain distribution with the accumulated plastic strain distribution obtained by finite element analyses using an elastic-perfectly plastic material. As a result, we confirmed that the formulated strain distribution can be used as the prediction of the plastic strain distribution for the cases with moderate distance of temperature travel. By considering the effect of this expansion, the formulated strain distribution can be used as conservative prediction of the accumulated plastic strain also for the cases with long distance temperature travel.

Keywords: Ratcheting, Plastic strain, Thermal stress, Traveling temperature distribution, Finite element analysis

1. Introduction

High-temperature components of fast breeder reactors (FBR) are designed based on elevated temperature design codes (ASME, 2015) (JSME, 2018). The elevated temperature design codes provide design evaluation method to prevent each failure mechanism. On these evaluation method, inelastic behaviors are estimated based on the result (e.g., thermal stress) of not inelastic but elastic finite element analysis (FEA). The prevention of excessive deformation due to thermal ratcheting is one of important issue in the design of high-temperature components of FBR. The current elevated temperature design codes provide the evaluation method to prevent excessive deformation due to Bree-type ratcheting (Bree, 1967). JSME code (JSME, 2018) provide also a limit for a range of membrane stress to ensure elastic shakedown behavior.

Meanwhile, a new type of thermal ratcheting derived from long-distance travel of temperature distribution was reported through an experimental study (Igari et al., 1990) that simulated a FBR vessel wall near the coolant surface. Since the current elevated temperature design codes (ASME, 2015) (JSME, 2018) have not considered this type of ratcheting, it is desired to develop additional design evaluation method to prevent this ratcheting mechanism and introduce it into the design codes. Considering the consistency within the design codes and the reduction of the calculation cost to help the parameter study on the practical design procedure, it is desirable to use the result of elastic FEA as the input for the new design evaluation method.
Several studies on this new type of ratcheting mechanism have been conducted. Wada et al. (1989, 1993) proposed an evaluation method for increment in plastic strain due to long distance travel of temperature distribution. Igari et al. (2000) proposed “mechanism-based” derivation of prediction equations for thermal ratcheting due to relatively short distance travel of “hot-spot-shaped” temperature distribution. These studies focused on mainly the evaluation of the magnitude of plastic strain caused on each cycle with sufficient conservativeness. As the result, when we consider dozens of transient cycles, their evaluation may contain large margin comparing with the actual accumulated strain.

Meanwhile, we had studied on a screening criterion for this ratcheting mechanism (Okajima et al., 2016, 2017) (Okajima, 2016). In order to prevent continuous accumulation of plastic strain due to this ratcheting mechanism, we proposed the criterion for the axial length of the region with full-section yielding, \( l_p \), by referring to the definition of local membrane stress in current design codes. We validated this screening criterion based on a number of FEA with various temperature travel model. Meanwhile, through these FEA, we have confirmed that there are some cases on which the accumulation of the plastic strain finally saturated, even if the travel distance of the temperature distribution exceeds the proposed criterion. It is desirable to allow such design conditions if the accumulated strain is not excessive. On these cases, larger hoop-membrane plastic strain caused at the center of yielding area than both ends, and this difference bring hoop residual stress also at the center of this area. Finally, sufficient residual hoop-membrane stress to show the elastic shakedown behavior is accumulated on entire of the region with plastic strain on the cylinder.

Because the residual stress is derived by only equilibrium of deformed cylinder, we thought that the hoop-membrane plastic strain distribution may strongly depends on the distribution of the residual stress that is required for the elastic shakedown behavior. When we achieve to validate it, we will be possible to predict the saturated distribution of the plastic strain based on the residual stress distribution that is required for the elastic shakedown behavior without detail consideration for interim process. This means that we can skip the detail consideration about the interim behavior before saturation. Actually, we performed several FEA with various initial strain that simulate various interim situation, and we confirmed that the shape of saturated plastic strain distribution has small dependency on the interim situation (Okajima, 2019). Therefore, the formulation of the plastic strain distribution that bring typical shape of residual stress distribution would be valuable information for the prediction of one derived from the temperature travel.

In this paper, based on the classical shell theory (Timoshenko and Krieger, 1959), we formulated the hoop-membrane plastic strain distribution that bring unified magnitude of the residual stress on the given area, and proposed to use the formulated plastic strain distribution as the prediction of the plastic strain derived from the repeating long-range-travel of temperature distribution. The plastic strain distribution is formulated as a polynomial of the fourth degree. In order to validate the prediction accuracy, the formulated distributions were compared with FEA results for the cylinder that subject to the travel of triangle-shape temperature distribution. As the result, for the case with short distance travel, we confirmed that the formulated distribution predicts the accumulated plastic strain distribution on FEA with high accuracy. Also for the case with long distance travel, by introducing procedures that estimate the expansion of the region with full-section yielding, the formulated distribution provide the prediction with rational safety margin.

2. Considering ratcheting mechanism

In this section, we explain the mechanism of thermal ratcheting derived from long-distance travel of temperature distribution.

2.1 Mechanism of the continuous accumulation of the plastic strain

In this subsection, we explain typical case with the continuous accumulation of the plastic strain due to long-distance temperature travel. As same with our previous papers (Okajima et al., 2016, 2017) (Okajima, 2016, 2019), we assume that thermal membrane stress is within the shakedown range in this paper. Because, JSME design and construction code (JSME, 2018) have prohibited excessive thermal membrane stress that exceeds the shakedown limit.

