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O-Integral evaluation for stress intensity factor of three-dimensional planar-crack with arbitrary shape

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Abstract

The numerical scheme of the O-integral is formulated for precisely evaluating the stress intensity factor of an embedded crack with arbitrary shape in an infinite elastic body. In this study, we evaluate the O-integral using an efficient numerical integration method by introducing an iso-parametric element and the Gauss–Legendre formula, which is typically used in the finite element method. To verify the numerical procedure introduced herein, the mode-I stress intensity factors for cracks are evaluated based on a circle, an ellipse, and a perturbed circle cracks. Result shows that the K results obtained using the proposed method are consistent with the exact solution. Therefore, fatigue crack propagation is successfully simulated using the O-integral for the elliptical crack.

Keywords: Three-dimensional crack, Stress intensity factor, Weight function, O-integral, Fatigue crack propagation

1. Introduction

The stress intensity factor of an arbitrarily shaped crack has been reported in many reports and summarized in a handbook (Tada, Paris and Irwin, 2000; Murakami et al., 1987). However, the crack shape, which can be treated exactly based on elasticity, is limited to two-dimensional or axisymmetric geometries. For three-dimensional cracks, the exact solutions for circular and elliptical cracks under a uniform stress field or linear-distributed stress field were determined using the potential function method (Sneddon and Lowengrub, 1969). Recently, the stress intensity factor for a circular crack with a disturbed crack front was solved using the perturbation method (Arai, 2018). As another approach to remove such limitations in three-dimensional cracks, a weight function method was developed. This method was first proposed by Bueckner (1970) to evaluate the stress intensity factor of two-dimensional cracks, and it was subsequently formulated as a three-dimensional crack by Rice (1972). He generalized the weight function formulation by including a three-dimensional planar crack in an elastic infinite body of domain Γ , as shown in Fig. 1(a).

Herein, the crack plane is denoted as Ω and its crack front $\partial\Omega$, as shown in Fig. 1(b). When two arbitrary points $Q' \in \partial\Omega$ selected along the crack front line and $Q \in \Omega$ laid on the crack plane are located, the stress intensity factor at the observed point Q' is expressed as

$$K_i(Q') = \int_{\partial\Omega} h_{ij}(Q, Q') \cdot t_j(Q) d\Omega(Q), \quad (1)$$

where the subscript i is I, II, and III for the stress intensity factors of modes I, II, and III, respectively. The function $h_{ij}(Q, Q')$ is a weight function related to the stress intensity factor of mode i at the observed point Q' when the

unit-concentrated load in the direction of the j -axis acts at point Q . Additionally, $t_j(Q)$ represents traction in the direction of j at a point Q on the crack plane located virtually in an elastic body with region Γ without a crack. If the outer region of the elastic body of region Γ is unbounded, then the following restriction is required:

$$h_{ij}(Q, Q') \rightarrow 0$$

It is noteworthy that the weight function h_{ij} must be determined for the prescribed crack shape; hence, another limitation is imposed in applying the weight function method to evaluate the stress intensity factor.

Fig.1 Three-dimensional crack embedded in elastic infinite body subjected to tensile loading

To overcome this limitation, Oore and Burns (1980) developed a generalized form of the weight function of the mode-I ($i = I$) stress intensity factor for a three-dimensional crack with arbitrary geometry.

$$h_{Iz}(Q, Q') = \frac{\sqrt{2}}{\pi} \frac{1}{\sqrt{f(Q)} \|Q - Q'\|^2}, \quad (2)$$

where $j = z$ is the component normal to the crack plane in the global coordinate defined on the crack plane, and

