Symmetries and their importance for statistical turbulence theory

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Abstract
The present article is intended to give a broad overview and present details on the Lie symmetry induced statistical turbulence theory put forward by the authors and various other collaborators over the last twenty years. For this is crucial to understand that our present text-book knowledge proclaims that Lie symmetries such as Galilean transformation lie at the heart of fluid dynamics. These important properties also carry over to the statistical description of turbulence, i.e. to the Reynolds stress transport equations and its generalization, the multi-point correlation equations (MPCE). Interesting enough, the MPCE admit a much larger set of symmetries, in fact infinite dimensional, subsequently named statistical symmetries. Apart from the MPCE also the two other known complete theories of turbulence, the Lundgren-Monin-Novikov (LMN) hierarchy of probability density functions and the Hopf functional theory, share this property of admitting both classical mechanical and statistical Lie symmetries. As the Galilean transformation illuminates fundamental properties of classical mechanics, the new statistical symmetries mirror key properties of turbulence such as intermittency and non-gaussianity. After an introduction to Lie symmetries have been given, these facts will be detailed for all three turbulence approaches i.e. MPCE, LMN and Hopf approach. From a practical point of view, these new symmetries have important consequences for our understanding of turbulent scaling laws. The symmetries form the essential foundation to construct exact solutions. Presently we detail this only for the infinite set of MPCE, which in turn are identified as classical and new turbulent scaling laws. Examples on various classical and new shear flow scaling laws including higher order moments will be presented. Even new scaling have been forecasted from these symmetries and in turn validated by DNS.

Key words: Turbulence, Scaling laws, Lie symmetries, Multipoint correlations, PDF methods, Hopf functional

1. Preface

The special importance of turbulence maybe comprehended by its ubiquity in innumerable natural and technical systems. Examples for natural turbulent flows are the atmospheric flow and the oceanic current which to calculate is a crucial point in climate research. Classical engineering application involving turbulence are the flows around airplanes or cars or turbulence within jet or reciprocating engines. Though supposed for more than hundred years, only with the advent of super computers it became apparent that the Navier-Stokes equations provide an excellent continuum mechanical model for turbulent flows. Still, the exclusive and direct application of the Navier-Stokes equations to practical flow problems at very high Reynolds numbers without invoking any additional assumptions is still several decades away.

However, in most applications it is not at all necessary to know all the detailed fluctuations of velocity and pressure present in turbulent flows but for the most part statistical measures are sufficient. This was in fact the key idea of O. Reynolds who was the first to suggest a statistical description of turbulence. The Navier-Stokes equations, however, constitute a non-linear and, due to the pressure Poisson equation, a non-local set of equations. As an immediate consequence of this the equations for the mean or expectation values for velocity and pressure lead to an infinite set of statistical equations, or, if truncated at some level of statistics, an un-closed system is generated.
For the full description of a turbulent field at a given time, all multipoint statistics up to infinite order need to be known. In turbulence research three complete statistical descriptions of turbulence are known, namely the infinite hierarchy of the multi-point correlation equations approach (so-called Friedmann-Keller hierarchy) Keller and Friedmann (1924), the infinite hierarchy of the multi-point probability density function (PDF) equations (Lundgren-Monin-Novikov (LMN) equations) Lundgren (1967); Monin (1967); Novikov (1968) and, finally, the Hopf functional approach, cf. Hopf (1952). Since all three approaches are infinite dimensional (though different in its mathematical appearance), any numerical approach for their solutions is excluded from the outset.

In order to obtain a much deeper insight into the statistical behavior of turbulence the authors have applied Lie symmetry group theory to the full infinite set of all the above mentioned statistical equations. Though by employing group theory we limit ourselves to the investigation of various canonical turbulent flow situations, still, the present authors are not aware of another methods that would allow more e.g. in terms of a greater complexity of flow geometries.

The present paper is an extensive but not exhaustive review of what has been done by the authors and their co-workers within the last almost twenty years. Several names should be mentioned who significantly contributed in one or the other way to the development of the theory. We have put them into alphabetical order according to their last names: Victor Avsarkisov, Alexei Cheviakov, Vladimir Grebenev, Silke Guenther, George Khujadze, Amirfarhang Mehdizadeh, Andreas Rosteck and Tanja Weller.

The results presented are based on the habilitation thesis of the first author Oberlack (2000), a larger set of dissertations: Mehdizadeh (2010), Avsarkisov (2013), Rosteck (2013), as well as several journal publications: Oberlack and Rosteck (2010), Rosteck and Oberlack (2011), Mehdizadeh and Oberlack (2010), Avsarkisov et al. (2014), Wachawczyk and Oberlack (2013), Wachawczyk et al. (2014). Still, some important new material on symmetries of the Hopf functional equation has been introduced in section 5.3.1.

The present review article is structured as follows. In section 2 we first revisit the Lie symmetries theory and its basic concepts using the 1D heat equation as a basic example. Only in the final subsection 2.7 the concept of a functional is introduced and the ”machinery” of Lie symmetry group is extended to functional differential equations needed to understand the Hopf functional approach. In the subsequent section 3 classical symmetries of Euler and Navier-Stokes equations are recalled. Next, in section 4 we present three methods for the full description of turbulent field, MPC equations, LMN hierarchy for PDFs and the Hopf approach. New statistical symmetries of these equations are discussed in Section 5. They lead to series of invariant solutions representing turbulent scaling laws, some of them are explored in Section 6.

2. Introduction to the theory of Lie groups

As already stated in the preface, the theory of Lie groups emerged to be one of the most important tools in understanding and especially in constructing exact solutions of differential equations. The first part of the present section discusses some examples that introduce the most important notions for the theory of Lie groups. The following subsections explain fundamental concepts more detailed and give related examples. The terms and the methods introduced herein form the basis for the mathematical proceeding in the sections 5 and 6.

In the past ten years many publications dealing with the theory of Lie groups and their application have been released. Several books give a good introduction to the methods of Lie groups without expecting prerequisites in group-theory. Just some fundamental basics about differential equations are required. In the order of increasing difficulty the following non-exhaustive list of books give an introduction to the theory of Lie groups: Hydon (2000), Bluman and Cole (1974), Bluman and Kumei (1989), Stephani (1989), Bluman et al. (2010) and Olver (2000). A very detailed review and an extensive body of results on the knowledge in the field of Lie group theory is given in the three books by Ibragimov (1994a), Ibragimov (1994b) and Ibragimov (1996).

2.1. Motivation and definitions

A differential equation is called autonomous if it does not contain the independent variable explicitly. In case of ordinary differential equations this property can immediately be used to reduce the order. The Blasius differential equation

\[ y''' + \frac{1}{2}yy'' = 0 \]  

(1)

is an example for a autonomous equation. Autonomy of a differential equation is equivalent to the property that the form
of the equation does not change under the transformation
\[ x^* = x + a, \quad y^* = y \]  
(2)
with \( x^* \) and \( y^* \) being the new set of variables and \( a \) representing a parameter. The transformation infers on how to reduce the order of an autonomous differential equation
\[ t(s) = y', \quad s = y, \]  
(3)
where basically the dependent variable \( y \) is introduced as a new, independent variable.

The transformation (2) is also called a symmetry, symmetry transformation or symmetry group of equation (1).

The term symmetry is taken from colloquial language and therein describes the virtual movement, e.g. rotation, of an object that does not change its shape. In terms of differential equations, the object corresponds to the differential equation and the virtual movement to a transformation of the dependent and independent variable (change of variables).

**Definition:** A symmetry is a transformation of the independent and dependent variables to a new set of variables as such that it does not change the form of a differential equation when writing it in terms of the new variables.

The symmetry (2) is also called a translation group because it is equivalent to a rigid movement when \( a \) changes. A reason for using the term “group” in the transformation 2 is given in subsection 2.2.

A mathematical formulation for the notion of symmetry can be defined as follows. The system of differential equations that has to be examined is
\[ F(x, y, y_1, y_2, \ldots, y_p) = 0, \]  
(4)
where \( x \) and \( y \) are the dependent and independent variables. \( y \) denotes the \( n \)th derivative of \( y \) with respect to \( x \) and \( p \) is the highest appearing order of differentiation. The transformation of variables
\[ x = \phi(x', y'), \quad y = \psi(x', y') \]  
(5)
is called a symmetry or symmetric transformation if the following equivalence is preserved:
\[ F(x, y, y_1, y_2, \ldots, y_p) = 0 \iff F(x^*, y^*, y^*_1, y^*_2, \ldots, y^*_p) = 0. \]  
(6)
That means that inserting transformation (5) into equation (4) does not change the equations’ shape when written in terms of \( x^* \) and \( y^* \). There are different kinds of symmetries, of which the ones with a continuous parameter are especially important for the construction of solutions.