Ordinarily, once plastic strain is caused by thermal stress, it also brings residual stress due to constraint against surrounding structure. When same magnitude of thermal stress is loaded, this residual stress prevents further plastic strain. This phenomenon is well-known as elastic shakedown behavior.
Fig. 1 Outline of the ratcheting mechanism derived from temperature travel. The accumulation of the plastic strain in the wide region disturbs the accumulation of the residual stress.

Figure 1 shows the model of a FBR vessel near the liquid sodium surface as an example of considering ratcheting mechanism. The high heat transfer coefficient between the liquid sodium and the vessel wall brings a sharp axial temperature gradient on the vessel wall. When we don’t control liquid sodium level, liquid level may increase in the start-up phase because of thermal expansion. Now, we assume very short time step with increase of liquid sodium level. Location of the temperature distribution slightly moves up together with liquid level, and temperature change causes in very narrow region. In this region, by the constraint from surrounding region, hoop plastic strain is caused to cancel the thermal expansion due to the temperature change. Though the plastic strain caused only in narrow region in the short time step, if the travel distance of liquid sodium level and temperature distribution is large, plastic strain finally caused in quite wide region. When we focus at center of this deformed region, sufficient residual stress is not accumulated because the plastic strain is caused also on the surrounding region.

Therefore, in cases with long-distance travel of temperature distribution, the elastic shakedown behavior is disturbed by the lack of the residual stress, and the plastic strain accumulates on each cycle continuously.

2.2 Mechanism of the saturation of the plastic strain accumulation

According to the FEA results on our previous papers (Okajima et al., 2016, 2017) (Okajima, 2016, 2019), in the case if neither magnitude nor travel distance of the temperature distribution were so massive, the accumulation of the plastic strain is finally saturated. In these cases, the saturated distributions of the plastic strain and residual stress showed following characteristics:

a) The residual stress distribution depended on the shape and travel distance of the temperature distribution. On the area where the temperature distribution passed, the residual stress reached nearly equal to difference between the maximum thermal membrane stress and the yielding stress. In other words, this residual stress is sufficient magnitude to reach elastic shakedown behavior when further travel of temperature distribution cause on this area.

b) Since the residual stress is derived from equilibrium among the cylinder with permanent deformation, the accumulated plastic strain distribution was characteristic shape to achieve the required distribution of the residual stress. Both ends of the region with full-section yielding is constrained by surrounding region, and reaches to shakedown due to the residual stress derived from this constrain. Meanwhile, larger plastic strain was accumulated at middle part of this region than the both ends, because this part was not constrained by the surrounding region. As the result, the difference of the magnitude of the accumulated plastic strain between the middle part and both ends also brought the sufficient residual stress to achieve elastic shakedown behavior at
the middle part.
Meanwhile, in the case with much longer distance of temperature travel, plastic strain was caused also on the surrounding region. Since the yielding of the surrounding region disturb the constrain at the both ends of the yielding region, the accumulated residual stress was not sufficient to reach the elastic shakedown, and the accumulation of the plastic strain continued without the saturation.

When we assume the saturation of the plastic strain accumulation, considering the characteristics of the saturated stress and strain distribution shown above, the saturated plastic strain distribution strongly depends on the residual stress distribution that is required to reach elastic shakedown. In fact, through FEA with one temperature travel model and various initial strain distribution that simulates plastic strain distribution in the middle of accumulating, we confirmed that the saturated distributions of plastic strain and residual stress have small path dependency (Okajima, 2019). This result suggests the possibility of the method that confirm whether elastic shakedown behavior can be achieved by the plastic strain distribution that satisfy strain limit.

3. Formulation of the circumferential plastic strain distribution

In this section, we have formulated distribution of the plastic strain distribution that brings typical shape of the residual stress distribution. Considering the characteristics shown in section 2, we propose the formulated plastic strain distribution as the prediction of the plastic strain derived from the repeating long-range-travel of temperature distribution.

3.1 Preconditions for the formulation

We have focused on following 2 parameters as dominant parameters of this ratcheting mechanism (Okajima, 2016)(Okajima et al., 2017): a) Maximum value of membrane stress intensity, \( \langle P_m + Q_m \rangle_{max} \), and b) the axial length of the full-section yield region, \( l_p \). In case if thermal membrane stress \( \sigma_{max} = \langle P_m + Q_m \rangle_{max} \) is loaded at specific point in thermal transient, the residual stress whose magnitude is \( \sigma_0 = \langle P_m + Q_m \rangle_{max} - \sigma_y \) is required to achieve the elastic shakedown behavior. Here, \( \sigma_y \) means yielding stress. We considered most pessimistic assumption on the required residual stress distribution, that is, largest value of \( \sigma_0 \) is required for entire region with full-section yield.

Figure 2 shows the concept of the formulation. Red lines and characters mean given parameter and assumption used as the input for the formulated equations.

![Fig. 2 Concept of the formulation. In this paper, we aim to formulate the plastic strain distribution that brings uniform residual stress on the given region.](image)

We assumed thin cylindrical structure that is subjected to hoop-membrane plastic strain. The detail of the assumptions for our formulation is as follows:

a) Region with full-section yielding is given parameter. No hoop-membrane plastic strain is caused at any position except for this given region.

b) The plastic strain distribution brings unified magnitude, \( \sigma_0 \), of hoop-membrane stress in the region with full-section yielding. This residual stress, \( \sigma_0 \), is smaller than the yielding stress, \( \sigma_y \).
c) Both of plastic strain and residual stress distributions are symmetry with central axis and horizontal plane $z=0$.