$$f(Q) = \int_{\partial\Omega} \frac{ds(P)}{\|Q - P\|^2}, \quad (3)$$

where $ds(P)$ is an infinitesimal measurement along the crack front, and point P is an arbitrary location along the crack front line. The weight function in Eq. (1), which is known as the “O-integral,” can be expressed as shown in Eq. (2). The applicability of the O-integral was demonstrated by solving the typical three-dimensional problem of circular and elliptical cracks (Oore and Burns, 1980). Other researchers have attempted to mathematically solve this O-integral formula (Oore and Burns, 1980; Livieri et al., 2005; Livieri and Segala, 2010). In another approach, a numerical scheme to derive the O-integral was developed (Desjardins, Burns and Thompson, 1991; Krasowsky et al., 1999; Ascenzi et al., 2002; Montenegro et al., 2006; Hachi et al., 2007). The integral term was evaluated using the simple discretization of the square element over the crack plane. Such a simple integral approach is guaranteed by the following theorem (Ascenzi, Pareschi and Segala, 2002): the O-integral has a finite value when point Q' laid on the crack plane approaches integral point P along the crack front line in the weight function of Eq. (2). It is noteworthy that the crack front line was segmented using the simple element of a square shape, resulting in a significant error.

The purpose of this study is to improve the estimation accuracy of the numerical integral involved in the O-integral using both the iso-parametric element and the Gauss–Legendre quadrature, which is typically used in the finite element technique.

2. Discretization for O-integral formula

The discretization method for evaluating the O-integral with high accuracy is explained in this section. We assumed an arbitrary curved line for the crack front line. To precisely assess the O-integral for such a curved crack front line, the

crack plane was segmented using an iso-parametric element.

First, the global coordinate $(x, y) \in \Omega$ was set on a three-dimensional crack plane, and the z -axis was placed in a direction normal to the crack plane. The associated crack region Ω was segmented using the triangle and quadrangle. For one of the elements, p is the number of nodes assigned to the element. The position in the global coordinate in one element can then be expressed by nodal coordinates $(x_i, y_i) \in \Omega$ ($i = 1, \dots, p$) as follows:

$$x = \sum_{i=1}^p N_i(\xi, \eta) x_i \quad y = \sum_{i=1}^p N_i(\xi, \eta) y_i, \quad (4)$$

where $N_i = N_i(\xi, \eta)$ is the shape function for interpolating the values in the elements with those of the nodal values, and (ξ, η) is the local coordinate system defined separately from the global coordinate system, where $|\xi| \leq 1, |\eta| \leq 1$. Therefore, Eq. (4) maps all physical quantities from the global coordinate system to the local coordinate system.

The infinitesimal area $d\Omega = dx dy$ in the crack plane is expressed as

$$d\Omega = \left(\frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \eta} - \frac{\partial y}{\partial \xi} \frac{\partial x}{\partial \eta} \right) d\xi d\eta, \quad (5)$$

where the small area in the crack plane is mapped to one in the local coordinate system.

Next, focusing on the element adjacent to the crack front line $\partial\Omega$, when the number of nodes assigned to the crack front point is q , the position $(x', y') \in \partial\Omega$ on this front line is expressed by nodal coordinates $(x'_i, y'_i) \in \partial\Omega$ ($i = 1, \dots, q$) as follows:

$$x' = \sum_{i=1}^q M_i(\xi) x'_i \quad y' = \sum_{i=1}^q M_i(\xi) y'_i, \quad (6)$$

where $M_i = M_i(\xi)$ is a shape function for interpolating values within the element edge with that of the nodal values assigned along the crack front line in the element adjacent to the crack front line, and variable ξ is the other local coordinate different from (ξ, η) introduced earlier, where $|\xi| \leq 1$.

