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Optimum tuning of damped Helmholtz silencer using an equivalent discrete model and considering open-end correction

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Abstract

This paper describes the optimum tuning of damped Helmholtz silencers. In this study, Helmholtz resonators are used as silencers that suppress acoustic resonance in host acoustic fields. Side branch silencers and Helmholtz silencers are commonly known as a type of vibration absorber; however, damped silencers have not been thoroughly studied thus far. Therefore, prior to this paper, we reported the optimum tuning of damped side branch silencers using modal analysis and two fixed point method. In this paper, we additionally describe the optimum tuning of damped Helmholtz silencers in a similar fashion as in our previous paper. The resonance mechanism of Helmholtz silencers is different from that of side branch silencers. The coupled vibration between the host acoustic field and Helmholtz silencer was theoretically analyzed using modal analysis in this study. An equivalent discrete model was obtained using the equations of motion using modal coordinate systems. Using the equivalent discrete model, the open-end correction of the neck of the Helmholtz silencer was considered, and the number of degrees of freedom of the equivalent discrete model was reduced to two to derive optimum tuning conditions using the two fixed point method. The optimum natural frequency ratio and loss factor of the Helmholtz silencer were derived using the vibration model with two degrees of freedom. The theoretical analysis was validated through simulations and experiments.

Keywords : Helmholtz resonator, Helmholtz silencer, Silencer, Dynamic vibration absorber, Acoustic field, Modal analysis, Open-end correction, Residual modes, Damping

1. Introduction

Dynamic vibration absorbers are often used to passively suppress vibration. These devices are particularly effective in suppressing the resonance peaks of vibration systems. When a vibration system is subjected to random excitation, or when the excitation frequency is variable, damped dynamic vibration absorbers are used. Dynamic vibration absorbers suppress the resonance of host vibration systems by using their own resonance. Similar to dynamic vibration absorbers, side branch silencers and Helmholtz silencers can suppress vibration of acoustic fields using their own resonance, and such acoustic silencers are usually used to suppress the acoustic resonance of the low-order acoustic mode in practical use (Chanaud, 1994; Ichiyanagi and Nishiumu, 2008; Ichiyanagi et al., 2013; Wu and Zhang, 2017); however, damped silencers have not been thoroughly studied thus far. Acoustic fields are often subjected to random excitation, and the excitation frequency is sometimes variable; however, studies on damped silencers seem to be insufficient. With this background, we presented the optimum tuning conditions for damped side branch silencers using an equivalent discrete model. Because the resonance mechanism of side branch silencers is different from that of Helmholtz silencers, the optimum tuning conditions for damped Helmholtz silencers must be separately derived. In the case of Helmholtz silencers, the neck and cavity of the silencer function as a mass and a spring, respectively. Helmholtz silencers are often referred to as Helmholtz resonators; however, they are referred to as silencers in this study because they are used to

suppress sound pressure in the host acoustic field. In this study, acoustic absorption materials are installed in the neck of the Helmholtz silencer to tune its damping. Selamet et al. (2005) reported a Helmholtz resonator lined with absorbing material; however, they installed acoustic absorption materials in the cavity of the Helmholtz resonator. Because the damping effect of the acoustic absorption materials result from the viscosity of the fiber surface, acoustic absorption materials have a damping effect on the mass rather than the spring. In addition, our method saves acoustic absorption material because the volume of the neck is smaller than that of the cavity. From these two perspectives, our method seems to be superior to the method proposed by Selamet et al.. The first purpose of this study is to derive the optimum tuning conditions for damped Helmholtz silencers. Using the analogy between dynamic vibration absorbers and Helmholtz silencers, the optimum tuning conditions for Helmholtz silencers are often described in a qualitative manner (Fukatsu et al., 2010; Maruyama et al., 2016); however, an exact two-degrees-of-freedom (2-DOF) analytical model is quantitatively obtained in this study. When we consider the optimum tuning of the natural frequency of a Helmholtz silencer, the effect of the open-end correction at both ends of the neck must be considered (Ingard, 1953; Panton and Miller, 1975; Rayleigh, 1945). Because the air near both ends of the neck vibrates together with the air in the neck, the effective mass of the neck is larger than the actual mass of the neck. Because the length of the neck of a Helmholtz silencer is usually shorter than the length of a side branch silencer, the effect of open-end correction on the optimum tuning of the natural frequency of a Helmholtz silencer is larger than that of a side branch silencer. Rayleigh theoretically derived the length of the open-end correction under two conditions: first, that the neck is installed in an infinite free space, and second, that the neck has an infinite flange (Rayleigh, 1945). Because these conditions do not match the case in which the condition that a Helmholtz silencer is installed in a finite enclosed space, these open-end corrections do not function well. For a finite enclosed space, Ingard described the length of the open-end correction of different aperture geometries; however, there is a problem that Ingard considered only the areas of the flange and cross-section surface. The height of the finite enclosed space also influences the length of the open-end correction; however, Ingard did not consider the height. To obtain the exact 2-DOF analytical model, and to perform more accurate open-end correction at both ends of the neck, modal analysis is applied to wave equations of acoustic fields in this study (Benaroya and Nagurka, 2009; Meirovitch, 2001; Nagamatsu, 1985; Yamada and Utsuno, 2020). Using the equations of motion using the modal coordinate systems, an equivalent discrete model is obtained. We can derive an exact 2-DOF analytical model using the equivalent discrete model. In addition, the effect of the open-end correction can be described by considering the effect of the non-related acoustic modes of the host acoustic field and cavity of the Helmholtz silencer. The optimum natural frequency ratio and loss factor of the damped Helmholtz silencer are derived using the 2-DOF model and two fixed point method (Ormondroyd and Den Hartog, 1928). The effectiveness of the proposed method and theoretical analysis is verified through simulations and experiments.

2. Theoretical analysis

In this study, two types of acoustic fields are considered as the host acoustic field: a one-dimensional acoustic tube and a cuboidal room. A one-dimensional acoustic tube with one end open is used as the analytical model in this study because the sound radiating from the acoustic tube is an industrial concern. The cuboidal room assumes an enclosed space, such as a cabin. There is no essential difference between the two types of host acoustic fields in the sense that the resonance of a certain low-order acoustic mode is suppressed by a Helmholtz silencer. However, the effect of the non-target acoustic modes is inherently greater in the three-dimensional acoustic field because of the presence of non-target acoustic modes at closer frequencies. An acoustic tube can be regarded as a one-dimensional acoustic field when the depth and height of the acoustic tube are sufficiently smaller than the wavelength of the sound wave. However, the acoustic modes of such an acoustic tube in the depth and height directions may affect the length of the open-end correction. Another purpose of describing both one-dimensional and three-dimensional acoustic fields is to investigate the effect of the acoustic modes in the depth and height directions on the open-end correction. The wave equations of the host acoustic field and cavity of the Helmholtz silencer and the equation of motion of the mass of the neck of the Helmholtz silencer are first derived, and modal analysis is applied to the wave equations and equation of motion (Yamada and Utsuno, 2020). An equivalent discrete model is obtained using the equations of motion with the modal coordinate systems. When the non-targeted acoustic modes of the host acoustic field and acoustic modes other than the (0, 0, 0) acoustic mode of the cavity are ignored, a 2-DOF analytical model is obtained; however, the effect of these ignored acoustic modes is not small. The effects of the non-targeted acoustic modes of the host acoustic field and

cavity determine the lengths of the open-end correction of the neck on the host acoustic field side and cavity side, respectively. Therefore, a 2-DOF equivalent discrete model considering the effect of these acoustic modes is described. Using the 2-DOF model and two fixed point method, the optimum natural frequency ratio and loss factor of the Helmholtz silencer are derived (Ormondroyd and Den Hartog, 1928).

2.1 Wave equations and modal analysis

The analytical models whose host acoustic fields are a one-dimensional acoustic tube and a three-dimensional cuboidal room are shown in Fig. 1(a) and (b), respectively. The necks of the Helmholtz silencers are filled with acoustic absorption materials in the analytical models shown in Fig. 1. In the analytical model shown in Fig. 1(a), the host acoustic field is excited by a piston at the left-hand end of the acoustic tube, whereas the host acoustic field is excited by a piston at the left front on the ceiling of the cuboidal room in the analytical model shown in Fig. 1(b). Because the cross-sectional lengths and height of the neck of the Helmholtz silencer are usually sufficiently smaller than the wavelength of the sound wave at the targeted excitation frequency, the air in the neck can be regarded as a mass. Therefore, the mass in the neck vibrates as a unit and behaves like a piston for the host acoustic field and cavity. Because the shapes of the neck and cavity are usually cuboids or cylinders, the cases in which both the neck and cavity are cuboids and cylinders are described in this study.

As shown in Fig. 2(a) and (b), the analytical models shown in Fig. 1(a) and (b) are divided into three substructures at both ends of the neck of the Helmholtz silencer using the substructure synthesis method (Nagamatsu and Okuma, 1991). The connecting faces in the host acoustic field and cavity are replaced with rigid walls and sound pressure sources (Yamada and Utsuno, 2020). In contrast, the mass of the neck is excited by the sound pressure at both ends. The sound pressures of the sound pressure sources in the host acoustic tube and cavity are determined by the displacement of the mass that corresponds to the air in the neck. The forces applied to the mass of the neck are determined by the sound pressure on the connecting faces in the host acoustic field and cavity. In the divided analytical models shown in Fig. 2(a) and (b), the pistons are similarly replaced with sound pressure sources and rigid walls.

