Topological derivative for an acoustic-elastic coupled system based on two-phase material model

Yuki NOGUCHI*, Takayuki YAMADA*, Takashi YAMAMOTO**, Kazuhiro IZUI* and Shinji NISHIWAKI*

* Department of Mechanical Engineering and Science, Graduate School of Engineering, Kyoto University
Kyoto daigaku-katsura C3, Nishikyo-ku, Kyoto 615-8540, Japan
E-mail: noguchi.yuuki.34s@st.kyoto-u.ac.jp

** Department of Mechanical Engineering, Kogakuin University
, 2665-1 Nakano, Hachioji, Tokyo 192-0015, Japan

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Abstract
This letter presents an explicit formulation for the topological derivative of a two-dimensional acoustic-elastic coupled system, expressed with a two-phase material model based on Biot’s theory. First, we briefly explain the two-phase material model in which the objective functional is assumed to be a domain integral of a certain function of velocity potential. The shape derivative of the objective functional is obtained using the usual Lagrangian formulation and we then construct the adjoint equation. Since it is known that the limit value of a shape derivative is equal to the topological derivative, asymptotic behavior for the boundary value problem for the state and adjoint variables is searched for and, based on the solutions, an explicit formulation of the topological derivative is thereby obtained. With the objective functional defined as the squared norm of the acoustic pressure, the topological derivatives for equivalent acoustic and elastic material domains are numerically compared with the numerical difference when a hole domain with a finite radius appears, using the FEM. The provided numerical examples demonstrate the validity of our topological derivative formulation and the procedure for calculating topological derivatives.

Key words: Topological derivative, Acoustic-elastic coupled system, Two-phase model, Structural design

1. Introduction

The topological derivative is the sensitivity of a cost function that measures a certain performance, such as structural stiffness, when an infinitesimal hole or inclusion appears in a homogeneous material domain. The concept of the topological derivative was first introduced in the field of structural optimization, for a shape optimization problem (Eschenauer et al. 1994). The distribution of a topological derivative indicates where the placement of a void domain or another material in a design domain most effectively minimizes an objective function, hence the topological derivative can be used to improve a performance in structural design problems. Recently, topological derivatives have been applied to level-set based shape optimization methods (Allaire et al. 2005; Allaire et al. 2011) and topology optimization methods (Otomori et al. 2014), and optimal configurations have been obtained for many physics problems such as elastic problems, heat conduction (Jing et al. 2015), and acoustic wave problems (Isakari et al. 2014). The use of topological derivatives is not limited to structural optimization problems. There is much research on inverse scattering problems using the concept of topological derivatives (Guzina and Bonnet 2004; Feijoo 2004a; Feijóo 2004b), and topological derivatives have been applied in a broad range of fields such as seismology, non-destructive testing, medical issues, and detection of subterranean materials.

Novotny et al, 2003 and Feijóo et al, 2003 revealed that the topological derivative is coincident with the limit value of a corresponding shape derivative. Consideration of such limit values requires solution of partial differential equations (PDEs) that describe asymptotic behavior in the vicinity of an infinitesimal hole. Having obtained topological derivatives in Poisson problems, topological derivatives for other PDEs were obtained based on their method, such as for the...
Two-phase material model

To represent an acoustic-elastic coupled system, we introduce a two-phase material model in which solid and inviscid compressible fluid phases are mixed. The mixed material is represented using a volume fraction $\varphi$ ($0 < \varphi < 1$), such as $M(1 - \varphi, \varphi)$, in which the volume fraction of the solid and fluid phases are $1 - \varphi$ and $\varphi$, respectively. This model can express acoustic or elastic material by taking the limit value of the volume fraction $\varphi$ to 1 or 0. Based on Biot’s theory (Biot, 1956a; Biot, 1956b), the governing equations for a two-phase material subject to a harmonic oscillation with angular frequency $\omega$ are expressed as

$$\frac{\partial \sigma_{ij}^{s}}{\partial x_{j}} + \omega^{2} \rho^{s} u_{i}^{s} = 0,$$  \hspace{1cm} (1)

$$\frac{\partial \sigma_{ij}^{f}}{\partial x_{j}} + \omega^{2} \rho^{f} u_{i}^{f} = 0,$$  \hspace{1cm} (2)

where $\sigma_{ij}^{s}$ and $\sigma_{ij}^{f}$ denote the stress tensor for the solid and fluid phase, respectively, and $u_{i}^{s}$ and $u_{i}^{f}$ are the solid and fluid phase displacements, respectively. $\rho^{s} = (1 - \varphi)\rho^{0}$ and $\rho^{f} = \varphi\rho^{0}$ represent the mass density of the solid and fluid phases, respectively, with $\rho^{0}$ denoting the mass density of the elastic and acoustic material, respectively. Assuming an isotropic linearly elastic and acoustic material, the constitutive equations for the stress tensors are

$$\sigma_{ij}^{s} = \lambda^{s} \varepsilon_{ij}^{s} + 2\mu^{s} \varepsilon_{ij}^{s}, \quad \sigma_{ij}^{f} = -\varphi \rho^{f} \delta_{ij} + \varphi \kappa^{f} \varepsilon_{ij}^{f},$$  \hspace{1cm} (3)