The notion of symmetry can be used for ordinary as well as for partial differential equations. A well known partial differential equation in classical physics that has many different symmetries is the one dimensional heat equation
\[ \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}. \]  
(7)

In addition to the translation groups in \( t, x \) and \( u \) thus adding a constant to one of the previous variables, the following symmetries
\[ t^* = \exp(2\alpha)t, \quad x^* = \exp(\alpha)x, \quad u^* = u \]  
(8)
and
\[ t^* = t, \quad x^* = x, \quad u^* = \exp(\beta)u \]  
(9)
can easily be recognized. Both are scaling symmetries.

From (8) we may construct a new independent variable that does not contain \( \alpha \)
\[ \eta = \frac{x}{\sqrt{t}}, \]  
(10)
Apparently this term is invariant under the scaling symmetry (8) since it does not change \( \eta \), i.e.
\[ \frac{x}{\sqrt{t}} = \frac{x^*}{\sqrt{t^*}}. \]  
(11)
(10) corresponds to the classic similarity variable of the heat equation (7).

By combining (8) and (9) another invariant may be generated

$$\phi = \frac{u}{r^n} \quad \text{with} \quad n = \frac{\beta}{2\alpha},$$

(12)

since

$$\frac{u}{r^n} = \frac{u^*}{r^n}$$

(13)

obviously is invariant under the transformation

$$t^* = \exp(2\alpha)t, \quad x^* = \exp(\alpha)x \quad \text{and} \quad u^* = \exp(\beta)u.$$  

(14)

Regarding $\phi$ in (12) as new independent variable and $\eta$ as similarity coordinate and employing this into equation (7) leads to the following, ordinary differential equation

$$\frac{d^2\phi}{dt^*} + \frac{1}{2\eta} \frac{d\phi}{d\eta} = n\phi = 0,$$

(15)

which has solutions in special functions.

The following introduces one of the six symmetries of equation (7). It is closely related to the fundamental solution of the equation (7)

$$t^* = t, \quad x^* = x + \delta t, \quad u^* = \exp\left(\frac{\delta x}{2} - \frac{\delta^2 t}{4}\right)u.$$  

(16)

This transformation points out that even relatively simple equations can have quite complicated symmetries that cannot be obtained intuitively.

Although different in character, all of the mentioned transformations have a common theoretical background. They represent continuous groups. This means they all have one (ore more) continuous parameter(s) and the mathematical property of a group. The notion of a transformation group is explained in the following section.

2.2. Transformation groups

A group is a mathematical set of objects, e.g. integers, matrices or, as in the present case, transformations, that are connected with a dual operator (an operator connecting two elements) and fulfill four conditions. These are (i) the closure of the group with respect to the operator, (ii) the set contains a unitary element, (iii) the set contains an inverse element and (iv) the defined operation is associative. The set of real numbers e.g. is a group with respect to addition. The addition of two real numbers is also a real number. The unitary element is zero and the inverse element of an arbitrary element $a$ is $-a$. The associativity of addition can of course be shown easily.

In the same manner the set of transformations $G_T$

$$T_\varepsilon : x^* = \phi(x; \varepsilon)$$

(17)

can be interpreted as a group depending on an arbitrary continuous parameter $\varepsilon \in \mathbb{R}$ and fulfilling the four conditions mentioned above. In this case $T_\varepsilon$ represents the transformation and $\varepsilon$ is called the group parameter. Transformations of the form (17) are called point transformations since every point $x$ is uniquely mapped to $x^*$. An extension of the notion of transformation are local transformations which means that $\phi$ contains derivatives and non-local transformations depending on integrals.

The mathematical form of the four conditions (i-iv) defining a transformation group is:

i. Combination of two transformations leads to a new transformation $T_{\varepsilon_1}T_{\varepsilon_2}x = T_{\varepsilon_2}T_{\varepsilon_1}x$ with $T_{\varepsilon_2} \in G_T$ (closure).

ii. There is a transformation $I \in G_T$ so that $IT_\varepsilon x = T_\varepsilon Ix = T_\varepsilon x$ (unitary element).

iii. Each transformation in $G_T$ has an inverse element so that $T_{-\varepsilon}^{-1}T_\varepsilon x = T_{-\varepsilon}T_\varepsilon^{-1}x = Ix = x$ (inverse element).

iv. Three transformations $T_{\varepsilon_i} \in G_T, i = 1, 2, 3$ fulfill the condition $T_{\varepsilon_1}(T_{\varepsilon_2}T_{\varepsilon_3})x = (T_{\varepsilon_1}T_{\varepsilon_2})T_{\varepsilon_3}x$ (associativity).

All of the above conditions are fulfilled by the transformations in subsection 2.1. As an example we examine the scaling symmetry (8). Closure can be proven by examining the transformation $T_{\alpha_2}T_{\alpha_1}$ thus catenating

$$t_2 = \exp(2\alpha_2)t_1, \quad x_2 = \exp(\alpha_2)x_1$$

(18)
and
\[ t_1 = \exp(2\alpha_1) t, \quad x_1 = \exp(\alpha_1)x \] (19)

This leads to a new transformation
\[ t_2 = \exp[2(\alpha_2 + \alpha_1)] t, \quad x_2 = \exp[(\alpha_2 + \alpha_1)]x \] (20)

which is also of the form (8) and the sum of the parameters \( \alpha_1 \) and \( \alpha_2 \) is a real number as well.

It is easy to comprehend that the unitary element of the group can be obtained by setting \( \alpha = 0 \) in (8) thus setting one of the group parameters in (20) to zero. The inverse element of the group is given by (8) when replacing \( \alpha \) with \(-\alpha\). This can also be seen in equation (20) by setting \( \alpha_1 = \alpha \) and \( \alpha_2 = -\alpha \). The associativity of (8) can be seen since applying (8) three times leads to a sum of group parameters which is obviously fulfilling the condition of associativity.

In a similar manner the group character of other transformations can be proven. As seen by means of a few examples in subsection 2.1 the transformation groups can be used to construct special solutions. The question that remains is how to obtain all transformation groups of a given differential equation.

An essentially untractable method is to introduce a complete indefinite point transformation (5) into the equation of interest (4) to then apply condition (6) and obtain all symmetry transformations. Even for simple differential equations this leads to a problem that is unsolvable.

The significant advantage of the technique of Lie groups is that an infinitesimal form of the transformation can be given that even when calculating symmetries of higher non-linear differential equations leads to a linear homogeneous system of differential equations for the transformation rule.

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An essentially untractable method is to introduce a complete indefinite point transformation (5) into the equation of interest (4) to then apply condition (6) and obtain all symmetry transformations. Even for simple differential equations this leads to an almost unmanageable and most important non-linear over-determined system of partial differential equations for the transformation rule.

The significant advantage of the technique of Lie groups is that an infinitesimal form of the transformation can be given that even when calculating symmetries of higher non-linear differential equations leads to a linear homogeneous system of differential equations for the transformation rule. This linear system of equations can usually be solved completely general employing simple integration heuristics.

2.3. Infinitesimal transformations

Since the following sections distinguish the independent and dependant variables \( x \) and \( y \) only the following symmetry transformations will be examined:
\[ x^* = \phi(x, y; \varepsilon) \quad \text{and} \quad y^* = \psi(x, y; \varepsilon) \] (21)

Furthermore and without loss of generality it is assumed that the unitary transformation corresponds to \( \varepsilon = 0 \):
\[ x^* = \phi(x, y; 0) = x \quad \text{and} \quad y^* = \psi(x, y; 0) = y \] (22)

Also it is assumed that the transformation rules for \( \phi \) and \( \psi \) are sufficiently smooth and thus invertible so that a Taylor expansion about \( \varepsilon = 0 \) can be done
\[ x^* = \phi(x, y; 0) + \frac{\partial \phi}{\partial \varepsilon} \bigg|_{\varepsilon=0} \varepsilon + O(\varepsilon^2) \quad \text{and} \quad y^* = \psi(x, y; 0) + \frac{\partial \psi}{\partial \varepsilon} \bigg|_{\varepsilon=0} \varepsilon + O(\varepsilon^2) \] (23)

The first term on each of the right hand sides can be replaced by (22) and terms of order \( O(\varepsilon) \) are formally replaced by \( \xi \) and \( \eta \):
\[ x^* = x + \xi(x, y)\varepsilon + O(\varepsilon^2) \quad \text{and} \quad y^* = y + \eta(x, y)\varepsilon + O(\varepsilon^2) \] (24)