### 3.2 Plastic strain distribution

As the first step, we discuss on the stress caused on the cylinder with plastic strain. We assumed thin cylinder whose axial length is infinite, radius is $r$, and thickness is $t$. When hoop plastic strain $\varepsilon_p(u)$ is caused on quite narrow area whose axial location $z$ is $u \leq z \leq u + dz$, it brings residual stress, $-E\varepsilon_p(u)$, at this area. Additionally, because of the equilibrium on the considering cylinder, this plastic strain makes the hoop stress also on the surrounding area. The magnitude of this hoop stress at the location $z$ depends on the distance from the area with plastic strain, $|z - u|$. Considering the above plastic strain brings an elastic load uniformly distributed along circular section, based on the thin cylinder theory (Timoshenko and Krieger, 1959), we can formulate this hoop stress at the axial location $z$ derived from the above plastic strain $\varepsilon_p(u)$ as follows:

$$
d\sigma_\theta(z) = \frac{1}{2} E \varepsilon_p(z) \left( \cos(\beta(z - u)) + \sin(\beta(z - u)) \right) \cdot E\varepsilon_p(u)\beta dz
$$

where

$$
\beta = \sqrt[3]{\frac{3(1 - \nu^2)}{r^2 t^2}}
$$

$E$ is Young’s modulus and $\nu$ is Poisson ratio. Igari et al. (2000) used same logic with eq.(1) for their formulation. When the plastic strain is caused on the large region on the cylinder, by the integration of eq.(1), we can formulate total hoop stress, $\sigma_\theta(z)$, derived from the plastic strain distribution as eq. (2).

$$
\sigma_\theta(z) = -E\varepsilon_p(z) + \int_{-\infty}^{z} \frac{E}{2} \exp\left(-\beta(z - u)\right) \left( \cos(\beta(z - u)) + \sin(\beta(z - u)) \right) \cdot E\varepsilon_p(u)\beta du
$$

$$
+ \int_{z}^{\infty} \frac{E}{2} \exp\left(-\beta(u - z)\right) \left( \cos(\beta(u - z)) + \sin(\beta(u - z)) \right) \cdot E\varepsilon_p(u)\beta du
$$

Then, we assume the symmetrical shape for the axial distribution of hoop membrane plastic strain $\varepsilon_p$ and stress $\sigma_\theta$. In other words, we assume following equation:

$$
\varepsilon_p(-z) = \varepsilon_p(z), \quad \sigma_\theta(-z) = \sigma_\theta(z)
$$

Additionally, we normalize axial location $Z$ and axial length of the region with full section yielding $l_p$ as follows:

$$
Z = \beta z, \quad L = \beta l_p
$$

As the result, eq.(2) is rearranged to eq.(5), which includes convolution.

$$
\sigma_\theta(Z) = -E\varepsilon_p(Z) + \frac{E}{2} \int_{-\infty}^{Z} \exp\left(-\left(Z + U\right)\right) \left( \cos\left(Z + U\right) + \sin\left(Z + U\right) \right) \cdot E\varepsilon_p(U) dU
$$

$$
+ \frac{E}{2} \int_{0}^{Z} \exp\left(-\left(Z - U\right)\right) \left( \cos\left(Z - U\right) + \sin\left(Z - U\right) \right) \cdot E\varepsilon_p(U) dU
$$

$$
+ \frac{E}{2} \int_{Z}^{\infty} \exp\left(-\left(U - Z\right)\right) \left( \cos\left(U - Z\right) + \sin\left(U - Z\right) \right) \cdot E\varepsilon_p(U) dU
$$

$$
- \frac{E}{2} \int_{-\infty}^{0} \exp\left(-\left(U + Z\right)\right) \left( \cos\left(U + Z\right) + \sin\left(U + Z\right) \right) \cdot E\varepsilon_p(U) dU
$$

This equation can be rearranged as follows:
\[
\sigma_0(Z) = \frac{E}{Z} \left[ -2\varepsilon_p(Z) + \exp(-Z) \cos(Z) \int_{0}^{\infty} \exp(-U) (\cos(U) + \sin(U)) \varepsilon_p(U) dU \\
+ \exp(-Z) \sin(Z) \int_{0}^{\infty} \exp(-U) (\cos(U) - \sin(U)) \varepsilon_p(U) dU \\
+ \int_{0}^{Z} \exp(-(Z - U))(\cos(Z - U) + \sin(Z - U)) \cdot \varepsilon_p(U) dU \\
+ \exp(Z) \cos(Z) \int_{0}^{\infty} \exp(-U) (\cos(U) + \sin(U)) \varepsilon_p(U) dU \\
- \exp(Z) \sin(Z) \int_{0}^{\infty} \exp(-U) (\cos(U) - \sin(U)) \varepsilon_p(U) dU \\
- \int_{0}^{Z} \exp(Z - U)(\cos(Z - U) - \sin(Z - U)) \cdot \varepsilon_p(U) dU \right]
\]

(6)

And, we replace integrals without location \( Z \) into simple variable as eq.(7) and eq.(8):

\[
A = \int_{0}^{\infty} \exp(-U)(\cos(U) + \sin(U)) \cdot \varepsilon_p(U) dU
\]

(7)

\[
B = \int_{0}^{\infty} \exp(-U)(\cos(U) - \sin(U)) \cdot \varepsilon_p(U) dU
\]

(8)

In order to solve the equation including convolution integration, eq.(6) is subjected Laplace transformation.

