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Dynamic response of a line crack perpendicular to the interface between a functionally graded piezoelectric strip and a homogeneous strip

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Abstract

In this paper, the dynamic fracture problem of a functionally graded piezoelectric material strip (FGPM strip) containing a crack perpendicular to the interface between the FGPM strip and a homogeneous layer is considered. It is assumed that the electro-elastic properties of the FGPM strip vary exponentially in the thickness direction, and that the crack faces are under a normal mechanical impact loading. The integral transform techniques and the dislocation density function are employed to reduce the problem to the solution of a singular integral equation. The dynamic stress intensity factors of the internal crack, the terminated crack and the edge crack are computed and are presented as a function of the normalized time for the various values of the nonhomogeneous and geometric parameters.

Keywords : Functionally graded piezoelectric material, Fracture mechanics, Elasticity, Normal crack, Stress intensity factor, Integral transform, Dynamic response

1. Introduction

For the application of the piezoelectric materials, Uchino et al. (1987) and Hall et al. (2005) fabricated a mono-morph actuators made from semi-conductive piezoelectric ceramics. Moreover, a new kind of material, such as a functionally graded piezoelectric material (FGPM) (Wu et al., 1996), has been developed to improve the reliability of the piezoelectric materials by expanding the concept of the well-known functionally graded materials (FGMs) to the piezoelectric materials. Thus, the fracture of FGPMs has received much attention, and several analytical studies concerned with the static fracture problems of FGPMs were reported (Li and Weng, 2002a; Wang and Zhang, 2004; Ueda, 2005a). One of the present authors has also studied the fracture problem of an actuator with an FGPM strip under a thermal load (Ueda et al., 2018; Mabuchi and Ueda, 2019) and an electric load (Ueda and Yamabata, 2020).

On the other hand, piezoelectric materials may be used under transient dynamic loads as well as electro-static or thermal loads in engineering applications. Therefore, it is very important to investigate the transient dynamic response of cracked piezoelectric materials. Although extensive studies have been made in the dynamic fracture analysis of homogeneous piezoelectric materials, relatively few studies have been made in the corresponding problem for FGPMs. Li and Weng (2002b) investigated the Yoffe-type moving center crack in a symmetric FGPM under an anti-plane mechanical loading and an in-plane electric loading. While Chen et al. (2003) studied the impact dynamic fracture mechanics for the unbounded FGPM, it is not suitable for material characterization. The present author also studied the impact response of an FGPM strip with a vertical crack (Ueda, 2005b). However, to our best knowledge, no one has systematically investigated the impact response of a crack in a piezoelectric actuator with functionally graded properties.

In this paper, we focused on the impact fracture problem of a monomorph actuator using an FGPM strip. The analytical laminate model of the monomorph actuator consists of an FGPM strip and a homogeneous elastic layer. The

problem of the normal crack in the FGPM strip is analyzed under the normal mechanical impact loads on the crack surfaces. Material properties are exponentially dependent on the distance from the interface between the FGPM strip and the homogeneous elastic layer. The Laplace-Fourier integral transform techniques and the dislocation density function are employed to reduce the problem to the solution of a singular integral equation (Sneddon and Lowengrub, 1969). The singular integral equation is solved by using the Gauss-Jacobi or Gauss-Chebyshev integration formulas (Erdogan et al., 1972). Numerical methods are then employed to obtain the time dependent solution by way of a Laplace inversion technique (Miller and Guy, 1966). The dynamic stress intensity factors at the tips of the internal crack, the terminated crack and the edge crack versus time are presented for various values of dimensionless parameters representing the crack size, the crack location and the material nonhomogeneity.

2. Formulation of the problem

An infinitely extended long FGPM strip of thickness h_1 containing a normal crack is bonded to an elastic layer of thickness h_2 as shown in Fig. 1. A set of Cartesian coordinates (x, y, z) is attached at the interface of the laminate and the principal axes are taken to coincide with the reference axes x , y and z , in which z is the poling axis and the laminate is in the plane strain conditions in the y -direction. The crack of length $2c$ is located along the z -axis from a to b ($b-a=2c$, $0 \leq a < b \leq h_1$).

Fig. 1 A normal crack in a piezoelectric actuator with functionally graded properties.

The material property parameters are taken to vary continuously along the z -direction inside the FGPM strip. The material properties of the FGPM strip, such as the elastic stiffness constants $c_{kli} \equiv c_{kli}(z)$, the piezoelectric constants $e_{kli} \equiv e_{kli}(z)$, the dielectric constants $\epsilon_{kki} \equiv \epsilon_{kki}(z)$ and the mass density $\rho_i \equiv \rho_i(z)$ ($i=1$), are one-dimensionally dependent as

$$(c_{kli}, e_{kli}, \epsilon_{kki}, \rho_i) = (c_{kl0}, e_{kl0}, \epsilon_{kk0}, \rho_0) \exp(\beta z) \quad (i=1), \quad (1)$$

where β is positive or negative constant, and the subscript 0 indicates the properties at the interface plane $z=0$. The material properties of the homogeneous elastic layer are the elastic stiffness constants c_{kli} and the mass density ρ_i ($i=2$), and the subscript $i=1, 2$ denotes the physical quantities of the FGPM strip and the homogeneous elastic layer, respectively. We suppose that the crack is opened suddenly under the action of the impact load $\sigma_0 H(t)$, where $H(t)$ is the Heaviside step function and t denotes time. For the case of the piezoelectric actuators, the electric potentials at the upper and lower surfaces of the FGPM strip should be fixed by the electrodes. In the following, the subscripts x, y, z will be used to refer to the direction of coordinates.

Let the displacement components and the electric potential be labeled by $u_{xi}(x, z, t)$, $u_{zi}(x, z, t)$ ($i=1, 2$) and $\phi_i(x, z, t)$. The stress components $\sigma_{xxi}(x, z, t)$, $\sigma_{zzi}(x, z, t)$, $\sigma_{xzi}(x, z, t)$ ($i=1, 2$) and the electric displacement components $D_{xi}(x, z, t)$, $D_{zi}(x, z, t)$ are found as

$$\begin{aligned}
& \left. \begin{aligned} \sigma_{xx1} &= c_{111} \frac{\partial u_{x1}}{\partial x} + c_{131} \frac{\partial u_{z1}}{\partial z} + e_{311} \frac{\partial \phi_1}{\partial z}, & \sigma_{zz1} &= c_{131} \frac{\partial u_{x1}}{\partial x} + c_{331} \frac{\partial u_{z1}}{\partial z} + e_{331} \frac{\partial \phi_1}{\partial z}, & \sigma_{zx1} &= c_{441} \left(\frac{\partial u_{x1}}{\partial z} + \frac{\partial u_{z1}}{\partial x} \right) + e_{151} \frac{\partial \phi_1}{\partial x}, \\ D_{x1} &= e_{151} \left(\frac{\partial u_{x1}}{\partial z} + \frac{\partial u_{z1}}{\partial x} \right) - \varepsilon_{111} \frac{\partial \phi_1}{\partial x}, & D_{z1} &= e_{311} \frac{\partial u_{x1}}{\partial x} + e_{331} \frac{\partial u_{z1}}{\partial z} - \varepsilon_{331} \frac{\partial \phi_1}{\partial z}, \end{aligned} \right\} \quad (2) \\
& \sigma_{xx2} = c_{112} \frac{\partial u_{x2}}{\partial x} + c_{132} \frac{\partial u_{z2}}{\partial z}, \quad \sigma_{zz2} = c_{132} \frac{\partial u_{x2}}{\partial x} + c_{332} \frac{\partial u_{z2}}{\partial z}, \quad \sigma_{zx2} = c_{442} \left(\frac{\partial u_{x2}}{\partial z} + \frac{\partial u_{z2}}{\partial x} \right). \quad (3)
\end{aligned}$$

Taking Eqs. (1) into consideration, the governing equations for the electromechanical fields of the FGPM strip and the homogeneous elastic layer may then be expressed as follows:

$$\begin{aligned}
& \left. \begin{aligned} c_{110} \frac{\partial^2 u_{x1}}{\partial x^2} + c_{440} \frac{\partial^2 u_{x1}}{\partial z^2} + (c_{130} + c_{440}) \frac{\partial^2 u_{z1}}{\partial x \partial z} + (e_{310} + e_{150}) \frac{\partial^2 \phi_1}{\partial x \partial z} + \beta \left[c_{440} \left(\frac{\partial u_{x1}}{\partial z} + \frac{\partial u_{z1}}{\partial x} \right) + e_{150} \frac{\partial \phi_1}{\partial x} \right] &= \rho_0 \frac{\partial^2 u_{x1}}{\partial t^2}, \\ c_{440} \frac{\partial^2 u_{z1}}{\partial x^2} + c_{330} \frac{\partial^2 u_{z1}}{\partial z^2} + (c_{130} + c_{440}) \frac{\partial^2 u_{x1}}{\partial x \partial z} + e_{150} \frac{\partial^2 \phi_1}{\partial x^2} + e_{330} \frac{\partial^2 \phi_1}{\partial z^2} + \beta \left(c_{130} \frac{\partial u_{x1}}{\partial x} + c_{330} \frac{\partial u_{z1}}{\partial z} + e_{330} \frac{\partial \phi_1}{\partial z} \right) &= \rho_0 \frac{\partial^2 u_{z1}}{\partial t^2}, \\ e_{150} \frac{\partial^2 u_{z1}}{\partial x^2} + e_{330} \frac{\partial^2 u_{z1}}{\partial z^2} + (e_{150} + e_{310}) \frac{\partial^2 u_{x1}}{\partial x \partial z} - \varepsilon_{110} \frac{\partial^2 \phi_1}{\partial x^2} - \varepsilon_{330} \frac{\partial^2 \phi_1}{\partial z^2} + \beta \left(e_{310} \frac{\partial u_{x1}}{\partial x} + e_{330} \frac{\partial u_{z1}}{\partial z} - \varepsilon_{330} \frac{\partial \phi_1}{\partial z} \right) &= 0, \end{aligned} \right\} \quad (4)
\end{aligned}$$

$$\begin{aligned}
& \left. \begin{aligned} c_{112} \frac{\partial^2 u_{x2}}{\partial x^2} + c_{442} \frac{\partial^2 u_{x2}}{\partial z^2} + (c_{132} + c_{442}) \frac{\partial^2 u_{z2}}{\partial x \partial z} &= \rho_2 \frac{\partial^2 u_{x2}}{\partial t^2}, \\ c_{442} \frac{\partial^2 u_{z2}}{\partial x^2} + c_{332} \frac{\partial^2 u_{z2}}{\partial z^2} + (c_{132} + c_{442}) \frac{\partial^2 u_{x2}}{\partial x \partial z} &= \rho_2 \frac{\partial^2 u_{z2}}{\partial t^2}. \end{aligned} \right\} \quad (5)
\end{aligned}$$