The crucial implication of Lie’s first theorem states that knowing the “infinitesimals” \( \xi \) and \( \eta \) in (24) uniquely determines the “global” form of the group transformation i.e. (21). From the global form of the transformation group (21) the infinitesimals can be obtained by simple differentiation
\[ \xi(x, y) = \frac{\partial \phi}{\partial \varepsilon} \bigg|_{\varepsilon=0} \quad \text{and} \quad \eta(x, y) = \frac{\partial \psi}{\partial \varepsilon} \bigg|_{\varepsilon=0} \] (25)

The reverse, i.e. the derivation of the global transformation (21) from the knowledge of the corresponding infinitesimals \( \xi \) and \( \eta \) results from Lie’s differential equations (Lie’s first theorem)
\[ \frac{dx^*}{d\varepsilon} = \xi(x^*, y^*) \quad \text{and} \quad \frac{dy^*}{d\varepsilon} = \eta(x^*, y^*) \] (26)
which have to satisfy the initial conditions
\[ \varepsilon = 0 : x^* = x \quad \text{and} \quad y^* = y . \]  

(27)

Note that a proof of the theorem is based on the conditions (i)-(iv) for the group transformations. It is obvious that arbitrary functions \( \phi \) and \( \Psi \) depending on a parameter, but not having group properties, cannot be expressed in terms of order \( O(\varepsilon) \) only. In order to exactly determine an arbitrary transformation that does not represent a transformation group in terms of a Taylor expansion all terms of the series have to be taken into account.

In addition to the continuous transformation groups many differential equations have finite transformation groups not depending on a group parameter. A known example is the reflection symmetry. The heat equation (7) e.g. is invariant under the transformation
\[ t^* = t \quad , \quad x^* = -x . \]  

(28)

This type of symmetry transformation will not be considered subsequently.

An example of the equivalence of global and infinitesimal forms of transformation groups are the scaling transformations (8) and (9) of the heat equation. Replacing \( \alpha \) with \( \varepsilon \) in (8), thus considering the transformation
\[ t^* = \exp(2\varepsilon)t \quad , \quad x^* = \exp(\varepsilon)x \quad , \quad u^* = u , \]  

(29)

leads, according to the definition of the infinitesimals in (25), to the terms
\[ \xi_t = \frac{\partial \exp(2\varepsilon)t}{\partial \varepsilon} \bigg|_{\varepsilon=0} = 2t , \quad \xi_x = \frac{\partial \exp(\varepsilon)x}{\partial \varepsilon} \bigg|_{\varepsilon=0} = x , \quad \eta_u = \frac{\partial u}{\partial \varepsilon} \bigg|_{\varepsilon=0} = 0 . \]  

(30)

Analogously, replacing \( \beta \) with \( \varepsilon \) in (9) leads to the transformation
\[ t^* = t \quad , \quad x^* = x \quad , \quad u^* = \exp(\varepsilon)u , \]  

(31)

which with (25) leads to the infinitesimals
\[ \xi_t = \frac{\partial t}{\partial \varepsilon} \bigg|_{\varepsilon=0} = 0 , \quad \xi_x = \frac{\partial x}{\partial \varepsilon} \bigg|_{\varepsilon=0} = 0 , \quad \eta_u = \frac{\partial \exp(\varepsilon)u}{\partial \varepsilon} \bigg|_{\varepsilon=0} = u . \]  

(32)

The subscripts of \( \xi \) and \( \eta \) point out the affiliation of the infinitesimals and do not denote a derivative. To obtain the global transformations the Lie equations (26) and (27) have to be integrated using (30) and (32) respectively. This results in the initial value problem for the infinitesimals (30)
\[ \frac{dr^*}{de} = 2r^* , \quad \frac{dx^*}{de} = x^* , \quad \frac{du^*}{de} = 0 \quad \text{with} \quad \varepsilon = 0 : t^* = t , \quad x^* = x , \quad u^* = u , \]  

(33)

which immediately leads to the transformation (29). Equivalently, for the infinitesimals (32) Lie’s differential equation is
\[ \frac{dr^*}{de} = 0 , \quad \frac{dx^*}{de} = 0 , \quad \frac{du^*}{de} = u^* \quad \text{with} \quad \varepsilon = 0 : t^* = t , \quad x^* = x , \quad u^* = u , \]  

(34)

with its solution representing (31).

From the idea of infinitesimal transformations one can conclude that it is obviously sufficient to know the infinitesimals of a symmetry transformation to calculate the similarity solution.

At the beginning of this section symmetries of differential equations where introduced meaning the invariance of a differential equation under a transformation. The notion of invariance can also be applied to functions. The following section deals with functions that are invariant under certain transformations. This examination uses the infinitesimal form of a transformation.

2.4. Invariant functions

Similar to the symmetry of differential equations invariant functions preserve their form under a given transformation. A given function \( f(x, y) \) is called invariant under a transformation
\[ x^* = \phi(x, y; \varepsilon) \quad \text{and} \quad y^* = \psi(x, y; \varepsilon) , \]  

(35)

provided that
\[ f(x, y) = f(x^*, y^*) . \]  

(36)
Since only classes of functions are to be examined that are invariant under group transformations the following adapts the notion of infinitesimal transformation to condition (36). For this the transformations (24) are inserted into (36) before doing a Taylor expansion about \( \varepsilon = 0 \):

\[
\begin{align*}
 f(x, y) &= f(x + \xi(x, y)\varepsilon + O(\varepsilon^2), y + \eta(x, y)\varepsilon + O(\varepsilon^2)) \\
 \Rightarrow f(x, y) &= f(x, y) + \varepsilon X f + \frac{\varepsilon^2}{2} X^2 f + O(\varepsilon^3),
\end{align*}
\]

(37)

where the operator \( X \) is defined as

\[
X = \xi_k \frac{\partial}{\partial x_k} + \eta_l \frac{\partial}{\partial y_l}
\]

(39)

and usually referred to as infinitesimal generator or simply generator.

Powers of \( X \) indicate repeated application of the operator and double indices indicate summation. On both sides of (37), \( f(x, y) \) can be eliminated leaving terms of order \( \varepsilon \) and higher. From the leading order the condition for invariance of \( f \) under the transformation (35) results in infinitesimal form:

\[
X f = \xi_k \frac{\partial f}{\partial x_k} + \eta_l \frac{\partial f}{\partial y_l} = 0.
\]

(40)

It can be proven that higher order terms do not have to be taken into account and that equation (40) is a necessary and sufficient condition for invariance under (35). Equation (40) represents a hyperbolic differential equation which can be solved with the method of characteristics. The characteristic system consisting of \( m \) independent and \( n \) dependent variables reads

\[
\frac{dx_1}{\xi_1} = \frac{dx_2}{\xi_2} = \cdots = \frac{dx_m}{\xi_m} = \frac{dy_1}{\eta_1} = \frac{dy_2}{\eta_2} = \cdots = \frac{dy_n}{\eta_n}.
\]

(41)

The solution of this system contains \( m + n + 1 \) constants of integration which are regarded as new independent variables of \( f \) and, here, represent a solution to equation (40). Thus (40) defines a function \( f(C_1, \ldots, C_{m+n-1}) \), which is invariant under a given group transformation (35). Here the invariant function is defined by its infinitesimals \( \xi, \eta \).

As an example we consider those functions \( f(x, t, u) \) that are invariant under the transformations (8) and (9). The infinitesimals of those transformations are given by (30) and (32) respectively. Using (40) leads to the following differential equation:

\[
2t \frac{\partial f}{\partial t} + x \frac{\partial f}{\partial x} = 0.
\]

(42)

The complete solution results from the characteristic system (41):

\[
f = f\left(u, \frac{x}{\sqrt{t}}\right).
\]

(43)

Assuming \( f \) has to be also invariant under the transformation (9), equation (40) and its infinitesimal form of the transformation (32) lead to the condition

\[
u \frac{\partial f}{\partial u} = 0.
\]

(44)

Obviously, \( f \) does not depend on \( u \) anymore and thus has the following form:

\[
f = f\left(\frac{x}{\sqrt{t}}\right).
\]

(45)

This section extends the notion of invariant functions to differential equations. Differential equations can be seen as function equations containing derivatives. However, presently we examine the inverse problem. In the above example we were looking for a function that is invariant under a given transformation. When examining differential equations the function is given and the corresponding invariant transformations are to be calculated, i.e. searched for.
2.5. Prolongations and the invariance of differential equations

The following section extends the idea of invariance from functions to differential equations. Two extensions of the definitions given in section 2.4 are introduced. The notion of a function has to be established in a way that derivatives can be taken into consideration as well. Furthermore the fact that instead of functions now equations are examined has to be considered.