In the case in which the host acoustic field is a one-dimensional acoustic tube, the wave equation of the host acoustic tube is given as

$$\rho_A \frac{\partial^2 \psi_A}{\partial t^2} = \kappa_A \frac{\partial^2 \psi_A}{\partial x_A^2} + \kappa_A w_p(t) \delta(x_A) - \kappa_A \frac{d_B(x_A) w_B(t)}{S_A} (H(x_A - x_{AL}) - H(x_A - x_{AR})), \quad (1)$$

where ρ_A is the density, κ_A is the bulk modulus, ψ_A is the displacement potential, x_A is the spatial coordinate, S_A is the cross-sectional area of the host acoustic field, t is the time, w_p is the displacement of the piston, d_B is the depth of the neck of the Helmholtz silencer, w_B is the displacement of the mass of the neck, δ is the Dirac delta function, H is the Heaviside step function, and x_{AL} and x_{AR} are the x_A coordinates of the left and right ends of the neck of the Helmholtz silencer, respectively. When the cross-sectional shape of the neck is rectangular, the depth d_B is constant. In contrast, the depth d_B depends on x_A when the shape is circular. The center of vibration of the left-hand piston is set to the origin of the x_A coordinate system, and the right-hand direction is the positive x_A direction. The upward direction corresponds to the positive direction of the displacement w_B . The equation of motion of the neck mass is expressed as

$$m_B^* \ddot{w}_B = \int_{S_B} p_A(x_A, t) dS_B - \int_{S_B} p_C|_{z_C=0} dS_B, \quad (2)$$

where m_B^* is the complex mass, S_B is the cross-sectional area of the neck, p_A and p_C are the sound pressures in the host acoustic field and cavity of the Helmholtz silencer, respectively, and z_C denotes the coordinate in the spatial coordinate system of the cavity of the Helmholtz silencer. The upper end of the neck is set to the origin of the z_C coordinate system, and the upward direction is set as the positive z_C direction. The complex mass m_B^* is expressed as follows:

$$m_B^* = m_B (1 - j\eta_B) = \rho_B^* S_B h_B, \quad \rho_B^* = \rho_B (1 - j\eta_B), \quad (3, 4)$$

where m_B is the real part of m_B^* , j is the imaginary unit, η_B is the loss factor, h_B is the height of the neck, ρ_B^* is the complex density of the air with the acoustic absorption material, and ρ_B is the real part of ρ_B^* . The wave equation

Fig. 1 Analytical models of host acoustic fields with a Helmholtz silencer. Figure 1(a) shows the analytical model whose host acoustic field is a one-dimensional tube. Figure 1(b) shows the analytical model whose host acoustic field is a three-dimensional cuboidal room.

Fig. 2 Analytical models divided into three substructures at the connecting faces at the upper and lower ends of the neck. Figure 2(a) shows the divided version of the analytical model shown in Fig. 1(a). Figure 2(b) shows the divided version of the analytical model shown in Fig. 1(b).

of the cavity of the Helmholtz silencer is expressed as

$$\rho_c \frac{\partial^2 \psi_c}{\partial t^2} = \kappa_c \nabla^2 \psi_c + \kappa_c w_B(t) H_{s_{bc}} \delta(z_c), \quad (5)$$

where ρ_c is the density, κ_c is the bulk modulus, ψ_c is the displacement potential of the cavity, and $H_{s_{bc}}$ is the function indicating the existence region of the neck of the Helmholtz silencer. The Laplacian ∇^2 and $H_{s_{bc}}$ are expressed as follows:

$$\nabla^2 = \begin{cases} \frac{\partial^2}{\partial x_c^2} + \frac{\partial^2}{\partial y_c^2} + \frac{\partial^2}{\partial z_c^2} & \text{(Case of cuboidal neck and cavity)} \\ \frac{\partial^2}{\partial r_c^2} + \frac{1}{r_c} \frac{\partial}{\partial r_c} + \frac{\partial^2}{\partial z_c^2} & \text{(Case of cylindrical neck and cavity)} \end{cases}, \quad (6)$$

$$H_{s_{bc}} = \begin{cases} (H(x_c - x_{cl}) - H(x_c - x_{cr}))(H(y_c - y_{cf}) - H(y_c - y_{cb})) & \text{(Case of cuboidal neck and cavity)} \\ H(R_B - r_c) & \text{(Case of cylindrical neck and cavity)} \end{cases}, \quad (7)$$

where x_c and y_c are the spatial coordinates of the cavity of the Helmholtz silencer when the cavity is cuboidal; r_c is the spatial coordinate for the cavity in the radial direction when the cavity is cylindrical; x_{cl} and x_{cr} are the x_c coordinates of the left and right ends of the neck in the cavity, respectively; y_{cf} and y_{cb} are the y_c coordinates of the forward and back ends of the neck in the cavity, respectively; and R_B is the radius of the neck when the shape of the neck is cylindrical. The left front of the cavity is set to the origin of the (x_c, y_c) coordinate system, and the positive directions of these coordinates are shown in Fig. 1(a), where the center of the cavity is set to the origin of the r_c coordinate system in the case of the cylindrical cavity, and the outward direction is the positive r_c direction. In the case in which the neck and cavity are cylindrical, the centers of the neck and cavity are aligned in this study. The

sound pressure values in the host acoustic field and cavity are respectively expressed as

$$p_A(x_A, t) = \rho_A \frac{\partial^2 \psi_A}{\partial t^2}, \quad p_C = \rho_C \frac{\partial^2 \psi_C}{\partial t^2}. \quad (8, 9)$$

The displacement potentials ψ_A and ψ_C are given by the superposition of the acoustic modes as follows:

$$\psi_A(x_A, t) = \sum_{j_A=1}^{\infty} \Psi_{A j_A}(x_A) \zeta_{A j_A}(t), \quad (10)$$

$$\begin{cases} \psi_C(x_C, y_C, z_C, t) = \sum_{j_{Cx}=0}^{\infty} \sum_{j_{Cy}=0}^{\infty} \sum_{j_{Cz}=0}^{\infty} \Psi_{C j_{Cx}}(x_C) \Psi_{C j_{Cy}}(y_C) \Psi_{C j_{Cz}}(z_C) \zeta_{C j_{Cx} j_{Cy} j_{Cz}}(t) & \text{(Case of cuboidal neck and cavity)} \\ \psi_C(r_C, z_C, t) = \sum_{j_{Cr}=0}^{\infty} \sum_{j_{Cz}=0}^{\infty} \Psi_{C j_{Cr}}(r_C) \Psi_{C j_{Cz}}(z_C) \zeta_{C j_{Cr} j_{Cz}}(t) & \text{(Case of cylindrical neck and cavity)} \end{cases}, \quad (11)$$

where $\Psi_{A j_A}$ is the eigenfunction of the displacement potential of the host acoustic tube, and $\Psi_{C j_{Cx}}$, $\Psi_{C j_{Cy}}$, $\Psi_{C j_{Cz}}$, and $\Psi_{C j_{Cr}}$ are the eigenfunctions of the displacement potential of the cavity in the x_C , y_C , z_C , and r_C directions, respectively; and, $\zeta_{A j_A}$, $\zeta_{C j_{Cx} j_{Cy} j_{Cz}}$ and $\zeta_{C j_{Cr} j_{Cz}}$ are the modal displacements of the host acoustic tube and cavity of the two cases, respectively. The subscripts, j_A , j_{Cx} , j_{Cy} , j_{Cz} , and j_{Cr} denote the orders of the acoustic mode.

Substituting Eq. (10) into the wave equation (1), multiplying both sides by $\Psi_{A i_A} / \rho_A$, and integrating over the entire range of the host acoustic tube yields the following equation of motion using modal coordinates:

$$M_{A i_A} \ddot{\zeta}_{A i_A} + K_{A i_A} \zeta_{A i_A} + \frac{c_A^2 E_{A i_A}}{S_A} w_B(t) = c_A^2 J_{A i_A} w_P(t), \quad (12)$$

$$M_{A i_A} = \int_0^{l_A} (\Psi_{A i_A}(x_A))^2 dx_A, \quad K_{A i_A} = -c_A^2 \int_0^{l_A} \frac{d^2 \Psi_{A i_A}}{dx_A^2} \Psi_{A i_A}(x_A) dx_A = \omega_{A i_A}^2 M_{A i_A}, \quad (13, 14)$$

$$E_{A i_A} = \int_{x_{AL}}^{x_{AR}} \Psi_{A i_A}(x_A) d_B(x_A) dx_A, \quad J_{A i_A} = \Psi_{A i_A}(0), \quad c_A = \sqrt{\kappa_A / \rho_A}, \quad \omega_{A i_A} = c_A k_{A i_A}, \quad (15-18)$$

where $M_{A i_A}$ is the modal mass, $K_{A i_A}$ is the modal stiffness, $E_{A i_A}$ is the influence coefficient of the neck on the host acoustic tube, $J_{A i_A}$ is the influence coefficient of the piston, c_A is the speed of sound, $\omega_{A i_A}$ is the natural angular frequency, $k_{A i_A}$ is the wave number, and l_A is the length of the host acoustic tube.

Using Eqs. (2) and (8)–(11), the equation of motion of the mass of the neck is transformed into the following equation:

$$\begin{cases} m_B^* \ddot{w}_B - \rho_A \sum_{j_A=1}^{\infty} E_{A j_A} \ddot{\zeta}_{A j_A} + \rho_C \sum_{j_{Cx}=0}^{\infty} \sum_{j_{Cy}=0}^{\infty} \sum_{j_{Cz}=0}^{\infty} E_{C j_{Cx} j_{Cy} j_{Cz}} \ddot{\zeta}_{C j_{Cx} j_{Cy} j_{Cz}} = 0 & \text{(Case of cuboidal neck and cavity)} \\ m_B^* \ddot{w}_B - \rho_A \sum_{j_A=1}^{\infty} E_{A j_A} \ddot{\zeta}_{A j_A} + \rho_C \sum_{j_{Cr}=0}^{\infty} \sum_{j_{Cz}=0}^{\infty} E_{C j_{Cr} j_{Cz}} \ddot{\zeta}_{C j_{Cr} j_{Cz}} = 0 & \text{(Case of cylindrical neck and cavity)} \end{cases}, \quad (19)$$

$$E_{C j_{Cx} j_{Cy} j_{Cz}} = \int_{x_{CL}}^{x_{CR}} \Psi_{C j_{Cx}}(x_C) dx_C \int_{y_{CL}}^{y_{CR}} \Psi_{C j_{Cy}}(y_C) dy_C \Psi_{C j_{Cz}}(0), \quad E_{C j_{Cr} j_{Cz}} = \int_0^{R_B} 2\pi r_C \Psi_{C j_{Cr}}(r_C) dr_C \Psi_{C j_{Cz}}(0), \quad (20, 21)$$

where $E_{C j_{Cx} j_{Cy} j_{Cz}}$ and $E_{C j_{Cr} j_{Cz}}$ are the influence coefficients of the neck on the cavity in the two cases, respectively.