The infinitesimal length ds along the crack front is expressed as

$$ds = \sqrt{(dx')^2 + (dy')^2} = \sqrt{\left(\frac{\partial x'}{\partial \xi} \right)^2 + \left(\frac{\partial y'}{\partial \xi} \right)^2} d\xi \quad (7)$$

Substituting Eq. (7) into Eq. (3) yields

$$f(x, y) \Rightarrow \sum_{m=1}^M \int_{-1}^1 \frac{1}{(x - x')^2 + (y - y')^2} \sqrt{\left(\frac{\partial x'}{\partial \xi} \right)^2 + \left(\frac{\partial y'}{\partial \xi} \right)^2} d\xi, \quad (8)$$

where M is the number of elemental edges along the crack front line. The Gauss–Legendre quadrature for Eq. (8) yields the following equation:

$$f(x, y) = \sum_{m=1}^M \sum_{n=1}^N \frac{\omega_n}{(x - x')^2 + (y - y')^2} \sqrt{\left(\frac{\partial x'}{\partial \xi} \right)^2 + \left(\frac{\partial y'}{\partial \xi} \right)^2} \bigg|_{\xi=\xi_n}, \quad (9)$$

where N is the order of integration, and ω_n and ξ_n are the n^{th} weights and positions, respectively. In this study, we used the Gauss–Legendre quadrature formula as weights.

The stress intensity factor at $Q'(x', y')$ was derived from Eq. (1), as follows:

$$K_I(x', y') = \frac{\sqrt{2}}{\pi} \sum_{k=1}^K \left[\sum_{i=1}^I \sum_{j=1}^J \frac{W_i W_j \sigma(x_k, y_k) \det[J]}{\sqrt{f(x_k, y_k) ((x_k - x')^2 + (y_k - y')^2)}} \right]_{\substack{x_k=\xi_i \\ y_k=\eta_j}}, \quad (10)$$

where

$$\det[J] \equiv \frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \eta} - \frac{\partial y}{\partial \xi} \frac{\partial x}{\partial \eta} \bigg|_{x_k, y_k}$$

Here, K is the number of elements in the crack plane; I and J are the integration orders along ξ and η axis; (ξ_i, η_j) is position at integration point; W_i and W_j are the weights corresponding to integration point, respectively.

3. Estimation results and discussion

The numerical program was developed based on Eqs. (10) and discretized using iso-parametric elements. In the developed program, the crack plane can be segmented into elements using a combination of eight-node quadrangle and six-node triangular iso-parametric elements, and the nodes can be generated uniformly over the crack plane such that the area of each element is almost uniform. All the crack shapes shown herein were segmented into 200 and 1440 segments in the radial and circumferential directions, respectively, and numerically integrated with an integration order of 10. These values were determined by verifying the convergence of K in advance.

3.1 Circular flat crack

First, the stress intensity factor of a circular flat crack with radius R under a uniform tensile stress σ_0 was analyzed. A comparison between the exact K solution (Smith, Kobayashi and Emery, 1967) and the results evaluated using our program are listed in Table 1. This list includes the approximate K solution obtained via the conventional numerical scheme (Oore and Burns, 1980), which uses a simple square-shaped element. The angle θ is the angle between the x -axis and the axis connecting the origin and the evaluation point when the origin of the global coordinate system is placed at the center of the circular crack. It is noteworthy that all the values of the stress intensity factor were normalized by $\sigma_0\sqrt{\pi R}$. The results show that the maximum error of the conventional numerical result was 1.5%, and that the evaluated results differed slightly depending on the angle θ . Meanwhile, the error of the results obtained using the present method was -0.19%, and the results were the same for all angles. In addition, it was confirmed that the results of the stress intensity factor under the linearly distributed tensile stress condition indicated similar accuracies in terms of identification.

Table 1 Stress intensity factor K of a circular flat crack with radius R under a uniform tensile stress σ_0 . Variation of K with the angle θ measured from the x -axis is listed. Herein, the conventional O-integral was evaluated by Oore & Burns scheme. It is shown that error for present is within 0.19%.