Referring to Fig. 1, it is assumed that $x=0$ is a plane of symmetry regarding to geometry and loading conditions. Thus, in analyzing the problem it is sufficient to consider one-half ($0 \leq x < \infty$) of the FGPM strip and the homogeneous elastic layer only. Also, the problem is assumed to have been reduced to a perturbation problem in which the crack surface tractions are the only nonzero external loads and the stresses in the laminate vanish for $x \rightarrow \infty$; therefore the symmetry and boundary conditions can be written as

$$\left. \begin{aligned} \sigma_{xx1}(0, z, t) &= -\sigma_0 H(t) \quad (a < z < b), \\ u_{x1}(0, z, t) &= 0 \quad (0 \leq z \leq a, b \leq z \leq h_1), \end{aligned} \right\} \quad (6)$$

$$\sigma_{zx1}(0, z, t) = 0, \quad D_{x1}(0, z, t) = 0, \quad (0 \leq z \leq h_1), \quad (7)$$

$$\sigma_{zx2}(0, z, t) = 0, \quad u_{x2}(0, z, t) = 0, \quad (-h_2 \leq z \leq 0), \quad (8)$$

$$\sigma_{zx1}(x, h_1, t) = 0, \quad \sigma_{zz1}(x, h_1, t) = 0, \quad \phi_1(x, h_1, t) = 0, \quad (0 \leq x < \infty), \quad (9)$$

$$\left. \begin{aligned} \sigma_{zx1}(x, 0, t) &= \sigma_{zx2}(x, 0, t), \quad \sigma_{zz1}(x, 0, t) = \sigma_{zz2}(x, 0, t), \\ u_{x1}(x, 0, t) &= u_{x2}(x, 0, t), \quad u_{z1}(x, 0, t) = u_{z2}(x, 0, t), \\ \phi_1(x, 0, t) &= 0, \end{aligned} \right\} \quad (0 \leq x < \infty), \quad (10)$$

$$\sigma_{zx2}(x, -h_2, t) = 0, \quad \sigma_{zz2}(x, -h_2, t) = 0, \quad (0 \leq x < \infty). \quad (11)$$

3. Method of the solution

Define a Laplace transform pair by

$$f^*(p) = \int_0^\infty f(t) \exp(-pt) dt, \quad f(t) = \frac{1}{2\pi i} \int_{Br} f^*(p) \exp(pt) dp, \quad (12)$$

where Br denotes the Bromwich path in pertinent complex planes, and “i” indicates the imaginary unit. The general solutions of the displacement components and the electric potential in the Laplace transform plane are obtained by solving the governing Eqs. (4) and (5) using the Laplace and Fourier integral transform techniques (Sneddon and Lowengrub, 1969):

$$\left. \begin{aligned} u_{x1}^*(x, z, p) &= \frac{i}{2\pi} \sum_{j=1}^3 \int_{-\infty}^{\infty} \frac{|s|}{s} p_{61j} A_{1j} \exp(|s| \gamma_{1j} x) \exp(-isz) ds + \frac{2}{\pi} \sum_{j=1}^6 \int_0^\infty p_{62j} B_{1j} \exp(s \gamma_{2j} z) \sin(sx) ds, \\ u_{z1}^*(x, z, p) &= \frac{1}{2\pi} \sum_{j=1}^3 \int_{-\infty}^{\infty} p_{71j} A_{1j} \exp(|s| \gamma_{1j} x) \exp(-isz) ds + \frac{2}{\pi} \sum_{j=1}^6 \int_0^\infty p_{72j} B_{1j} \exp(s \gamma_{2j} z) \cos(sx) ds, \\ \phi_1^*(x, z, p) &= \frac{1}{2\pi} \sum_{j=1}^3 \int_{-\infty}^{\infty} p_{51j} A_{1j} \exp(|s| \gamma_{1j} x) \exp(-isz) ds + \frac{2}{\pi} \sum_{j=1}^6 \int_0^\infty p_{52j} B_{1j} \exp(s \gamma_{2j} z) \cos(sx) ds, \end{aligned} \right\} \quad (13)$$

$$\left. \begin{aligned} u_{x2}^*(x, z, p) &= -\frac{1}{2\pi} \int_{-\infty}^{\infty} [\gamma_1 A_{21} \exp(-|s| \gamma_1 x) + A_{22} \exp(-|s| \gamma_2 x)] \exp(-isz) ds \\ &\quad + \frac{2}{\pi} \int_0^\infty \{ [B_{21} \exp(-s \gamma_1 z) - B_{22} \exp(s \gamma_1 z)] + \gamma_2 [B_{23} \exp(-s \gamma_2 z) - B_{24} \exp(s \gamma_2 z)] \} \sin(sx) ds, \\ u_{z2}^*(x, z, p) &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{s}{|s|} [A_{21} \exp(-|s| \gamma_1 x) + \gamma_2 A_{22} \exp(-|s| \gamma_2 x)] \exp(-isz) ds \\ &\quad + \frac{2}{\pi} \int_0^\infty \{ \gamma_1 [B_{21} \exp(-s \gamma_1 z) + B_{22} \exp(s \gamma_1 z)] + [B_{23} \exp(-s \gamma_2 z) + B_{24} \exp(s \gamma_2 z)] \} \cos(sx) ds, \end{aligned} \right\} \quad (14)$$

where $A_{1j} \equiv A_{1j}(s, p)$ ($j=1, 2, 3$), $B_{1j} \equiv B_{1j}(s, p)$ ($j=1, 2, \dots, 6$) and $A_{2j} \equiv A_{2j}(s, p)$ ($j=1, 2$), $B_{2j} \equiv B_{2j}(s, p)$ ($j=1, 2, 3, 4$) are the unknown functions to be solved, and the known functions $\gamma_{1j} \equiv \gamma_{1j}(s, p)$ and $\gamma_{2j} \equiv \gamma_{2j}(s, p)$ are shown in Appendix A of the previous paper (Ueda, 2005b). The known functions $p_{k1j} \equiv p_{k1j}(s, p)$ ($j=1, 2, 3, k=5, 6, 7$), $p_{k2j} \equiv p_{k2j}(s, p)$ ($j=1, 2, \dots, 6, k=5, 6, 7$) and $\gamma_j \equiv \gamma_j(s, p)$ ($j=1, 2$) are given in Appendix A. Substituting Eqs. (13) and (14) into Eqs. (2) and (3), one obtains the stress and electric displacement expressions.

The problem may be reduced to a singular integral equation by defining the following new unknown function $G^*(z, p)$:

$$G^*(z, p) = \begin{cases} \frac{\partial}{\partial z} u_{x1}^*(0, z, p) & (a < z < b), \\ 0 & (0 \leq z \leq a, b \leq z \leq h_1). \end{cases} \quad (15)$$

Making use of the first boundary condition (6) with Eqs. (7)-(11), we have the following singular integral equation for the determination of the unknown function $G^*(\xi, p)$:

$$\int_a^b G^*(\xi, p) \left[\frac{1}{\xi - z} + \sum_{i=1}^4 M_i(\xi, z, p) \right] d\xi = \exp(-\beta z) \frac{\pi}{\Re[Z^\infty]} \frac{\sigma_0}{p} \quad (a < z < b), \quad (16)$$

where the constant Z^∞ and the kernel functions $M_i(\xi, z, p)$ ($i=1, 2, 3, 4$) are given in Appendix B. The singular integral Eq. (16) for $0 \leq a < b < h_1$ is to be solved with the following subsidiary conditions obtained from the second boundary condition (6).

$$\int_a^b G^*(\xi, p) d\xi = 0. \quad (17)$$

To solve the singular integral Eq. (16) and the additional Eq. (17) by using the Gauss-Jacobi or Gauss-Chebyshev integration formulas (Erdogan, et al., 1972), we introduce the following function $\Phi^*(\xi, p)$:

$$G^*(\xi, p) = \left\{ \begin{array}{ll} \frac{c}{[(\xi - a)(b - \xi)]^{1/2}} \Phi^*(\xi, p) & (0 < a < b < h_1 : \text{Internal crack}), \\ \frac{c}{(\xi - a)^\Omega (b - \xi)^{1/2}} \Phi^*(\xi, p) & (0 = a < b < h_1 : \text{Terminated crack}), \\ \left(\frac{b - \xi}{\xi - a} \right)^{1/2} \Phi^*(\xi, p) & (0 < a < b = h_1 : \text{Edge crack}), \end{array} \right\} \quad (18)$$

where Ω is the order of stress singularity at the crack tip ($z = a = 0$) and is given by the root of the following equation (Ting and Hoang, 1984).

$$|\Delta| = 0. \quad (19)$$

In Eq. (19), Δ is a square matrix of order 10 and $|\Delta|$ is the determinant of the square matrix Δ . The elements of Δ are $\Delta_{1,1}(\Omega), \Delta_{1,2}(\Omega), \dots, \Delta_{10,10}(\Omega)$ and are given in Appendix C.