When the neck and cavity are cuboidal, substituting Eq. (11) into the wave equation (5), multiplying both sides by $\Psi_{C i_{Cx}} \Psi_{C i_{Cy}} \Psi_{C i_{Cz}} / \rho_C$, and integrating over the entire range of the cavity yields the following equation of motion using modal coordinates:

$$M_{C i_{Cx} i_{Cy} i_{Cz}} \ddot{\zeta}_{C i_{Cx} i_{Cy} i_{Cz}} + K_{C i_{Cx} i_{Cy} i_{Cz}} \zeta_{C i_{Cx} i_{Cy} i_{Cz}} - c_C^2 E_{C i_{Cx} i_{Cy} i_{Cz}} w_B(t) = 0, \quad c_C = \sqrt{\kappa_C / \rho_C}, \quad (22, 23)$$

$$M_{C i_{Cx} i_{Cy} i_{Cz}} = \int_0^{l_C} (\Psi_{C i_{Cx}}(x_C))^2 dx_C \int_0^{d_C} (\Psi_{C i_{Cy}}(y_C))^2 dy_C \int_0^{h_C} (\Psi_{C i_{Cz}}(z_C))^2 dz_C, \quad K_{C i_{Cx} i_{Cy} i_{Cz}} = \omega_{C i_{Cx} i_{Cy} i_{Cz}}^2 M_{C i_{Cx} i_{Cy} i_{Cz}}, \quad (24, 25)$$

where $M_{C i_{Cx} i_{Cy} i_{Cz}}$ is the modal mass, $K_{C i_{Cx} i_{Cy} i_{Cz}}$ is the modal stiffness, c_C is the speed of sound, $\omega_{C i_{Cx} i_{Cy} i_{Cz}}$ is the natural angular frequency of the cavity, and l_C , d_C , and h_C are the lengths of the cavity in the x_C , y_C , and z_C directions, respectively. When the neck and cavity are cylindrical, substituting Eq. (11) into the wave equation (5), multiplying both sides by $r_C \Psi_{C i_{Cr}} \Psi_{C i_{Cz}} / \rho_C$, and integrating over the entire range of the cavity yields the following

equation of motion using modal coordinates:

$$M_{C_{i_C} i_{Cz}} \ddot{\xi}_{C_{i_C} i_{Cz}} + K_{C_{i_C} i_{Cz}} \xi_{C_{i_C} i_{Cz}} - c_C^2 E_{C_{i_C} i_{Cz}} w_B(t) = 0, \quad (26)$$

$$M_{C_{i_C} i_{Cz}} = \int_0^{R_C} r_C \left(\Psi_{C_{i_C}}(x_C) \right)^2 dr_C \int_0^{h_C} \left(\Psi_{C_{i_Cz}}(z_C) \right)^2 dz_C, \quad K_{C_{i_C} i_{Cz}} = \omega_{C_{i_C} i_{Cz}}^2 M_{C_{i_C} i_{Cz}}, \quad (27, 28)$$

where $M_{C_{i_C} i_{Cz}}$ is the modal mass, $K_{C_{i_C} i_{Cz}}$ is the modal stiffness, $\omega_{C_{i_C} i_{Cz}}$ is the natural angular frequency of the cavity, and R_C is the radius of the cylindrical cavity.

Similar to the case where the host acoustic field is a one-dimensional acoustic tube, the wave equation of the three-dimensional cuboidal room and the equation of motion of the mass of the neck of the Helmholtz silencer are respectively given as

$$\rho_A \frac{\partial^2 \psi_A}{\partial t^2} = \kappa_A \left(\frac{\partial^2 \psi_A}{\partial x_A^2} + \frac{\partial^2 \psi_A}{\partial y_A^2} + \frac{\partial^2 \psi_A}{\partial z_A^2} \right) + \kappa_A w_p(t) H_p(x_A, y_A) \delta(z_A - h_A) - \kappa_A w_B(t) H_{S_{BA}} \delta(z_A - h_A), \quad (29)$$

$$m_B^* \ddot{w}_B = \int_{S_B} p_A|_{z_A=h_A} dS_B - \int_{S_B} p_C|_{z_C=0} dS_B, \quad (30)$$

where x_A , y_A , and z_A are the spatial coordinates of the cuboidal room, as shown in Fig. 1(b); H_p and $H_{S_{BA}}$ are the functions indicating the existence regions of the piston and neck of the Helmholtz silencer, respectively; and h_A is the height of the cuboidal room. The front lower left corner of the cuboidal room is set to the origin of the (x_A, y_A, z_A) coordinates, and the positive directions of the coordinates are shown in Fig. 1(b). The symbols common to the one-dimensional acoustic tube are not explained here. As shown in Fig. 1(b), when the shape of the piston is rectangular, H_p is expressed as

$$H_p(x_A, y_A) = H(l_p - x_A) H(d_p - y_A), \quad (31)$$

where l_p and d_p are the lengths of the piston in the x_A and y_A directions, respectively. Similarly, when the shape of the neck of the Helmholtz silencer is rectangular, $H_{S_{BA}}$ is given as

$$H_{S_{BA}}(x_A, y_A) = (H(x_A - x_{AL}) - H(x_A - x_{AR}))(H(y_A - y_{AF}) - H(y_A - y_{AB})), \quad (32)$$

where x_{AL} and x_{AR} are the x_A coordinates of the left and right sides of the rectangular neck of the Helmholtz silencer, respectively, and y_{AF} and y_{AB} are the y_A coordinates of the forward and back sides of the rectangular neck of the Helmholtz silencer, respectively. When the shape of the neck of the Helmholtz silencer is circular, $H_{S_{BA}}$ is expressed as

$$H_{S_{BA}}(x_A, y_A) = H(R_B^2 - (x_A - x_{AC})^2 - (y_A - y_{AC})^2), \quad (33)$$

where x_{AC} and y_{AC} are the x_A and y_A coordinates of the center of the circular neck of the Helmholtz silencer, respectively. The wave equation of the cavity of the Helmholtz silencer is given by Eq. (5). The displacement potential ψ_A is expressed by the superposition of the acoustic modes as

$$\psi_A(x_A, y_A, z_A, t) = \sum_{j_{Ax}=0}^{\infty} \sum_{j_{Ay}=0}^{\infty} \sum_{j_{Az}=0}^{\infty} \Psi_{A j_{Ax}}(x_A) \Psi_{A j_{Ay}}(y_A) \Psi_{A j_{Az}}(z_A) \xi_{A j_{Ax} j_{Ay} j_{Az}}(t), \quad (34)$$

where $\Psi_{A j_{Ax}}$, $\Psi_{A j_{Ay}}$, and $\Psi_{A j_{Az}}$ are the eigenfunctions of the displacement potential of the three-dimensional cuboidal room in the x_A , y_A , and z_A directions, respectively, and $\xi_{A j_{Ax} j_{Ay} j_{Az}}$ is the modal displacement of the three-dimensional cuboidal room. The subscripts j_{Ax} , j_{Ay} , and j_{Az} denote the orders of the acoustic mode.

Substituting Eq. (34) into the wave equation (29), multiplying both sides by $\Psi_{A i_{Ax}} \Psi_{A i_{Ay}} \Psi_{A i_{Az}} / \rho_A$, and integrating over the entire range of the host acoustic field yields the following equation of motion using modal coordinates:

$$M_{A i_{Ax} i_{Ay} i_{Az}} \ddot{\xi}_{A i_{Ax} i_{Ay} i_{Az}} + K_{A i_{Ax} i_{Ay} i_{Az}} \xi_{A i_{Ax} i_{Ay} i_{Az}} + c_A^2 E_{A i_{Ax} i_{Ay} i_{Az}} w_B(t) = c_A^2 J_{A i_{Ax} i_{Ay} i_{Az}} w_p(t), \quad (35)$$

$$M_{A i_{Ax} i_{Ay} i_{Az}} = \int_0^{l_A} \left(\Psi_{A i_{Ax}}(x_A) \right)^2 dx_A \int_0^{d_A} \left(\Psi_{A i_{Ay}}(y_A) \right)^2 dy_A \int_0^{h_A} \left(\Psi_{A i_{Az}}(z_A) \right)^2 dz_A, \quad K_{A i_{Ax} i_{Ay} i_{Az}} = \omega_{A i_{Ax} i_{Ay} i_{Az}}^2 M_{A i_{Ax} i_{Ay} i_{Az}}, \quad (36, 37)$$

$$E_{A i_{Ax} i_{Ay} i_{Az}} = \begin{cases} \int_{x_{AL}}^{x_{AR}} \Psi_{A i_{Ax}}(x_A) dx_A \int_{y_{AF}}^{y_{AB}} \Psi_{A i_{Ay}}(y_A) dy_A \Psi_{A i_{Az}}(h_A) & \text{(Case of cuboidal neck)} \\ \int_{S_B} \Psi_{A i_{Ax}}(x_A) \Psi_{A i_{Ay}}(y_A) dS_B \Psi_{A i_{Az}}(h_A) & \text{(Case of cylindrical neck)} \end{cases}, \quad (38)$$

$$J_{A i_{Ax} i_{Ay} i_{Az}} = \int_0^{l_p} \Psi_{A j_{Ax}}(x_A) dx_A \int_0^{d_p} \Psi_{A j_{Ay}}(y_A) dy_A \Psi_{A j_{Az}}(h_A), \quad (39)$$

where $M_{A i_{Ax} i_{Ay} i_{Az}}$ is the modal mass, $K_{A i_{Ax} i_{Ay} i_{Az}}$ is the modal stiffness, $E_{A i_{Ax} i_{Ay} i_{Az}}$ is the influence coefficient of the neck on the cuboidal room, $J_{A i_{Ax} i_{Ay} i_{Az}}$ is the influence coefficient of the piston on the cuboidal room, $\omega_{A i_{Ax} i_{Ay} i_{Az}}$ is the natural angular frequency of the cuboidal room, and l_A and d_A are the lengths of the cuboidal room in the x_A and y_A directions, respectively.