Angle θ [deg]	Exact K	Conventional O-integral	Present O-integral	Error [%] for present
0	0.6366	0.6443	0.6356	-0.19
45	0.6366	0.6403	0.6356	-0.19
90	0.6366	0.6460	0.6356	-0.19

3.2 Elliptical flat crack

The stress intensity factor of an elliptical crack under uniform tensile stress σ_0 is listed in Table 2. This table also includes the exact solution for comparison (Shah and Kobayashi, 1971). The elliptical shape has an aspect ratio of (RB/RA) , where RA is half the length of the major axis, and RB is one for the minor axis. The definition of angle θ is the same as that for the circular crack, and all values of the stress intensity factor are normalized by $\sigma_0\sqrt{\pi RA}$. The results show that the stress intensity factor can be evaluated accurately up to an aspect ratio of 0.6. However, the estimation error increases as the aspect ratio decreases, i.e., the elliptical crack shape becomes flatter.

Additionally, the stress intensity factor of an elliptical crack under a linearly distributed stress expressed as

$$\sigma = \sigma_0 \left(1 + \frac{y}{RB}\right) \quad (11)$$

is listed in Table 3, where the y -axis is along the minor axis of the ellipse. As in the case of uniform tensile stress, the stress intensity factor can be evaluated accurately up to an aspect ratio of 0.6. However, the error increased as the aspect ratio decreased. Because the maximum error of the conventional O-integral was 8% at an aspect ratio of 0.8, it can be inferred that the present method for the O-integral can be used to assess the stress intensity factor with satisfactory accuracy compared with the conventional procedure.

3.3 Circular flat crack with slightly perturbed crack front

In this section, we consider a circular flat crack with a slightly perturbed crack front line, as shown in Fig. 2, in order to show applicability of this method to more complicate crack problem. The crack front shape interested here is expressed by the radius $R(\theta)$ varying with angle θ as follows:

$$R(\theta) = R - \varepsilon\{1 + \cos(m\theta)\}, \quad (12)$$

where ε is the small amplitude around the reference circular crack front, and m is the order for determining the disturbed shape of the crack front line. For such a perturbed circular crack problem, the following distributed traction is assumed to be subjected to the crack surface as follows:

$$\sigma = \sigma_0 \left\{ \left(\frac{x}{x_0} \right)^k + 1 \right\}, \quad (13)$$

where σ_0 is the reference traction, and k is the order for determining the distributed traction normal to the crack plane. In this case, the reference length x_0 was set to 10. The exact solution for the stress intensity factor under distributed stress was provided by Arai (2018).

Fig. 2 Circular flat crack with slightly perturbed crack front subjected to distributed traction normal to crack plane

Table 2 Stress intensity factor K of an elliptical crack under uniform tensile stress σ_0 . Influence of the minor axis RB in the elliptical shape on K value is shown. Herein, the exact solution was given by Shah and Kobayashi. It is found that the K is evaluated accurately up to an aspect ratio of 0.6. However, the error increased as the aspect ratio decreased.

RB	Angle θ [deg]	Exact K	Present O-integral	Error [%]
0.8	0	0.5641	0.5662	0.36
	45	0.6006	0.6004	-0.03
	90	0.6307	0.6255	-0.82
0.6	0	0.4701	0.4775	1.58
	45	0.5519	0.5564	0.83
	90	0.6068	0.6004	-1.06
0.4	0	0.3476	0.3621	4.17
	45	0.4807	0.4896	1.85
	90	0.5495	0.5465	-0.55
0.2	0	0.1904	0.2093	9.94
	45	0.3625	0.3708	2.30
	90	0.4256	0.4408	3.58

Table 3 Stress intensity factor K of an elliptical crack under a linearly distributed stress. Influence of the minor axis RB in the elliptical shape on K value is shown. Herein, the exact solution was also given by Shah and Kobayashi. It is found that the K is evaluated accurately up to an aspect ratio of 0.6 as well as uniform tensile stress result.