The key idea for finding symmetries of differential equations, which will be detailed below, is the extension of the infinitesimal transformation (24) to differentials. For this we need to introduce some preliminary definitions and notations.

To define invariance of a partial differential equation or a system of differential equations in a generalized form we may first need to write it into a more compact form. As already mentioned above $x$ and $y$ represent vectors of length $m$ and $n$. The derivatives of order $p$ of $y$ with respect to $x$ are abbreviated as follows:

$$ y_{k,j_1,j_2,...,j_p} = \frac{\partial^p y_k}{\partial x_{j_1} \partial x_{j_2} \cdots \partial x_{j_p}}. \tag{46} $$

All derivatives of a given order $p$ are gathered in new vectors and written in the following compact way:

$$ y_1 = \left[ y_{k,j_p} \right], \quad y_2 = \left[ y_{k,j_p,j_q} \right], \quad y_3 = \left[ y_{k,j_p,j_q,j_r} \right], \ldots \quad \text{with} \quad k = 1, \ldots, m \quad \text{and} \quad j_p, j_q, j_r, \ldots = 1, 2, \ldots, n. \tag{47} $$

The brackets indicate a summation of all possible combinations of $k$, $j_p$, $j_q$ and $j_r$, which have to be extended accordingly for higher order derivatives. Since ordinary differential equations are commutative, meaning $y_{k,j_p,j_q} = y_{k,j_q,j_p}$, the amount of vector elements reduces according to the restriction $j_p \leq j_q \leq \ldots$.

Furthermore we need to define a new differential operator since we need to examine derivatives of functions of the form $G\left[ x, y, y_1, y_2, \ldots, y_p \right]$ with respect to $x$.

The motivation for the new operator is given by means of a simple example. The derivative of $H(x_1, x_2, y_1, y_2, y_{11}, y_{12}, y_{22}, \ldots)$ with respect to $x_1$ would, in classic notation, look as follows due to the implicit dependency of the depending variable $y$ on $x_1$ and $x_2$:

$$ \frac{\partial H}{\partial x_1} = \left( \frac{\partial H}{\partial y_1} y_{11} + \frac{\partial H}{\partial y_2} y_{12} + \frac{\partial H}{\partial y_{11}} y_{11} + \frac{\partial H}{\partial y_{12}} y_{12} + \frac{\partial H}{\partial y_{22}} y_{22} + \ldots \right). \tag{48} $$

Obviously, this nomenclature is not unique since $\partial H/\partial x_1$ does not clearly point out whether just the explicit dependencies of $x_1$ in $H$ are dealt with or the implicit dependencies as well.

Since in the generalized case the vectors $y, y_1, \ldots$ depend on $x$ implicitly we define the derivative with respect to $x_i$ as a complete differential operator

$$ \frac{D}{Dx_i} = \frac{\partial}{\partial x_i} + y_k \frac{\partial}{\partial y_k} + y_{k,l} \frac{\partial}{\partial y_{k,l}} + \ldots. \tag{49} $$

The above definition prevents confusion when calculating derivatives with respect to $x$. Accordingly, a derivative of the form $\partial/\partial x_i$ affects only terms that explicitly depend on $x$. All implicit dependencies on $x$ in $y, y_1, \ldots$ are covered by the form of (49). This may be shown for the differentiation of a dependent variable $y_m$ with respect to an independent variable $x_i$.

$$ y_{m,j} = \frac{D y_m}{D x_i} = \frac{\partial y_m}{\partial x_i} + y_k \frac{\partial y_m}{\partial y_k} + y_{k,l} \frac{\partial y_m}{\partial y_{k,l}} + \ldots. \tag{50} $$

Since $y_m$ does not depend on $x_i$ explicitly but just implicitly only the second term on the right hand side contributes to the differentiation in (50), where the identity $\partial y_m/\partial y_k = \delta_{mk}$. Of course, this can be extended to higher order derivatives

$$ y_{m,j} = \frac{D y_{m,j}}{D x_j}, \quad y_{m,j,k} = \frac{D y_{m,j,k}}{D x_k}, \quad \ldots. \tag{51} $$

For the transformation of differential equations it is not sufficient to examine the transformations in their algebraic form

$$ x^* = \phi(x, y; e) \quad \text{and} \quad y^* = \psi(x, y; e) \tag{52} $$

but it is also necessary to calculate the occurring differentials according to the chain rule for transformations. For this purpose we further introduce the operator

$$ \frac{D}{Dx_i} = \frac{\partial}{\partial x_i} + y_{k,i} \frac{\partial}{\partial y_k} + y_{k,j} \frac{\partial}{\partial y_{k,j}} + \ldots. \tag{53} $$
which, synonymously to (49), defines the differentiation with respect to $x'^i$. For example, according to (50) and (51) follows:

$$
y'_{m,i} = \frac{D'}{D'x_j} y'_m , \quad y'_{m,j} = \frac{D'}{D'x_j} y'_{m,i} , \quad y'_{m,ij} = \frac{D'}{D'x_k} y'_{m,ijk} , \ldots .
$$

(54)

For $\frac{D}{Dx_i}$ the chain rule for transformations together with (52) leads to

$$
\frac{D}{Dx_i} = \frac{D\phi_m}{Dx_i} \frac{D'}{D'x^m} .
$$

(55)

We note that for reasons of uniqueness the operator $\frac{D'}{D'x_j}$ only contains terms of the form $y'_{1}, y'_{2}, \ldots$ and not $y$ or its differentials with respect to $x'$. It is obvious that this is only a matter of definitions and solely depends on which terms $\frac{D'}{D'x_j}$ should affect. Since in the following these are only terms containing $y'_{1}, y'_{2}, \ldots$, (53) is the only appropriate definition.

Using the rule of differentiation given in (55) for the dependent variables in (52) leads to

$$
\frac{D\phi_k}{Dx_i} = \frac{D\phi_m}{Dx_i} \frac{D'y'_{k}}{D'x^m} .
$$

(56)

The last term on the right hand side can be replaced according to (54) which leads to

$$
\frac{D\phi_k}{Dx_i} = y'_{m,k} \frac{D\phi_m}{Dx_i} ,
$$

(57)

where the terms on the right hand side are exchanged to avoid misunderstandings concerning the effect of the operator $\frac{D}{Dx_i}$. (57) is the implicit definition of a first derivative in the coordinate system ($x', y'$). It is obvious that it is going to be quite difficult to use this for large systems of differential equations.

In section 2.3 we have already shown that in order to determine a group transformation only its infinitesimal form

$$
x'^i = x_i + \xi_i e + O(e^2) \quad \text{and} \quad y'^m = y_m + \eta_m e + O(e^2)
$$

(58)

is necessary. This will now be extended to derivatives of the form

$$
y'_{m,i} = y_{m,i} + \xi_{m,i} e + O(e^2) ,
$$

(59)

$$
y'_{m,ij} = y_{m,ij} + \xi_{m,ij} e + O(e^2) .
$$

(60)

$$
\vdots \quad \vdots \quad \vdots
$$

These new infinitesimals $\xi_{m,i} \ldots$ are interpreted as unknown functions depending on $\xi$ and $\eta$ only and to be determined in the following. Furthermore, note that the indices of $\xi_{m,i} \ldots$ do not contain any comma but merely a semicolon after the first position. This indicates that the first index corresponds to the $m^{th}$ vector element of $y$, whereas all other indices correspond to the $i^{th}$ and $j^{th}$ derivative of $y_{m,i,j} \ldots$ respectively. It is particularly important that all indices of $\xi_{m,i} \ldots$ after the semicolon do not represent derivatives.

Now the transformation rule (57) will be used to determine the unknown infinitesimals $\xi_{m,i} \ldots$. For this purpose we insert (58), (59) into (57), replacing $\phi$ and $\psi$ with the infinitesimals $x'$ and $y'$ respectively. We obtain

$$
\frac{D}{Dx_i} \left( y_k + \eta_k e + O(e^2) \right) = \left( y_{k,m} + \xi_{k,m} e + O(e^2) \right) \frac{D}{Dx_i} \left( x_m + \xi_m e + O(e^2) \right)
$$

(61)

which can be decomposed into equations of order $e^0$

$$
\frac{D\eta_k}{Dx_i} = y_{k,m} \frac{Dx_m}{Dx_i}
$$

(62)

and $e^1$

$$
\frac{D\eta_k}{Dx_i} = \xi_{k,m} \frac{Dx_m}{Dx_i} + y_{k,m} \frac{D\xi_m}{Dx_i}
$$

(63)
Using definition (49) it can be shown that (62) represents an identity by employing \( \frac{\partial \xi_0}{\partial y_i} = \delta_{ij} \). With this equation (63) can be solved for \( \xi_{k,j} \)

\[
\xi_{k,j} = \frac{D\eta_k}{Dx_i} - y_{k,m} \frac{D\xi_m}{Dx_i},
\]

thus the first infinitesimal form of the transformation of differentials of order one in (59) is known, provided the infinitesimals \( \varepsilon \) and \( \eta \) are given.