Using Eqs. (8), (9), (11), (30), and (34), the equation of motion of the neck mass is transformed into the following equation:

$$\begin{cases} m_B^* \ddot{w}_B - \rho_A \sum_{j_{Ax}=0}^{\infty} \sum_{j_{Ay}=0}^{\infty} \sum_{j_{Az}=0}^{\infty} E_{A j_{Ax} j_{Ay} j_{Az}} \ddot{\xi}_{A j_{Ax} j_{Ay} j_{Az}} + \rho_C \sum_{j_{Cx}=0}^{\infty} \sum_{j_{Cy}=0}^{\infty} \sum_{j_{Cz}=0}^{\infty} E_{C j_{Cx} j_{Cy} j_{Cz}} \ddot{\xi}_{C j_{Cx} j_{Cy} j_{Cz}} = 0 & \text{(Case of cuboidal cavity)} \\ m_B^* \ddot{w}_B - \rho_A \sum_{j_{Ax}=0}^{\infty} \sum_{j_{Ay}=0}^{\infty} \sum_{j_{Az}=0}^{\infty} E_{A j_{Ax} j_{Ay} j_{Az}} \ddot{\xi}_{A j_{Ax} j_{Ay} j_{Az}} + \rho_C \sum_{j_{Cx}=0}^{\infty} \sum_{j_{Cz}=0}^{\infty} E_{C j_{Cx} j_{Cz}} \ddot{\xi}_{C j_{Cx} j_{Cz}} = 0 & \text{(Case of cylindrical cavity)} \end{cases}. \quad (40)$$

In the cases of the cuboidal neck and cavity and cylindrical neck and cavity, the equations of motion of the cavity using modal coordinates are given by Eqs. (22) and (26), respectively. From Eqs. (12), (19), (22), (26), (35), and (40), there is no essential difference between the two analytical models shown in Fig. 1(a) and (b).

2.2 Equivalent discrete model

Because there is no essential difference between the equations of motion using the two analytical models shown in Fig. 1(a) and (b), and the equations of the one-dimensional acoustic tube are simpler, the equivalent discrete model is described using the analytical model of the one-dimensional acoustic tube.

The force applied to the neck mass in the upper direction by the host acoustic field is expressed as

$$f_U = \int_{S_B} p_A(x_A, t) dS_B = \rho_A \sum_{j_A=1}^{\infty} E_{A j_A} \ddot{\xi}_{A j_A} = \rho_A \sum_{j_A=1}^{\infty} E_{A j_A} \left(-\omega^2 \right) \frac{c_A^2 J_{A j_A} w_p(t) - c_A^2 E_{A j_A} w_B(t) / S_A}{-\omega^2 M_{A j_A} + K_{A j_A}}, \quad (41)$$

where ω is the excitation angular frequency, and only harmonic vibration is considered. In this study, only harmonic vibration is considered, and free vibration is ignored. Similarly, the force applied to the neck mass in the downward direction by the cavity is given as

$$f_D = \int_{S_B} p_C|_{z_C=0} dS_B = \begin{cases} \rho_C \sum_{j_{Cx}=0}^{\infty} \sum_{j_{Cy}=0}^{\infty} \sum_{j_{Cz}=0}^{\infty} E_{C j_{Cx} j_{Cy} j_{Cz}} \ddot{\xi}_{C j_{Cx} j_{Cy} j_{Cz}} & \text{(Case of cuboidal neck and cavity)} \\ \rho_C \sum_{j_{Cx}=0}^{\infty} \sum_{j_{Cz}=0}^{\infty} E_{C j_{Cx} j_{Cz}} \ddot{\xi}_{C j_{Cx} j_{Cz}} & \text{(Case of cylindrical neck and cavity)} \end{cases}, \quad (42)$$

where $\ddot{\xi}_{C j_{Cx} j_{Cy} j_{Cz}}$ and $\ddot{\xi}_{C j_{Cx} j_{Cz}}$ are respectively expressed as

$$\ddot{\xi}_{C j_{Cx} j_{Cy} j_{Cz}} = \left(-\omega^2 \right) \frac{c_C^2 E_{C j_{Cx} j_{Cy} j_{Cz}} w_B(t)}{-\omega^2 M_{C j_{Cx} j_{Cy} j_{Cz}} + K_{C j_{Cx} j_{Cy} j_{Cz}}}, \quad \ddot{\xi}_{C j_{Cx} j_{Cz}} = \left(-\omega^2 \right) \frac{c_C^2 E_{C j_{Cx} j_{Cz}} w_B(t)}{-\omega^2 M_{C j_{Cx} j_{Cz}} + K_{C j_{Cx} j_{Cz}}}. \quad (43, 44)$$

Because there is no essential difference between the two cases in Eq. (42), the equivalent discrete model is described using the case of a cuboidal neck and cavity. From Eqs. (41) and (42), the equivalent discrete model can be drawn, as shown in Fig. 3(a). Here, $w_{h j_A}$ and $w_{e j_{Cx} j_{Cy} j_{Cz}}$ are the equivalent displacements of the j_A -th acoustic mode of the host acoustic field and the (j_{Cx}, j_{Cy}, j_{Cz}) acoustic mode of the cavity, respectively, and their positive direction is the upper direction. The equivalent mass and equivalent spring constant of the j_A -th acoustic mode, and the force applied to equivalent mass are respectively given as

$$m_{h_{j_A}} = \frac{\kappa_A E_{A_{j_A}}^2}{S_A K_{A_{j_A}}}, \quad k_{h_{j_A}} = \frac{\kappa_A E_{A_{j_A}}^2}{S_A M_{A_{j_A}}}, \quad f_{h_{j_A}} = \frac{\kappa_A E_{A_{j_A}} J_{A_{j_A}} \ddot{w}_p(t)}{K_{A_{j_A}}}. \quad (45-47)$$

The equivalent mass and equivalent spring constant of the (j_{Cx}, j_{Cy}, j_{Cz}) acoustic mode of the cavity are respectively given as

$$m_{c_{j_{Cx} j_{Cy} j_{Cz}}} = \frac{\kappa_C E_{C_{j_{Cx} j_{Cy} j_{Cz}}}^2}{K_{C_{j_{Cx} j_{Cy} j_{Cz}}}}, \quad k_{c_{j_{Cx} j_{Cy} j_{Cz}}} = \frac{\kappa_C E_{C_{j_{Cx} j_{Cy} j_{Cz}}}^2}{M_{C_{j_{Cx} j_{Cy} j_{Cz}}}}. \quad (48, 49)$$

Because the modal stiffness of the $(0, 0, 0)$ acoustic mode of the cavity is zero, the equivalent mass of the $(0, 0, 0)$ acoustic mode is infinite. Therefore, a rigid wall is drawn in Fig. 3(a) instead of the equivalent mass of the $(0, 0, 0)$ acoustic mode. The natural frequency of the Helmholtz silencer is determined by the mass of the neck and the equivalent spring constant of the $(0, 0, 0)$ acoustic mode of the cavity. If only the targeted acoustic mode of the host acoustic field and the $(0, 0, 0)$ acoustic mode of the cavity are considered and the other acoustic modes are ignored, the equivalent discrete model is given as a 2-DOF vibration system. Using the 2-DOF equivalent discrete model and two fixed point method, the optimum frequency ratio and optimum loss factor of the Helmholtz silencer can be derived; however, the effect of the non-target acoustic modes is not sufficiently small. Using the equivalent discrete model shown in Fig. 3(a), the effects of the non-target acoustic modes of the host acoustic field and cavity of the Helmholtz silencer are categorized into two types depending on their natural frequencies. When the natural frequencies of non-target acoustic modes are sufficiently lower than the natural frequency of the targeted acoustic mode of the host acoustic field, only their springs function because their mass points hardly vibrate. In contrast, when the natural frequencies of non-target acoustic modes are sufficiently higher than the natural frequency of the targeted acoustic mode, only their masses function because their springs hardly expand or contract and their masses vibrate together with the mass of the neck. Based on these lines of thought, the equivalent discrete model shown in Fig. 3(a) can be simplified as shown in Fig. 3(b). Here, the i_A -th acoustic mode of the host acoustic field is the targeted acoustic mode. Note that the forces applied to the equivalent masses of the residual acoustic modes of the host acoustic field are ignored in the equivalent discrete model shown in Fig. 3(b). Because the forces applied to the equivalent masses of the residual acoustic modes of the host acoustic field hardly affect the optimum natural frequency ratio, these forces are ignored for simplicity. The discrete model shown in Fig. 3(b) is given as a 2-DOF vibration system. Therefore, the optimum natural frequency ratio and loss factor can be derived using the two fixed point method. The lengths of the open-end correction at both ends of the neck are not required to derive the optimum natural frequency ratio and optimum loss factor using the two fixed point method; however, the lengths are formulated here for reference. The residual higher-order acoustic modes of the host acoustic field function as additional masses to the neck mass. Similarly, the residual higher-order acoustic modes of the cavity function as additional masses to the neck mass. These additional masses substantially increase the length of the neck; however, the residual lower-order acoustic modes of the host acoustic field substantially decrease the length of the neck because the residual lower-order acoustic modes function as additional springs to the neck mass. From the total mass of the additional masses that corresponds to the residual acoustic modes of the cavity, the length of the open-end correction of the neck at the end of the cavity side is

Fig. 3 Equivalent discrete models of the host acoustic field and Helmholtz silencer. Figure 3(a) shows the equivalent discrete model of the whole vibration system. Figure 3(b) shows a 2-DOF equivalent discrete model considering the effect of residual acoustic modes of the host acoustic field and cavity as springs and masses. Figure 3(c) shows a simplified 2-DOF equivalent discrete model.

