How do bubbles reduce the speed of sound in a bubbly liquid in a duct?

Junya KAWAHARA*, Masao WATANABE* and Kazumichi KOBAYASHI*

* Division of Mechanical and Space Engineering, Hokkaido University
N13W8, Kita-ku, Sapporo, 060-8628, Japan
E-mail: junyakawahara@frontier.hokudai.ac.jp

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Abstract
The purpose of this study is to clarify how bubbles reduce the speed of sound in a bubbly liquid in a duct, in which a homogeneous medium can no longer be assumed. Getting inspired by the explanation for the origin of refractive indices in the field of optics, we theoretically examine pressure wave propagation in a square duct filled with a compressible liquid containing only a spherical bubble. Theoretical examination reveals that even a single bubble in a square duct can delay the phase of the input pressure wave, causing an apparent reduction in the speed of sound. Based on this result, we can define the speed of sound in a bubbly liquid under the assumption of a homogeneous medium by considering the phase delay caused by radial oscillations of bubbles aligned in the square duct in the limit of numerous bubbles.

Key words: Bubble, Bubbly liquid, Bubble dynamics, Pressure wave, Speed of sound

1. Introduction

Pressure wave propagation in a liquid containing bubbles in a duct is important for the development of mechanical engineering applications (Kameda and Matsumoto, 1996; Matsukuma et al., 2013; Minato, 2002; Ohtani et al., 2002). Recently, Kaneko et al. proposed a technique to generate microbubbles utilizing a venturi tube (Kaneko et al., 2012; Uesawa et al., 2012). They explained the mechanism of microbubble generation in a venturi tube based on a well-known theory of supersonic flow in a Laval nozzle. Because the speed of sound in a bubbly liquid is much lower than that in a pure liquid, the velocity of a bubbly flow easily reaches the speed of sound in a bubbly liquid in the throat of a venturi tube. Subsequently, with the rapid pressure recovery in the diverging area, several or a few bubbles injected into the converging area collapse violently in the diverging area. The collapse of these bubbles generates numerous microbubbles in the diverging area. The mechanism of microbubble generation can be explained on the basis of the reduction in the speed of sound in the liquid containing bubbles.

In previous studies (Kaneko et al., 2012; van Wijngaarden, 1972; Wilson and Roy, 2008; Wood, 1930), the mechanism of the reduction in the speed of sound was explained in a bubbly liquid, in which a homogeneous medium could be assumed. In general, the speed of sound $a$ is defined as $a = 1/\sqrt{\kappa}$, where $\rho$ is the density and $\kappa$ is the compressibility; hence, we need to derive these quantities for a bubbly liquid. The most important parameter is the volume fraction of the gas phase, i.e., the void fraction $\beta$ that is given by $\beta = V_g/(V_l + V_g)$, where $V$ is the volume and the subscripts $l$ and $g$ refer to the liquid and gas phases, respectively. The mixture density and compressibility are $\rho_m = (1 - \beta)\rho_l + \beta\rho_g$ and $\kappa_m = (1 - \beta)\kappa_l + \beta\kappa_g$, respectively, where the subscript $m$ denotes the effective quantities pertaining to the mixture. Therefore, the ratio of the speed of sound in a pure liquid to that in a mixture, $\chi$, can be written as follows:

$$\chi = \frac{a_l}{a_m} = \left[ (1 - \beta)^2 + \frac{\beta^2 a_l^2}{a_g^2} + \beta(1 - \beta) \frac{\rho_l^2 a_g^2 + \rho_g^2 a_l^2}{\rho_l \rho_g a_l^2} \right]^{1/2}. \tag{1}$$