\[
\mathcal{L}[\sigma_0(Z)] = \frac{E}{2} \left[ -2\mathcal{L}[\varepsilon_p(Z)] + \mathcal{L}[A \cdot \cos(Z)(\exp(-Z) + \exp(Z))] + B \cdot \sin(Z)(\exp(-Z) - \exp(Z))] \right. \\
+ \mathcal{L}[\exp(-Z)(\cos(Z) + \sin(Z)) - \exp(Z)(\cos(Z) - \sin(Z))] \cdot \mathcal{L}[\varepsilon_p(Z)] \]

\[
\therefore \mathcal{L}[\sigma_0(Z)] = \frac{E}{2} \left[ \mathcal{L}[\varepsilon_p(Z)] \left[ -2 + \frac{(s + 1) + 1}{(s + 1)^2 + 1} - \frac{(s - 1) - 1}{(s - 1)^2 + 1} \right] + A \left\{ \frac{s + 1}{(s + 1)^2 + 1} + \frac{s - 1}{(s - 1)^2 + 1} \right\} \right. \\
+ B \left\{ \frac{1}{(s + 1)^2 + 1} - \frac{1}{(s - 1)^2 + 1} \right\} \]

(9)

\[
\therefore \mathcal{L}[\varepsilon_p(Z)] = \frac{E}{(s^4 + 4)} \left[ -s^4 \mathcal{L}[\varepsilon_p(Z)] + AS^3 - 2BS \right]

(10)

\[
\therefore \mathcal{L}[\sigma_0(Z)] = - \left( 1 + \frac{4}{s^4} \right) \mathcal{L}[\sigma_0(Z)] + \frac{A}{s} + \frac{2B}{s^3}

(11)

(12)

By the inverse Laplace transformation of eq.(12), we can obtain the distribution of plastic strain as eq.(13).

\[
\varepsilon_p(Z) = - \frac{\sigma_0(Z)}{E} \left( 1 + \frac{Z^4}{4} \right) - 2 \int_{0}^{Z} \int_{0}^{U_4} \int_{0}^{U_3} \int_{0}^{U_2} \sigma_0(U_1) dU_1 dU_2 dU_3 dU_4 + A - BZ^2

(13)

As shown in Fig.2, on the region with full-section yielding \((Z \leq L/2)\), \(\sigma_0(Z) = \sigma_0\). By substituting it for eq.(13), we can obtain the plastic strain distribution as eq.(14):

\[
\varepsilon_p(Z) = - \frac{\sigma_0}{E} \left( 1 + \frac{Z^4}{6} \right) + A - BZ^2 \quad (Z \leq \frac{L}{2})
\\
\varepsilon_p(Z) = 0 \quad (Z > \frac{L}{2})

(14)

As a next step, we formulate \( A \) and \( B \), which remain in eq.(14). We can rearrange eq.(7) to eq.(15) and eq. (8) to eq.(16) based on partial integration.
In order to vanish \( A \), we obtain eq.(17) by eq. (15) * [exp(L/2)\{sin(L/2)\}] + eq. (16) * [exp(L/2)\{cos(L/2)\}].

\[
\begin{align*}
A &= \left[ \exp \left( -\frac{L}{2} \right) \right] \left\{ -\cos \left( \frac{L}{2} \right) \right\} \cdot \varepsilon_p \left( \frac{L}{2} \right) - (-1) \cdot \varepsilon_p (0) \\
&= \frac{1}{2} \left[ \exp \left( -\frac{L}{2} \right) \right] \left\{ -\cos \left( \frac{L}{2} \right) - \sin \left( \frac{L}{2} \right) \right\} \cdot \frac{d\varepsilon_p (U)}{dU} \bigg|_{U=L/2} - 0 \cdot \varepsilon_p (0) \\
&= \frac{1}{2} \left[ \exp \left( -\frac{L}{2} \right) \right] \left\{ -\cos \left( \frac{L}{2} \right) - \sin \left( \frac{L}{2} \right) \right\} \cdot \frac{d^2 \varepsilon_p (U)}{(dU)^2} \bigg|_{U=L/2} + 0 \cdot \varepsilon_p (0) \\
&= \frac{1}{4} \left[ \exp \left( -\frac{L}{2} \right) \right] \left\{ -\cos \left( \frac{L}{2} \right) + \cos \left( \frac{L}{2} \right) \right\} \cdot \frac{d^3 \varepsilon_p (U)}{(dU)^3} \bigg|_{U=L/2} + 0 \cdot \varepsilon_p (0) \\
&= \frac{1}{4} \left[ \exp \left( -\frac{L}{2} \right) \right] \left\{ -\cos \left( \frac{L}{2} \right) - \sin \left( \frac{L}{2} \right) \right\} \cdot \frac{d^4 \varepsilon_p (U)}{(dU)^4} \bigg|_{U=L/2} + 0 \cdot \varepsilon_p (0)
\end{align*}
\]

\[
B = \left[ \exp \left( -\frac{L}{2} \right) \right] \left\{ \sin \left( \frac{L}{2} \right) \right\} \cdot \varepsilon_p \left( \frac{L}{2} \right) - 0 \cdot \varepsilon_p (0)
\]