The stress intensity factors $K_{IA} \equiv K_{IA}(t)$ at $z = a$ and $K_{IB} \equiv K_{IB}(t)$ at $z = b$ may be evaluated as

$$K_{IA} = \left\{ \begin{array}{ll} -Z^\infty \exp(\beta a) (\pi c)^{1/2} \frac{1}{2\pi i} \int_{Br} \frac{\Phi^*(a, p)}{p} \exp(pt) dp & (0 < a < b < h_1), \\ -\mu^* Z^\infty \exp(\beta a) \pi^{1/2} c^\Omega \frac{1}{2\pi i} \int_{Br} \frac{\Phi^*(a, p)}{p} \exp(pt) dp & (0 = a < b < h_1), \\ -Z^\infty \exp(\beta a) (4\pi c)^{1/2} \frac{1}{2\pi i} \int_{Br} \frac{\Phi^*(a, p)}{p} \exp(pt) dp & (0 < a < b = h_1), \end{array} \right\} \quad (20)$$

$$K_{IB} = \left\{ \begin{array}{ll} Z^\infty \exp(\beta b) (\pi c)^{1/2} \frac{1}{2\pi i} \int_{Br} \frac{\Phi^*(b, p)}{p} \exp(pt) dp & (0 < a < b < h_1), \\ 2^{1/2-\Omega} Z^\infty \exp(\beta b) (\pi c)^{1/2} \frac{1}{2\pi i} \int_{Br} \frac{\Phi^*(b, p)}{p} \exp(pt) dp & (0 = a < b < h_1), \\ 0 & (0 < a < b = h_1), \end{array} \right\} \quad (21)$$

where the constant μ^* is given by

$$\mu^* = \frac{c_{442}}{\sin(\pi \Omega)} \sum_{k=1}^3 (\gamma_{1k}^\infty)^{-\Omega} (S_{1k} + S_{2k}). \quad (22)$$

In Eq. (22), the constant γ_{1k}^∞ is given in Appendix B and the constants S_{1k}, S_{2k} ($k=1,2,3$) are also shown in Appendix D. The dynamic stress intensity factors K_{IA} and K_{IB} can be found from Eqs. (20) and (21) by using a numerical Laplace inversion scheme (Miller and Guy, 1966).

4. Numerical results and discussion

To consider the effects of the normalized crack location parameter $(a+b)/h_1$, the normalized crack length parameters c/h_1 , b/h_1 , $(h_1 - a)/h_1$ and the normalized nonhomogeneous parameter βh_1 on the time dependences of the dynamic stress intensity factors K_{IA} and K_{IB} , the solutions of the singular integral Eq. (16) and the additional

Eq. (17) have been computed numerically. For the purpose of numerical illustrations, the electro-elastic properties of cadmium selenide with the following properties (Ashida and Tauchert, 1998) are used as the properties of the FGPM strip at the plane $z = 0$.

$$\left. \begin{aligned} c_{110} &= 7.41 \times 10^{10} [\text{N/m}^2], & c_{130} &= 3.93 \times 10^{10} [\text{N/m}^2], \\ c_{330} &= 8.36 \times 10^{10} [\text{N/m}^2], & c_{440} &= 1.32 \times 10^{10} [\text{N/m}^2], \\ e_{310} &= -0.16 [\text{C/m}^2], & e_{330} &= 0.347 [\text{C/m}^2], \\ e_{150} &= -0.138 [\text{C/m}^2], & & \\ \varepsilon_{110} &= 0.825 \times 10^{-10} [\text{C/Vm}], & \varepsilon_{330} &= 0.903 \times 10^{-10} [\text{C/Vm}]. \end{aligned} \right\} \quad (23)$$

The properties of titanium (Ti) with following properties are also used as the properties of the elastic layer.

$$c_{112} = c_{332} = \frac{2(1-\nu)}{1-2\nu} \mu, \quad c_{132} = \frac{2\nu}{1-2\nu} \mu, \quad c_{442} = \mu, \quad (24)$$

where the constants μ and ν are the modulus of transverse elasticity and Poisson's ratio, and the values of them are $\mu = 4.26 \times 10^{10} [\text{N/m}^2]$, $\nu = 0.28$ (JSME, 1991)

4.1. Internal crack

In the first set of calculations, to examine the validity of the analytical methods and the accuracy of the numerical results, we consider two special cases of this problem. The first one is the case of $c/h_1 \rightarrow 0.0$, $(a+b)/2h_1 = 0.5$ and $\beta h_1 = 0.0$, and this case corresponds to the previous study (Shindo et al. 1999). The material properties of BaTiO₃ and PZT-6B are given in the previous study. Figure 2 indicates the comparison of the present results (solid lines) and the previous results (circles) on the dynamic normalized stress intensity factor $K_{IA}/\sigma_0(\pi c)^{1/2}$ versus the normalized time $c_p t/c$, where $c_p = (c_{440}/\rho_0)^{1/2}$, and both of the results are consistent with each other.

Fig. 2 The comparison of the present results and the previous results on the stress intensity factor K_{IA} for $\beta h_1 = 0.0$, $c/h_1 \rightarrow 0.0$, $(a+b)/2h_1 = 0.5$ and $h_2/h_1 = 0.0$.

The second one is the case of the simple FGPM strip ($h_2/h_1 = 0.0$). We study the effect of the normalized nonhomogeneous parameter βh_1 on the dynamic stress intensity factors K_{IA} and K_{IB} for this case. Figures 3(a) and 3(b) show the time dependences of the normalized stress intensity factors $(K_{IA}, K_{IB})/\sigma_0(\pi c)^{1/2}$. It is supposed that the values of the parameters are $c/h_1 = 0.1$, $(a+b)/2h_1 = 0.5$ and $\beta h_1 = 1.0, 0.0, -1.0$. In these figures, the solid, broken and dashed lines indicate the results for $\beta h_1 = 1.0, 0.0, -1.0$, respectively. Note that the values of those intensity factors rise sharply at first, reach maximum values and then decrease and approach the static values with increasing $c_p t/c$. The value of $K_{IA}/\sigma_0(\pi c)^{1/2}$ decreases with increasing βh_1 , and the opposite is true for $K_{IB}/\sigma_0(\pi c)^{1/2}$. Because of anti-symmetry, the values of $(K_{IA}, K_{IB})/\sigma_0(\pi c)^{1/2}$ for $\beta h_1 = 1.0$, respectively, coincide with the values of $(K_{IB}, K_{IA})/\sigma_0(\pi c)^{1/2}$ for $\beta h_1 = -1.0$, and, of course, $[K_{IA}]_{\beta h_1=0.0} = [K_{IB}]_{\beta h_1=0.0}$. These facts can guarantee the validity of the analytical methods and the accuracy of the numerical results.

Fig. 3(a) The effect of the nonhomogeneous parameter βh_1 on the stress intensity factor K_{IA} for the simple FGPM plate ($h_2/h_1=0.0$) with $(a+b)/2h_1=0.5$ (internal crack).

Fig. 3(b) The effect of the nonhomogeneous parameter βh_1 on the stress intensity factor K_{IB} for the simple FGPM plate ($h_2/h_1=0.0$) with $(a+b)/2h_1=0.5$ (internal crack).

In the next set of calculations, we consider the case of the laminate that consists of the FGPM strip and the homogeneous elastic layer. Figures 4(a) and 4(b) indicate the influence of βh_1 on the time dependences of K_{IA} and K_{IB} of the internal crack for $c/h_1=0.1$ and $(a+b)/2h_1=0.2$ with $h_2/h_1=1.0$ and 0.0 . While the increase of βh_1 induces the reduction of the value of K_{IA} , the opposite is true for the case of K_{IB} . This trend is the same as the case of $h_2/h_1=0.0$. The values of K_{IA} and K_{IB} for the case of $h_2/h_1=1.0$ are lower than the case of $h_2/h_1=0.0$, and this tendency of K_{IA} is remarkable. Generally speaking, the elastic layer has an effect on the decrease of the stress intensity factors.

Fig. 4(a) The effect of the nonhomogeneous parameter βh_1 on the stress intensity factor K_{IA} for $c/h_1=0.1$ and $(a+b)/2h_1=0.2$ with $h_2/h_1=0.0, 1.0$ (internal crack).

Fig. 4(b) The effect of the nonhomogeneous parameter βh_1 on the stress intensity factor K_{IB} for $c/h_1 = 0.1$ and $(a+b)/2h_1 = 0.2$ with $h_2/h_1 = 0.0, 1.0$ (internal crack).

Fig. 5(a) The effect of the crack length parameter c/h_1 on the stress intensity factor K_{IA} for $\beta h_1 = 1.0$ and $(a+b)/2h_1 = 0.5$ with $h_2/h_1 = 1.0$ (internal crack).