where $\varepsilon_{ij}^{s}$ and $\varepsilon_{ij}^{f}$ are the strain tensors of the solid and fluid phases, respectively. $\lambda^{s}$ and $\mu^{s}$ are Lame’s first and second constant of the solid phase. $\rho^{f} = -\kappa^{f} \varepsilon_{ij}^{f}$ represents the pressure of the fluid phase, where $\kappa^{f}$ is the bulk modulus of the acoustic material incorporating the fluid phase. Taking the divergence of Eq. (2) and using a velocity potential $\psi^{f}$ that is related to $\rho^{f}$ according to $\rho^{f} = -i\omega\psi^{f}$, the fluid phase is governed by the following Helmholtz equation,

$$\frac{\varphi^{2}}{\rho^{f}} \frac{\partial^{2} \psi^{f}}{\partial x_{i} \partial x_{i}} + \omega^{2} \frac{\varphi^{2}}{R^{f}} \psi^{f} = 0,$$  \hspace{1cm} (4)

where $R^{f}$ is the radius of the fluid phase.
where \( R' = \varphi K' \). We define mixed stress tensor \( \sigma_{ij}' \) and mixed displacement \( u_i' \) as follows:

\[
\sigma_{ij}' = \sigma_{ij}^s + \sigma_{ij}^f, \quad u_i' = (1 - \varphi)u_i^s + \varphi u_i^f.
\]  

(5)

On the boundaries between different two-phase materials, the displacement of solid phase \( u_i^s \) and the velocity potential \( \psi^f \) satisfy the continuity conditions, respectively. Furthermore, traction of the mixed stress \( \sigma_{ij}^f n_j \) and \( i\omega\varphi(u_i^f - u_i^s)n_i \), where \( n_i \) represents a normal unit vector, also satisfy continuity conditions. Based on these continuity conditions, we do not have to impose any boundary conditions, such as the coupling boundary conditions that are usually imposed in an acoustic-elastic coupled system, on the boundaries between different two-phase materials. In contrast to the application of Biot’s model for poroelastic material in previous research, our model does not deal with poroelastic material and therefore no coupling parameters or damping term are required. The only adjustable parameter in the acoustic-elastic coupled system here is the volume fraction \( \varphi \).

3. Derivation of the topological derivative

3.1. Definition of topological derivative

Topological derivative \( D_T F \) is generally defined as a sensitivity of an objective functional \( F \) when an infinitesimal hole domain \( \Omega_i \), whose radius is \( \varepsilon > 0 \), appears at a certain position \( x \) in \( \Omega \), as follows:

\[
D_T F = \lim_{\varepsilon \to 0} \frac{F(\Omega \setminus \Omega_i) - F(\Omega)}{V(\varepsilon)},
\]  

(6)

where the function \( V(\varepsilon) \) is defined so that the limit for \( \varepsilon \) exists and it is usually related to the measure of the hole domain \( \Omega_i \). We set \( V(\varepsilon) = -\pi \varepsilon^2 \), as in the cited work (Caprio and Rapún 2008a; Caprio and Rapún 2008b). It is known that the topological derivative is related to the shape derivative (Novotny et al, 2003; Feijóo et al, 2003) as follows: The shape derivative of \( F \) is defined as follows,

\[
DF(\Omega) \cdot V = \left. \frac{d}{d\tau} F(\phi_\tau(\Omega_i)) \right|_{\tau=0},
\]  

(7)

where deformation \( \phi_\tau \) is written as follows,

\[
\phi_\tau(x) = x + \tau V(x),
\]

The vector field \( V \) points in the direction opposite that of the inward-pointing normal unit vector \( n' \) on \( \partial \Omega_i \), such as \( V_i = V_n n'_i(x) \), where \( V_n \) has a negative constant scalar value. To relate shape derivative to topological derivative, \( V \) is extended to \( \Omega \) so that it attenuates near the boundary \( \partial \Omega_i \). Then topological derivative is given as follows,

\[
D_T F = \lim_{\varepsilon \to 0} \frac{1}{V'(\varepsilon)V_n} \left. \frac{d}{d\tau} F(\phi_\tau(\Omega_i)) \right|_{\tau=0},
\]  

(8)

where \( V'(\varepsilon) \) is the derivative of \( V(\varepsilon) \) with respect to \( \varepsilon \).

![Fig. 1 Acoustic elastic coupled system model.](image-url)
3.2. Acoustic-elastic coupled system settings

We derive the topological derivative for the acoustic-elastic coupled system shown in Fig. 1. The model consists of an acoustic material domain \( \Omega_{\text{ac}} \) governed by the Helmholtz equation and a two-phase material domain \( \Omega \setminus \Omega_{\text{out}} \), where \( \Omega \) represents the entire design domain. Furthermore, the two-phase material domain is divided into an acoustic region \( \Omega_{\text{ac}} \) and an elastic region \( \Omega_{\text{el}} \). A plane acoustic wave \( p^f = p_0 \exp(-i k x) \) propagates along the \( x \)direction, with \( p_0 \) the amplitude and \( k \) the wave number of the plane wave, respectively. The presence of the elastic cylinder \( \Omega_{\text{el}} \) causes the incident wave to scatter. To simulate an unbounded system, a non-reflecting boundary condition for \( p^f \) is imposed on \( \Gamma_{\text{in}} \) and \( \Gamma_{\text{out}} \), as follows:

\[
\frac{1}{\rho^a} \frac{\partial \rho^a}{\partial x_i} n_i = \frac{1}{\rho^a} \left( i k a \phi^a - i k a n_j n_j \phi^a \right) \quad \text{on} \quad \Gamma_{\text{in}}, \quad (9)
\]