\[
\begin{align*}
&= \frac{1}{2} \left[ \exp \left( -\frac{L}{2} \right) \right] \left\{ -\sin \left( \frac{L}{2} \right) \right\} \cdot \frac{d\varepsilon_p (U)}{dU} \bigg|_{U=L/2} - 0 \cdot \varepsilon_p (0) \\
&= \frac{1}{2} \left[ \exp \left( -\frac{L}{2} \right) \right] \left\{ -\sin \left( \frac{L}{2} \right) \right\} \cdot \frac{d^2 \varepsilon_p (U)}{(dU)^2} \bigg|_{U=L/2} + 0 \cdot \varepsilon_p (0) \\
&= \frac{1}{4} \left[ \exp \left( -\frac{L}{2} \right) \right] \left\{ -\sin \left( \frac{L}{2} \right) \right\} \cdot \frac{d^3 \varepsilon_p (U)}{(dU)^3} \bigg|_{U=L/2} + 0 \cdot \varepsilon_p (0) \\
&= \frac{1}{4} \left[ \exp \left( -\frac{L}{2} \right) \right] \left\{ -\sin \left( \frac{L}{2} \right) \right\} \cdot \frac{d^4 \varepsilon_p (U)}{(dU)^4} \bigg|_{U=L/2} + 0 \cdot \varepsilon_p (0)
\end{align*}
\]

In order to vanish \( A \), we obtain eq.(17) by eq. (15) * [exp(L/2)\{sin(L/2)\}] + eq. (16) * [exp(L/2)\{cos(L/2)\}].

\[
\begin{align*}
A &= \left[ \exp \left( -\frac{L}{2} \right) \right] \left\{ -\cos \left( \frac{L}{2} \right) \right\} \cdot \varepsilon_p \left( \frac{L}{2} \right) - (-1) \cdot \varepsilon_p (0) \\
&= \frac{1}{2} \left[ \exp \left( -\frac{L}{2} \right) \right] \left\{ -\cos \left( \frac{L}{2} \right) - \sin \left( \frac{L}{2} \right) \right\} \cdot \frac{d\varepsilon_p (U)}{dU} \bigg|_{U=L/2} + \frac{1}{4} \cdot d^2 \varepsilon_p (U) \bigg|_{U=L/2} \\
&= \frac{1}{2} \left[ \exp \left( -\frac{L}{2} \right) \right] \left\{ -\cos \left( \frac{L}{2} \right) + \cos \left( \frac{L}{2} \right) \right\} \cdot \frac{d^2 \varepsilon_p (U)}{(dU)^2} \bigg|_{U=L/2} + \frac{1}{4} \cdot d^2 \varepsilon_p (U) \bigg|_{U=L/2} \\
&= \frac{1}{4} \left[ \exp \left( -\frac{L}{2} \right) \right] \left\{ -\cos \left( \frac{L}{2} \right) - \sin \left( \frac{L}{2} \right) \right\} \cdot \frac{d^3 \varepsilon_p (U)}{(dU)^3} \bigg|_{U=L/2} + \frac{1}{4} \cdot d^3 \varepsilon_p (U) \bigg|_{U=L/2} \\
&= \frac{1}{4} \left[ \exp \left( -\frac{L}{2} \right) \right] \left\{ -\cos \left( \frac{L}{2} \right) - \sin \left( \frac{L}{2} \right) \right\} \cdot \frac{d^4 \varepsilon_p (U)}{(dU)^4} \bigg|_{U=L/2} + \frac{1}{4} \cdot d^4 \varepsilon_p (U) \bigg|_{U=L/2}
\end{align*}
\]

\[
B = \left[ \exp \left( -\frac{L}{2} \right) \right] \left\{ \sin \left( \frac{L}{2} \right) \right\} \cdot \varepsilon_p \left( \frac{L}{2} \right) - 0 \cdot \varepsilon_p (0)
\]

\[
\begin{align*}
&= \frac{1}{2} \left[ \exp \left( -\frac{L}{2} \right) \right] \left\{ -\sin \left( \frac{L}{2} \right) \right\} \cdot \frac{d\varepsilon_p (U)}{dU} \bigg|_{U=L/2} - 0 \cdot \varepsilon_p (0) \\
&= \frac{1}{2} \left[ \exp \left( -\frac{L}{2} \right) \right] \left\{ -\sin \left( \frac{L}{2} \right) \right\} \cdot \frac{d^2 \varepsilon_p (U)}{(dU)^2} \bigg|_{U=L/2} + 0 \cdot \varepsilon_p (0) \\
&= \frac{1}{4} \left[ \exp \left( -\frac{L}{2} \right) \right] \left\{ -\sin \left( \frac{L}{2} \right) \right\} \cdot \frac{d^3 \varepsilon_p (U)}{(dU)^3} \bigg|_{U=L/2} + 0 \cdot \varepsilon_p (0) \\
&= \frac{1}{4} \left[ \exp \left( -\frac{L}{2} \right) \right] \left\{ -\sin \left( \frac{L}{2} \right) \right\} \cdot \frac{d^4 \varepsilon_p (U)}{(dU)^4} \bigg|_{U=L/2} + 0 \cdot \varepsilon_p (0)
\end{align*}
\]