given as

$$\delta_c = \frac{m_{\delta_c}}{\rho_B S_B}, \quad m_{\delta_c} = \sum_{j_{Cz}=1}^{\infty} m_{c00j_{Cz}} + \sum_{j_{Cy}=1}^{\infty} \sum_{j_{Cz}=0}^{\infty} m_{c0j_{Cy}j_{Cz}} + \sum_{j_{Cx}=1}^{\infty} \sum_{j_{Cy}=0}^{\infty} \sum_{j_{Cz}=0}^{\infty} m_{cj_{Cx}j_{Cy}j_{Cz}}, \quad (50, 51)$$

where m_{δ_c} is the total mass of the additional masses corresponding to the residual acoustic modes of the cavity. In contrast, the length of the open-end correction of the neck at the end of the host acoustic field side is complicated because the effect of the additional spring must be considered. When the i_A -th acoustic mode of the host acoustic field is the targeted acoustic mode, the total spring constants of the additional springs and total mass of the additional masses that correspond to the lower-order and higher-order residual acoustic modes of the host acoustic field are respectively given as

$$k_{\delta_h} = \sum_{j_A=1}^{i_A-1} k_{hj_A}, \quad m_{\delta_h} = \sum_{j_A=i_A+1}^{\infty} m_{hj_A}. \quad (52, 53)$$

Because springs can be regarded as negative masses with frequency dependence, the negative mass at the natural frequency of the targeted acoustic mode of the host acoustic field is used to derive the length of the open-end correction in this study. Using this concept, the length of the open-end correction of the neck at the end of the host acoustic field side is expressed as

$$\delta_h = \frac{m_{\delta_h} - k_{\delta_h} / \omega_{Ai_A}^2}{\rho_B S_B}. \quad (54)$$

The substantial length of the neck is larger than the height of the neck h_B ; however, the volumes of the host acoustic field and cavity do not have to be reduced depending on the length of the open-end correction at both ends of the neck. Partial overlap between the substantial neck and host acoustic field or cavity is acceptable because the increase in the length of the neck is determined by the residual acoustic modes. From Eqs. (38), (45), (53), and (54), the residual acoustic modes where particles vibrate in the height direction have a greater effect on the additional mass than the other residual acoustic modes. In addition, the additional mass of the higher-order acoustic modes is smaller.

Some of the natural frequencies of the residual acoustic modes of the host acoustic field are not sufficiently lower or higher than the natural frequency of the targeted acoustic mode of the host acoustic field, especially in the case of the three-dimensional acoustic field. In such cases, the residual acoustic modes cannot be regarded as simple springs or masses. In contrast, the natural frequencies of the residual acoustic modes of the cavity of the Helmholtz silencer are sufficiently higher than the natural frequency of the targeted acoustic mode of the host acoustic field. Therefore, the residual acoustic modes of the cavity can be simply regarded as masses in the usual case. In this study, the effects of the residual acoustic modes that cannot be regarded as springs or masses are considered using dynamic spring constants or dynamic masses at the natural frequency of the targeted acoustic mode. When the i_A -th acoustic mode of the host acoustic field is targeted, the dynamic spring constant and dynamic mass of the j_A -th acoustic mode at the natural frequency of the i_A -th acoustic mode are respectively expressed as

$$k_{hdj_A} = \frac{-\omega_{Ai_A}^2 m_{hj_A} k_{hj_A}}{-\omega_{Ai_A}^2 m_{hj_A} + k_{hj_A}} = \frac{k_{hj_A}}{1 - (\omega_{Aj_A} / \omega_{Ai_A})^2}, \quad m_{hdj_A} = \frac{m_{hj_A} k_{hj_A}}{-\omega_{Ai_A}^2 m_{hj_A} + k_{hj_A}} = \frac{m_{hj_A}}{1 - (\omega_{Aj_A} / \omega_{Ai_A})^2}. \quad (55, 56)$$

The additional springs and masses shown in Fig. 3(b) should be replaced with the springs and masses with spring constants and masses given by Eqs. (55) and (56), respectively. Usually, the residual acoustic modes of the cavity can be simply regarded as masses; however, when the residual acoustic modes cannot be regarded as simple masses, dynamic masses should be considered in a similar fashion to Eq. (56).

2.3 Optimum tuning using two fixed point method

The equivalent discrete model shown in Fig. 3(b) is simplified as shown in Fig. 3(c). Here, m_h , k_h , f_h , m_s^* , k_s , w_h , and w_s are respectively given as

$$m_h = m_{hi_A}, \quad k_h = k_{hi_A}, \quad f_h = f_{hi_A}, \quad m_s^* = m_s (1 - j\eta_s), \quad k_s = k_{c000} + k_{\delta_h}, \quad (57-61)$$

$$w_h = w_{hi_A}, \quad w_s = w_B, \quad m_s = m_B + m_{\delta_h} + m_{\delta_c}, \quad \eta_s = \eta_B m_B / m_s, \quad (62-65)$$

where the additional spring constant and additional masses must be either simple or dynamic. Using the simplified equivalent discrete model and two fixed point method, the optimum natural frequency ratio and loss factor of the Helmholtz silencer are derived in this section (Ormondroyd and Den Hartog, 1928).

Following Section 2.2, the analytical model of the one-dimensional acoustic field is used as a representative in this section. The nondimensional sound pressure in the host acoustic field is expressed as

$$\frac{p_A(x_A, t)}{\rho_A c_A \dot{w}_p} = \frac{\rho_A \Psi_{Ai_A}(x_A) \ddot{\zeta}_{Ai_A}(t)}{\rho_A c_A \dot{w}_p} = \frac{j g c_A \Psi_{Ai_A}(0) \Psi_{Ai_A}(x_A) k_h (w_h - w_s)}{\sqrt{M_{Ai_A} K_{Ai_A}} f_h}, \quad g = \frac{\omega}{\omega_{Ai_A}}, \quad (66, 67)$$

where g is the excitation frequency ratio, and only the sound pressure generated by the targeted acoustic mode is considered. From Eq. (66), the following transfer function is used as the evaluation index to suppress the sound pressure in the host acoustic field:

$$G_p = \frac{j g k_h (W_h - W_s)}{F_h}, \quad (68)$$

where W_h and W_s are the complex amplitudes of the displacements w_h and w_s , respectively, and F_h is the amplitude of the excitation force f_h . G_p is nondimensional and is obtained by removing several constant coefficients from Eqs. (66).

When the sound radiated from the apertures of the host acoustic field should be reduced, the particle velocity should be reduced. In this case, the nondimensional particle velocity should be considered. The nondimensional particle velocity is given as

$$\frac{\dot{w}_A(x_A, t)}{\dot{w}_p} = \frac{-\frac{d\Psi_{Ai_A}(x_A)}{dx_A} \zeta_{Ai_A}(t)}{w_p} = \frac{-c_A^2 \Psi_{Ai_A}(0) \frac{d\Psi_{Ai_A}(x_A)}{dx_A} k_h (w_h - w_s)}{K_{Ai_A} f_h}, \quad (69)$$

where only the particle velocity of the targeted acoustic mode of the host acoustic field was considered. From Eq. (69), the following transfer function is used as the evaluation index to reduce the particle velocity in the host acoustic field:

$$G_v = \frac{k_h (W_h - W_s)}{F_h}. \quad (70)$$

G_v is nondimensional and is obtained by removing several constant coefficients from Eq. (69).

Using the simplified equivalent discrete model, the transfer functions G_p and G_v are respectively derived as

$$G_p = \frac{j g k_h (W_h - W_s)}{F_h} = j g \frac{f^2 - g^2 + j \eta_s g^2}{g^4 - g^2 (1 + f^2 + f \gamma) + f^2 + j \eta_s g^2 (1 - g^2)}, \quad (71)$$

$$G_v = \frac{k_h (W_h - W_s)}{F_h} = \frac{f^2 - g^2 + j \eta_s g^2}{g^4 - g^2 (1 + f^2 + f \gamma) + f^2 + j \eta_s g^2 (1 - g^2)}, \quad (72)$$

$$f = \frac{\omega_s}{\omega_h}, \quad \gamma = \sqrt{\frac{m_h k_h}{m_s k_s}}, \quad \omega_h = \sqrt{\frac{k_h}{m_h}} = \omega_{Ai_A}, \quad \omega_s = \sqrt{\frac{k_s}{m_s}}, \quad (73-76)$$

where f is the natural frequency ratio, γ is the characteristic impedance ratio, ω_h is the natural angular frequency of the targeted acoustic mode of the host acoustic field, and ω_s is the natural angular frequency of the Helmholtz silencer, considering the effect of the residual acoustic modes of the host acoustic field and cavity of the Helmholtz silencer. The simplified equivalent discrete model and transfer functions G_p and G_v are identical to those obtained in the study of the damped side branch silencer (Yamada et al., 2021). Therefore, the optimum natural frequency ratio and optimum loss factor can be derived in the same manner.

Using two fixed point method, we can obtain the optimum natural frequency ratio and loss factor of the vibration absorber that minimize the maximum amplitude in the frequency domain. The magnitudes of the transfer functions G_p

and G_v have two fixed points that are independent of the loss factor. The optimum natural frequency ratios can be derived such that the amplitudes at the two fixed points become equal. Subsequently, the optimum loss factors are derived such that the amplitude is approximately maximized at the two fixed points. The optimum natural frequency ratio f_{opt} , optimum loss factor η_{sopt} , excitation frequency ratios at the two fixed points A and B, and amplitude at the two fixed points are respectively derived as

$$f_{\text{opt}} = \begin{cases} 1 & (\text{Case of } G_p) \\ (-\gamma + \sqrt{16 + \gamma^2})/4 & (\text{Case of } G_v) \end{cases}, \quad \eta_{\text{sopt}} = \begin{cases} \sqrt{3\gamma(4 + \gamma)}/2 & (\text{Case of } G_p) \\ \sqrt{\gamma(3\sqrt{16 + \gamma^2} + 5\gamma)}/2 & (\text{Case of } G_v) \end{cases}, \quad (77, 78)$$

$$g_{A, B} = \begin{cases} \sqrt{4 + \gamma \mp \sqrt{\gamma(8 + \gamma)}}/2 & (\text{Case of } G_p) \\ \sqrt{1 \mp \sqrt{\gamma(\sqrt{16 + \gamma^2} - \gamma^2)}}/2 & (\text{Case of } G_v) \end{cases}, \quad |G_p|_{g=g_{A, B}} = \sqrt{\frac{2}{\gamma}}, \quad |G_v|_{g=g_{A, B}} = 2\sqrt{\frac{2}{\gamma\sqrt{16 + \gamma^2} - \gamma^2}}, \quad (79-81)$$

where the optimum loss factor η_{sopt} is defined by the root mean square of the values of the two loss factors derived under the condition where the amplitude becomes maximum at the two fixed points, and $g_{A, B}$ are the excitation frequency ratios at two fixed points A and B.