Fig. 5(b) The effect of the crack length parameter c/h_1 on the stress intensity factor K_{IB} for $\beta h_1 = 1.0$ and $(a+b)/2h_1 = 0.5$ with $h_2/h_1 = 1.0$ (internal crack).

Figures 5(a) and 5(b) show the influence of c/h_1 on the time dependences of K_{IA} and K_{IB} for $\beta h_1 = 1.0$ and $(a+b)/2h_1 = 0.5$ with $h_2/h_1 = 1.0$. In these figures, the solid, broken and dashed lines indicate the results for $c/h_1 = 0.1, 0.2, 0.3$, respectively. As c/h_1 increase, the value of $K_{IA}/\sigma_0(\pi c)^{1/2}$ decreases and the value of $K_{IB}/\sigma_0(\pi c)^{1/2}$ increases. This phenomenon can be easily understood by the effects of the elastic layer and the free surface. However it is noted that the values of $(K_{IA}, K_{IB})/\sigma_0(\pi c)^{1/2}$ are normalized by the half of the crack length c .

For example, when the normalizing crack length c' is fixed for c at $c/h_1 = 0.1$, the values of $(K_{IA}, K_{IB})/\sigma_0(\pi c')^{1/2}$ are

$$\left. \begin{aligned} (K_{IA}, K_{IB})/\sigma_0(\pi c')^{1/2} &= \sqrt{2}(K_{IA}, K_{IB})/\sigma_0(\pi c)^{1/2} & (c/h_1 = 0.2) \\ (K_{IA}, K_{IB})/\sigma_0(\pi c')^{1/2} &= \sqrt{3}(K_{IA}, K_{IB})/\sigma_0(\pi c)^{1/2} & (c/h_1 = 0.3) \end{aligned} \right\} \quad (25)$$

4.2. Terminated crack

In the third set of calculations, we study the stress intensity factors of the terminated crack ($a/h_1 = 0, b/h_1 < 1.0$). The order of stress singularity Ω at the crack tip ($z = a = 0$) given by Eq. (19) is $\Omega = 0.434240$. Figures 6(a) and 6(b) indicate the effect of the nonhomogeneous parameter βh_1 on the time dependences of the normalized stress intensity factors $K_{IA}/\sigma_0\pi^{1/2}c^\Omega$ and $K_{IB}/\sigma_0(\pi c)^{1/2}$. It is supposed that the geometric parameters are $b/h_1 = 0.2$ and $h_2/h_1 = 1.0$. In these figures, the solid, broken and dashed lines indicate the results for $\beta h_1 = 1.0, 0.0, -1.0$, respectively. Figures 7(a) and 7(b) are the same as Figs. 6(a) and 6(b) for $b/h_1 = 0.4$. To make the effect of the crack length on $K_{IA}/\sigma_0\pi^{1/2}c^\Omega$ and $K_{IB}/\sigma_0(\pi c)^{1/2}$ clear, the new nondimensional stress intensity factors $K_{IA}/\sigma_0\pi^{1/2}c'^\Omega$ and $K_{IB}/\sigma_0(\pi c')^{1/2}$ defined by Eq.(25) are also used in Figs.7(a) and 7(b), where the normalizing crack length c' is fixed for c at $b/h_1 = 0.2$. The tendency of the time dependences of the stress intensity factors are similar to ones for the internal crack shown in the previous figures. The magnitude of the normalized stress intensity factors and the effect of βh_1 increase as b/h_1 increases.

Fig. 6(a) The effect of the nonhomogeneous parameter βh_1 on the stress intensity factor K_{IA} for $a/h_1 = 0.0$ and $b/h_1 = 0.2$ with $h_2/h_1 = 1.0$ (terminated crack).

Fig. 6(b) The effect of the nonhomogeneous parameter βh_1 on the stress intensity factor K_{IB} for $a/h_1 = 0.0$ and $b/h_1 = 0.2$ with $h_2/h_1 = 1.0$ (terminated crack).

Fig. 7(a) The effect of the nonhomogeneous parameter βh_1 on the stress intensity factor K_{IA} for $a/h_1 = 0.0$ and $b/h_1 = 0.4$ with $h_2/h_1 = 1.0$ (terminated crack).

Fig. 7(b) The effect of the nonhomogeneous parameter βh_1 on the stress intensity factor K_{IB} for $a/h_1 = 0.0$ and $b/h_1 = 0.4$ with $h_2/h_1 = 1.0$ (terminated crack).

4.3. Edge crack

Fig. 8 The effect of the nonhomogeneous parameter βh_1 on the stress intensity factor K_{IA} for $(h_1 - a)/h_1 = 0.1$ and $b/h_1 = 1.0$ with $h_2/h_1 = 1.0$ (edge crack).

The final set of calculations, we consider the case of the edge crack ($a/h_1 > 0.0$, $b/h_1 = 1.0$, $c = (h_1 - a)/2$). Figure 8 shows the influence of βh_1 on $K_{IA} / (\sigma_0 (2\pi c)^{1/2})$ for $(h_1 - a)/h_1 = 0.1$ with $h_2/h_1 = 1.0$. In this figure,

the solid, broken and dashed lines indicate the results for $\beta h_1 = 1.0, 0.0, -1.0$, respectively. The increase in the parameter βh_1 induces decrease in the peak value of the stress intensity factor. The inertial effect for the case of the edge crack is more remarkable than the case of the internal crack. Figure 9 indicates the effect of crack length parameter $(h_1 - a)/h_1$ on the time dependence of the normalized dynamic stress intensity factor $K_{IA}/\sigma_0(2\pi c')^{1/2}$ for $\beta h_1 = -1.0$ with $h_2/h_1 = 1.0$. In this figure, the solid, broken and dashed lines indicate the results for $(h_1 - a)/h_1 = 0.1, 0.05, 0.01$, respectively, and c' is fixed for c at $(h_1 - a)/h_1 = 0.1$. The value of $K_{IA}/\sigma_0(2\pi c')^{1/2}$ increases with increasing the crack length $(h_1 - a)/h_1$. Because the rigidity near the free surface of the FGPM decreases with increasing the crack length and decreasing the parameter βh_1 , the amplitude of stress intensity factor of the edge crack may become larger than ones of the internal or terminated crack.

Fig. 9 The effect of the crack length parameter $(h_1 - a)/h_1$ on the stress intensity factor K_{IA} for $\beta h_1 = -1.0$ and $b/h_1 = 1.0$ with $h_2/h_1 = 1.0$ (edge crack).

5. Conclusion

The impact response of the cracked functionally graded piezoelectric strip bonded to the homogeneous elastic layer is studied. The effects of the elastic layer, the crack length, the crack location and the material nonhomogeneity on the fracture behavior are considered. The following facts can be found from the numerical results.

- (1) The numerical results of K_{IA} and K_{IB} for the simple FGPM strip have some anti-symmetrical properties. Therefore, the validity of the analysis and the accuracy of the numerical results may be guaranteed.
- (2) The distinct overshooting phenomenon can be observed not only for the case of the edge crack but also for the case of the internal and terminated cracks, and these facts may suggest the importance of this transient analysis.
- (3) Especially, the normalized stress intensity factor K_{IB} at the crack tip near the free surface becomes large. On the other hand, the normalized stress intensity factor K_{IA} at the crack tip near the interface is reduced by the elastic layer. However, because the values of the stress intensity factors depend on the crack length, the designers should be attentive to this matter.
- (4) Generally, the decrease of βh_1 is beneficial for reducing the stress intensity factor K_{IA} . However, the opposite is true for K_{IB} .

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Appendix A

The known functions $p_{k1j} \equiv p_{k1j}(s, p)$ ($j=1,2,3, k=1,2,...,8$) are given by

$$\left. \begin{aligned} p_{11j} &= c_{110}\gamma_{1j}a_{1j} - c_{130} - e_{310}b_{1j}, & p_{21j} &= e_{150}(\gamma_{1j} + a_{1j}) - \varepsilon_{110}\gamma_{1j}b_{1j}, \\ p_{31j} &= c_{440}(\gamma_{1j} + a_{1j}) + e_{150}\gamma_{1j}b_{1j}, & p_{41j} &= c_{130}\gamma_{1j}a_{1j} - c_{330} - e_{330}b_{1j}, \\ p_{51j} &= b_{1j}, & p_{61j} &= a_{1j}, \\ p_{71j} &= 1, & p_{81j} &= e_{310}\gamma_{1j}a_{1j} - e_{330} + \varepsilon_{330}b_{1j}, \end{aligned} \right\} (j=1,2,3) \quad (A.1)$$

where the functions $\gamma_{1j} \equiv \gamma_{1j}(s, p)$, $a_{1j} \equiv a_{1j}(s, p)$ and $b_{1j} \equiv b_{1j}(s, p)$ ($j=1,2,3$) are given in Appendix A of the previous paper (Ueda, 2005b), and the known functions $p_{k2j} \equiv p_{k2j}(s, p)$ ($j=1,2,...,6, k=1,2,...,8$) are given by

$$\left. \begin{aligned} p_{12j} &= c_{110}a_{2j} - \gamma_{2j}(c_{130} - e_{310}b_{2j}), & p_{22j} &= e_{150}(\gamma_{2j}a_{2j} - 1) - \varepsilon_{110}b_{2j}, \\ p_{32j} &= c_{440}(\gamma_{2j}a_{2j} - 1) + e_{150}b_{2j}, & p_{42j} &= c_{130}a_{2j} + \gamma_{2j}(c_{330} - e_{330}b_{2j}), \\ p_{52j} &= -b_{2j}, & p_{62j} &= a_{2j}, \\ p_{72j} &= 1, & p_{82j} &= e_{310}a_{2j} + \gamma_{2j}(e_{330} + \varepsilon_{330}b_{2j}), \end{aligned} \right\} (j=1,2,\dots,6) \quad (\text{A.2})$$

where the functions $\gamma_{2j} \equiv \gamma_{2j}(s, p)$, $a_{2j} \equiv a_{2j}(s, p)$ and $b_{2j} \equiv b_{2j}(s, p)$ ($j=1,2,\dots,6$) are also given in Appendix A of the previous paper (Ueda, 2005b).