In order to vanish \( A \), we obtain eq.(17) by eq. (15) * [exp(L/2)\{sin(L/2)\}] + eq. (16) * [exp(L/2)\{cos(L/2)\}].
\[ B + \frac{1}{2}(-2B) = \exp\left(-\frac{L}{2}\right)\left(\frac{L}{2}\right) \cdot \left[ -\frac{\sigma_0}{E} \left(1 + 6 \frac{L}{2}\right) + A - B \left(\frac{L}{2}\right)^2 \right] \]

\[
\begin{align*}
- \frac{1}{2} & \left[ \exp\left(-\frac{L}{2}\right) \left(-\frac{\cos(L)}{2} - \sin\left(\frac{L}{2}\right)\right) \cdot \left[ -\frac{2\sigma_0}{E} \left(\frac{L}{2}\right)^3 - 2B \left(\frac{L}{2}\right)^2 \right] \right] \\
+ \frac{1}{2} & \left[ \exp\left(-\frac{L}{2}\right) \left(\frac{\cos(L)}{2}\right) \cdot \left[ -\frac{4\sigma_0}{E} \left(\frac{L}{2}\right) \right] \right] \\
- \frac{1}{4} & \left[ \exp\left(-\frac{L}{2}\right) \left(-\frac{\cos(L)}{2} + \sin\left(\frac{L}{2}\right)\right) \cdot \left[ -\frac{4\sigma_0}{E} \left(\frac{L}{2}\right) \right] \right] \\
+ \frac{1}{4} & \left[ \exp\left(-\frac{L}{2}\right) \left(-\frac{\sin(L)}{2}\right) \cdot \left[ -\frac{4\sigma_0}{E} \right] \right] \\
\end{align*}
\] (20)

\[
\therefore \sin\left(\frac{L}{2}\right) \cdot A = \frac{\sigma_0}{E} \left[ \sin\left(\frac{L}{2}\right) \cdot \left[ 1 + 6 \frac{L}{2}\right] + 6 \frac{L}{2} - \frac{L}{2} - 1 \right] + \cos\left(\frac{L}{2}\right) \cdot \left[ \frac{3}{2} \frac{L}{2} + \frac{L}{2} + \frac{L}{2} \right] \\
+ B \left[ \sin\left(\frac{L}{2}\right) \cdot \left[ \left(\frac{L}{2}\right)^2 + \left(\frac{L}{2}\right) \right] + \cos\left(\frac{L}{2}\right) \cdot \left[ \left(\frac{L}{2}\right)^2 + 1 \right] \right] \\
= \frac{\sigma_0}{E} \left[ \sin\left(\frac{L}{2}\right) \cdot \left[ \frac{1}{6} \left(\frac{L}{2}\right)^4 + \frac{1}{3} \left(\frac{L}{2}\right)^3 - \left(\frac{L}{2}\right) - 1 \right] + \cos\left(\frac{L}{2}\right) \cdot \left[ \frac{3}{2} \left(\frac{L}{2}\right) + \left(\frac{L}{2}\right) \right] \right] \\
\therefore A = \frac{-\sigma_0}{E} \left[ \frac{L^4}{96} + \frac{L^3}{12} + \frac{L^2}{4} + \frac{L}{2} \right] \\
\] (22)

By substituting eq.(19) and eq.(22) for eq.(14), we can predict the plastic strain distribution that bring uniform residual stress \(\sigma_0\) for the region with full-section yielding.

### 3.3 Residual stress distribution on surrounding region

As wrote in subsection 3.1, we assumed no plastic strain caused on surrounding region \((Z>L/2)\). By substituting \(\varepsilon_p(Z) = 0\) for eq.(13), we can obtain residual stress distribution on the surrounding region. We used a general solution for the residual stress distribution shown as eq.(23), considering \(\sigma_0(Z) \to 0\) when \(Z \to \infty\).

\[
\begin{align*}
\sigma_0(Z) &= \sigma_0 \quad \text{for } Z \leq \frac{L}{2} \\
\sigma_0(Z) &= -\sigma_0 \cdot \exp\left(-\left(Z - \frac{L}{2}\right)\right) \left\{ C_1 \cos\left(Z - \frac{L}{2}\right) + C_2 \sin\left(Z - \frac{L}{2}\right) \right\} \quad \text{for } Z > \frac{L}{2}
\end{align*}
\] (23)

For \(Z > L/2\), eq.(24) to (27) were obtained by the integration of eq.(23). In these equations, \(Z_E = Z - L/2\) is used to shrink the description.

\[
\begin{align*}
\int_0^{Z} \frac{\sigma_0(U_1)}{\sigma_0} \, dU_1 &= \int_{L/2}^{Z} \frac{\sigma_0(U_1)}{\sigma_0} \, dU_1 + \int_{0}^{L/2} \frac{\sigma_0(U_1)}{\sigma_0} \, dU_1 \\
&= \left[ \exp(-Z_E) \left( \frac{C_1 + C_2}{2} \cos(Z_E) + \frac{-C_1 + C_2}{2} \sin(Z_E) \right) - \frac{C_1 + C_2}{2} \right] + \frac{L}{2}
\end{align*}
\] (24)
By substituting eq.(23) and eq.(27) for eq.(13), eq.(28) was obtained.

\[ -C_1 + \frac{L^4}{96} + \left( C_1 - C_2 + \frac{L^3}{12} \right) Z_E + \left( C_2 + \frac{L^2}{4} \right) Z_E^2 - \frac{C_1 + C_2 - L}{3} Z_E^3 - \frac{E}{\sigma_0^2} \left( A - B \left( Z_E + \frac{L}{2} \right)^2 \right) = 0 \]  

(28)

In order to fulfill eq.(28) for \( Z_E > 0 \), following simultaneous equations must be satisfied.

\[
\begin{align*}
-C_1 + C_2 - L &= 0 \\
C_2 + \frac{L^2}{4} + \frac{E}{\sigma_0} B &= 0 \\
C_1 - C_2 + \frac{L^3}{12} + \frac{E}{\sigma_0} BL &= 0 \\
-C_1 + \frac{L^4}{96} - \frac{E}{\sigma_0^2} \left( A - BL^2 \right) &= 0
\end{align*}
\]  