Equation (1) is known as Wood’s equation (van Wijngaarden, 1972; Wilson and Roy, 2008; Wood, 1930). From Eq. (1), the mechanism by which the speed of sound in a bubbly liquid is reduced from that in a pure liquid in the context of...
Is it merely a coincidence? The purpose of this study is to answer this question by presenting a theoretical basis for the reduction in the speed of sound in a bubbly liquid, or rather, a liquid containing gas bubbles that cannot be considered as a homogeneous continuum medium. We investigate pressure wave propagation in a bubbly liquid that is composed of spherical gas bubbles suspended in a liquid medium in a square duct. Pressure waves cause radial oscillations of the spherical gas bubbles, which generate pressure fields. Therefore, the pressure waves are scattered by the bubbles. We assume that the bubble size is much smaller than the wavelength of the pressure wave; hence, we restrict ourselves to low-frequency wave propagation in a bubbly liquid (Wilson and Roy, 2008). Then, the bubbles become acoustic scatterers that exhibit a low-frequency resonance known as the Minnaert resonance (Minnaert, 1933). A number of investigators have studied pressure wave propagation including acoustic scattering generated by bubbles in a bubbly liquid, in which a homogeneous medium can be practically assumed (Caflisch et al., 1985; Carstensen and Foldy, 1947; Commander and Prosperetti, 1989; Foldy, 1945; Fuster et al., 2014). Based on their studies, we will clarify the mechanism of the reduction in the speed of sound in a liquid containing bubbles in a square duct.

To do so, we recall that the speed of electric waves in a material with a refractive index $n$ is $c/n$, where $c$ is the speed of light. We are greatly inspired by the explanation of this phenomenon given by “The Origin of the Refractive Index” in Feynman et al. (1963). We have determined that we can have the same discussion for the origin of $\chi$ as was done for $n$; to this end, we hereafter refer to $\chi$ as the refractive index of a bubbly liquid. Recently, Leroy et al. (2009) experimentally and theoretically studied the transmission of ultrasound through a single layer of bubbles based on Feynman’s study. Here, we investigate pressure wave propagation in a liquid containing spherical bubbles in a square duct instead of electric wave propagation in a material. In particular, we discuss the reduction in the speed of sound in a liquid containing spherical bubbles under the following physical assumptions: 1. the total pressure field can always be represented by the summation of the pressure fields generated by all radial oscillations of the bubbles, and 2. the pressure field generated by a single bubble is given by its radial oscillation evaluated with retardation at a speed of $a_t$. 

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**Fig. 1** (a) A single spherical bubble placed in a square duct filled with a compressible liquid. (b) A schematic of the distribution of imaginary bubbles in the $y-z$ plane.
2. Radial oscillation of a single bubble in a square duct

We investigate pressure wave propagation in a square duct with the side of $D$ [Fig. 1(a)] filled with a compressible liquid that contains a single bubble placed at the point $B$. The input plane linear wave propagates along the $x$ direction of the duct. We suppose that the input plane pressure wave $p_0 + p_{in}(t)$ is measured at the center of the bubble, where $t$ is the time, $p_0$ is the undisturbed pressure, and $p_{in}(t)$ is the oscillatory component of the wave. The input wave has an angular frequency $\omega$ and a wavelength $\lambda$, where $\omega \lambda = 2\pi a_i$, and has the following form: $p_0 + p_{in}(t) = p_0 + P_0 e^{i\omega t}$, where $P_0$ is the amplitude of the input wave ($P_0/p_0 \ll 1$).

Now, we consider the pressure field generated at the point $Q$ by the radial oscillation of the spherical bubble. In this study, the following assumptions are made: (1) the center of the bubble is fixed in the duct, (2) the radial oscillation of the bubble is spherically symmetric, (3) the flow field in the liquid is irrotational, and (4) the terms of the attenuation due to the acoustic radiation of order $1/\alpha t$ are neglected. Then, the liquid velocity potential $\phi$ for the flow field generated by the radial oscillation of a single bubble at the distance $r$ can be written as $\phi(t) = -R^2(t - r/a_i)R(t - r/a_i)/r$ (Keller and Kolodner, 1956; Prosperetti and Lezzi, 1986; Takahira et al., 1995), where $R(t)$ is the bubble radius. It should be emphasized here that all attenuations, including the acoustic radiation of order $1/\alpha t$, are ignored, but the delay time is considered (where the delay time is defined as the finite amount of time that it takes for a pressure change to propagate a given distance with the speed of sound in a pure liquid being $a_i$). We further assume that the bubble is filled with a non-condensable gas that follows the polytropic process without phase changes; then, the radial oscillation of the bubble is given by

$$R(t)R(t) + \frac{3}{2} R^2(t) = \frac{1}{p_t} \left\{ p_0 \left[ \frac{R_0}{R(t)} \right]^3 - p_s(t) \right\}, \quad (2)$$