2.4 Size of Helmholtz silencer

Because the size of the Helmholtz silencer should be small for practical use, the size is discussed in this section based on the assumption that the optimum natural frequency ratio and loss factor are used. Because there is no essential difference between the two analytical models shown in Fig. 1(a) and (b) and between the shapes of the necks and cavities, the analytical model shown in Fig. 1(a) with the cuboidal neck and cavity is used as a representative in this section. In addition, the effect of the residual acoustic modes of the host acoustic field and cavity of the Helmholtz silencer is ignored for simplicity. The undamped natural angular frequency of the Helmholtz silencer and the characteristic impedance ratio are respectively expressed as

$$\omega_s = \sqrt{\frac{k_s}{m_s}} = \sqrt{\frac{k_{c000}}{m_B}} = \sqrt{\frac{S_B \kappa_C}{\rho_B h_B l_C d_C h_C}}, \quad \gamma = \sqrt{\frac{m_h k_h}{m_s k_s}} = \frac{m_h \omega_h}{m_s \omega_s} = \frac{\kappa_A E_{A i_A}^2 \omega_h}{S_A K_{A i_A}} \frac{1}{\rho_B S_B h_B \omega_s} = \frac{\kappa_A}{f S_A} \frac{E_{A i_A}^2}{K_{A i_A}} \frac{1}{\rho_B S_B h_B}. \quad (82, 83)$$

Here, the values of κ_A , S_A , and ω_h are predetermined by the material properties of the host acoustic field. Because the cavity of the Helmholtz silencer is usually filled with air, the value of κ_C is also predetermined. In addition, the values of f and ω_s are determined using the optimum natural frequency ratio. The value of ρ_B depends on the acoustic absorption material. When the material properties of the acoustic absorption material are determined, the value of ρ_B is determined. Under these conditions, S_B , h_B , l_C , d_C , and h_C are the parameters that can be tuned. From Eqs. (14) and (15), $E_{A i_A}^2/K_{A i_A}$ is approximately proportional to S_B^2/ω_h^2 . From Eqs. (80), (81), and (83), the performance of the Helmholtz silencer depends mostly on S_B/h_B . As the value of S_B/h_B increases, the maximum sound pressure in the frequency domain is reduced. However, the volume of the cavity $l_C d_C h_C$ must be increased as the value of S_B/h_B increases because the value of ω_s must be maintained. In other words, the performance of the Helmholtz silencer is determined by the volume of the cavity $l_C d_C h_C$. Although the neck is usually a small percentage of the total volume of the Helmholtz silencer, the volume of the neck can be reduced while maintaining the value of S_B/h_B . If the size of the neck is reduced, the particle velocity in the neck may increase, resulting in velocity-squared damping (Ingard and Ising, 1967); however, this nonlinear phenomenon is not considered in this study. Throughout the discussion in Section 2.4, the effect of the open-end correction due to the residual acoustic modes of the host acoustic field and cavity is ignored; however, if that effect is considered, the optimum volume of the cavity is slightly affected by the value of S_B .

3. Simulation

Two types of simulation results are presented in this section. First, to validate the optimum natural frequency ratio and optimum loss factor derived in Section 2.3, the simulation results using the 2-DOF equivalent discrete model

shown in Fig. 3(c) are presented briefly. Second, the simulation results using the multi-DOF model are presented to investigate the effectiveness of considering the residual acoustic modes of the host acoustic field and cavity of the Helmholtz silencer. Because the effect of the residual acoustic modes of the host acoustic field depends strongly on the density of the acoustic modes of the host acoustic field, the simulation results obtained using both analytical models shown in Fig. 1(a) and (b) are presented in the case of the simulations using the multi-DOF model. To investigate the effect of the residual acoustic modes of the host acoustic field that are ignored when the host acoustic field is regarded as a one-dimensional acoustic field, the simulation results using the three-dimensional acoustic field instead of the one-dimensional acoustic field are also shown. The simulations presented in this section are based on the theoretical analysis described in Section 2, and the validity of the theoretical analysis is presented in Section 4 through a comparison of the simulation and experimental results.

3.1 Simulation of 2-DOF model

The simulation results of the magnitude plots of the transfer functions G_p and G_v are shown in Fig. 4(a) and (b), respectively. Here, the optimum frequency ratio f_{opt} , seven values of the loss factor η_s , and the characteristic impedance ratio $\gamma = 0.02$ were used. The optimum natural frequency ratios and loss factors used in the simulations are listed in Table 1. In both cases, the magnitudes at the two fixed points are equal, and the magnitude almost reaches its maximum at the two fixed points when the optimum loss factor $\eta_{s\text{opt}}$ is used.

Fig. 4 Simulation results of the magnitude plots of the transfer functions G_p and G_v using the optimum frequency ratio f_{opt} , seven values of loss factor η_s , and the characteristic impedance ratio $\gamma = 0.02$.

Table 1 Optimum natural frequency ratios and loss factors used in the simulations in Section 3.1.

	f_{opt}	$\eta_{s\text{opt}}$
G_p	1	0.174
G_v	0.995	0.174

3.2 Simulation of infinite-DOF model

The material properties of the simulation using the analytical model shown in Fig. 1(a) are listed in Table 2. Here, x_{AO} is the x_{A} coordinate of the observation point, and the depth and height of the host acoustic tube are 0.05 and 0.1 m, respectively. The shapes of the neck and cavity are cuboidal in these simulations. The first acoustic mode of the host acoustic field was targeted for suppression in these simulations. The Helmholtz silencer was installed in the center of the host acoustic tube in the depth direction. To investigate the effect of the residual acoustic modes on the open-end correction, the theoretical values of the height h_c , loss factor η_B , and lengths of the open-end corrections δ_h and δ_c using different numbers of acoustic modes are shown in Fig. 5. Here, n_{max} is the highest modal order. The first through n_{max} th acoustic modes of the host acoustic field and $(0, 0, 0)$ through $(n_{\text{max}}, n_{\text{max}}, n_{\text{max}})$ acoustic modes of the cavity were used in the case of the one-dimensional acoustic field, and the $(0, 0, 0)$ through $(n_{\text{max}}, n_{\text{max}}, n_{\text{max}})$ acoustic modes of the host acoustic field and cavity were used in the case of the three-dimensional acoustic field. The optimum height h_c derived by the proposed method decreases with an increase in n_{max} . The difference in the open-end corrections on the host acoustic field side between the results using constant residual masses and constant dynamic masses is small. In contrast, the difference in the open-end corrections on the host acoustic field side between the results obtained using the one-dimensional and three-dimensional acoustic fields is significant. Therefore, the

residual acoustic modes of the host acoustic field in the depth and height directions should be considered to precisely estimate the length of the open-end correction. As described in Section 2.2, the residual masses of the acoustic modes, where the particles vibrate in the height direction, are relatively large. The lengths of the open-end corrections almost converged for $n_{\max} > 300$ in the case of the three-dimensional acoustic field. Therefore, $n_{\max}$ should be greater than 300. In the following simulations, the three-dimensional acoustic field was used as the host acoustic field, and the (0, 0, 0) through (500, 500, 500) acoustic modes of the host acoustic field and cavity were used. However, the computation time for coupled vibration analysis became too long as the number of degrees of freedom increased. First, the residual stiffness and residual mass were proposed to consider the effects of higher-order vibration modes without increasing the number of degrees of freedom (Nagamatsu, 1985). Therefore, simulations using this concept were conducted in this study. Prior to the simulation using the (0, 0, 0) through (500, 500, 500) acoustic modes, to investigate the effectiveness of lowering the degrees of freedom, simulations were conducted using the (0, 0, 0) through (20, 20, 20) acoustic modes of the host acoustic tube and cavity, where the (0, 0, 0) through (n, n, n) acoustic modes of the host acoustic tube and cavity had the same number of degrees of freedom as vibration systems, and the remaining acoustic modes had no degrees of freedom and were added to the neck as additional mass. The simulation results for $n = 1, 2, 5, 10$, and 20 are presented in Fig. 6, where $P_A(x_A, y_A, z_A)$ is the complex amplitude of the sound pressure $p_A(x_A, y_A, z_A, t)$ and W_p is the real amplitude of the displacement of the piston $w_p(t)$. In these simulations, the additional mass of the residual acoustic modes of the host acoustic field and cavity was derived using a constant dynamic mass. In the case of $n = 20$, all acoustic modes have degrees of freedom. From these simulation results, the effect of lowering the degrees of freedom by using the residual mass on the frequency response function was found to be small. To investigate the effect of the higher-order residual acoustic modes as the residual masses, simulations using different numbers of higher-order residual acoustic modes as residual masses were also conducted. The simulation results are shown in Fig. 7. The (0, 0, 0) through (n, n, n) acoustic modes of the host acoustic tube and cavity were used in these simulations. In these simulations, the (0, 0, 0) through (20, 20, 20) acoustic modes of the host acoustic tube and cavity had the same number of degrees of freedom as vibration systems, and the remaining acoustic modes had no degrees of freedom and were added to the neck as additional mass. The optimum values of h_c and η_B shown in Fig. 5 were used for each case. For example, for $n = 500$, $h_c = 0.094$ [m] was used. The theoretical optimum height of the cavity decreases with an increase in the number of residual acoustic modes; however, the shapes of the magnitude plot of the nondimensional sound pressure did not change significantly. The simulation results of the magnitude plots of the nondimensional sound pressure using several methods to derive the lengths of the open-end corrections are shown in Fig. 8. In these simulations, the (0, 0, 0) through (500, 500, 500) acoustic modes of the host acoustic field and cavity were used, and the (0, 0, 0) through (20, 20, 20) acoustic modes of the host acoustic tube and cavity had the same number of degrees of freedom as vibration systems. Because the length of the host acoustic tube is 0.85 m and the speed of sound is 340 m/s, the natural frequency of the $(i_A, 0, 0)$ acoustic mode of the host acoustic field is given as

Table 2 Material properties of the simulation conducted using the analytical model shown in Fig. 1(a).