(29)

As the solution of eq.(29), eq.(30) and eq.(31) were formulated for \( C_1 \) and \( C_2 \) in eq.(23).

\[
C_1 = \frac{1}{6} L^2 + \frac{2}{3} L - \frac{1}{3} + \frac{2}{3} \left( \frac{1}{L + 2} \right)
\]  

(30)
When the residual stress on the surrounding region exceeds the yielding stress \( \sigma_y \), full-section yielding is caused on this region and the precondition a) shown in subsection 3.1 is violated. Therefore, we should not use eq.(14) when \( |\sigma_0 \cdot C_1| > \sigma_y \). Figure 3 shows this application limit of required residual stress \( \sigma_0 \) for eq.(14) with one for the existing method proposed by Igari et al. (2000). Based on eq.(23), we can apply eq.(14) to the wider range of conditions than the existing method. When \( l_p = \sqrt{r t} \), which is screening criterion proposed on our previous paper (Okajima et al., 2017), maximum residual stress on surrounding region is nearly equal to \( \sigma_0 \).

\[
C_2 = -\frac{1}{6}L^2 + \frac{1}{3}L + \frac{1}{3} - 2 \left( \frac{1}{L+2} \right)
\]
(31)

Fig. 3 Stress limit of application of the formulated plastic strain distribution. Ideally, the proposed method in this paper may apply to wider range of conditions than the conventional method. However, we should consider also the effect of the expansion of \( l_p \) due to superposition of the residual stress and the thermal stress on the surrounding region.

The above discussion has considered only residual stress derived from the permanent deformation of the cylinder. However, the surrounding region is subjected to also thermal stress derived from the traveling temperature distribution. The full-section yielding may be caused by combination of the thermal stress and the residual stress given by eq.(23), (30) and (31). In order to estimate this yield of surrounding region conservatively, we propose iteration procedure as follows:

- **a)** Through elastic FEA, spatial distribution of minimum/maximum value of hoop stress derived from traveling temperature distribution is evaluated for each position.
- **b)** Evaluate the full-section yielding region as the region where the maximum hoop-membrane stress exceeds the yielding stress, \( \sigma_y \).
- **c)** Using the axial length of the full-section yielding region, \( l_p \), calculate \( C_1 \) based on eq.(30).
- **d)** Evaluate the full-section yielding region again. On this process, we define this region as where the maximum hoop-membrane stress exceeds \( \sigma_y - |\sigma_0 \cdot C_1| \).
- **e)** Repeat c) and d) for several times. If the expansion of the yielding region saturates, we can use the saturated value of \( l_p \) for the prediction of the plastic strain distribution. Otherwise, if \( |\sigma_0 \cdot C_1| \) exceeds \( \sigma_y \), we cannot use the formulated distribution for the prediction of the accumulated plastic strain.

4. Comparison with the accumulated plastic distribution derived from the repeating long-range-travel of temperature distribution

In this section, we validate that the formulated distribution of plastic strain can be used for the prediction of the accumulated one derived from repeating temperature travel. The validation is based on the comparison with the plastic strain distribution calculated by the procedure shown in section 3 with the accumulated plastic strain obtained by elastic-plastic Finite Element Analysis.
4.1 Condition for Finite Element Analysis

In this subsection, we explain about the conditions used for the Finite Element Analysis. The FEM code FINAS Ver.21 and the 8-node axisymmetric solid element were used in these analyses. As same with our previous papers, we assumed elastic perfectly-plastic material, which is simple and conservative constitutive model. These FEAs ignored temperature dependency of mechanical properties and used ones for 316FR stainless steel at 550 °C.

Figure 4 shows the model of cylinder used in FEAs. This cylinder that consists of 21857 nodes and 6720 elements is subjected to temperature travel shown in Fig.5. Uniform temperature was given on all nodes on each horizontal section. We defined to use this temperature distribution because it brings similar shape of residual stress distribution with the assumption shown in Fig. 2. Therefore, this temperature distribution brings pessimistic case of residual stress distribution as discussed in subsection 3.1, meanwhile it is not realistic temperature distribution.

This temperature distribution and its travel cause thermal stress shown in Fig.6. As shown in Fig.6(a), the temperature distribution itself cause plastic strain on only limited area. However, the travel of this distribution spread plastic strain on large area. We analyzed 7 cases with various travel distance shown in Table 1. We analysed until 30 cycles of thermal transient for each case. At the end of each cycle, we evaluate stress and strain on circumferential direction at all nodes and obtain membrane stress and strain as the average value on the horizontal section.

![Fig. 4 Cylinder model for Finite Element Analysis. Elastic-perfectly Plastic material is assumed for this cylinder.](image)

![Fig. 5 Model of the travel of temperature distribution. Triangle-shaped distribution simulate the hot-spot model, which is used for the formulation by Igari et al. A cycle of the temperature travel consists of 3 phases, generate, travel, and vanish.](image)
Fig. 6 Result of elastic FEA. Dot line means yielding stress. Though the triangle-shaped temperature distribution brings plastic strain to only limited area on the cylinder, its travel brought large region with full-section yielding.