where $R_0$ is the initial undisturbed bubble radius, $\gamma$ is the polytropic constant, and $p_s(t)$ is defined as the pressure in the liquid at the location of the bubble center in the absence of the bubble (Prosperetti and Lezzi, 1986). By substituting $p_0 + p_{in}(t)$ into $p_s(t)$, we can write $R(t)$ as a function of time: $R(t) = R_0 - \omega t e^{-\omega t}([p_s R_0(\omega_0^2 - \omega^2)])$, with $\omega_0^2 = 3 \gamma p_0/(\rho_0 R_0^3)$ ($\omega/\omega_0 \ll 1$). Note that the pressure field at $Q$ at the instant $t$ is given by the radial oscillation of the bubble at the earlier time $t' = t - x/a_i$, where $x/a_i$ is the time it takes for the pressure waves to propagate the distance $x$ from the point $B$ to the point $Q$. We can calculate the pressure field generated at $Q$ using the following:

$$P_B = -p_t \left[ \frac{\partial \phi(t)}{\partial t} \right] = p_t \omega^2 S_0 e^{-i\omega t} \frac{\omega_0^2 e^{-i\omega t} }{x} . \quad (3)$$

The radiative wave generated by the bubble is reflected off the rigid walls of the duct. We can calculate the reflected wave by calculating the pressure field due to the oscillations of imaginary bubbles, i.e., by using the method of images. We place infinitely many imaginary bubbles into a two-dimensional array in the $y - z$ plane; hence, we investigate the layer of bubbles (Domenico, 1982; Leroy et al., 2009; Surov, 1999; Timofeev et al., 1985), as shown in Fig. 1(b). Then, we consider only the case of a duct with a sufficiently large $D$; thus, the number density of bubbles on the $y - z$ plane, $\eta$, which is defined as $\eta = D^{-2}$ is very close to 0. We take advantage of this assumption to avoid $p_s(t)$ being modified by contributions from imaginary bubbles.

Now, we find the total pressure field at $Q$ by adding the pressure fields generated by the real and imaginary bubbles. We can observe the pressure field only at finite $Q$; hence, we have to add the contributions from the bubbles within a circle of radius $L$ from $B$ in the $y - z$ plane, as shown in Fig. 1(b). We define the summation of the pressure fields generated by these bubbles as $P_{tot}$. The total pressure field generated at $Q$ is given as the summation of $P_{in}$ and $P_{hs}$ (Fuster and Colonius, 2011; Takahira et al., 1995):

$$P_{tot}(t) = P_0 e^{i\omega t} e^{-i\omega t} + \sum_{i,j \in C, |x| < L} \frac{\rho t R_0^2 \omega^2 \omega_0^2 e^{-i\omega t} e^{-i\omega t} r_i^2 r_j^2}{r_i r_j} = (1 - i\Delta \theta) P_0 e^{i\omega t} e^{-i\omega t} = P_0 e^{-i\Delta \theta} e^{i\omega t} e^{-i\omega t}, \quad (4)$$

with

$$\Delta \theta = \omega \left( \frac{2 \pi a_i R_0 \eta}{\omega_0^2 - \omega^2} \right) \ll 1, \quad (5)$$

where $\zeta_{ij}$ is the distance from $B$ to the point of the $(i, j)$ imaginary bubble and $r_{ij}^2 = x^2 + \zeta_{ij}^2$. We follow the same procedure as Feynman et al. (1963) in performing the summation in Eq. (4).

It should be emphasized here that \( e^{i\Delta \theta} \) is the most important term in the present study. As Feynman et al. (1963) stated: multiplying an oscillating function \( e^{i\omega t} \) by a factor \( e^{i\Delta \theta} \) is equivalent to changing the phase of the oscillation by the angle \( -\Delta \theta \). This has retarded the phase by the amount \( \Delta \theta \), i.e., \( \omega[2\pi a_l R_0 \eta/(\omega_0^2 - \omega^2)] \).

