An extensional formulation for a diffusive solution of the level-set equation by considering a relation to the scalar conservation equation

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Abstract

This paper investigates a relation between the scalar conservation equation and the level-set equation based on Liu's local interface speed model for a premixed combustion flame. A new model formulation is introduced for the source term of the conservation equation in three-dimensional interface phenomena, which gives the solution of the level-set equation coupled with a re-initialization procedure for the physical interface problems governed by conservation law. It may be an extensional formulation for a diffusive solution of the level-set equation.

Key words : Level-set equation, Conservation equation, Re-initialization procedure, Local interface speed model

1. Introduction

Level-set approach is widely applied to interface problems in physical simulations (Osher et.al. 2002), and is also investigated as a powerful tool of mathematical analysis (Giga et.al. 2015). In practical numerical simulation, the level-set equation becomes often unstable because it allows multiple solutions for defining the same interface profile. Therefore an additional approach, called "re-initialization" (Russo et.al. 2000, Olsson et.al. 2005) is usually applied to determine the unique diffusive solutions. Though this approach is useful practically, its physical background isn't clear especially for a type of physical interface solution governed by conservation law of thermo-fluid problems such as a combustion flame or fluid phase interface. This paper investigates a relation between the scalar conservation equation and the level-set equation based on a local interface speed model for a premixed combustion flame (Liu et.al. 2011 called Liu's model below) which was proposed by author and co-worker.

2. Relation between level-set equation and physical solution of conservation law

When a level-set equation is applied to analysis of physical interface phenomena, its target interface is define by the iso-surfaces of a representing scalar value which may be governed by a kind of conservation equation. Assuming such a scalar, u distributes continuously along both sides of the interface and has different values u_- and u_+ in the far fields divided by the interface, an "interface" defined by the iso-surface $u = u_0$ ($u_- \leq u_0 \leq u_+$) should be expressed by level-set equation. For example, if the interface becomes a linear combination of constant growing flow and mean curvature flow, the governing equation is given by

$$\frac{\partial u}{\partial t} = -V|\nabla u| \quad (1),$$

by the interface speed $V = s(u_0) + c(u_0)\kappa$, where the speed factors, $s(u_0)$ and $c(u_0)$ are the interface speed of constant growing flow and the speed coefficient of mean curvature flow at the interface $u = u_0$, and κ is the mean curvature.

Because of “geometrical” characteristics of the level-set equation (Chen et.al. 1991), the time development of the interface should be independent on any monotonic scalar function value $u^* = G(u)$ sharing the same interface of $u^* = u_0^* = G(u_0)$ as $u = u_0$, so the arbitrary initial profile of u^* can be applied to the same interface solution. In the practical applications of the level-set equation, a signed distance function (Russo et.al. 2000) or its monotonic function, for example, hyperbolic function (Olsson et.al. 2005), is often applied for resetting the initial profile (as called “re-initialization procedure”) in order to keep a stable solution for calculations carried out over long time. In those methods, the solution of the level-set function $u^* = G(u)$ is reset to the fixed initial profile (a signed distance function or hyperbolic function) which keeps the same iso-surface by solving additional differential equation in an appropriate period of the time marching step of the level-set equation (1).

Now, we consider interface phenomena which are described by a conservation equation of the scalar value $u^* = G(u)$. If a physical interface exists stably in the thin layer, the scalar value u^* should be profiled between a finite thickness, where any value u^* in the range $u_-^* = G(u_-) < u^* < G(u_+) = u_+^*$ can represent the interface position. So we want to derive a condition which satisfies the level-set equation (2) in the arbitrary iso-surfaces of $u = u_0$ ($u^* = u_0^* = G(u_0)$) by insisting their initial solutions, and also we expect that the condition may remain stable under appropriate physical conditions. As an example of such solutions, Liu et.al (2011) proposed a model based on the thermal conservation equation for a premixed combustion flame which is generated at the interface of unburnt low temperature and burnt high temperature regions and propagates (or grows rapidly) to the former direction by a constant speed. They assumed a hyperbolic profile of temperature through the thin flame interface at a constant speed. They considered the conservation equation of temperature as a level-set function u^* , then proposed a model by assumption of a hyperbolic profile of temperature through the thin flame interface, which gives a source term by combustion heat release as

$$\dot{q} = \rho_0 S^* |\nabla u^*| \quad (2),$$

where $\rho_0 S^*$ is a linear function of u^* . According to combustion theory, they explained a physical mean of S^* as “local flame speed” by analogy of flame speed, S_L which is measured by the convection velocity balancing to stable flame propagation. Here density ρ_0 is defined as the given constant at the velocity measurement position (usually in the unburnt region). In the conservation equation, the source term $\dot{q}$ modeled by the local flame speed S^* and the diffusion flux by ∇u^* consists of the flame propagation by “observed” flame speed, S_L in the premixed combustion. This model of local flame speed $\rho_0 S^*$ by the a linear function of u^* derives Burgers’ equation in the one-dimensional plane flame case, which indicates a common mathematical background as models for shock wave of compressible flow and narrow water wave etc., which are governed by the conservation laws in fluid dynamics and considered as typical interface phenomena.