The known functions $\gamma_j \equiv \gamma_j(s, p)$ and the constants c_j ($j=1,2$) are given by

$$\gamma_j = \left[1 + \left(\frac{p}{c_j s} \right)^2 \right]^{1/2} \quad (j=1,2), \quad c_1 = \left[\frac{2(1-\nu)}{1-2\nu} \left(\frac{c_{442}}{\rho_2} \right) \right]^{1/2}, \quad c_2 = \left(\frac{c_{442}}{\rho_2} \right)^{1/2}, \quad (\text{A.3})$$

where ν denotes Poisson's ratio.

Appendix B

The constant Z^∞ is given by

$$Z^\infty = \lim_{s \rightarrow \infty} Z(s, p), \quad Z(s, p) = \sum_{j=1}^3 p_{11j} \delta_j \quad (\text{B.1})$$

where $\delta_j \equiv \delta_j(s, p)$ ($j=1,2,3$) are

$$\delta_1 = \frac{\rho_{22}\rho_{33}}{a_{11}\rho_{22}\rho_{33} - a_{12}\rho_{21}\rho_{33} - a_{13}\rho_{22}\rho_{31}}, \quad \delta_j = -\frac{\rho_{j1}}{\rho_{jj}} \delta_1 \quad (j=2,3) \quad (\text{B.2})$$

with

$$\left. \begin{aligned} \rho_{2j} &\equiv \rho_{2j}(s, p) = p_{313}p_{21j} - p_{213}p_{31j} \\ \rho_{3j} &\equiv \rho_{3j}(s, p) = p_{312}p_{21j} - p_{212}p_{31j} \end{aligned} \right\} (j=1,2,3) \quad (\text{B.3})$$

The kernel functions $M_i(\xi, z, p)$ ($i=1,2,3,4$) are

$$\left. \begin{aligned} M_1(\xi, z, p) &= \int_0^\infty \left\{ \frac{\Re[Z(s, p)]}{\Re[Z^\infty]} - 1 \right\} \sin[s(\xi - z)] ds \\ M_2(\xi, z, p) &= \int_0^\infty \frac{\Im[Z(s, p)]}{\Re[Z^\infty]} \cos[s(\xi - z)] ds \\ M_3(\xi, z, p) &\equiv M_3(\xi, z) = -\frac{2}{\Re[Z^\infty]} \int_0^\infty m^\infty(s, \xi, z) ds \\ M_4(\xi, z, p) &= -\frac{2}{\Re[Z^\infty]} \int_0^\infty [m(s, \xi, z, p) - m^\infty(s, \xi, z)] ds \end{aligned} \right\} \quad (\text{B.4})$$

where

$$\left. \begin{aligned}
m(s, \xi, z, p) &= \sum_{i=1}^8 \sum_{j=1}^3 p_{12j} r_{ji} Q_i \exp(s \gamma_{2j} z) + \sum_{i=1}^8 \sum_{j=4}^6 p_{12j} r_{ji} Q_i \exp\{-s \gamma_{2j} (h_1 - z)\}, \\
m^\infty(s, \xi, z) &= \sum_{k=1}^3 \left\{ \sum_{i=1}^3 p_{i+2,1k}^\infty \left[\sum_{j=1}^3 F_{ijk} \exp(-s \theta_{1jk}) + \sum_{j=4}^6 F_{ijk} \exp(-s \theta_{3jk}) \right] \right. \\
&\quad + \sum_{i=4}^5 p_{i-1,1k}^\infty \left[\sum_{j=1}^3 F_{ijk} \exp(-s \theta_{2jk}) + \sum_{j=4}^6 F_{ijk} \exp(-s \theta_{4jk}) \right] \\
&\quad - p_{51k}^\infty \sum_{j=1}^3 \left[F_{6jk} \exp(-s \theta_{2jk}) + F_{6jk} \exp(-s \theta_{4jk}) \right] \\
&\quad + p_{61k}^\infty \sum_{j=1}^3 \left[F_{7jk} \exp(-s \theta_{1jk}) - F_{7jk} \exp(s \gamma_{2j}^\infty z) \right] \\
&\quad + p_{61k}^\infty \sum_{j=4}^6 \left[F_{7jk} \exp(-s \theta_{3jk}) - F_{7jk} \exp(-s \gamma_{2j}^\infty (h_1 - z)) \right] \\
&\quad \left. - p_{71k}^\infty \left[\sum_{j=1}^3 F_{8jk} \exp(-s \theta_{1jk}) + \sum_{j=4}^6 F_{8jk} \exp(-s \theta_{3jk}) \right] \right\},
\end{aligned} \right\} \quad (B.5)$$

with

$$\left. \begin{aligned}
F_{ijk} &= \begin{cases} \frac{1}{2} p_{12j}^\infty r_{ji}^\infty \delta_k^\infty & (i = 1, 2, 4, 5) \\ \frac{1}{2 \gamma_{1k}^\infty} p_{12j}^\infty r_{ji}^\infty \delta_k^\infty & (i = 3, 6, 7, 8) \end{cases} & (j = 1, 2, \dots, 6), \\
\theta_{1jk} &= -\frac{\xi}{\gamma_{1k}^\infty} - \gamma_{2j}^\infty z, \quad \theta_{2jk} = -\frac{h_1 - \xi}{\gamma_{1k}^\infty} - \gamma_{2j}^\infty z & (j = 1, 2, 3), \\
\theta_{3jk} &= -\frac{\xi}{\gamma_{1k}^\infty} + \gamma_{2j}^\infty (h_1 - z), \quad \theta_{4jk} = -\frac{h_1 - \xi}{\gamma_{1k}^\infty} + \gamma_{2j}^\infty (h_1 - z) & (j = 4, 5, 6),
\end{aligned} \right\} \quad (k = 1, 2, 3), \quad (B.6)$$

$$\left. \begin{aligned}
p_{i1j}^\infty &= \lim_{s \rightarrow \infty} p_{i1j} \quad (i = 1, 2, \dots, 8, \quad j = 1, 2, 3), & p_{i2j}^\infty &= \lim_{s \rightarrow \infty} p_{i2j} \quad (i = 1, 2, \dots, 8, \quad j = 1, 2, \dots, 6), \\
\gamma_{1j}^\infty &= \lim_{s \rightarrow \infty} \gamma_{1j} \quad (j = 1, 2, 3), & \gamma_{2j}^\infty &= \lim_{s \rightarrow \infty} \gamma_{2j} \quad (j = 1, 2, \dots, 6), \\
r_{ji}^\infty &= \lim_{s \rightarrow \infty} r_{ji} \quad (i, j = 1, 2, \dots, 10), & \delta_j^\infty &= \lim_{s \rightarrow \infty} \delta_j \quad (j = 1, 2, 3).
\end{aligned} \right\} \quad (B.7)$$

The functions $r_{ij} \equiv r_{ij}(s, p)$ ($i, j = 1, 2, \dots, 10$) are the elements of a square matrix $\mathbf{R} = \mathbf{W}^{-1}$ of order 10. The elements $w_{i,j} \equiv w_{i,j}(s, p)$ ($i, j = 1, 2, \dots, 10$) of the square matrix $\mathbf{W}$ are given by

$$\left. \begin{aligned}
w_{i-2,j} &= p_{i2j} & (i = 3, 4, 5), \\
w_{i+1,j} &= p_{i2j} \exp(s \gamma_{2j} h_1) & (i = 3, 4, 5), \\
w_{i+1,j} &= p_{i2j} & (i = 6, 7),
\end{aligned} \right\} \quad (j = 1, 2, \dots, 6), \quad (B.8)$$

$$\left. \begin{aligned}
w_{1,7} &= 2c_{442}\gamma_1, & w_{1,8} &= 2c_{442}\gamma_1, \\
w_{1,9} &= c_{442}(1+\gamma_2^2), & w_{1,10} &= c_{442}(1+\gamma_2^2), \\
w_{2,7} &= c_{442}(1+\gamma_2^2), & w_{2,8} &= -c_{442}(1+\gamma_2^2), \\
w_{2,9} &= 2c_{442}\gamma_2, & w_{2,10} &= -2c_{442}\gamma_2, \\
w_{7,7} &= -1, & w_{7,8} &= 1, \\
w_{7,9} &= -\gamma_2, & w_{7,10} &= \gamma_2, \\
w_{8,7} &= -\gamma_1, & w_{8,8} &= -\gamma_1, \\
w_{8,9} &= -1, & w_{8,10} &= -1, \\
w_{9,7} &= 2\gamma_1 \exp(s\gamma_1 h_2), & w_{9,8} &= 2\gamma_1 \exp(-s\gamma_1 h_2), \\
w_{9,9} &= (1+\gamma_2^2) \exp(s\gamma_2 h_2), & w_{9,10} &= (1+\gamma_2^2) \exp(-s\gamma_2 h_2), \\
w_{10,7} &= (1+\gamma_2^2) \exp(s\gamma_1 h_2), & w_{10,8} &= -(1+\gamma_2^2) \exp(-s\gamma_1 h_2), \\
w_{10,9} &= 2\gamma_2 \exp(s\gamma_2 h_2), & w_{10,10} &= -2\gamma_2 \exp(-s\gamma_2 h_2).
\end{aligned} \right\} \quad (B.9)$$