Analogously to the above procedure all other infinitesimals for higher order derivatives can be derived. Without proof we introduce the following recursive formula which computes infinitesimals of higher order:

\[
\xi_{k,j_{1},...j_{s}} = \frac{D\xi_{k,j_{1},...j_{s-1}}}{Dx_i} - y_{k,m,j_{1},...j_{s-1}} \frac{D\xi_m}{Dx_i} \text{ for } s > 1.
\]

At this point it becomes obvious why the complete differential operator (49) was introduced. To derive (64), merely derivatives with respect to \( x_i \) and \( y_j \) were necessary, whereas the infinitesimals on the right hand side of (65) also contain \( \xi_{k,j_{1},...j_{s-1}} \) which explicitly contain members of the vectors \( y, y, \ldots \).

With these definitions available the notion of invariance for algebraic functions will be extended to differential equations. The symmetry of a system of differential equations is defined similarly to the invariance of a function in (36).

As already defined in subsection 2.1 a system of differential equations - in the following referred to as differential equation - is invariant under the transformation (52) if

\[
F(x, y, y_1, y_2, \ldots, y_p) = 0 \iff F(x', y', y_1', y_2', \ldots, y_p') = 0
\]

holds, where \( p \) represents the highest order of differentiation. Assuming that the vector function \( F \) is differentiable with respect to all arguments \( x, y, y_1, \ldots \) allows us to introduce the transformations (58) and (59) into the right one of the equations (66). A Taylor expansion of the latter equation about \( \epsilon = 0 \) leads to

\[
F(x, y, y_1, y_2, \ldots, y_p') + \epsilon \left[ \xi_{j_1} \frac{\partial}{\partial x_{j_1}} + \eta_{j_1} \frac{\partial}{\partial y_{j_1}} + \xi_{j_{2,1}} \frac{\partial}{\partial y_{j_{2,1}}} + \cdots + \xi_{j_{p,1}} \frac{\partial}{\partial y_{j_{p,1}}} \right] F(x, y, y_1, y_2, \ldots, y_p') + O(\epsilon^2) = 0 \quad (67)
\]

The left of the equations (66) can be inserted into (67) and using the fact that only terms of order \( \epsilon \) have to be considered to determine the group transformation, as already discussed in subsection 2.3, leads to

\[
\left. \left[ \xi_{j_1} \frac{\partial}{\partial x_{j_1}} + \eta_{j_1} \frac{\partial}{\partial y_{j_1}} + \xi_{j_{2,1}} \frac{\partial}{\partial y_{j_{2,1}}} + \cdots + \xi_{j_{p,1}} \frac{\partial}{\partial y_{j_{p,1}}} \right] F \right|_{\epsilon = 0} = 0 \quad (68)
\]

The above equation is a necessary and sufficient condition for the invariance of a differential equation under a transformation group defined by its infinitesimals \( \xi_i \) and \( \eta_j \). The equation to the right of the vertical line of the index indicates that for solving this problem the equation of interest \( F = 0 \) and its differential consequences have to be used. The meaning of this notation is made clear with the following example.

The operator

\[
\left( \frac{\partial^n}{\partial x_i^n} \right) X = \xi_{j_1} \frac{\partial}{\partial x_{j_1}} + \eta_{j_1} \frac{\partial}{\partial y_{j_1}} + \xi_{j_{2,1}} \frac{\partial}{\partial y_{j_{2,1}}} + \cdots + \xi_{j_{p,1}} \frac{\partial}{\partial y_{j_{p,1}}} \quad (69)
\]

is called \( p^n \) prolongation of the operator (generator)

\[
X = \xi_{j_1} \frac{\partial}{\partial x_{j_1}} + \eta_{j_1} \frac{\partial}{\partial y_{j_1}} \quad (70)
\]

Obviously, the question from equation (66) has been transformed to condition (68) under the assumption that only transformations that form a group are permitted. The fact that equation (68) is much easier to solve i.e. the computation of \( \xi_i \) and \( \eta_j \) and the resulting \( \xi_{j_{p,1}} \), than the original problem (66) is demonstrated in the following example for the heat equation (7).

Note that when calculating the infinitesimals (65) all variables in the operator (69) have to be regarded as independent parameters. This is not only important for all variables \( y \) but also for all terms from the vectors \( y, y, \ldots \). Accordingly, the handling of condition (68) becomes very easy and elegant since using the operator (69) merely demands to apply differentiations to a differential equation.
For practical reasons we rename the following variables in equation (7):

\[ x \mapsto x_1, \quad t \mapsto x_2, \quad u \mapsto y_1. \]  

(71)

which leads to

\[
\frac{\partial y_1}{\partial x_2} - \frac{\partial^2 y_1}{\partial x_1^2} = 0 \quad \text{or} \quad y_{1,2} - y_{1,1,11} = 0. 
\]

(72)

Applying condition (68) to equation (72) reads as follows:

\[
\left[ \zeta_{1,2} \frac{\partial}{\partial y_{1,2}} + \zeta_{1,11} \frac{\partial}{\partial y_{1,11}} \right] (y_{1,2} - y_{1,11}) = 0 \quad \text{on} \quad y_{1,2} - y_{1,11} = 0. 
\]

(73)

Since equation (72) only contains the terms \( y_{1,2} \) and \( y_{1,11} \) all remaining terms of operator (69) in (73) are not given explicitly as they vanish anyway. Differentiation of (73) leads to

\[
\zeta_{1,2} - \zeta_{1,11} = 0 \quad \text{on} \quad y_{1,2} - y_{1,11} = 0. 
\]

(74)

The repeated statement of the investigated heat equation on the right of the equations (73) and (74) has the same meaning as the equation stated on the right of the vertical line in (68). The reason for this will become more obvious further below.

Since (72) contains the variables \( x_1, x_2 \) and \( y_1 \) only we look for infinitesimals of the form \( \xi_1(x_1, x_2, y_1), \xi_2(x_1, x_2, y_1) \) and \( \eta_1(x_1, x_2, y_1) \). Equation (74) only contains the infinitesimals \( \zeta_{1,2} \) and \( \zeta_{1,11} \). The recursion formula (65) states that for \( \zeta_{1,11} \) the term \( \zeta_{1,1} \) has to be calculated as well. The infinitesimals of the derivatives result from the equations (64) and (65)

\[
\xi_{1,1} = \frac{\partial \xi_1}{\partial x_1} + y_{1,11} \left[ \frac{\partial \xi_1}{\partial y_{1,11}} + \frac{\partial \xi_1}{\partial y_{1,11}} + y_{1,12} \left[ \frac{\partial \xi_1}{\partial y_{1,11}} + \frac{\partial \xi_1}{\partial y_{1,11}} - y_{1,12} \left[ \frac{\partial \xi_1}{\partial y_{1,11}} + \frac{\partial \xi_1}{\partial y_{1,11}} \right] \right] \right]. 
\]

(75)

and

\[
\xi_{1,2} = \frac{\partial \xi_2}{\partial x_2} + y_{1,12} \left[ \frac{\partial \xi_2}{\partial y_{1,12}} + \frac{\partial \xi_2}{\partial y_{1,12}} + y_{1,12} \left[ \frac{\partial \xi_2}{\partial y_{1,12}} + \frac{\partial \xi_2}{\partial y_{1,12}} - y_{1,12} \left[ \frac{\partial \xi_2}{\partial y_{1,12}} + \frac{\partial \xi_2}{\partial y_{1,12}} \right] \right] \right]. 
\]

(76)

and

\[
\eta_{1,11} = \frac{\partial \eta_1}{\partial x_1} + y_{1,11} \left[ \frac{\partial \eta_1}{\partial y_{1,11}} + \frac{\partial \eta_1}{\partial y_{1,11}} + y_{1,12} \left[ \frac{\partial \eta_1}{\partial y_{1,11}} + \frac{\partial \eta_1}{\partial y_{1,11}} - y_{1,12} \left[ \frac{\partial \eta_1}{\partial y_{1,11}} + \frac{\partial \eta_1}{\partial y_{1,11}} \right] \right] \right]. 
\]

(77)