3. Deriving an extended level-set equation by conservation law

Below shall we consider a generalized formulation of Liu’s model in a three-dimensional case? We consider the scalar u^* as an intensive mount like temperature which satisfies the conservation equation with material density ρ as

$$\rho \frac{\partial u^*}{\partial t} = -\rho \mathbf{v} \cdot \nabla u^* + \nabla \cdot \mathbf{j} + \dot{q} \quad (3),$$

where the right hand side are the convection flux by material velocity $\mathbf{v}$, the diffusion flux modeled by the gradient model as $\mathbf{j} = D \nabla u^*$ (diffusion factor: D) and the source term $\dot{q}$ by combustion heat release, respectively. We want to find a condition for which it also becomes a solution of the level-set equation (1) under the appropriate initial conditions.

According to a flamelet concept of thin flame theory, a local solution of scalar u^* can be approximated as a small element of a one-dimensional plane flame. Then applying Liu’s model (2) by the local “interface” speed $\rho_0 S^*$ renamed for the general interface phenomena, the following condition is derived along the coordinate ξ normal to the interface as

$$(\rho_0 S^* - \rho_0 S_L) \frac{du^*}{d\xi} + \frac{d}{d\xi} \left(D \frac{du^*}{d\xi} \right) = 0, \text{ where } \left(\frac{du^*}{d\xi} \right) \mathbf{n} = \nabla u^*, \quad \mathbf{n} = \frac{\nabla u^*}{|\nabla u^*|} \quad (4),$$

and its integral form,

$$\int (\rho_0 S^* - \rho_0 S_L) du^* + D \frac{du^*}{d\xi} = 0 \quad (5),$$

where $\rho \mathbf{v} \cdot \mathbf{n} = \rho_0 S_L$ is assumed constant by the conservation law of one-dimensional flow in the steady plane flame. Under the simplified boundary condition as $u^* = u_-^* = 0$ and $u^* = u_+^* = 1$ for unburnt and burnt sides, respectively and $|\nabla u^*| = du^*/d\xi = 0$ in both sides, a linear function of local interface speed S^* is given by $\rho_0 S^* = 2\rho_0 S_L u^*$ as following to the Liu's linear function approximation, then the solution is derived as

$$F \equiv \int (\rho_0 S^* - \rho_0 S_L) du^* = \rho_u S_L u^* (u^* - 1), \text{ where } \frac{\rho_u S_L}{2D} = \left[\frac{\partial u^*}{\partial \xi} \right]_{\max} = \frac{1}{\delta} \quad \text{at } u^* = \frac{1}{2} \quad (6).$$

Here, δ indicates a length scale of the interface thickness of the profile u^* . It is noted in the above formulation that the function F gives the profile u^* in the coordinate ξ normal to the interface, which is the condition for identification of the solution of the conservation equation (3) to a diffusive solution of the level-set equation (1) (Ishida et.al. 2012). Substituting the model formulation (6) to equation (3), Liu's model is derived as one-dimensional formulation as

$$\rho \frac{\partial u^*}{\partial t} = -\rho \mathbf{v} \cdot \frac{\partial u^*}{\partial \xi} + \frac{\partial}{\partial \xi} \left(D \frac{\partial u^*}{\partial \xi} \right) + \left(\frac{dF}{du^*} \right) \frac{\partial u^*}{\partial \xi} \quad (7).$$

Here the source term formulation (2) is modeled by the function F as

$$\dot{q} = \rho_u S_L |\nabla u^*| = \left(\frac{dF}{du^*} \right) \frac{\partial u^*}{\partial \xi} \quad (8).$$

For the three-dimensional extension of Liu's model, the second term of the right hand side of formulation (7) should be conducted by the diffusion term in the conservation equation (3). By using an identity of vector equations as

$$\nabla \cdot (D \nabla u^*) \equiv \mathbf{n} \cdot \nabla (D |\nabla u^*|) + D |\nabla u^*| (\nabla \cdot \mathbf{n})$$

$$\nabla \cdot F \mathbf{n} \equiv \mathbf{n} \cdot \nabla F + F (\nabla \cdot \mathbf{n}) \quad (9),$$

a couple of the first terms in these equations (8) correspond to the last two terms of formulation (7) which relate to the constant growing flow, while another couple of the second terms which may relate the mean curvature flow. Considering these two relation between the source terms by the function F and the diffusion ones by the absolute gradient $|\nabla u^*|$, three-dimensional extended formulation of Liu's model (8) should be modeled by these two coupling relations by a common term $(F + D |\nabla u^*|)$.