\[
\frac{1}{\rho^a} \frac{\partial \rho^a}{\partial x_i} n_i = i \frac{k a}{\rho^a} n_i \quad \text{on} \quad \Gamma_{\text{out}}, \quad (10)
\]

where \( \phi^a = -\frac{\rho^a}{\omega p_0} \exp(-i k x) \), and \( k_{\text{in}} \) is a wave vector whose components are \( k_{\text{in}1} = k_a \) and \( k_{\text{in}2} = 0 \). The coupling conditions on \( \Gamma_{\text{c}} \) are expressed as

\[
u^a_i n^a_i = i \frac{\partial \rho^a}{\partial \phi^a} n^a_i \quad \text{on} \quad \Gamma_{\text{c}}, \quad (11)
\]

\[
\sigma^{\alpha \beta}_{ij} \nu^a_j = -i o \omega n^a_i \quad \text{on} \quad \Gamma_{\text{c}}, \quad (12)
\]

\[
\psi^f = \phi^a \quad \text{on} \quad \Gamma_{\text{c}}, \quad (13)
\]

where \( n^a_i \) and \( n^a_i \) are outward normal unit vectors for the two-phase material region and acoustic material region, respectively.

In this study, the topological derivative is calculated in the two-phase material domain \( \Omega \setminus \Omega_{\text{out}} \). We set the hole domain \( \Omega_{\text{c}} \) in \( \Omega_{\text{ac}} \) to calculate the topological derivative for the equivalent acoustic domain. Using the same procedure, we can obtain a topological derivative for the equivalent elastic domain \( \Omega_{\text{el}} \) by setting the hole domain \( \Omega_{\text{c}} \) in \( \Omega_{\text{el}} \), but shall limit our explanation here to the first case, to obtain the topological derivative for equivalent acoustic domain. In this case, the governing equations for the system are as follows:

\[
\frac{\partial \sigma^{\alpha (m)}_{ij}}{\partial x_j} + \omega^2 \rho^{(m)} u^i_f = 0 \quad \text{in} \quad \Omega_{\text{in}}, \quad (14)
\]

\[
\left( \frac{\varphi^2}{R^2} \right)^{(m)} + \frac{\partial^2 \varphi^f}{\partial x_i \partial x_j} + \omega^2 \left( \frac{\varphi^2}{R^2} \right)^{(m)} \psi^f = 0 \quad \text{in} \quad \Omega_{\text{in}}, \quad (15)
\]

\[
\frac{1}{\rho^a} \frac{\partial \rho^a}{\partial x_i} n_i + \omega^2 \frac{1}{K^2} \psi^a = 0 \quad \text{in} \quad \Omega_{\text{out}}, \quad (16)
\]

where superscripted quantities with \( m = i, \epsilon, \text{elastic} \) represent quantities of \( \Omega_{\text{c}}, \Omega_{\text{i}}, \text{and } \Omega_{\text{el}}, \) respectively, where \( \Omega_{\text{c}} = \Omega_{\text{ac}} \setminus \Omega_{\text{out}} \). The solutions \( (u^i_f , \psi^f) \) are expressed as \( (u^i_-, \psi^f_-) \) in \( \Omega_{\text{in}} \), \( (u^i_+, \psi^f_+) \) in \( \Omega_{\text{c}} \), and \( (u^i_+, \psi^f_-) \) in \( \Omega_{\text{el}} \). The continuity conditions on \( \Gamma \), the boundary of a small hole domain \( \Omega_{\text{c}} \), are expressed as

\[
u^i_- = u^i_+ \quad \text{on} \quad \Gamma, \quad (17)
\]

\[
\psi^f_- = \psi^f_+ \quad \text{on} \quad \Gamma, \quad (18)
\]

\[
\sigma^{\alpha \beta}_{ij}(u^i_- - u^i_+, \psi^f_- n_j - \sigma^{\alpha \beta}_{ij}(u^i_+, \psi^f_+) n_j) = 0 \quad \text{on} \quad \Gamma, \quad (19)
\]

\[
i o \varphi^f_-(u^i_- - u^i_+, \psi^f_+ n_j - i o \varphi^f_+(u^i_+ - u^i_-) n_j) = 0 \quad \text{on} \quad \Gamma, \quad (20)
\]

and the other boundary conditions are as follows:

- continuity condition on \( \partial \Omega_{\text{el}} \)
- non-reflecting condition Eq. (9) and (10),
- coupling condition Eq. (11), (12), and (13),

where \( \Gamma \) is the boundary of a small hole domain \( \Omega_{\text{c}} \). We assume that the objective functional \( F \) is set in \( \Omega_{\text{out}} \) and is a functional of acoustic velocity potential \( \psi^a \) and its complex conjugate \( \psi^{*a} \), as follows:

\[
F(\Omega_c) = \int_{\Omega_c} \rho f(\psi^a, \psi^{*a}) d\Omega,
\]

where \( f \) is density function for \( F \).
3.3. Explicit formula for the topological derivative

To obtain the explicit formula for the topological derivative using definition (8), the following procedures are required.