Table 1 Analysed cases

<table>
<thead>
<tr>
<th>Case</th>
<th>Maximum temperature difference, $\Delta T$ [°C]</th>
<th>Mises stress (membrane) [MPa]</th>
<th>Travel distance of the temperature distribution, $l_0$ [mm]</th>
<th>Conservative estimation of $l_p$ [mm] (Based on subsection 3.3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case 1</td>
<td>50</td>
<td>140</td>
<td>500</td>
<td>593</td>
</tr>
<tr>
<td>Case 2</td>
<td></td>
<td></td>
<td>600</td>
<td>707</td>
</tr>
<tr>
<td>Case 3</td>
<td></td>
<td></td>
<td>800</td>
<td>937</td>
</tr>
<tr>
<td>Case 4</td>
<td></td>
<td></td>
<td>1000</td>
<td>1173</td>
</tr>
<tr>
<td>Case 5</td>
<td></td>
<td></td>
<td>1100 (exceed the limit)</td>
<td></td>
</tr>
<tr>
<td>Case 6</td>
<td></td>
<td></td>
<td>1150 (exceed the limit)</td>
<td></td>
</tr>
<tr>
<td>Case 7</td>
<td></td>
<td></td>
<td>1200 (exceed the limit)</td>
<td></td>
</tr>
</tbody>
</table>

4.2 Result

Figure 7 shows the comparison of the accumulated hoop-membrane strain evaluated through FEA. On Cases 1, 2, accumulation of the strain saturated before less than 10th cycle. On Case 3, it saturated at 18th cycle. Also on case 4, the increment of plastic strain on 30th cycle was about $7 \times 10^{-1}$ mm/mm and it was less than 1/1000 times of the total value of accumulated strain. Therefore, we can expect that significant increase of the plastic strain will not be caused by further dozen cycles of temperature travel for this case. We can evaluate conservative $l_p$ through the iteration process shown in subsection 3.3 for these cases. Therefore, the elastic shakedown behavior is observed on all cases that we can apply eq.(14) for the prediction of plastic strain. Meanwhile, on Case 5, the increment of plastic strain on 30th cycle was about $2 \times 10^{-6}$ mm/mm and it was larger than 1/1000 times of the total value.
Fig. 7 Comparison of accumulated hoop-membrane strain between various travel distance of temperature distribution. The accumulation of the plastic strain is finally saturated in all cases that we can apply eq.(14). Meanwhile, though the increment of the plastic strain decreases for each cycle, the accumulation had not finished on Cases 5, 6, and 7, at least until 30 cycle.

Fig. 8 FEA result on Case 2. Equation (14) can predict the accumulated plastic strain and residual stress distribution with high accuracy.
Fig. 9 FEA result on Case 4. Though the area with plastic strain expanded for each cycle at early stage, the accumulation of plastic strain was finally saturated. By using conservative $l_p$ estimated by the procedure shown in subsection 3.3, we can predict the accumulated plastic strain with rational safety margin.

Fig. 10 FEA result on Case 7. The plastic strain was accumulated continuously until at least 30th cycle. In this case, eq. (14) cannot predict the accumulated plastic strain derived from temperature travel, because we cannot define conservative $l_p$ by the iteration process.

Figure 8 shows FEA results on Case 2. For Cases 1 and 2, since the expansion of the yield region is limited, eq.(14) using travel distance of temperature distribution $l_n$ as $l_p$ (red-broken line in Fig.8) provide high-accuracy prediction of the accumulated plastic strain distribution. When we estimate $l_p$ by iteration procedure shown in subsection 3.3, eq.(14) provide the prediction with rational margin (black-broken line in Fig.8). This margin is derived from the overestimation of the residual stress on the expanded region of $l_p$ ($-50 < z < 0$, $600 < z < 650$).

Figure 9 shows FEA results on Case 4. For Cases 3 and 4, plastic strain caused also on surrounding area of temperature travel, and the area with full-section yield was expanded cycle by cycle. As a result, when we use travel distance $l_n$ as $l_p$, eq.(14) underestimate the plastic strain distribution for these cases (red-broken line in Fig.9). Meanwhile, we can estimate conservative $l_p$ based on the procedures shown in subsection 3.3 for these cases. By using this conservative $l_p$, eq. (14) can predict the accumulated plastic strain distribution with rational safety margin (black-broken line in Fig.9).

Figure 10 shows FEA results on Case7. On Cases 5, 6 and 7, the accumulation of plastic strain continued until at least 30th cycle. The region with plastic strain spread on the axial direction cycle by cycle, and it finally reached the location without compressive thermal stress due to temperature distribution. On these cases, we could not estimate $l_p$ by iteration procedure shown in subsection 3.3. Therefore, we cannot use eq. (14) for the case with excessive distance of
temperature travel.

As the result of this validation study, we confirmed that we can use eq.(14) for the prediction of the accumulated plastic strain as long as we can estimate conservative \( l_p \) based on the procedures shown in subsection 3.3. Meanwhile, it is desirable to make detail discussion about the applicability of this iteration process for other shape of temperature distribution.

5. Conclusion

We have formulated the plastic strain distribution that bring uniform residual stress in the given region. Through FEA for the cylinder made with elastic-perfectly plastic material, we confirmed that the formulated strain distribution can be used as the prediction of the plastic strain distribution accumulated by the repeating moderate distance travel of triangle-shaped temperature distribution.

In this formulation, the region with plastic strain is assumed as known parameter. On FEA results of the cases with long-distance travel of temperature, this region expanded with the repetition of temperature travel. We proposed also the procedures to estimate the magnitude of this expansion conservatively. By using this estimation procedure, the formulated strain distribution can be used as conservative prediction of the accumulated plastic strain also for the cases with long distance temperature travel. The detail discussion about applicability of this procedure to various temperature distribution is future issue.

References