If we assume a relation between the level-set equation (1) and the conservation equation (3) under the condition

$$F + D |\nabla u^*| = 0 \quad (10),$$

the conservation equation (4) is modified to

$$\begin{aligned} \rho \frac{\partial u^*}{\partial t} = & -(\rho \mathbf{v} - \rho \mathcal{S}(u_0) \mathbf{n}) \cdot \nabla u^* + \rho \mathcal{C}(u_0) |\nabla u^*| (\nabla \cdot \mathbf{n}) \\ & - \frac{\rho \mathcal{C}(u_0)}{D} (F + D |\nabla u^*|) (\nabla \cdot \mathbf{n}) + \nabla \cdot (F + D |\nabla u^*|) \mathbf{n} \end{aligned} \quad (11),$$

where the three-dimensional extended formulation of Liu's model (8) is expressed by

$$\dot{q} = \rho \mathcal{S}(u_0) \mathbf{n} \cdot \nabla u^* + \frac{\rho \mathcal{S}(u_0)}{D} \mathbf{n} \cdot \nabla F + \left(1 - \frac{\rho \mathcal{S}(u_0)}{D} \right) \nabla \cdot F \mathbf{n} \quad (12).$$

When the profile of u^* satisfies the condition (10), the third and fourth terms of the right hand side of equation (11) becomes zero, then equation (11), that is, the conservation equation (3) becomes equivalent to the level-set equation (1) by the speed factors $s(u_0)$ and $c(u_0)$ under the zero material velocity condition ($\rho \mathbf{v} = 0$) to express the constant growing flow induced and the mean curvature flow. Here assuming the physical solution of $\rho s(u_0)$ and $\rho c(u_0)/D$ are independent on the conversion function $u_0^* = G(u_0)$, a motion for any iso-surface by $u^* = u_0^*$ (that is $u = u_0$) can be expressed by a unique global level-set equation (11) with the profile condition (10), which can define the model formulation (12) of the conservation equation (3), too.

As Oshima et.al. (2011) indicated, the pseudo time marching equation for the re-initialization procedure introduced for stabilizing the level-set solutions can be evaluated as a local solution of interface thickness which is coupled with a global solution of interface shape to derive the well-defined governing equation of the physical profile. In the new model equation (11) with the profile condition (10), the third term in the proportion to mean curvature and the fourth term in the same as Olsson's procedure are added to the level-set equation (1) with the constant growing flow by $s(u_0)$ and the mean curvature flow by $c(u_0)$. Here the last two terms of equation (9) are rewritten in the normal coordinate ξ as

$$\begin{aligned} & -\beta(F + D|\nabla u^*|)(\nabla \cdot \mathbf{n}) + \nabla \cdot (F + D|\nabla u^*|)\mathbf{n} \\ & = \beta \frac{d}{d\xi} \left\{ \left(\frac{F}{|\nabla u^*|} + D \right) \frac{du^*}{d\xi} \right\} + (1 - \beta) \nabla \cdot \left\{ \left(\frac{F}{|\nabla u^*|} + D \right) \nabla u^* \right\}, \text{ where } \beta = \frac{\rho c(u_0)}{D} \end{aligned} \quad (13),$$

therefore, if $0 \leq \beta \leq 1$, when the gradient is steeper than the target condition $F + D|\nabla u^*| = 0$, the two terms of formulation (13) induce positive diffusion effects which make the solution converge into the target condition by one-dimensional (only normal to the iso-surface) and isotropic ways. In the contrary as the gradient is smoother than the condition, they induce negative diffusion. As Ishida et.al. (2009) has indicated that the one-dimensional formulation (7) of Liu's model gives a diffusive solution of the level-set equation, so its three-dimensional formulation (11) is considered as an extended formulation for a diffusive solution of the original level-set equation (1).

As considered above, the additional terms (13) to the level-set equation (1) (that is, third and fourth terms in the equation (11)) are essentially the same meaning of re-initialization procedures in the two different ways by the non-conservation form (Russo et.al 2000) and by the conservation form (Olsson et.al 2005). Moreover, the new formulation (11) derived from the scalar conservation equation (3) not only expresses a physical meaning of the re-initialization procedure in the numerical simulation of the level-set equation, and also enables to avoid a numerical instability of the original level-set equation by the physical oriented viscous term from the conservation equation.

4. Conclusive remarks

Concluding the above, the extended formulation (11) for a diffusive solution of the level-set equation is derived by introducing a three-dimensional extension of Liu's local interface speed model, which also gives the essentially same solution of the level-set equation coupled with the re-initialization procedures. It expresses a physical meaning of the re-initialization procedure related to the scalar conservation law, and also enables to avoid a numerical instability of the original level-set equation by introducing the physical oriented viscous term.

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