**Step 1:** Shape derivative derivation

The shape derivative is calculated based on the usual Lagrangian method, with details available in the references (Allaire et al., 2011; Allaire et al., 2014). Here, we show the outline of the shape derivative derivation. We define the Lagrangian as follows:

$$L(\tau, u^{\tau}, \eta^{\tau}, \eta^\tau, w^{\tau}, \tilde{w}) = L(\tau, W, \tilde{W})$$

$$= \int_{\Omega_{out}} f(\eta^\tau, \eta_m^\tau)d\Omega + 2\sum_{i} \left[ a(\Omega_{x_i}; W, \tilde{W}) - b(W, \tilde{W}) + I(\Gamma_{x_i}; W, \tilde{W}) \right],$$

where \( W = (u^{\tau}, \eta^{\tau}, \eta^\tau) \) and \( \tilde{W} = (w^{\tau}, \tilde{\eta}^{\tau}, \tilde{\eta}) \). The explicit forms of \( a(\Omega_{x_i}; W, \tilde{W}), b(W, \tilde{W}), \) and \( I(\Gamma_{x_i}; W, \tilde{W}) \) are written as

$$a(\Omega_{x_i}; W, \tilde{W}) = \int_{\Omega_{x_i}} \left( \alpha^{(i)}(w^{\tau}) \right) \eta_m^{\tau} d\Omega + \int_{\Omega_{x_i}} \frac{1}{\rho^\tau} \left( \phi^{E} \right) \eta_m^{\tau} d\Omega + \int_{\Omega_{x_i}} \frac{1}{\rho^\tau} \left( \phi^{E} \right) \eta_m^{\tau} d\Omega$$

$$b(W, \tilde{W}) = \int_{\Gamma_{x_i}} \left( \phi^{E} \right) \eta_m^{\tau} d\Gamma + \int_{\Gamma_{x_i}} \frac{1}{\rho^\tau} \left( \phi^{E} \right) \eta_m^{\tau} d\Gamma + \int_{\Gamma_{x_i}} \frac{1}{\rho^\tau} \left( \phi^{E} \right) \eta_m^{\tau} d\Gamma$$

$$I(\Gamma_{x_i}; W, \tilde{W}) = \int_{\Gamma_{x_i}} \left( \phi^{E} \right) \eta_m^{\tau} d\Gamma + \int_{\Gamma_{x_i}} \frac{1}{\rho^\tau} \left( \phi^{E} \right) \eta_m^{\tau} d\Gamma + \int_{\Gamma_{x_i}} \frac{1}{\rho^\tau} \left( \phi^{E} \right) \eta_m^{\tau} d\Gamma$$

where \( \Omega_{x_i} = \Omega \setminus (\Omega_{\tau_{x_i}} \cup \Omega_{out} \cup \Omega_{\text{elastic}}) \) and the subscript \( \tau \) stands for the quantity with respect to the deformed domain, based on the deformation mapping (8). The optimality conditions are as follows:

$$\sum_{x_i} \frac{\partial L(\tau, W, \tilde{W})}{\partial W_i} = 0,$$

$$\sum_{x_i} \frac{\partial L(\tau, W, \tilde{W})}{\partial W_i} = 0,$$

$$\tilde{W} = U = (u^{\tau}, \phi^{\tau}, \psi^\tau)$$ satisfies Eq. (27), and \( \tilde{W} = P \) satisfies Eq. (26) when \( P = (u^{\tau}, \zeta^{\tau}, \xi^{\tau}) \) is the solution of the following adjoint equations:

$$\frac{\partial \sigma_{ij}^{(m)}(\psi^\tau)}{\partial x_j} + \omega^2 \rho^{(m)} u_i^\tau = 0$$ in \( \Omega_{out} \).
\[
\frac{\left(\varphi^2\right)_{r}^{(m)}}{\rho^2} \frac{\partial^2 \xi_j}{\partial x_3 \partial x_3} + \omega^2 \left(\frac{\varphi^2}{R^2}\right) \psi_j = 0 \quad \text{in } \Omega_m, \tag{29}
\]
\[
\frac{1}{\rho^2} \frac{\partial^2 \xi_i}{\partial x_3 \partial x_3} + \omega^2 \frac{1}{K^2} \xi_i + \frac{\partial f(\psi_i, \psi_{in})}{\partial \psi_i} = 0 \quad \text{in } \Omega_{out}, \tag{30}
\]
\[
v_j^c = v_j \quad \text{on } \Gamma, \tag{31}
\]
\[
\zeta_j^c = \zeta_j \quad \text{on } \Gamma, \tag{32}
\]
\[
\sigma_{ij}^{(\psi)}(v_{ij}^c, \xi_j^c) n_i - \sigma_{ij}^{(\psi)}(v_{ij}^c, \xi_j^c) n_j = 0 \quad \text{on } \Gamma, \tag{33}
\]
\[
\imath \omega \phi^{(i)}(v_j^c - v_i^c) n_i - \imath \omega \phi^{(e)}(v_j^c - v_i^c) n_i = 0 \quad \text{on } \Gamma, \tag{34}
\]
non-reflecting condition (without incident waves) on \(\Gamma_{in} \cup \Gamma_{out},\)
coupling condition on \(\Gamma_c.\)