3. Reduction in the speed of sound in a bubbly liquid in a square duct

Now, we further consider this phase delay using Fig. 2(a). The solid (blue) curve in Fig. 2(a) represents \( P_{in} \). This input pressure wave oscillates the bubble; then, \( P_{bs} \) is generated [the chain (red) curve]. This generated wave has a phase delay of \( \Delta \xi \). Note that only the propagating wave in the positive \( x \) direction is drawn in Fig. 2(a). The dashed (purple) curve is the superposition of the blue and red curves, \( P_{tot} \). Note that the total pressure wave delays in space by \( \Delta \xi \), as shown in Fig. 2(a). The delay distance \( \Delta \xi \) is related to the phase delay \( \Delta \theta \) as follows:

\[
\Delta \xi = \frac{\lambda}{2\pi} \Delta \theta. \tag{6}
\]

Note that once \( a_l, R_0, \eta, \omega, \) and \( \omega_0 \) are determined, \( \Delta \xi \) can be calculated as a constant, i.e., \( \Delta \xi \) is independent of \( x \) and \( t \).

We investigate the role of \( \Delta \xi \) in the reduction in the speed of sound in a liquid containing a large number of bubbles to determine whether our model can derive Eq. (1). We suppose that a large number of bubbles are evenly aligned on the center line of the square duct with a separation of \( \Delta b \), as shown in Fig. 2(b). We discuss the wavelength of the total pressure wave in the media. The wavelength is defined as the distance between consecutive corresponding points of the same phase; then, the wavelength of the input pressure wave, \( \lambda \), is shown in Fig. 2(b). Figure 2(b) illustrates that the delay distance \( \Delta \xi \) causes a reduction in the wavelength; namely, the apparent wavelength of the total pressure wave, \( \lambda_m \), is less than \( \lambda \). It should be emphasized here that the angular frequency of the total pressure wave has the same angular frequency of the input pressure wave that can be understood from Eq. (4). Therefore, \( \lambda_m < \lambda \) leads to the result that \( a_m < a_l \) because \( a_m = \omega \lambda_m/(2\pi) \). This is the fundamental mechanism of the apparent reduction in the speed of sound in a bubbly liquid. It should be emphasized here that the phase speed of the linear pressure wave propagating in the duct is the speed of sound in a bubbly liquid in this study inspired by Feynman et al. (1963).

Now, let us examine the apparent wavelength of the total pressure wave, \( \lambda_m \). Suppose that the first bubble [the leftmost bubble in Fig. 2(b)] is placed on the origin; then, the \( x \)-coordinate of the \( k \)-th bubble, \( x_k \), is \((k-1)\Delta b\). We elaborate on how these \( k \) bubbles reduce the wave propagation distance defined as the product of the wavelength and the number of waves. We set the reference point at \(-\Delta x_L\), where \( 0 < \Delta x_L < \Delta b \), to avoid phase discontinuity at the origin. Then, we adopt the distance from \(-\Delta x_L \) to \( x_k + \Delta x_k \) as the wave propagation distance of the total pressure wave, \( d_m \), where \( 0 < \Delta x_R < \Delta b \):
\(d_m = \Delta x_L + \Delta x_R + (k-1)\Delta b\). We should note that the difference between \(d_m\) and the wave propagation distance of the input pressure wave, \(d\), is the summation of the delay distances caused by the \(k\) bubbles. Therefore, \(d = d_m + k\Delta \xi\), which can be understood graphically from Figs. 2(a) and (b). Now, we introduce the most essential assumption that \(k \gg 1\); then, the refractive index of the bubbly liquid, \(\chi_m\), can be obtained from the following:

\[
\chi_m = \frac{\alpha_s}{d_m} = \frac{\lambda}{\Delta x_m} = \frac{d}{d_m} = 1 + \frac{k}{\Delta x_L + \Delta x_R + (k-1)\Delta b} \left(\frac{\lambda \Delta \theta}{2\pi}\right) \approx 1 + \frac{\lambda \Delta \theta}{2\pi \Delta b} > 1.
\] (7)

Notice here that the volume fraction of bubbles, \(\beta_m\), can be formally given as

\[
\beta_m = \frac{4\pi \rho_0^3}{3\lambda b}.
\] (8)

Then, with the use of Eq. (5), Eq. (7) can be rewritten as

\[
\chi_m = 1 + \frac{\omega L}{2\pi \Delta b} \left(\frac{2\pi a_s R_0 a}{\omega^2 \rho - \omega^2}\right) \approx 1 + \frac{\mu \Delta \beta_m}{2 \rho_0 \rho g},
\] (9)

where the conditions are as follows: \(\omega/\omega_0 \ll 1\) and \(\gamma = 1\) are used. Note that the right-hand side of Eq. (9) can be also derived from Eq. (1) when the relation \(p_0 = \rho_d a_g^2\) is used and the conditions are as follows: \(\beta \ll (\lambda / \rho g)/(\lambda / \rho g) \ll 1\) are imposed in Eq. (1).