Inserting the infinitesimals (76) and (77) into equation (74) leads to a differential equation in \( \xi_1, \xi_2 \) and \( \eta_1 \)

\[
\frac{\partial \xi_1}{\partial x_2} - \frac{\partial^2 \eta_1}{\partial x_1^2} + y_{1,1} \left[ -2 \frac{\partial \xi_1}{\partial x_1 y_1} - \frac{\partial^2 \xi_1}{\partial x_1^2} y_{1,1} + y_{1,2} \left[ 2 \frac{\partial \xi_1}{\partial x_1} - \frac{\partial^2 \xi_1}{\partial x_1^2} + \frac{\partial^2 \xi_1}{\partial x_1 y_1} \right] \right] + y_{1,11} \left[ \frac{\partial \xi_1}{\partial y_{1,11}} + \frac{\partial^2 \xi_1}{\partial x_1 y_1} y_{1,11} + y_{1,12} \left[ 2 \frac{\partial \xi_1}{\partial x_1} \right] \right] + y_{1,12} \left[ \frac{\partial \xi_1}{\partial y_{1,12}} + \frac{\partial^2 \xi_1}{\partial x_1 y_1} y_{1,12} \right] = 0. 
\]

(78)

In the derivation of the above equation the term \( y_{1,11} \) has been replaced by \( y_{1,2} \) according to the heat equation (72). With this the meaning of the repeated statement of the equations of interest behind the vertical line in (68) and (74) respectively becomes obvious. It is to illustrate that after all differentiations some parts of the equation - here it is \( y_{1,11} \) - can be replaced by the respective remainder of the equation. Not doing so will lead to more restrictions, i.e. more equations for the wanted variables \( \xi_1, \xi_2 \) and \( \eta_1 \) and, in turn, to a curtailed set of symmetries. In principle, (78) is called the set of determining equations, though a more convenient form will be given below.
In order to unscramble equation (78) we may note that the infinitesimals in (78) are functions of \( x_1, x_2 \) and do not contain any derivatives of \( y_1 \). Since equation (78) has to hold for arbitrary \( y_{1,1}, y_{1,2}, \ldots \) the respective coefficients for \( y_{1,1}, y_{1,2}, y_1^3, y_1^4, y_1^{1,1}, y_1^{1,2}, y_{1,1}^{1,2}, y_{1,2}^{1,1}, y_{1,2}^{1,2}, y_{1,1}^{1,1} \) have to be set to zero.

To simplify the structure of the resulting determining system the coefficients of \( y_{1,2} \), i.e. \( \frac{\partial \xi_2}{\partial y_1} = 0 \), and \( y_{1,1} \), i.e. \( \frac{\partial \xi_1}{\partial y_1} = 0 \), are inserted into the remaining equations. For the coefficient of \( y_{1,1} \) this results in another monomial of the form \( \frac{\partial \xi_1}{\partial y_1} = 0 \) which is also inserted into the remaining system. The resulting equations are:

\[
\begin{align*}
\frac{\partial \xi_1}{\partial y_1} &= 0, \\
\frac{\partial \xi_2}{\partial x_1} &= 0, \\
\frac{\partial \xi_2}{\partial y_1} &= 0, \\
\frac{\partial^2 \eta_1}{\partial y_1^2} &= 0, \\
2 \frac{\partial \xi_1}{\partial x_1} - \frac{\partial \xi_2}{\partial x_2} &= 0, \\
\frac{\partial^2 \xi_1}{\partial x_1^2} - \frac{\partial \xi_1}{\partial x_2} - 2 \frac{\partial^2 \eta_1}{\partial x_1 \partial y_1} &= 0, \\
\frac{\partial \eta_1}{\partial x_2} - \frac{\partial^2 \eta_1}{\partial x_1^2} &= 0.
\end{align*}
\]  

(79)–(82)

Equations (79)–(82) can be integrated to obtain

\[
\xi_1 = \xi_1(x_1, x_2), \quad \xi_2 = \xi_2(x_2) \quad \text{and} \quad \eta_1 = f_1(x_1, x_2)y_1 + f_2(x_1, x_2). \tag{86}
\]

The above result is inserted into equation (83) and after one integration with respect to \( x_1 \) we obtain

\[
\xi_1 = \frac{1}{2} \frac{d^2 \xi_2(x_2)}{dx_2^2} x_1 + g(x_2). \tag{87}
\]

Using both of the above results in (84) and integrating the result with respect to \( x_1 \) leads to

\[
f_1 = -\frac{1}{8} \frac{d^3 \xi_2(x_2)}{dx_2^3} x_1^2 - \frac{1}{2} \frac{d \xi_2(x_2)}{dx_2} x_1 + h(x_2). \tag{88}
\]

Inserting all intermediate results (86)–(88) into equation (85) leads to

\[
y_1 \left( -\frac{1}{8} \frac{d^3 \xi_2(x_2)}{dx_2^3} x_1^2 - \frac{1}{2} \frac{d^2 \xi_2(x_2)}{dx_2^2} x_1 + \frac{d \xi_2(x_2)}{dx_2} + \frac{1}{4} \frac{d^2 \xi_2(x_2)}{dx_2^2} \right) + \frac{d f_2(x_1, x_2)}{dx_2} - \frac{d^2 f_2(x_1, x_2)}{dx_2^2} = 0. \tag{89}
\]

The above equation also has to hold for arbitrary \( x_1, x_2 \) and \( y_1 \). Accordingly the coefficient of \( y_1 \) has to vanish independently of the following two terms. Furthermore, the above parenthesis expression has the form of a polynomial in \( x_1 \) forcing all coefficients of \( x_1^i \) with \( i = 0, 1, 2 \) to zero. From this the complete integrals for \( \xi_2, g \) and \( h \) can be computed. Together with (86)–(88) the final result for the infinitesimals becomes

\[
\begin{align*}
\xi_1 &= a_1 + a_4 x_1 + a_5 x_1 x_2 + a_6 x_2, \\
\xi_2 &= a_2 + 2a_4 x_2 + a_5 x_2^2, \\
\eta_1 &= y_1 \left[ a_3 - \frac{a_5}{2} \left( \frac{x_1^2}{2} + x_2 \right) \right] + f_2(x_1, x_2),
\end{align*}
\]

(90)–(92)

where \( f_2(x_1, x_2) \) is a solution of \( \frac{d f_2(x_1, x_2)}{dx_2} - \frac{d^2 f_2(x_1, x_2)}{dx_2^2} = 0 \) and \( a_1 \)–\( a_6 \) represent arbitrary parameters. It is easy to verify that the results (90)–(92) satisfy not only equations (79)–(85) but also equation (78). The equation for \( f_2 \) is obviously identical to the investigated heat equation. It can be shown (see e.g. Bluman et al. (2010)) that this is a consequence of the linear character of the heat equation and occurs synonymously for all linear differential equations. For this reason
we do not examine \( f_2 \) in the following but it should be noted that the appearance of \( f_2 \) being a solution of the original heat equation is a manifestation of the superposition principle of linear differential equations. Merely the non-trivial transformation groups are considered for further calculations according to the parameters \( a_1-\ldots-a_6 \).

Since the infinitesimals, e.g. (90)-(92), can always be represented by a sum of different irreducible elements the operator
\[
X = \xi_i \frac{\partial}{\partial x_i} + \eta_j \frac{\partial}{\partial y_j}
\] (93)
can also be decomposed into its elements. These can also be regarded as elements of a linear vector space. For this reason it is common to specify a list of operators which indicate the invariance of the differential equation of interest. For the heat equation these are
\[
X_1 = \frac{\partial}{\partial x_1}, \quad X_2 = \frac{\partial}{\partial x_2}, \quad X_3 = y_1 \frac{\partial}{\partial y_1}, \quad X_4 = x_1 \frac{\partial}{\partial x_1} + 2x_2 \frac{\partial}{\partial x_2}, \quad X_5 = x_1x_2 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} - \left( \frac{x_1^2}{4} + \frac{x_2^2}{2} \right)y_1 \frac{\partial}{\partial y_1}, \quad X_6 = x_2 \frac{\partial}{\partial x_1} - \frac{x_1}{2} y_1 \frac{\partial}{\partial y_1},
\] where the indices of \( X_i \) and \( a_i \) in (90)-(92) accord. Each arbitrary linear combination of the operators \( X_1-\ldots-X_6 \) and their respective prolongations represent an invariant transformation of the heat equation.

The infinitesimals (90)-(92) contain also the transformation groups (30) and (32) derived in subsection 2.3. Here these correspond to the operators \( X_4 \) and \( X_3 \) and the parameters \( a_4 \) and \( a_3 \) respectively for the remaining constants being equal to zero.