The functions $Q_i \equiv Q_i(s, \xi, p)$ ($i=1, 2, \dots, 8$) are

$$\begin{aligned}
Q_1 &= \frac{1}{2} \sum_{j=1}^3 \Re \left[\frac{p_{31j}^\infty \delta_j^\infty}{\gamma_{1j}^\infty} \exp\left(\frac{s\xi}{\gamma_{1j}^\infty}\right) \right] \\
&\quad - \frac{s}{\pi} \sum_{j=1}^3 \int_0^\infty \left\{ \Re \left[\frac{p_{31j} \delta_j}{\gamma_{1j}^2} \frac{1}{\eta^2 + (s/\gamma_{1j})^2} \right] - \Re \left[\frac{p_{31j}^\infty \delta_j^\infty}{\gamma_{1j}^{\infty 2}} \frac{1}{\eta^2 + (s/\gamma_{1j}^\infty)^2} \right] \right\} \cos(\eta\xi) d\eta \\
&\quad + \frac{s}{\pi} \sum_{j=1}^3 \int_0^\infty \Im \left[\frac{p_{31j} \delta_j}{\gamma_{1j}^2} \frac{1}{\eta^2 + (s/\gamma_{1j})^2} \right] \sin(\eta\xi) d\eta,
\end{aligned} \quad (B.10)$$

$$\begin{aligned}
Q_2 &= \frac{1}{2} \sum_{j=1}^3 \Re \left[\frac{p_{41j}^\infty \delta_j^\infty}{\gamma_{1j}^\infty} \exp\left(\frac{s\xi}{\gamma_{1j}^\infty}\right) \right] \\
&\quad + \frac{1}{\pi} \sum_{j=1}^3 \int_0^\infty \left\{ \Re \left[\frac{p_{41j} \delta_j}{\gamma_{1j}} \frac{\eta}{\eta^2 + (s/\gamma_{1j})^2} \right] - \Re \left[\frac{p_{41j}^\infty \delta_j^\infty}{\gamma_{1j}^\infty} \frac{\eta}{\eta^2 + (s/\gamma_{1j}^\infty)^2} \right] \right\} \sin(\eta\xi) d\eta \\
&\quad + \frac{1}{\pi} \sum_{j=1}^3 \int_0^\infty \Im \left[\frac{p_{41j} \delta_j}{\gamma_{1j}} \frac{\eta}{\eta^2 + (s/\gamma_{1j})^2} \right] \cos(\eta\xi) d\eta,
\end{aligned} \quad (B.11)$$

$$\begin{aligned}
Q_3 &= \frac{1}{2} \sum_{j=1}^3 \Re \left[p_{51j}^\infty \delta_j^\infty \exp\left(\frac{s\xi}{\gamma_{1j}^\infty}\right) \right] \\
&\quad - \frac{s}{\pi} \sum_{j=1}^3 \int_0^\infty \left\{ \Re \left[\frac{p_{51j} \delta_j}{\gamma_{1j}} \frac{1}{\eta^2 + (s/\gamma_{1j})^2} \right] - \Re \left[\frac{p_{51j}^\infty \delta_j^\infty}{\gamma_{1j}^\infty} \frac{1}{\eta^2 + (s/\gamma_{1j}^\infty)^2} \right] \right\} \cos(\eta\xi) d\eta \\
&\quad + \frac{s}{\pi} \sum_{j=1}^3 \int_0^\infty \Im \left[\frac{p_{51j} \delta_j}{\gamma_{1j}} \frac{\eta}{\eta^2 + (s/\gamma_{1j})^2} \right] \sin(\eta\xi) d\eta,
\end{aligned} \quad (B.12)$$

$$\begin{aligned}
Q_4 = & \frac{1}{2} \sum_{j=1}^3 \Re \left[\frac{p_{31j}^{\infty} \delta_j^{\infty}}{\gamma_{1j}^{\infty}} \exp \left(\frac{s(h_1 - \xi)}{\gamma_{1j}^{\infty}} \right) \right] \\
& - \frac{s}{\pi} \sum_{j=1}^3 \int_0^{\infty} \left\{ \Re \left[\frac{p_{31j} \delta_j}{\gamma_{1j}^2} \frac{1}{\eta^2 + (s/\gamma_{1j})^2} \right] - \Re \left[\frac{p_{31j}^{\infty} \delta_j^{\infty}}{\gamma_{1j}^{\infty 2}} \frac{1}{\eta^2 + (s/\gamma_{1j}^{\infty})^2} \right] \right\} \\
& \quad \times \cos[\eta(h_1 - \xi)] d\eta \\
& - \frac{s}{\pi} \sum_{j=1}^3 \int_0^{\infty} \Im \left[\frac{p_{31j} \delta_j}{\gamma_{1j}^2} \frac{1}{\eta^2 + (s/\gamma_{1j})^2} \right] \sin[\eta(h_1 - \xi)] d\eta,
\end{aligned} \tag{B.13}$$

$$\begin{aligned}
Q_5 = & \frac{1}{2} \sum_{j=1}^3 \Re \left[\frac{p_{41j}^{\infty} \delta_j^{\infty}}{\gamma_{1j}^{\infty}} \exp \left(\frac{s(h_1 - \xi)}{\gamma_{1j}^{\infty}} \right) \right] \\
& + \frac{1}{\pi} \sum_{j=1}^3 \int_0^{\infty} \left\{ \Re \left[\frac{p_{41j} \delta_j}{\gamma_{1j}^2} \frac{\eta}{\eta^2 + (s/\gamma_{1j})^2} \right] - \Re \left[\frac{p_{41j}^{\infty} \delta_j^{\infty}}{\gamma_{1j}^{\infty 2}} \frac{\eta}{\eta^2 + (s/\gamma_{1j}^{\infty})^2} \right] \right\} \\
& \quad \times \sin[\eta(h_1 - \xi)] d\eta \\
& - \frac{1}{\pi} \sum_{j=1}^3 \int_0^{\infty} \Im \left[\frac{p_{41j} \delta_j}{\gamma_{1j}^2} \frac{\eta}{\eta^2 + (s/\gamma_{1j})^2} \right] \cos[\eta(h_1 - \xi)] d\eta,
\end{aligned} \tag{B.14}$$

$$\begin{aligned}
Q_6 = & -\frac{1}{2} \sum_{j=1}^3 \Re \left[p_{51j}^{\infty} \delta_j^{\infty} \exp \left(\frac{s(h_1 - \xi)}{\gamma_{1j}^{\infty}} \right) \right] \\
& + \frac{s}{\pi} \sum_{j=1}^3 \int_0^{\infty} \left\{ \Re \left[\frac{p_{51j} \delta_j}{\gamma_{1j}} \frac{1}{\eta^2 + (s/\gamma_{1j})^2} \right] - \Re \left[\frac{p_{51j}^{\infty} \delta_j^{\infty}}{\gamma_{1j}^{\infty}} \frac{1}{\eta^2 + (s/\gamma_{1j}^{\infty})^2} \right] \right\} \\
& \quad \times \cos[\eta(h_1 - \xi)] d\eta \\
& + \frac{s}{\pi} \sum_{j=1}^3 \int_0^{\infty} \Im \left[\frac{p_{51j} \delta_j}{\gamma_{1j}} \frac{1}{\eta^2 + (s/\gamma_{1j})^2} \right] \sin[\eta(h_1 - \xi)] d\eta,
\end{aligned} \tag{B.15}$$

$$\begin{aligned}
Q_7 = & -\frac{1}{2} \sum_{j=1}^3 \Re \left[p_{61j}^{\infty} \delta_j^{\infty} \left\{ 1 - \exp \left(\frac{s\xi}{\gamma_{1j}^{\infty}} \right) \right\} \right] \\
& - \frac{s^2}{\pi} \sum_{j=1}^3 \int_0^{\infty} \frac{1}{\eta} \left\{ \Re \left[\frac{p_{61j} \delta_j}{\gamma_{1j}^2} \frac{1}{\eta^2 + (s/\gamma_{1j})^2} \right] - \Re \left[\frac{p_{61j}^{\infty} \delta_j^{\infty}}{\gamma_{1j}^{\infty 2}} \frac{1}{\eta^2 + (s/\gamma_{1j}^{\infty})^2} \right] \right\} \sin(\eta\xi) d\eta \\
& - \frac{s^2}{\pi} \sum_{j=1}^3 \int_0^{\infty} \frac{1}{\eta} \Im \left[\frac{p_{61j} \delta_j}{\gamma_{1j}^2} \frac{1}{\eta^2 + (s/\gamma_{1j})^2} \right] \cos(\eta\xi) d\eta,
\end{aligned} \tag{B.16}$$

$$\begin{aligned}
Q_8 = & -\frac{1}{2} \sum_{j=1}^3 \Re \left[p_{71j}^{\infty} \delta_j^{\infty} \exp \left(\frac{s\xi}{\gamma_{1j}^{\infty}} \right) \right] \\
& + \frac{s}{\pi} \sum_{j=1}^3 \int_0^{\infty} \left\{ \Re \left[\frac{p_{71j} \delta_j}{\gamma_{1j}} \frac{1}{\eta^2 + (s/\gamma_{1j})^2} \right] - \Re \left[\frac{p_{71j}^{\infty} \delta_j^{\infty}}{\gamma_{1j}^{\infty}} \frac{1}{\eta^2 + (s/\gamma_{1j}^{\infty})^2} \right] \right\} \cos(\eta\xi) d\eta \\
& - \frac{s}{\pi} \sum_{j=1}^3 \int_0^{\infty} \Im \left[\frac{p_{71j} \delta_j}{\gamma_{1j}} \frac{1}{\eta^2 + (s/\gamma_{1j})^2} \right] \sin(\eta\xi) d\eta.
\end{aligned} \tag{B.17}$$