Now, we obtain the most remarkable result that the refractive index of a bubbly liquid [Eq. (9), which has been well-validated experimentally] can be derived by considering the phase delay caused by radial oscillations of numerous bubbles aligned in the square duct. This result clearly shows that the phase delay caused by bubble oscillation is the essentially significant physical process that contributes to the reduction in the speed of sound in a bubbly liquid in a duct. The role of bubble oscillations in the reduction in the speed of sound is qualitatively the same as that of electron oscillations in the reduction in the speed of light in a material with a refractive index of \(n > 1\).

4. Can only a single bubble reduce the speed of sound in a square duct?

Now, we are ready to discuss the following: can only a single bubble reduce the speed of sound in a square duct? We consider a liquid volume whose length is \(\Delta x\). By substituting \(k = 1\) and \(\Delta x_L = \Delta x_R = \Delta x/2\) in Eq. (7), we obtain the refractive index of the liquid volume that contains only a single bubble, \(\chi_s\), which is also the ratio of the speed of sound in the pure liquid to that in the liquid volume that contains only a single bubble:

\[
(\chi_s - 1) \Delta x = \frac{\lambda \Delta \theta}{2\pi}.
\] (10)

The right-hand side of Eq. (10) is a well-defined positive constant, where \(\Delta \theta\) can be calculated with the use of Eq. (5); hence, \(\chi_s > 1\). This result, \(\chi_s > 1\), shows that only a single bubble can reduce the speed of sound in a square duct.

Now, we further consider the refractive index of \(\chi_s\). We should recall that \(\Delta x\) was arbitrarily defined; since \(\chi_s\) is defined by the use of Eq. (10), we cannot avoid any arbitrariness in the definition of \(\chi_s\). In other words, we cannot untangle the dependency between \(\chi_s\) and \(\Delta x\) in a liquid containing only a single spherical bubble in the square duct; only the product \((\chi_s - 1) \Delta x\) can be properly defined from Eq. (10). The failure of uniquely defining \(\Delta x\) in Eq. (10) is equivalent to the failure in uniquely defining the void fraction of a liquid containing only a single bubble in a square duct, \(\beta_s\). If \(\beta_s\) had been properly defined, \(\Delta x\) could have also been defined; then, \(\chi_s\) could also have been defined without ambiguity using Eq. (10).

We recall that in the definition of \(\chi_m\) in Eq. (7), we implicitly assumed \(\chi_m\) to be a well-defined constant in a volume. In other words, we assumed that the bubbly liquid is a homogeneous continuum medium; hence, \(\beta_m\) was properly defined without ambiguity. However, a volume of liquid containing only a single bubble is far from a homogeneous continuum medium; rather, it is a discrete medium for which \(\beta_s\) cannot be uniquely defined. Hence, it is mathematically inadequate to discuss the refractive index of a liquid containing only a single bubble, as is understood from Eq. (10). It should be emphasized that the refractive index is essentially a value defined only in a properly defined continuum medium.

5. Conclusions

We have investigated pressure wave propagation in a square duct filled with a compressible liquid that contains spherical bubbles. We have clarified that even a single bubble in a square duct can indeed delay the phase of an input pressure...
wave, thus causing an apparent reduction in the speed of sound. This is the most fundamental and essential physical process governing pressure wave propagation in a bubbly liquid in a duct. However, the refractive index of a liquid containing only a single bubble in a square duct cannot be uniquely defined, because the void fraction cannot be uniquely defined. It should be emphasized that the refractive index is essentially a value defined only within a homogeneous continuum medium; we conclude that the speed of sound in a liquid containing only a single bubble in a square duct itself is neither a physically nor a mathematically meaningful quantity. However, we can observe an apparent reduction in the speed of sound in a liquid that contains a finite number of bubbles, for which the void fraction can be properly defined.

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References