From the infinitesimals (90)-(92) the global transformations can be computed according to Lie’s differential equations (26) and (27). For the infinitesimals corresponding to the parameters \( a_4 \) and \( a_3 \) this has already been demonstrated in the equations (30)-(34) together with (8), (9). In further examples we compute the transformations corresponding to the parameters \( a_5 \) and \( a_6 \). Parameter \( a_5 \) belongs to the infinitesimals
\[
\xi_1 = x_1x_2, \quad \xi_2 = x_2^2, \quad \eta_1 = \left( \frac{x_1^2}{4} + \frac{x_2^2}{2} \right)y_1
\] (100)
which, using Lie’s differential equations (26) and (27), leads to the following global transformations:
\[
x^*_1 = \frac{x_1}{1 - \varepsilon x_2}, \quad x^*_2 = \frac{x_2}{1 - \varepsilon x_2}, \quad y^*_1 = \frac{y_1 \exp \left( \frac{1}{4 - \varepsilon x_2} \right)}{\sqrt{1 - \varepsilon x_2}}.
\] (101)
Analogously the infinitesimals of the parameters \( a_6 \) are
\[
\xi_1 = x_2, \quad \xi_2 = 0, \quad \eta_1 = \frac{y_1 x_1}{2}
\] (102)
which, using Lie’s differential equations, leads to
\[
x^*_1 = x_1 + \varepsilon x_2, \quad x^*_2 = x_2, \quad y^*_1 = y_1 \exp \left( \frac{\varepsilon}{2} \left( x_1 + \varepsilon x_2 \right) \right).
\] (103)
This already is the symmetry transformation for the heat equation given in (16).

Since now all symmetries, i.e. their infinitesimal forms, have been calculated in the following subsection will show how to obtain group invariant solutions thereof, which, if only scaling is involved, are referred to as similarity solutions. It is important to note that we do this without the need to compute the global form of the transformations.
2.6. Invariant solutions (similarity solutions)

Using a heuristic procedure subsection 2.1 emphasized the following: If a function is invariant under a transformation and this transformation also is a symmetry of a partial differential equation, this function can be regarded as a new dimensionally reduced similarity coordinate of the differential equation. The example used in 2.1 consisted of the scale symmetry \( r^p = \exp(2\alpha_1 t), x^p = \exp(\alpha_1 x) \) of the heat equation (7) and the corresponding similarity coordinate \( \eta = \sqrt{\frac{t}{\alpha_1 x}} \) which is an invariant under the latter scaling symmetry.

Based on the above idea and using the application of infinitesimal transformations in subsection 2.4 the notion of invariant functions has been defined in equation (40). The example of the heat equation showed that the similarity coordinate \( \eta \) can also be expressed by means of the infinitesimal transformation and the operator \( X \) respectively.

According to the notion of an invariant function we may similarly define invariant solutions. For this we consider the operator (generator)

\[
X = \xi_k \frac{\partial}{\partial x_k} + \eta_l \frac{\partial}{\partial y_l}
\]

(104)

and its prolongation which are assumed to be an infinitesimal transformation of the differential equation

\[
F(x, y, y_1, \ldots, y_p, y) = 0
\]

(105)

Based on this \( y = \Theta(x) \) is called an invariant solution of (105), if:

i. \( y - \Theta(x) \) is an invariant function with respect to \( X \) and

ii. \( y = \Theta(x) \) is a solution of (105).

According to the definition of an invariant function (40) condition (i) leads to the equations

\[
X[y - \Theta(x)] = 0 \quad \text{on} \quad y = \Theta(x).
\]

Using (104) the equations (106) may be written as hyperbolic system of differential equations

\[
\xi_k(x, \Theta) \frac{\partial \Theta}{\partial x_k} = \eta(x, \Theta)
\]

(107)

with the corresponding characteristic equations

\[
\frac{dx_1}{\xi_1(x, y)} = \frac{dx_2}{\xi_2(x, y)} = \cdots = \frac{dx_m}{\xi_m(x, y)} = \frac{dy_1}{\eta_1(x, y)} = \frac{dy_2}{\eta_2(x, y)} = \cdots = \frac{dy_n}{\eta_n(x, y)},
\]

(108)

where \( \Theta \) has been replaced by \( y \).

The system has \( m + n - 1 \) solutions which are regarded as new variables. Generally, it is advisable to regard the \( m - 1 \) solutions of the \( m \) equations on the left hand side as new variables and to equate the \( n \) expressions on the right hand side with any one of the \( m \) expressions on the left hand side. After the integration we regard the resulting constants as new independent variables of the differential equation (105).

In order to verify consistency, in the second step (ii) the new variables have to be introduced into the original system, which now has one less variable.

To emphasize the notion of an invariant solution we return to the example of the heat equation and construct some group invariant solutions. For this purpose we introduce the infinitesimals (90)–(92) into the characteristic equation (108)

\[
\frac{dx_1}{a_1 + a_1 x_1 + a_5 x_1 x_2 + a_6 x_2} = \frac{dx_2}{a_2 + 2a_1 x_2 + a_5 x_1^2} = \frac{dy_1}{y_1 \left[a_3 - \frac{a_5}{2} \left(x_1^2 + x_2^2\right) + \frac{a_6}{2} x_1^3\right]}.
\]

(109)

Different parameter combinations result in different invariant solutions. Here, only a few will be derived as an example. A complete survey of all invariant solutions of the heat equation can be found in Bluman and Cole (1974).

For the left hand side of equation (109) the assumption that all \( a_i \neq 0 \) and \( a_5^2 = a_2 a_5 \) leads to the complete integral

\[
\delta_1 = \frac{a_5 x_1 + a_6}{a_5(a_4 + a_5 x_2)}.
\]

(110)
with $\delta_1$ representing the constant of integration. Using (110) allows to solve the right hand side of equation (109)

$$y_1 = \frac{H_1}{(a_4 + a_5 x_2)^3} \exp \left( \frac{1}{4} \frac{a_2^2 a_1^2 x_2}{a_5} + \frac{1}{48} \frac{K}{a_5 x_2 + a_4} + \frac{1}{4} \frac{L \delta_1 + M}{a_5 x_2 + a_4} \right)$$

with $K = \frac{a_1^2 a_5^2 + a_1^2 a_5^2 - 2 a_1 a_4 a_2 a_6}{a_5}, L = a_4 a_6 - a_1 a_5, M = \frac{-a_2^2 - 4 a_3 a_5 - 2 a_4 a_6}{a_5}$ (111)

with $H_1$ representing a second constant of integration. Inserting $\delta_1$ and $H_1$ as new independent and dependant variables into equation (72) leads to an ordinary differential equation which has a general solution in the terms of Airy functions.

In contrast to the rather complicated similarity solution of the above example the assumption $a_2 = a_4 = a_5 = 0$ leads to the known solution of the heat equation with a point source as initial condition. Analogous to the above example the integration of the left and right hand side of (109) leads to the solutions

$$\delta_2 = x_2$$

and

$$y_1 = H_2 \exp \left( \frac{1}{4} \frac{x_1^2 - 4 a_3 a_6}{a_5} \frac{x_1}{\delta_2 + \frac{a_1}{a_6}} \right)$$

with $\delta_2$ and $H_2$ representing the integration constants. $\delta_2$ and $H_2$ are now regarded as independent and dependent variables respectively and inserted into equation (72) lead to a differential equation of order one

$$\frac{dH_2(\delta_2)}{d\delta_2} = \frac{2 a_3 a_6}{a_5} \left( \frac{a_1}{a_6} \right)^2 \frac{\delta_2 + \frac{a_1}{a_6}}{2 \left( \frac{a_1}{a_6} \right)^2} H_2(\delta_2).$$

The solution of this equation together with (113) represents the point source mentioned above,

$$y_1 = \frac{P}{\sqrt{x_2 + \frac{a_1}{a_6}}} \exp \left( \frac{1}{4} \frac{x_1 - 2 a_3 a_6}{a_5} \frac{x_1}{x_2 + \frac{a_1}{a_6}} \right).$$

with $P$ representing the source strength.

Finally, we want to explain the notion of a symmetry breaking by means of the above exemplary solutions. For large parts of this paper this notion is of special importance. The solution (111) of the heat equation contains a number of undetermined parameters $a_i$, the group parameters of the transformation groups which can be used to adapt to different boundary or initial conditions. In most cases one or more parameters are set to zero to fit a given condition. For the group transformation corresponding to this parameter this means that the transformation is no longer conform to the given boundary or initial condition. This case is referred to as symmetry breaking since one or more symmetries are violated by external conditions. The assumption $a_2 = a_4 = a_5 = 0$ for the construction of the second solution introduced a number of symmetry breakings causing the corresponding solution (115) only to satisfy a limited amount of initial conditions.