The semi-infinite integral of the kernel function $M_3(\xi, z)$ in Eqs. (B.4) is practicable and the result is

$$\begin{aligned}
M_3(\xi, z) &= -\frac{2}{\Re[Z^\infty]} \int_0^\infty m^\infty(s, \xi, z) ds \\
&= -\frac{2}{\Re[Z^\infty]} \sum_{k=1}^3 \left[\sum_{i=1}^3 P_{i+2,1k}^\infty \left(\sum_{j=1}^3 \frac{F_{ijk}}{\theta_{1jk}} + \sum_{j=4}^6 \frac{F_{ijk}}{\theta_{3jk}} \right) + \sum_{i=4}^5 P_{i-1,1k}^\infty \left(\sum_{j=1}^3 \frac{F_{ijk}}{\theta_{2jk}} + \sum_{j=4}^6 \frac{F_{ijk}}{\theta_{4jk}} \right) \right. \\
&\quad + P_{51k}^\infty \left(\sum_{j=1}^3 \frac{F_{6jk}}{\theta_{2jk}} + \sum_{j=4}^6 \frac{F_{6jk}}{\theta_{4jk}} \right) \\
&\quad - P_{61k}^\infty \left(\sum_{j=1}^3 \frac{F_{7jk}}{\theta_{1jk}} + \sum_{j=4}^6 \frac{F_{7jk}}{\theta_{3jk}} + \sum_{j=1}^3 \frac{F_{7jk}}{\gamma_{2j}^\infty z} - \sum_{j=4}^6 \frac{F_{7jk}}{\gamma_{2j}^\infty (h_1 - z)} \right) \\
&\quad \left. - P_{71k}^\infty \left(\sum_{j=1}^3 \frac{F_{8jk}}{\theta_{1jk}} + \sum_{j=4}^6 \frac{F_{8jk}}{\theta_{3jk}} \right) \right]. \tag{B.18}
\end{aligned}$$

Appendix C

The elements $\Delta_{i,j} \equiv \Delta_{i,j}(\Omega)$ ($i, j = 1, \dots, 10$) of Δ are

$$\left. \begin{aligned}
&\Delta_{1,1} = \text{Re}[\lambda_{z11}(-1)^{-\Omega}], & \Delta_{1,2} = \text{Im}[\lambda_{z11}(-1)^{-\Omega}], & \Delta_{1,3} = \text{Re}[\lambda_{z12}(-1)^{-\Omega}], \\
&\Delta_{1,4} = \text{Im}[\lambda_{z12}(-1)^{-\Omega}], & \Delta_{1,5} = \text{Re}[\lambda_{z13}(-1)^{-\Omega}], & \Delta_{1,6} = \text{Im}[\lambda_{z13}(-1)^{-\Omega}], \\
&\Delta_{1,k} = 0 \ (k = 7 \sim 10), & \Delta_{2,1} = \text{Re}[\lambda_{zx11}(-1)^{-\Omega}], & \Delta_{2,2} = \text{Im}[\lambda_{zx11}(-1)^{-\Omega}], \\
&\Delta_{2,3} = \text{Re}[\lambda_{zx12}(-1)^{-\Omega}], & \Delta_{2,4} = \text{Im}[\lambda_{zx12}(-1)^{-\Omega}], & \Delta_{2,5} = \text{Re}[\lambda_{zx13}(-1)^{-\Omega}], \\
&\Delta_{2,6} = \text{Im}[\lambda_{zx13}(-1)^{-\Omega}], & \Delta_{2,k} = 0 \ (k = 7 \sim 10), & \Delta_{3,1} = \text{Re}[\lambda_{x11}p_{11}^{-\Omega}], \\
&\Delta_{3,2} = \text{Im}[\lambda_{x11}p_{11}^{-\Omega}], & \Delta_{3,3} = \text{Re}[\lambda_{x12}p_{12}^{-\Omega}], & \Delta_{3,4} = \text{Im}[\lambda_{x12}p_{12}^{-\Omega}], \\
&\Delta_{3,5} = \text{Re}[\lambda_{x13}p_{13}^{-\Omega}], & \Delta_{3,6} = \text{Im}[\lambda_{x13}p_{13}^{-\Omega}], & \Delta_{3,7} = -\text{Re}[\lambda_{x21}p_{21}^{-\Omega}], \\
&\Delta_{3,8} = -\text{Im}[\lambda_{x21}p_{21}^{-\Omega}], & \Delta_{3,9} = -\text{Re}[\lambda_{x22}p_{22}^{-\Omega}], & \Delta_{3,10} = -\text{Im}[\lambda_{x22}p_{22}^{-\Omega}], \\
&\Delta_{4,1} = \text{Re}[\lambda_{zx11}p_{11}^{-\Omega}], & \Delta_{4,2} = \text{Im}[\lambda_{zx11}p_{11}^{-\Omega}], & \Delta_{4,3} = \text{Re}[\lambda_{zx12}p_{12}^{-\Omega}], \\
&\Delta_{4,4} = \text{Im}[\lambda_{zx12}p_{12}^{-\Omega}], & \Delta_{4,5} = \text{Re}[\lambda_{zx13}p_{13}^{-\Omega}], & \Delta_{4,6} = \text{Im}[\lambda_{zx13}p_{13}^{-\Omega}], \\
&\Delta_{4,7} = -\text{Re}[\lambda_{zx21}p_{21}^{-\Omega}], & \Delta_{4,8} = -\text{Im}[\lambda_{zx21}p_{21}^{-\Omega}], & \Delta_{4,9} = -\text{Re}[\lambda_{zx22}p_{22}^{-\Omega}], \\
&\Delta_{4,10} = -\text{Im}[\lambda_{zx22}p_{22}^{-\Omega}], & \Delta_{5,1} = \text{Re}[q_{x11}p_{11}^{1-\Omega}], & \Delta_{5,2} = \text{Im}[q_{x11}p_{11}^{1-\Omega}], \\
&\Delta_{5,3} = \text{Re}[q_{x12}p_{12}^{1-\Omega}], & \Delta_{5,4} = \text{Im}[q_{x12}p_{12}^{1-\Omega}], & \Delta_{5,5} = \text{Re}[q_{x13}p_{13}^{1-\Omega}], \\
&\Delta_{5,6} = \text{Im}[q_{x13}p_{13}^{1-\Omega}], & \Delta_{5,7} = -\text{Re}[q_{x21}p_{21}^{1-\Omega}], & \Delta_{5,8} = -\text{Im}[q_{x21}p_{21}^{1-\Omega}], \\
&\Delta_{5,9} = -\text{Re}[q_{x22}p_{22}^{1-\Omega}], & \Delta_{5,10} = -\text{Im}[q_{x22}p_{22}^{1-\Omega}], & \Delta_{6,1} = \text{Re}[q_{z11}p_{11}^{1-\Omega}], \\
&\Delta_{6,2} = \text{Im}[q_{z11}p_{11}^{1-\Omega}], & \Delta_{6,3} = \text{Re}[q_{z12}p_{12}^{1-\Omega}], & \Delta_{6,4} = \text{Im}[q_{z12}p_{12}^{1-\Omega}], \\
&\Delta_{6,5} = \text{Re}[q_{z13}p_{13}^{1-\Omega}], & \Delta_{6,6} = \text{Im}[q_{z13}p_{13}^{1-\Omega}], & \Delta_{6,7} = -\text{Re}[q_{z21}p_{21}^{1-\Omega}], \\
&\Delta_{6,8} = -\text{Im}[q_{z21}p_{21}^{1-\Omega}], & \Delta_{6,9} = -\text{Re}[q_{z22}p_{22}^{1-\Omega}], & \Delta_{6,10} = -\text{Im}[q_{z22}p_{22}^{1-\Omega}], \\
&\Delta_{7,k} = 0 \ (k = 1 \sim 6), & \Delta_{7,7} = \text{Re}[\lambda_{zx21}], & \Delta_{7,8} = \text{Im}[\lambda_{zx21}], \\
&\Delta_{7,9} = \text{Re}[\lambda_{zx22}], & \Delta_{7,10} = \text{Im}[\lambda_{zx22}], & \Delta_{8,k} = 0 \ (k = 1 \sim 6), \\
&\Delta_{8,7} = \text{Re}[q_{z21}], & \Delta_{8,8} = \text{Im}[q_{z21}], & \Delta_{8,9} = \text{Re}[q_{z22}], \\
&\Delta_{8,10} = \text{Im}[q_{z22}], & \Delta_{9,1} = \text{Re}[q_{\phi11}(-1)^{-\Omega}], & \Delta_{9,2} = \text{Im}[q_{\phi11}(-1)^{-\Omega}], \\
&\Delta_{9,3} = \text{Re}[q_{\phi12}(-1)^{-\Omega}], & \Delta_{9,4} = \text{Im}[q_{\phi12}(-1)^{-\Omega}], & \Delta_{9,5} = \text{Re}[q_{\phi13}(-1)^{-\Omega}], \\
&\Delta_{9,6} = \text{Im}[q_{\phi13}(-1)^{-\Omega}], & \Delta_{9,k} = 0 \ (k = 7 \sim 10), & \Delta_{10,1} = \text{Re}[\lambda_{xD11}p_{11}^{-\Omega}], \\
&\Delta_{10,2} = \text{Im}[\lambda_{xD11}p_{11}^{-\Omega}], & \Delta_{10,3} = \text{Re}[\lambda_{xD12}p_{12}^{-\Omega}], & \Delta_{10,4} = \text{Im}[\lambda_{xD12}p_{12}^{-\Omega}], \\
&\Delta_{10,5} = \text{Re}[\lambda_{xD13}p_{13}^{-\Omega}], & \Delta_{10,6} = \text{Im}[\lambda_{xD13}p_{13}^{-\Omega}], & \Delta_{10,k} = 0 \ (k = 7 \sim 10),
\end{aligned} \right\} \tag{C.1}$$