Using \(W = U\) and \(\hat{W} = P,\) the shape derivative is obtained by differentiating Lagrangian \(L\) with respect to \(r\) as follows:
\[
DJ \cdot V = 2\Re \left[ \frac{d}{dr}(\Omega_{\tau}, U, P) \right]_{r=0} \tag{35}
\]
Using polar coordinate \((r, \theta),\) which has the origin at the center of small circle domain \(\Omega_c\) and continuity conditions on \(\Gamma,\)
finally the shape derivative is obtained as follows:
\[
DF \cdot V = 2\Re \left[ -(V_n) \int_{\Gamma_c} (I_1 + I_2 + I_3 + I_4) d\Gamma \right], \tag{36}
\]
where,
\[
I_1 = \frac{1}{2\mu + \lambda} \frac{\partial \sigma_{ij}^{(\psi)}(u^c, \xi_j)}{\partial x_3} - \frac{\partial \sigma_{ij}^{(\psi)}(u^c, \xi_j)}{\partial x_3} \left(\frac{\varphi^2}{R^2}\right)_{r}^{(m)} \tag{37}
\]
\[
I_2 = \alpha^2 \left(1 - \frac{\partial^2 \varphi_{ij}^c}{\partial \theta^2} \right) \frac{\partial \psi_{ij}^c}{\partial x_3} - \alpha^2 \left(\frac{\partial^2 \varphi_{ij}^c}{\partial \theta^2} \right) \frac{\partial \psi_{ij}^c}{\partial x_3} \frac{\partial \varphi_{ij}^c}{\partial \theta} + \frac{\alpha^2}{\alpha^2} \imath \omega \phi^{(i)} - \phi^{(e)}(\frac{\partial \psi_{ij}^c}{\partial x_3}) \frac{\partial \varphi_{ij}^c}{\partial \theta} \frac{\partial \varphi_{ij}^c}{\partial \theta} + \frac{\partial \varphi_{ij}^c}{\partial x_3} \psi_{ij}^c \tag{38}
\]
\[
I_3 = \alpha^2 \left(1 - \frac{\partial^2 \varphi_{ij}^c}{\partial \theta^2} \right) \frac{\partial \psi_{ij}^c}{\partial x_3} + \alpha^2 \left(\frac{\partial^2 \varphi_{ij}^c}{\partial \theta^2} \right) \frac{\partial \psi_{ij}^c}{\partial x_3} \frac{\partial \varphi_{ij}^c}{\partial \theta} + \frac{\alpha^2}{\alpha^2} \imath \omega \phi^{(i)} - \phi^{(e)}(\frac{\partial \psi_{ij}^c}{\partial x_3}) \frac{\partial \varphi_{ij}^c}{\partial \theta} \frac{\partial \varphi_{ij}^c}{\partial \theta} + \frac{\partial \varphi_{ij}^c}{\partial x_3} \psi_{ij}^c \tag{39}
\]
where \(\alpha^m = (\frac{\alpha}{\rho})^{(m)} (m = i, e)\) and the brackets \([ \ ]\) represent the difference between the inner and outer domain, e.g., \([c] = e^i - e^e.\)

**Step 2:** Asymptotic behavior of boundary value problems

To obtain topological derivative using definition (8), the solutions and gradients in the infinitesimal inner domain \(\Omega_c\) must be evaluated. Here, we examine asymptotic behavior of state variables for the following approximated boundary value problem:
\[
\frac{\partial \sigma_{ij}^{(m)}(u_j^{(1)\text{str}})}{\partial x_3} + \omega^2 \rho^{(m)} u_i^{(1)\text{str}} = 0 \quad \text{in } \Omega_m, \tag{41}
\]
\[
\left(\frac{\varphi^2}{R^2}\right)_{r}^{(m)} \frac{\partial^2 \psi_i^{(1)\text{str}}}{\partial x_3 \partial x_3} + \omega^2 \left(\frac{\varphi^2}{R^2}\right) \psi_i^{(1)\text{str}} = 0 \quad \text{in } \Omega_m, \tag{42}
\]
\[
u_i^{(1)\text{str}} = u_i^{(1)\text{str}} \quad \text{on } \Gamma, \tag{43}
\]
\[
\psi_i^{(1)\text{str}} = \phi_i^{(1)\text{str}} \quad \text{on } \Gamma, \tag{44}
\]
\[
\sigma_{ij}^{(\psi)}(u_i^{(1)\text{str}}, \psi_i^{(1)\text{str}}) n_i - \sigma_{ij}^{(\psi)}(u_i^{(1)\text{str}}, \psi_i^{(1)\text{str}}) n_j = 0 \quad \text{on } \Gamma, \tag{45}
\]
\[
\imath \omega \phi^{(i)}(u_i^{(1)\text{str}} - u_i^{(1)\text{str}}) n_i - \imath \omega \phi^{(e)}(u_i^{(1)\text{str}} - u_i^{(1)\text{str}}) n_i = 0 \quad \text{on } \Gamma, \tag{46}
\]
Radiating condition for scattered field,
where \( m = i, e \), and \( \Omega_e = \mathbb{R}^2 \setminus \Omega_i \). We assume that the solutions of this boundary problem can be decomposed using the following solutions before the hole domain \( \Omega_i \) appears:

\[
\begin{align*}
    u^{(e)}(x) &= u^*(x)\chi_{\Omega_e} + \hat{u}^{(e)}(x), \\
    \phi^{(e)}(x) &= \psi^*(x)\chi_{\Omega_e} + \hat{\phi}^{(e)}(x).
\end{align*}
\]