RB	Angle θ [deg]	Exact K	Present O-integral	Error [%]
0.8	0	0.5641	0.5662	0.36
	45	0.8704	0.8858	1.77
	90	1.0288	1.0260	-0.27
0.6	0	0.4701	0.4775	1.58
	45	0.7851	0.8189	4.31
	90	0.9670	0.9688	0.19
0.4	0	0.3476	0.3621	4.17
	45	0.6706	0.7154	6.68
	90	0.8546	0.8674	1.49
0.2	0	0.1904	0.2093	9.94
	45	0.4966	0.5343	7.60
	90	0.6469	0.6928	7.10

Table 4 Stress intensity factor K of circular flat crack with slightly perturbed crack front for $k = 0$. Influence of the small amplitude ε and the order m for determining the perturbed shape of the crack front line on K value is shown. It is found that the results obtained using the present method agreed well with the exact solutions for the results of small amplitudes.

m	ε	Angle θ [deg]	Exact K	Present O-integral	Error [%]
4	0.010	0	0.6387	0.6380	-0.10
		45	0.6148	0.6240	1.49
		90	0.6387	0.6380	-0.10
	0.025	0	0.6527	0.6434	-1.43
		45	0.5951	0.6082	2.20
		90	0.6527	0.6433	-1.43
	0.050	0	0.6692	0.6522	-2.54
		45	0.5605	0.5816	3.76
		90	0.6692	0.6521	-2.55
6	0.010	0	0.6476	0.6413	-0.97
		45	0.6286	0.6309	0.37
		90	0.6084	0.6206	2.01
	0.025	0	0.6714	0.6513	-3.00
		45	0.6264	0.6238	-0.40
		90	0.5781	0.5992	3.64
	0.050	0	0.6933	0.6668	-3.82
		45	0.6114	0.6075	-0.65
		90	0.5239	0.5628	7.42

Table 5 Stress intensity factor K of circular flat crack with slightly perturbed crack front for $k = 3$. Influence of the small amplitude ε and the order m for determining the perturbed shape of the crack front line on K value is shown. It is found that the results obtained using the present method also agreed with the exact solutions for the results of small amplitudes.

m	ε	Angle θ [deg]	Exact K	Present O-integral	Error [%]
4	0.010	0	0.6390	0.6383	-0.10
		45	0.6150	0.6241	1.49
		90	0.6387	0.6380	-0.10
	0.025	0	0.6530	0.6436	-1.44
		45	0.5951	0.6083	2.20
		90	0.6527	0.6433	-1.43
	0.050	0	0.6696	0.6524	-2.56
		45	0.5606	0.5817	3.75
		90	0.6692	0.6521	-2.55
6	0.010	0	0.6480	0.6417	-0.97
		45	0.6288	0.6311	0.37
		90	0.6084	0.6206	2.01
	0.025	0	0.6718	0.6516	-3.00
		45	0.6265	0.6240	-0.40
		90	0.5781	0.5992	3.64
	0.050	0	0.6936	0.6671	-3.83
		45	0.6116	0.6076	-0.65
		90	0.5239	0.5628	7.42

All the results of the stress intensity factor were normalized by $\sigma_0\sqrt{\pi R}$. A comparison between the exact K solution and the results evaluated using the O-integral is listed in Tables 4 and 5. For the results of small amplitudes, the results obtained using the present method agreed well with the exact solutions for all shapes of distributed traction. However, as the amplitude increased, the difference between the results obtained using the present method and the exact solutions increased, reaching a maximum error of approximately 7% for the highest amplitude at order $m = 6$. Therefore, the O-integral should be applied with caution for crack shapes with excessively large amplitudes.

4. Fatigue crack growth simulation

4.1 Numerical procedure

The fatigue crack growth process was simulated based on the stress intensity factor evaluated using the O-integral. It was assumed that the fatigue crack developed further based on the Paris law, which can identify the crack size propagating per δN cycle. The associated incremental crack size δa can be expressed as follows:

$$\delta a = \delta N \cdot C \left[\Delta K_I(Q') \right]^n, \quad (14)$$

where

$$\Delta K_I(Q') = K_{I \max} - K_{I \min}, \quad (15)$$

which is the stress intensity range defined as the difference in the stress intensity factor between the maximum and minimum values during one cycle, and C and n are material constants identified from the crack propagation data.