where $\lambda_{x1L}, \lambda_{z1L}, \lambda_{zx1L}, \lambda_{xD1L}, \lambda_{zD1L}$ ($L=1, 2, 3$) and $\lambda_{x2L}, \lambda_{z2L}, \lambda_{zx2L}$ ($L=1, 2$) are given by

$$\left. \begin{aligned} \lambda_{x1L} &= c_{110}q_{x1L} + (c_{130}q_{z1L} + e_{310}q_{\phi1L})p_{1L}, \\ \lambda_{z1L} &= c_{130}q_{x1L} + (c_{330}q_{z1L} + e_{330}q_{\phi1L})p_{1L}, \\ \lambda_{zx1L} &= c_{440}(p_{1L}q_{x1L} + q_{z1L}) + e_{150}q_{\phi1L}, \\ \lambda_{xD1L} &= e_{150}q_{z1L} - (\varepsilon_{110}q_{\phi1L} - e_{150}q_{x1L})p_{1L}, \\ \lambda_{zD1L} &= e_{310}q_{x1L} - (\varepsilon_{330}q_{\phi1L} - e_{310}q_{z1L})p_{1L}, \end{aligned} \right\} \quad (L=1,2,3), \quad (C.2)$$

$$\left. \begin{aligned} \lambda_{x2L} &= c_{112}q_{x2L} + c_{132}p_{2L}q_{z2L}, \\ \lambda_{z2L} &= c_{132}q_{x2L} + c_{332}p_{2L}q_{z2L}, \\ \lambda_{zx2L} &= c_{442}(p_{2L}q_{x2L} + q_{z2L}), \end{aligned} \right\} \quad (L=1,2). \quad (C.3)$$

The values of p_{1L} ($L=1,2,3$) make the determinant of coefficient matrix of the following homogeneous simultaneous equations vanish and $q_{x1L}, q_{z1L}, q_{\phi1L}$ ($L=1,2,3$) are the associated solutions of the simultaneous equations.

$$\begin{bmatrix} c_{110} + c_{440}p_{1L}^2 & (c_{130} + c_{440})p_{1L} & (e_{130} + e_{150})p_{1L} \\ (c_{130} + c_{440})p_{1L} & c_{440} + c_{330}p_{1L}^2 & e_{150} + e_{330}p_{1L}^2 \\ (e_{130} + e_{150})p_{1L} & e_{150} + e_{330}p_{1L}^2 & -\varepsilon_{110} - \varepsilon_{330}p_{1L}^2 \end{bmatrix} \begin{bmatrix} q_{x1L} \\ q_{z1L} \\ q_{\phi1L} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (L=1,2,3). \quad (C.4)$$

Similarly, the homogeneous simultaneous equations to determine the values of p_{2L} and q_{x2L}, q_{z2L} ($L=1,2$) are

$$\begin{bmatrix} c_{112} + c_{442}p_{2L}^2 & (c_{132} + c_{442})p_{2L} \\ (c_{132} + c_{442})p_{2L} & c_{442} + c_{332}p_{2L}^2 \end{bmatrix} \begin{bmatrix} q_{x2L} \\ q_{z2L} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (L=1,2). \quad (C.5)$$

Appendix D

The constants S_{1k}, S_{2k} ($k=1,2,3$) are

$$\left. \begin{aligned} S_{1k} &= \sum_{j=1}^3 (2r_{B3j}^{\infty} + r_{B4j}^{\infty}) \operatorname{Re} [p_{j+2,1k}^{\infty} \delta_k^{\infty}], \\ S_{2k} &= \sum_{j=7}^8 (2r_{B3j}^{\infty} + r_{B4j}^{\infty}) \operatorname{Re} [p_{j-1,1k}^{\infty} \gamma_{1k}^{\infty} \delta_k^{\infty}], \end{aligned} \right\} \quad (k=1,2,3), \quad (D.1)$$

where

$$r_{B3j}^{\infty} = \begin{cases} \sum_{i=1}^3 p_{62i}^{\infty} r_{2ij}^{\infty} & (j=1 \sim 6,8), \\ \sum_{i=1}^3 p_{62i}^{\infty} r_{2ij}^{\infty} + 1 & (j=7), \end{cases} \quad (D.2)$$

$$r_{B4j}^{\infty} = \begin{cases} \sum_{i=1}^3 (p_{62i}^{\infty} + p_{72i}^{\infty}) r_{2ij}^{\infty} & (j=1 \sim 6), \\ \sum_{i=1}^3 (p_{62i}^{\infty} + p_{72i}^{\infty}) r_{2ij}^{\infty} + 1 & (j=7), \\ \sum_{i=1}^3 (p_{62i}^{\infty} + p_{72i}^{\infty}) r_{2ij}^{\infty} - 1 & (j=8), \end{cases} \quad (D.3)$$

with

$$r_{21j}^{\infty} = \frac{\lambda_{222}^{\infty} \lambda_{R1j}^{\infty} - \lambda_{212}^{\infty} \lambda_{R2j}^{\infty}}{\lambda_{212}^{\infty} \lambda_{221}^{\infty} - \lambda_{222}^{\infty} \lambda_{211}^{\infty}}, \quad r_{22j}^{\infty} = \frac{\lambda_{221}^{\infty} \lambda_{R1j}^{\infty} - \lambda_{211}^{\infty} \lambda_{R2j}^{\infty}}{\lambda_{211}^{\infty} \lambda_{222}^{\infty} - \lambda_{221}^{\infty} \lambda_{212}^{\infty}}, \quad (j=1 \sim 8), \quad (\text{D.4})$$

$$r_{23j}^{\infty} = \left\{ \begin{array}{ll} -\frac{g_{231}^{\infty} r_{21j}^{\infty} + g_{232}^{\infty} r_{22j}^{\infty}}{g_{233}^{\infty}} & (j=1 \sim 2, 4 \sim 8), \\ -\frac{g_{231}^{\infty} r_{21j}^{\infty} + g_{232}^{\infty} r_{22j}^{\infty}}{g_{233}^{\infty}} + 1 & (j=3), \end{array} \right\} \quad (\text{D.5})$$

$$\left. \begin{array}{l} \lambda_{21j}^{\infty} = g_{223}^{\infty} g_{21j}^{\infty} - g_{213}^{\infty} g_{22j}^{\infty}, \\ \lambda_{22j}^{\infty} = g_{233}^{\infty} g_{21j}^{\infty} - g_{213}^{\infty} g_{23j}^{\infty}, \end{array} \right\} \quad (j=1, 2), \quad (\text{D.6})$$

and the non-zero components of λ_{Rij}^{∞} ($i=1, 2, j=1 \sim 8$) are

$$\left. \begin{array}{ll} \lambda_{R11}^{\infty} = -g_{223}^{\infty}, & \lambda_{R12}^{\infty} = -g_{213}^{\infty}, \\ \lambda_{R17}^{\infty} = -g_{223}^{\infty} g_1^{\infty} - g_{213}^{\infty} g_2^{\infty}, & \lambda_{R18}^{\infty} = -g_{223}^{\infty} g_2^{\infty} - g_{213}^{\infty} g_1^{\infty}, \\ \lambda_{R21}^{\infty} = -g_{233}^{\infty}, & \lambda_{R23}^{\infty} = -g_{213}^{\infty}, \\ \lambda_{R27}^{\infty} = -g_{233}^{\infty} g_1^{\infty}, & \lambda_{R28}^{\infty} = -g_{233}^{\infty} g_2^{\infty}. \end{array} \right\} \quad (\text{D.7})$$

In the above equations, g_{2ij}^{∞} ($i, j=1, 2, 3$) and g_i^{∞} ($i=1, 2$) are given by

$$\left. \begin{array}{l} g_{21j}^{\infty} = p_{32j}^{\infty} - g_1^{\infty} p_{62j}^{\infty} + g_2^{\infty} p_{72j}^{\infty}, \\ g_{22j}^{\infty} = p_{42j}^{\infty} + g_2^{\infty} p_{62j}^{\infty} - g_1^{\infty} p_{72j}^{\infty}, \\ g_{23j}^{\infty} = p_{52j}^{\infty}, \end{array} \right\} \quad (j=1, 2, 3), \quad (\text{D.8})$$

$$g_1^{\infty} = 2c_{442} \frac{2(1-\nu)}{3-4\nu}, \quad g_2^{\infty} = 2c_{442} \frac{1-2\nu}{3-4\nu}. \quad (\text{D.9})$$