ρ_A, ρ_C	1.2 kg/m ³	ρ_B	1.2 kg/m ³	(x_{AL}, x_{AR})	(0.6, 0.61) m
κ_A, κ_C	138720 Pa	l_B, d_B	0.01 m	(x_{CL}, x_{CR})	(0.045, 0.055) m
l_A	0.85 m	h_B	0.02 m	(y_{CF}, y_{CB})	(0.045, 0.055) m
S_A	0.05×0.1 m ²	l_C, d_C	0.1 m	x_{AO}	0 m

Fig. 5 Theoretical optimum values of the height h_c and loss factor η_B , and lengths of the open-end corrections δ_h and δ_c using the acoustic tube as the host acoustic field and different numbers of acoustic modes.

Fig. 6 Simulation results of the magnitude plots of the nondimensional sound pressure using the (0, 0, 0) through (20, 20, 20) acoustic modes of the host acoustic tube and cavity, where the (0, 0, 0) through (n, n, n) acoustic modes of the host acoustic tube and cavity have the same number of degrees of freedom as vibration systems, and the remaining acoustic modes have no degrees of freedom and are added to the neck as additional mass. Figure 6(a) shows the entire graphs, and Figure 6(b) shows an enlarged view near the peak in Figure 6(a).

Fig. 7 Simulation results of the magnitude plots of the nondimensional sound pressure using the (0, 0, 0) through (n, n, n) acoustic modes of the host acoustic tube and cavity, where the (0, 0, 0) through (20, 20, 20) acoustic modes of the host acoustic tube and cavity have the same number of degrees of freedom as vibration systems, and the remaining acoustic modes have no degrees of freedom and are added to the neck as additional mass.

Fig. 8 Simulation results of the magnitude plots of the nondimensional sound pressure using various combinations of optimum values when the acoustic tube was used as the host acoustic field.

$100(2i_A - 1)$ [Hz] in these simulations. The simulation results obtained using two types of conventional open-end corrections are shown in Fig. 8 (Ingard, 1953, Rayleigh, 1945). One of the lengths of the conventional open-end correction is given by (Rayleigh, 1945)

$$\delta = \frac{8}{3\pi} \sqrt{\frac{S_B}{\pi}}. \quad (84)$$

The above correction is derived under the condition in which the silencer has an infinite flange in a semi-infinite space. Note that the length given by Eq. (84) corresponds to the open-end correction on one side of the neck. Because we should consider the open-end correction for both sides of the neck, we should use twice the length of Eq. (84). Ingard proposed another conventional open-end correction (Ingard, 1953). The height of the cavity h_c was tuned by considering the lengths of the open-end corrections in the simulations. The theoretical values of the height h_c and loss factor η_B , as well as lengths of the open-end corrections δ_h and δ_c are listed in Table 3. As shown in Fig. 8, the

effect of residual acoustic modes is not negligible. The height of the cavity should be modified by considering the open-end correction; however, the two types of conventional open-end corrections are not effective. Because the flange of the Helmholtz silencer cannot be regarded as infinite, the length of the open-end correction given by Eq. (84), is ineffective. In contrast, the open-end correction proposed by Ingard is not effective because the heights of the host acoustic field and cavity are not considered. In these simulations, the difference between the two types of proposed methods using a simple residual mass and constant dynamic mass was small.

Simulations using the analytical model of the three-dimensional acoustic field shown in Fig. 1(b) were conducted. The material properties of the simulation using the analytical model shown in Fig. 1(b) are listed in Table 4. Here, (x_{AO}, y_{AO}, z_{AO}) denote the (x_A, y_A, z_A) coordinates of the observation point, and both the cross-sectional shapes of the neck and cavity of the Helmholtz silencer are rectangular. To investigate the effect of the residual acoustic modes on the open-end corrections, the theoretical values of the height h_C , loss factor η_B , and lengths of the open-end corrections δ_h and δ_c using different numbers of acoustic modes are shown in Fig. 9. Here, $n_{\max}$ is the highest modal order. The $(0, 0, 0)$ through $(n_{\max}, n_{\max}, n_{\max})$ acoustic modes of the host acoustic field and cavity were used in these calculations. The lengths of the open-end corrections were theoretically derived using the additional constant dynamic mass. The lengths of the open-end corrections almost converged for $n_{\max} > 300$. In the following simulations, the $(0, 0, 0)$ through $(500, 500, 500)$ acoustic modes of the host acoustic field and cavity were used, and the $(0, 0, 0)$ through $(20, 20, 20)$ acoustic modes of the host acoustic field and cavity had the same number of degrees of freedom as vibration systems. The simulation results of the magnitude plots of the nondimensional sound pressure at the observation point are shown in Fig. 10. The $(1, 0, 0)$ acoustic mode of the host acoustic field was suppressed in these simulations. The natural frequency of the $(1, 0, 0)$ acoustic mode of the host acoustic field is 200 Hz because the length l_A is 0.85 m in these simulations. In the vicinity of the $(1, 0, 0)$ acoustic mode of the host acoustic field, there are $(0, 1, 0)$, $(1, 1, 0)$, $(0, 0, 1)$, and $(0, 1, 1)$ acoustic modes with natural frequencies of 170, 262, 283, and 330 Hz,

Table 3 Theoretical values of the height h_C and loss factor η_B , and the lengths of the open-end corrections δ_h and δ_c in the simulation using the analytical model shown in Fig. 1(a).

	h_C [mm]	η_B	δ_h [mm]	δ_c [mm]
Without considering residual acoustic modes	146	0.452	0	0
Proposed method (constant dynamic mass)	93.9	0.561	6.79	4.39
Proposed method (constant residual mass)	94.5	0.559	6.59	4.39
Infinite flange (Eq. (84))	99.0	0.547	4.79	4.79
Ingard method	99.6	0.545	5.29	4.11

Table 4 Material properties of the simulation conducted using the analytical model shown in Fig. 1(b).

ρ_A, ρ_C	1.2 kg/m ³	ρ_B	1.2 kg/m ³	(x_{AL}, x_{AR})	(0.1, 0.12) m
κ_A, κ_C	138720 Pa	l_B, d_B	0.02 m	(y_{AF}, y_{AB})	(0.1, 0.12) m
l_A	0.85 m	h_B	0.03 m	(x_{CL}, x_{CR})	(0.04, 0.06) m
d_A	1 m	l_C, d_C	0.1 m	(y_{CF}, y_{CB})	(0.04, 0.06) m
h_A	0.6 m	l_P, d_P	0.04 m	(x_{AO}, y_{AO}, z_{AO})	(0, 0, 0) m

Fig. 9 Theoretical optimum values of the height h_C and loss factor η_B , and lengths of the open-end corrections δ_h and δ_c using the cuboidal room as the host acoustic field and different numbers of acoustic modes.

respectively. Similarly to Fig. 8, the simulation results obtained using two types of conventional open-end corrections are also shown in Fig. 10. The height of the cavity h_c was tuned by considering the length of the open-end correction in the simulations. The theoretical values of the height h_c , loss factor η_B , and lengths of the open-end corrections δ_h and δ_c are listed in Table 5. Because the acoustic modes are relatively dense, the effect of the residual acoustic modes is large in this case. The two types of conventional open-end corrections were also ineffective in this case. In addition, the simulation results of the two types of proposed methods using a simple residual mass and constant dynamic mass are slightly different.

Fig. 10 Simulation results of the magnitude plots of the nondimensional sound pressure using various combinations of optimum values when the cuboidal room was used as the host acoustic field.

Table 5 Theoretical values of the height h_c and loss factor η_B , and lengths of the open-end corrections δ_h and δ_c in the simulation using the analytical model shown in Fig. 1(b).

	h_c [mm]	η_B	δ_h [mm]	δ_c [mm]
Without considering residual acoustic modes	97.6	0.0696	0	0
Proposed method (constant dynamic mass)	62.1	0.0876	9.34	7.81
Proposed method (constant residual mass)	62.3	0.0873	9.19	7.81
Infinite flange (Eq. (84))	59.6	0.0891	9.58	9.58
Ingard method	63.4	0.0864	9.19	7.00

4. Experiment

To validate the theoretical analysis, two types of experiments corresponding to the analytical models shown in Fig. 1(a) and (b) were conducted in this study. A cylindrical acoustic tube and cuboidal acoustic room were used as the host acoustic field. The second acoustic mode was targeted to be suppressed in the experiment using the cylindrical acoustic tube, whereas in the other experiment, the $(1, 0, 0)$ acoustic mode was targeted to be suppressed. Schematics of the two types of experimental apparatus are shown in Fig. 11(a) and (b). The two-microphone method was used to evaluate the nondimensional sound pressure (Chung and Blaser, 1980). Microphones 1 and 2 in Fig. 11(a) and (b) were used for the two-microphone method. In contrast, microphone 3 was installed at the observation point to directly evaluate the sound pressure. In the experiment using the cuboidal room, an additional acoustic tube was used to measure the particle displacement at the excitation surface using the two-microphone method. In this case, the excitation surface was the connecting face between the additional acoustic tube and cuboidal room. Both the cross-sectional shapes of the neck and cavity of the Helmholtz silencer were circular in these experiments, and the cylindrical neck and cavity were arranged concentrically. The cross-sectional shape of the additional acoustic tube for the two-microphone method was square, as shown in Fig. 11(b). The material properties of the two types of experimental apparatus are listed in Tables 6 and 7, respectively. Here, R_A in Table 6 is the radius of the host acoustic tube, and l_1 and l_2 are the lengths shown in Fig. 11(a) and (b). The lengths l_1 and l_2 are associated with the two-microphone method. The complex density ρ_B^* of air with acoustic absorption material was identified using the specific acoustic impedance of the acoustic absorption material with equal bulk density. The specific acoustic impedances of the acoustic absorption material with equal bulk density were measured in a preliminary experiment using the two-microphone method. The length l_A of the host acoustic tube listed in Table 6 includes the open-end correction. In this case, the length of the open-end correction is given as $0.6R_A$ because the host acoustic tube was placed in a hemi-anechoic room (Rayleigh, 1945). Therefore, the actual length of the host acoustic tube was 0.705 m.