Furthermore, we expand \( \hat{u}^{(e)} \) and \( \hat{\phi}^{(e)} \) exponentially by \( \epsilon \), yielding \( \hat{u}^{(e)}(\xi) = \hat{u}^{(1)}(\xi) + \epsilon \hat{u}^{(2)}(\xi) + O(\epsilon^2) \) and \( \hat{\phi}^{(e)}(\xi) = \hat{\phi}^{(1)}(\xi) + \epsilon \hat{\phi}^{(2)}(\xi) + O(\epsilon^2) \), and \( \xi = \frac{x - x_0}{\epsilon} \) is scaled according to the radius \( \epsilon \), where \( x_0 \) denotes the center of \( \Omega_i \). The boundary value problem for \( \hat{u}^{(1)} \) and \( \hat{\phi}^{(1)} \) then becomes

\[
\begin{align*}
    \frac{\partial}{\partial \xi_j} \left[ \sigma^{(m)}_{ij}(\xi) \hat{u}^{(1)\epsilon} \right] &= 0 \quad \text{in} \quad \Omega_m(\xi), \\
    \left( \frac{\varphi^2}{\rho_i} \right)^{(m)} \frac{\partial^2 \hat{\phi}^{(1)\epsilon}}{\partial \xi_i \partial \xi_j} &= 0 \quad \text{in} \quad \Omega_m(\xi), \\
    \hat{u}^{(1)\epsilon} - u^{(1)\epsilon} &= u^*(x_0) \quad \text{on} \quad \Gamma(\xi), \\
    \hat{\phi}^{(1)\epsilon} - \psi^{(1)\epsilon} &= \psi^*(x_0) \quad \text{on} \quad \Gamma(\xi), \\
    \sigma^{(0)}_{ij}(\hat{u}^{(1)\epsilon}) n_j - \sigma^{(0)}_{ij}(u^{(1)\epsilon}) n_j &= 0 \quad \text{on} \quad \Gamma(\xi), \\
    \left( \frac{\varphi^2}{\rho_i} \right)^{(0)} \frac{\partial \hat{\phi}^{(1)\epsilon}}{\partial \xi_i} n_i - \left( \frac{\varphi^2}{\rho_i} \right)^{(0)} \frac{\partial \psi^{(1)\epsilon}}{\partial \xi_i} n_i &= 0 \quad \text{on} \quad \Gamma(\xi).
\end{align*}
\]

Radiating condition for \( \hat{u}^{(1)} \) and \( \hat{\phi}^{(1)} \).

The solutions to this problem are therefore obtained as follows:

\[
\begin{align*}
    \hat{u}^{(1)} &= u^*(x_0)\chi_B, \\
    \hat{\phi}^{(1)} &= \psi^*(x_0)\chi_B,
\end{align*}
\]

where \( B = \Omega(\xi) \) represents a unit sphere, which corresponds to \( \Omega_i \) after scaling by \( \xi \). Next, the boundary value problem for \( \hat{u}^{(2)} \) and \( \hat{\phi}^{(2)} \) are given as follows:

\[
\begin{align*}
    \frac{\partial}{\partial \xi_j} \left[ \sigma^{(m)}_{ij}(\hat{u}^{(2)\epsilon}) \right] &= 0 \quad \text{in} \quad \Omega_m(\xi), \\
    \left( \frac{\varphi^2}{\rho_i} \right)^{(m)} \frac{\partial^2 \hat{\phi}^{(2)\epsilon}}{\partial \xi_i \partial \xi_j} &= 0 \quad \text{in} \quad \Omega_m(\xi), \\
    \hat{u}^{(2)\epsilon} - u^{(2)\epsilon} &= \hat{u}^*(x_0) \quad \text{on} \quad \Gamma(\xi), \\
    \hat{\phi}^{(2)\epsilon} - \phi^{(2)\epsilon} &= \phi^*(x_0) \quad \text{on} \quad \Gamma(\xi), \\
    \sigma^{(0)}_{ij}(\hat{u}^{(2)\epsilon}) n_j - \sigma^{(0)}_{ij}(u^{(2)\epsilon}) n_j &= \sigma^{(0)}_{ij}(u^{(1)\epsilon}) n_j + i\omega \phi^{(1)\epsilon} \hat{\psi}^{(1)\epsilon} n_j + i\omega \phi^{(1)\epsilon} \psi^{(1)\epsilon} n_j \quad \text{on} \quad \Gamma(\xi), \\
    \left( \frac{\varphi^2}{\rho_i} \right)^{(0)} \frac{\partial \hat{\phi}^{(1)\epsilon}}{\partial \xi_i} n_i - \left( \frac{\varphi^2}{\rho_i} \right)^{(0)} \frac{\partial \phi^{(1)\epsilon}}{\partial \xi_i} n_i &= \left( \frac{\varphi^2}{\rho_i} \right)^{(0)} \frac{\partial \hat{\phi}^{(1)\epsilon}}{\partial \xi_i} n_i - \left( \frac{\varphi^2}{\rho_i} \right)^{(0)} \frac{\partial \phi^{(1)\epsilon}}{\partial \xi_i} n_i + i\omega \phi^{(1)\epsilon} \hat{\psi}^{(1)\epsilon} n_i + i\omega \phi^{(1)\epsilon} \psi^{(1)\epsilon} n_i \quad \text{on} \quad \Gamma(\xi).
\end{align*}
\]

Radiating condition for \( \hat{u}^{(2)} \) and \( \hat{\phi}^{(2)} \).