Fig. 3 Crack tip shape before and after crack propagation under cyclic loading and the definition of crack extension direction.

Figure 3 shows the crack tip shape before and after crack propagation under cyclic loading. This illustration shows that the solid and dashed lines are the crack front line before and after advancing, respectively. The origin of the global coordinate system (x, y) was set in the crack plane. When the angle φ is expressed counterclockwise from the x -axis, and the distance from the coordinate origin to the crack tip is expressed as $r(\varphi)$, point Q' is

$$x = r(\varphi)\cos\varphi, \quad y = r(\varphi)\sin\varphi \quad (16)$$

Subsequently, the normal vector perpendicular to the tangent to the crack front line at point Q' is

$$\vec{n} = (\sin\theta, \cos\theta) = \left(\frac{-\left(\frac{dy}{dx}\right)}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}, \frac{1}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}} \right) \quad (17)$$

The crack front line propagated along this direction with amount $\delta a \cdot \vec{n}$. Because the crack front line is discretized, the associated incremental crack size is expressed as

$$\delta a_x = -\delta a \frac{\left(\frac{\Delta y}{\Delta x}\right)}{\sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}}, \quad \delta a_y = \delta a \frac{1}{\sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}}, \quad (18)$$

where Δx and Δy are the distances between the adjoining nodes along the crack front line, as shown in Fig. 4.

Fig. 4 Crack extension procedure

4.2 Numerical model and results

In this section, we consider an elliptical crack problem. The analysis conditions are presented in Table 6. Here, the tensile stress σ_0 was set to 1.0, and the material constants set herein were a hypothetical value to confirm the function of O-integral. The number of divisions and the order of integration of the Gaussian–Legendre quadrature are the same as those presented in Section 3.

Table 6 Analysis condition for elliptical crack problem

RA	RB	σ_{max}	σ_{min}	Loading case	C	m
1.0	0.8	1	0	$\sigma = \sigma_0$	1.5	3
				$\sigma = \sigma_0 \times \left(1 + \frac{y}{RB}\right)$		

Figure 5 shows the analysis results for uniform stress distribution. It was discovered that the amount of crack growth per cycle was larger on the short axis than that on the long axis. In this case, the crack shape appeared to be more circular as the number of cycles increased, which is a reasonable result from the viewpoint of linear fracture mechanics.

Figure 6 shows the analysis results of the linear stress distribution. In this case, the amount of crack growth was larger at the evaluation point angle of 90° than at other crack edges, and the crack shape developed into a chestnut shape as the number of cycles increased.

Fig. 5 Crack propagation analysis for uniform stress distribution. It is found that the amount of crack growth per cycle is larger on the short axis than that on the long axis, which is a reasonable result from the viewpoint of linear fracture mechanics.

Fig. 6 Crack propagation analysis for distributed stress distribution. It is found that the amount of crack growth is larger at the evaluation point angle of 90° than at other crack edges, and the crack shape developed into a chestnut shape as the number of cycles increased.

5. Conclusion

Herein, we proposed an efficient method for the numerical integration of the O-integral by incorporating the concept of finite elements. The stress intensity factor for internal cracks of various shapes was evaluated using this method. Furthermore, by combining the proposed method with the Paris rule, a numerical simulation of the fatigue crack growth was performed.

First, a formulation and algorithm for the numerical integration of the O-integral based on the concept of finite

elements was developed. Subsequently, the stress intensity factors for circular, elliptical, and slightly perturbed circular cracks were evaluated using an analytical program based on this algorithm. The results were confirmed to be consistent with the exact solutions and analytical results. Next, an extended analysis program was developed to account for the Paris law. It was demonstrated that this program can simulate fatigue crack growth for cracks with elliptical initial shapes subjected to uniform and linear distributed loading.

As the future works, the present method will be applied to the internal crack problems for pipes and pressure vessels used in power plants. In addition, the O-integral will be extended to apply not only to Mode I condition but also to mixed mode condition.

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