Glass wool was used as the acoustic absorption material in these experiments. The bulk densities of the glass wool were 5.5 kg/m^3 and 2.2 kg/m^3 , respectively. As can be seen from Tables 6 and 7, the loss factor of the glass wool was almost proportional to the bulk density of the glass wool. Therefore, the loss factor was tuned with reference to the bulk density in the experiment. Because a glass bottle was used as the Helmholtz silencer in these experiments, a taper was present between the neck and cavity. Therefore, the tapered part was distributed into the neck and cavity under the condition that the total volume and total height were maintained. The height of the tapered part was 17 mm, and the distributed additional heights of the neck and cavity were 6 and 11 mm, respectively. The heights of the neck and cavity listed in Tables 6 and 7 include these additional heights. The same glass bottle was used in these experiments; however, the heights of the neck were different, as listed in Tables 6 and 7. This is because the height of the neck is determined by the sum of the thickness of the hole in the wall of the host acoustic field, the height of the neck of the glass bottle, and the additional height due to the tapered part. The thicknesses of the holes in the walls of the host acoustic fields were 27 and 20 mm, respectively. The height of the cavity was tuned using clay in these experiments.

In these experiments, the nondimensional sound pressure values at the observation points were obtained using the following equation:

$$\frac{P_3}{\rho_A c_A j \omega W_p} = \frac{P_3}{P_1} \frac{\rho_A c_A j \omega W_p}{P_1}, \quad \frac{\rho_A c_A j \omega W_p}{P_1} = \begin{cases} \frac{(P_2/P_1) \cos k l_1 - \cos k(l_1 + l_2)}{-j \sin k l_2} & \text{(Acoustic tube)} \\ \frac{(P_2/P_1) \cos k(l_1 + l_2) - \cos k l_1}{-j \sin k l_2} & \text{(Cuboidal room)} \end{cases}, \quad k = \frac{\omega}{c_A}, \quad (85-87)$$

where P_1 , P_2 , and P_3 are the complex amplitudes of the sound pressure at microphones 1–3, respectively, and k is the wave number. In the experiment using the cuboidal acoustic room, W_p is the amplitude of the particle displacement on the excitation surface.

Fig. 11 Schematics of the two types of experimental apparatus. Figure 11(a) and (b) shows the schematics of the experimental apparatus using a cylindrical acoustic tube and a cuboidal room as the host acoustic field, respectively.

Table 6 Material properties of the experimental apparatus using the cylindrical acoustic tube.

ρ_A, ρ_C	1.19 kg/m^3	ρ_B^*	$1.29(1-0.23j) \text{ kg/m}^3$	h_C	0.035 m
κ_A, κ_C	141000 Pa	h_B	0.063 m	R_C	0.031 m
l_A	0.731 m	R_B	0.011 m	x_{AC}	0.600 m
R_A	0.044 m	(l_1, l_2)	$(0.070, 0.300) \text{ m}$	x_{AO}	0.675 m

Table 7 Material properties of the experimental apparatus using the cuboidal room.

ρ_A, ρ_C	1.19 kg/m^3	ρ_B^*	$1.26(1-0.10j) \text{ kg/m}^3$	h_C	0.048 m
κ_A, κ_C	141000 Pa	h_B	0.056 m	R_C	0.031 m
l_A	0.525 m	R_B	0.011 m	x_{AC}	0.021 m
d_A	0.300 m	l_p, d_p	0.040 m	y_{AC}	0.282 m
h_A	0.290 m	(l_1, l_2)	$(0.093, 0.080) \text{ m}$	(x_{AO}, y_{AO}, z_{AO})	$(0.400, 0.225, 0.290) \text{ m}$

4.1 Reduction of acoustic resonance in acoustic tube

The experimental results of the magnitude plots of the nondimensional sound pressure at the observation point are shown in Fig. 12(a). In Fig. 12(a), three types of experimental results are shown: the experimental result using the damped Helmholtz silencer, the experimental result using the undamped Helmholtz silencer, and the experimental result without using the Helmholtz silencer. The simulation results of the magnitude plots of the nondimensional sound pressure using the material properties of the experimental apparatus are shown in Fig. 12(b). In this simulation, the circular cross-section of the host acoustic tube was replaced with a square cross-section of equal area for simplicity. The (0, 0, 0) through (200, 200, 200) acoustic modes of the host acoustic tube and (0, 0) through (200, 200) acoustic modes of the cylindrical cavity of the Helmholtz silencer were used in these simulations, and the (0, 0, 0) through (20, 20, 20) acoustic modes of the host acoustic tube and (0, 0) through (20, 20) acoustic modes of the cylindrical cavity had the same number of degrees of freedom as vibration systems. Using the same procedure as in Section 3, the number of acoustic modes was determined to be sufficiently large. The sound pressure was evaluated at the surface of the side wall of the acoustic tube at the observation point in these experiments and simulations. The theoretical optimum values of the height of the cavity and loss factor of the neck, as well as lengths of the open-end corrections are listed in Table 8. The theoretical optimum values of the height and loss factor were used instead of the actual experimental values of the experimental apparatus in the simulations. In addition, the theoretical optimum values of the Helmholtz silencer were derived using the constant dynamic mass and constant dynamic spring constant; the theoretical optimum values and lengths of the open-end corrections using the other methods are also listed in Table 8 for reference. There is a difference because the loss factor of the host acoustic tube was ignored; however, the experimental and simulation results of the magnitude plots agree well. In addition, the theoretical optimum values agree well with the experimentally tuned values listed in Table 6. The length of the open-end correction using Eq. (84) was approximately 1.5 times larger than those of the proposed methods on the cavity side; however, the difference in the optimum heights of the cavity is small because the difference in the sum of the open-end corrections at both ends is reduced by the difference on the host acoustic tube side.

Fig. 12 Experimental and simulation results of the magnitude plots of the nondimensional sound pressure obtained using the experimental apparatus shown in Fig. 11(a). Figure 12(a) and (b) shows the experimental and simulation results, respectively.

Table 8 Theoretical values of the height h_c and loss factor η_B , and lengths of the open-end corrections δ_h and δ_c when the cylindrical acoustic tube was used as the host acoustic field in the experiment.

	h_c [mm]	η_B	δ_h [mm]	δ_c [mm]
Without considering residual acoustic modes	44.4	0.226	0	0
Proposed method (constant dynamic mass)	35.3	0.254	10.1	6.07
Proposed method (constant residual mass)	35.7	0.252	9.23	6.07
Infinite flange (Eq. (84))	34.3	0.257	9.34	9.34
Ingard method	36.0	0.251	9.60	5.21

4.2 Reduction of acoustic resonance in cuboidal room

The experimental results of the magnitude plots of the nondimensional sound pressure at the observation point are

shown in Fig. 13(a). The simulation results using the material properties of the experimental apparatus are shown in Fig. 13(b). The (0, 0, 0) through (200, 200, 200) acoustic modes of the host cuboidal room and (0, 0) through (200, 200) acoustic modes of the cylindrical cavity of the Helmholtz silencer were used in this simulation, and the (0, 0, 0) through (20, 20, 20) acoustic modes of the host cuboidal room and (0, 0) through (20, 20) acoustic modes of the cylindrical cavity had the same number of degrees of freedom as vibration systems. The number of acoustic modes was determined to be sufficiently large. The theoretical optimum values of the height of the cavity and loss factor of the neck, as well as lengths of the open-end corrections are listed in Table 9. The theoretical optimum values of the height and loss factor were used instead of the actual experimental values of the experimental apparatus in the simulation. In addition, the theoretical optimum values of the Helmholtz silencer were derived using the constant dynamic mass, and the theoretical optimum values and lengths of the open-end corrections using the other methods are also listed in Table 9 for reference. There is a difference because the loss factor of the host cuboidal room was ignored; however, the experimental and simulation results of the magnitude plots agree well. In addition, the theoretical optimum values agree well with the experimentally tuned values listed in Table 7. In this case, the theoretical optimum values of the height and loss factor derived by the conventional method assuming an infinite flange are almost the same as those of the proposed method; however, the lengths of the open-end corrections of these two methods on each side of the neck do not agree well. The sums of the lengths of the open-end corrections of these two methods on both sides fortuitously agreed well.

Fig. 13 Experimental and simulation results of the magnitude plots of the nondimensional sound pressure obtained using the experimental apparatus shown in Fig. 11(b). Figure 13(a) and (b) shows the experimental and simulation results, respectively.

Table 9 Theoretical values of the height h_c and loss factor η_B , and lengths of the open-end corrections δ_h and δ_c when the cuboidal room was used as the host acoustic field in the experiment.

	h_c [mm]	η_B	δ_h [mm]	δ_c [mm]
Without considering residual acoustic modes	59.3	0.108	0	0
Proposed method (constant dynamic mass)	44.7	0.124	11.7	6.58
Proposed method (constant residual mass)	45.0	0.124	11.2	6.58
Infinite flange (Eq. (84))	44.5	0.124	9.34	9.34
Ingard method	47.5	0.120	8.77	5.21

5. Conclusion

The optimum tuning conditions for damped Helmholtz silencers were derived in this study. The equivalent discrete model of the coupled vibration system including the host acoustic field, neck of the Helmholtz silencer, and cavity of the Helmholtz silencer was obtained using modal analysis. The lengths of the open-end correction on both sides of the neck of the Helmholtz silencer were theoretically derived using the equivalent discrete model. The length of the open-end correction on the host acoustic field side was determined by the residual acoustic modes of the host acoustic field. In contrast, the length of the open-end correction on the cavity side was determined by the residual acoustic modes of the cavity of the Helmholtz silencer. Considering the effect of the open-end correction, the optimum natural frequency ratio and optimum loss factor of the Helmholtz silencer were derived using the two fixed point method. Based on the assumption that the optimum values are used, the size of the Helmholtz silencer was theoretically

investigated, and it was found that the sound suppression performance depends on the volume of the cavity. It was also found that the neck volume can be reduced if the ratio between the cross-sectional area and neck height is maintained at a certain value. The theoretical analysis regarding the optimum tuning conditions and open-end correction was validated through simulations and experiments.

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