Using polar coordinates \((r, \theta)\), the inner solutions of this problem are obtained as follows (Barber, 2010):

\[
\begin{align*}
    \hat{u}^{(2)\epsilon} &= \frac{1}{2\mu^{(0)}} \left[ A_1 (\epsilon^{(0)}) - 1 \right] r - 2D_3 r \cos 2\theta - 2E_3 r \sin 2\theta, \\
    \hat{u}^{(2)} &= \frac{1}{2\mu^{(0)}} \left[ -\frac{A_4}{r} + 2D_3 r \sin 2\theta - 2E_3 r \cos 2\theta \right], \\
    \hat{\phi}^{(2)\epsilon} &= Ar \sin \theta + Br \cos \theta,
\end{align*}
\]

with the coefficients in the above expressions expressed as follows:

\[
A = \frac{2(\frac{\varphi^2}{\rho_i})^{(0)} \phi^*(x_0) a}{(\frac{\varphi^2}{\rho_i})^{(0)} + (\frac{\varphi^2}{\rho_i})^{(0)}},
\]

where \( m = i, e \), and \( \Omega_e = \mathbb{R}^2 \setminus \Omega_i \).
\[ B = \frac{2(\frac{b}{a})^3 e_{ij}(x_0)_{ij} + (\varphi^{(0)} - \varphi^{(0)})i\omega u_i'(x_0)}{(\frac{b}{a})^3} \]  
\[ A_1 = \frac{2\mu^{(0)}\mu^{(0)}[u_{1,1}(x_0) + u_{2,2}(x_0)] + \mu^{(0)}[i2\omega(\varphi^{(0)} - \varphi^{(0)})\psi^f(x_0) + \sigma^{(0)}_{11}(x_0) + \sigma^{(0)}_{22}(x_0)]}{4\mu^{(0)} + 2(k - 1)\mu^{(0)}} \]  
\[ A_4 = A_4' = \frac{\mu^{(0)}\mu^{(0)}[-u_{1,2}(x_0) + u_{2,1}(x_0)]}{\mu^{(0)} - \mu^{(0)}} \]  
\[ D_3 = \frac{\kappa^{(0)}\mu^{(0)}\sigma^{(0)}_{12}(x_0) - \sigma^{(0)}_{11}(x_0) - 2\mu^{(0)}\mu^{(0)}[u_{1,1}(x_0) - u_{2,2}(x_0)]}{4\kappa^{(0)}\mu^{(0)} + 4\mu^{(0)}} \]  
\[ E_3 = \frac{-\mu^{(0)}[\kappa^{(0)}\sigma^{(0)}_{12}(x_0) + \mu^{(0)}[u_{1,2}(x_0) + u_{2,1}(x_0)]]}{2\kappa^{(0)}\mu^{(0)} + 2\mu^{(0)}} \]  

where \( k^{(in)} = 3 - 4\nu^{(in)} \) in case of plane strain and \( \nu^{(in)} \) is Poisson's ratio. Following the same procedure for the adjoint variable, similar asymptotic behavior is obtained.

**Step3:** Calculate the explicit formula

Substituting these formulas into \( DF \cdot V \) and integrating each term, the explicit formula for the topological derivative is obtained based on the relationship between the topological derivative and shape derivative (8). The following relationship for state variables and their gradients are used when taking the limit of \( \epsilon \),

\[ u^{(e)}(x) = u^{(e)}(x_0), \quad \psi^{(e)}(x) = \psi^{(e)}(x_0), \]  
\[ \frac{\partial u^{(e)}}{\partial x_j} = \frac{\partial u^{(e)}}{\partial x_j}(x), \quad \frac{\partial \psi^{(e)}}{\partial x_j} = \frac{\partial \psi^{(e)}}{\partial x_j}(x) \]  

The same relationships are used for the adjoint variables and the shape derivative then takes following form:

\[ DF \cdot V = 2\mathbb{R} \left[ (-V)\epsilon \int_0^{2\pi} (I_1 + I_2 + I_3 + I_4) d\theta \right] \]  

where,

\[ I_1 = -\frac{1}{2\mu + \lambda} \left[ \sigma^{(0)}_{rr}(\hat{u}^{(2)})\sigma^{(0)}_{rr}(\hat{u}^{(2)}) \right] + \frac{\lambda}{2\mu + \lambda} \left[ \sigma^{(0)}_{rr}(\hat{u}^{(2)})\epsilon_{00}^{(e)} - (\hat{u}^{(2)}) \right] + \sigma^{(0)}_{rr}(\hat{u}^{(2)})\epsilon_{00}^{(e)} \]  
\[ I_2 = \alpha'(1 - \alpha') \frac{\partial \hat{u}^{(2)}}{\partial \epsilon_{rr}^{(e)}} + \alpha' \frac{\partial \hat{u}^{(2)}}{\partial \epsilon_{rr}^{(e)}} - \alpha' \frac{\partial \hat{u}^{(2)}}{\partial \epsilon_{rr}^{(e)}} + \alpha_1 \frac{\partial \hat{u}^{(2)}}{\partial \epsilon_{rr}^{(e)}} - \alpha_1 \frac{\partial \hat{u}^{(2)}}{\partial \epsilon_{rr}^{(e)}} \]  
\[ I_3 = \alpha'(1 - \alpha') \frac{\partial \hat{u}^{(2)}}{\partial \epsilon_{rr}^{(e)}} - \alpha_1 \frac{\partial \hat{u}^{(2)}}{\partial \epsilon_{rr}^{(e)}} - \alpha_1 \frac{\partial \hat{u}^{(2)}}{\partial \epsilon_{rr}^{(e)}} + \alpha_1 \frac{\partial \hat{u}^{(2)}}{\partial \epsilon_{rr}^{(e)}} + \alpha_1 \frac{\partial \hat{u}^{(2)}}{\partial \epsilon_{rr}^{(e)}} \]  
\[ I_4 = \alpha'(1 - \alpha') \frac{\partial \hat{u}^{(2)}}{\partial \epsilon_{rr}^{(e)}} + \alpha_1 \frac{\partial \hat{u}^{(2)}}{\partial \epsilon_{rr}^{(e)}} + \alpha_1 \frac{\partial \hat{u}^{(2)}}{\partial \epsilon_{rr}^{(e)}} - \alpha_1 \frac{\partial \hat{u}^{(2)}}{\partial \epsilon_{rr}^{(e)}} \]  

where \( u_i'(x_0) = u_i'(x_0) \cos \theta + u_i'(x_0) \sin \theta, \) and \( u_i'(x_0) = -u_i'(x_0) \sin \theta + u_i'(x_0) \cos \theta, \) while \( v_i'(x_0) \) and \( v_i'(x_0) \) take the same formulas. To calculate the integral of \( \theta \), Eq. (66) through Eq. (73) are used. The following formulas are especially convenient.

\[ \int_0^{2\pi} \epsilon_{00}^{(e)}(\hat{u}^{(2)}) d\theta = \frac{\pi(\epsilon - 1)A_1}{\mu^{(0)}}, \]  
\[ \int_0^{2\pi} \sigma_{rr}^{(0)}(\hat{u}^{(2)}) d\theta = 4\pi A_1, \]  
\[ \int_0^{2\pi} \sigma_{rr}^{(0)}(\hat{u}^{(2)}) d\theta = 4\pi(2A_1 D_1 + D_3 D_3 + E_3 E_3), \]  

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respectively. In the two-phase material domain, air and rubber are approximately expressed as elements are prevented from overlapping geometrical features, such as the circumferences of circles. We use air as the

\[ F = \int_{\Omega_0} \rho \dot{u}^2 d\Omega = \int_{\Omega_0} \rho \dot{u}^2 d\Omega. \]  

To obtain the topological derivative for the equivalent elastic domain, we only have to exchange the roles of the inner and outer domains that appear in the above formulas. If the inner \( \Omega_i \) and outer \( \Omega_e \) domains are filled with different combination of equivalent acoustic materials, sound-hard and sound-soft limits can be considered by taking the limit values of the parameters defined in the inner domain, indicated with superscript \( i \). The proposed topological derivative is valid for the sound-hard condition expressed as \( (\frac{\rho c}{\rho c})^0 = \alpha^0 \to 0 \) and invalid for the sound-soft condition expressed as \( (\frac{\rho c}{\rho c})^0 = \alpha^0 \to \infty \).

4. Numerical examples

We confirm the validity of obtained topological derivative by comparing it with the numerical differences. The state and adjoint variables are calculated using FEM analysis under the assumption of a plane strain state. The analysis domain, as shown in Fig. 1, is discretized with roughly 200000 2nd order triangular elements. To reduce numerical errors, the finite elements are prevented from overlapping geometrical features, such as the circumferences of circles. We use air as the acoustic material, where the mass density and speed of sound are 1.2 kg/m\(^3\) and 344 m/s, respectively. Rubber is used as the elastic material, with Young’s modulus, mass density, and Poisson’s ratio set to 175 MPa, 1877 kg/m\(^3\), and 0.4, respectively. In the two-phase material domain, air and rubber are approximately expressed as \( M(1 - \bar{\varepsilon}, \bar{\varepsilon}) \) and \( M(\bar{\varepsilon}, 1 - \bar{\varepsilon}) \), where \( \bar{\varepsilon} \) is set to 10\(^{-9}\). The frequency of the incident acoustic wave is set to 1000 Hz. The objective function is defined as the squared norm of the acoustic pressure in \( \Omega_{\text{out}} \) as follows:

\[ F = \int_{\Omega_{\text{out}}} \rho \dot{u}^2 d\Omega = \int_{\Omega_{\text{out}}} \rho \dot{u}^2 d\Omega. \]  

The infinitesimal hole that corresponds to the definition of the topological derivative is replaced by a hole domain \( \Omega_0 \) filled with an equivalent elastic material with a radius of 10\(^{-4}\) in \( \Omega_{\text{acoustic}} \), or, analogously, by a hole domain \( \Omega_0 \) filled with an equivalent acoustic material with a radius of 10\(^{-4}\) in \( \Omega_{\text{elastic}} \). This allows us to calculate the following numerical difference, \( D_T F' \):

\[ D_T F' = \frac{F_{\Omega_0} - F_0}{-\pi \varepsilon^2}. \]  

where \( F_{\Omega_0} \) represents the objective functional value after the appearance of hole domain \( \Omega_0 \), and \( F_0 \) represents the value without the hole. Figure 2 shows plots of the topological derivative and the differences between it and (a) the equivalent acoustic region, and (b) the equivalent elastic region. Both values are plotted in the range of (a) \( x = 0.11 \) m to \( x = 0.39 \) m, and (b) \( x = 0.41 \) m to \( x = 0.59 \) m along the \( y = 0.5 \) m line. The congruency of these plots demonstrates the validity of the proposed topological derivative.
5. Conclusion

This letter explained the derivation of a formula for calculating the topological derivative for a two dimensional acoustic-elastic coupled system, expressed using a two-phase model. The two-phase model was briefly explained and the procedure for calculating the topological derivative in the two-phase model was described. Numerical examples demonstrated the validity of our derived topological derivative.

References


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