### 2.7. Extension of Lie symmetry groups to functional differential equations

For further purposes we present here an extension of the Lie group theory towards functional equations with functional derivatives. This part is needed for further understanding of Sections 4.5 and 5.3 where the functional Hopf equation for turbulence is considered. In this Section we will mainly refer to our previous works (Oberlack and Waclawczyk (2006), Waclawczyk and Oberlack (2013)) where this extension was introduced and applied to find symmetries of the Hopf formulation of the Burgers equation. We consider such functional differential equations, which may be regarded as extensions of partial differential equations. The key idea is that the discrete set of independent variables $(v_1, v_2, \ldots, v_n)$ in a partial equation is replaced by a continuous set of variables denoted by $[v(x)]$, Lévy and Pellegrino (1951). Partial derivatives with respect to $v_i$ are replaced by functional derivatives, denoted by $\delta f/\delta v_i$. In order to illustrate this extension we introduce the example

$$\frac{\partial f}{\partial t} = \sum_{i=1}^{n} \frac{\partial f}{\partial v_i} \delta v_i$$

(116)
where \( f = f(v_1, v_2, \ldots, v_n, t) \). Taking the continuum limit we obtain
\[
\frac{\partial f}{\partial t} = \int v(x) \frac{\delta f}{\delta v(x)} \, dx \tag{117}
\]
where \( f = f([v(x)], t) \) is a an unknown functional. The exact definition of the functional derivative \( \delta f/\delta v(x) \) is given e.g. in Lévy and Pellegrino (1951), Gelfand and Fomin (1963). However, in the present work we will use the following formula
\[
\frac{\delta f([v(x)])}{\delta v(x')} = \frac{\partial f([v(x)])}{\partial v(x')} \bigg|_{x'} = \lim_{\epsilon \to 0} \frac{f([v(x) + \epsilon \delta(x - x')]) - f([v(x)])}{\epsilon} \tag{118}
\]
Although somewhat less rigorous, the above definition is particularly suitable for calculations. For the sake of example we use the definition (118) to compute the derivative of the following functional \( f([v(x)]) = \int A(x)v(x)dx \) where \( A(x) \) is a given function. According to equation (118) the functional derivative of \( f \) reads
\[
\frac{\delta f([v(x)])}{\delta v(x')} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[ \int A(x)(v(x) + \epsilon \delta(x - x'))dx - \int A(x)v(x)dx \right] = A(x') \tag{119}
\]
In Oberlack and Wacławczyk (2006) we studied the extension of the partial differential equation for a scalar function \( \Phi(v_1, \ldots, v_n, t) \) of \( n + 1 \) independent variables. The general form of the differential equation describing \( \Phi \) reads:
\[
F(v_1, \ldots, v_n, t, \Phi, \Phi, \Phi_1, \ldots, \Phi_q) = 0 \tag{120}
\]
where \( \Phi \) denote the \( k \)-th derivatives of the function \( \Phi \) with respect to any possible combination of independent variables and \( q \) is the highest order of derivative present in Eq. (120). In the continuum limit \( t \) becomes a function of the continuous variable \( x \). The following considerations can be generalised for the vector forms of \( v, x \) and \( t \), however, we do not present this general case here, in order to keep the notation as simple as possible.

The considered function \( \Phi \) becomes a functional \( \Phi = \Phi([v(x)], t) \) and the partial differential equation (120) becomes a functional differential equation:
\[
F([v(x)], t, \Phi, \Phi, \Phi_1, \ldots, \Phi_q) = 0 \tag{121}
\]
Here again, \( \Phi \) denotes all possible derivatives of order \( k \), which can include partial derivatives with respect to \( t \) and functional derivatives with respect to \( v(x) \). The following, equivalent notation will be used for the first functional derivatives:
\[
\Phi_{x(x)} = \frac{\delta \Phi}{\delta v(x)} = \frac{\partial \Phi}{\partial v(x)}dx \tag{122}
\]
Higher order derivatives can be expressed in an analogous way. The latter notations was originally used by Hopf (1952) and can be more convenient in some situations, e.g., to denote second order partial derivative, partial with respect to \( t \) and functional with respect to \( v(x) \)
\[
\Phi_{x(x')} = \frac{\partial^2 \Phi}{\partial v(x) \partial t} \tag{123}
\]
For further purposes we solve a simple functional equation by the method of characteristics. For the sake of clarity we will first present necessary formulae for the partial differential equation (120) and introduce their counterparts in the continuum limit (121). The two approaches will also be called “classical” and “continuum” formulation, respectively. Let us consider the following hyperbolic equation in a classical and continuum formulation
\[
\Phi \frac{\partial F}{\partial \Phi} + \sum_{i=1}^n \frac{\partial F}{\partial v_i} = 0 \quad \rightarrow \quad \Phi \frac{\partial F}{\partial \Phi} + \int_a^b \frac{\partial F}{\partial v(x')} \, dx' = 0 \tag{123}
\]
where \( F = F(\Phi, v_1, \ldots, v_n) \) in the classical formulation and \( F = F(\Phi, [v(x)]) \) in the continuum limit, \( v_i \) and \( v(x) \) constitute sets of independent variables and \( \Phi \) is a dependent variable: \( \Phi = \Phi(v_1, \ldots, v_n) \) or \( \Phi = \Phi([v(x)]) \). The characteristic equations of (123) are
\[
\frac{d \Phi}{\Phi} = dv_1 = \cdots = dv_n \quad \rightarrow \quad \frac{d \Phi}{\Phi} = d\delta v(x) \quad \text{for each} \quad x \in (a, b) \tag{124}
\]
which determine \( n \) integration constants \( C_i \) in the classical formulation and an infinite set of integration constants \( C(x) \) in the continuum formulation. The constants may be employed as new (dependent and independent) variables of \( F \).
Corresponding solutions of Eqs. (123) have the forms $F = F(C_1, \ldots, C_n)$ and $F = F(C_1, (C(x)))$. An example of a possible solution of the characteristic system (124) is presented below. We may, e.g., consider the equations

$$\frac{d\Phi}{\Phi} = dt_1, \quad dt_1 = dt_2, \quad dt_2 = dt_3, \quad \ldots, \quad dt_{n-1} = dt_n$$

(125)

to obtain the following constants of integration

$$C_1 = \Phi \exp [-v(t_1)], \quad C_2 = v_1 - v_2, \quad C_3 = v_2 - v_3, \quad \ldots, \quad C_n = v_{n-1} - v_n \ ,$$

(126)

which have their counterparts in the continuum formulation,

$$C_1 = \Phi \exp [-v(x_1)], \quad C(x) = \frac{dv(x)}{dx} \int dx$$

(127)

where $x_1$ is a fixed point in the domain $x \in (a, b)$. Hence, a functional that constitutes a solution of Eq. (123) in its continuum limit may be written, e.g., as

$$F = F\left(\Phi \exp [-v(x_1)], \frac{dv(x)}{dx} \int dx\right) \ .$$

(128)

If (128) is introduced into (123) we obtain

$$\frac{\partial F}{\partial C_1} \exp [-v(x_1)] + \frac{\partial F}{\partial C(x)} \int dx \frac{d}{dx} \left[\delta(x-x')dx'\right] - \frac{\partial F}{\partial C_1} \Phi \exp [-v(x_1)] \int dx \delta(x'-x_1)dx' = 0$$

where the first and last term cancel, while the derivative in the second term equals zero.

2.7.1. Finite and infinitesimal transformations In Oberlack and Waclawczyk (2006) the classical symmetry method used to analyse the partial differential equations (120) was extended towards functional differential equation (121). The aim was to derive such transformations of variables $\Phi^*, t^*, v^*(x)$, which do not change the form of the considered Eq. (121)

$$F([v^*(x)], t^*, \Phi^*, v^*, \Phi^*, \ldots, \Phi^*) = 0 \ .$$

(129)

The variable $x$ was not transformed since it constitutes a continuous “counting” parameter, such as a summation index in the classical counterpart. We considered only such transformations of variables which constitute Lie groups, i.e. they depend on a continuous parameter $\varepsilon$ and satisfy group properties presented in Section 2.2.

The comparison of a finite one-parameter Lie point transformation for the classical and continuum formulation was presented in Oberlack and Waclawczyk (2006) in form of a table 1.

<table>
<thead>
<tr>
<th>Table 1</th>
<th>Comparison of one-parameter Lie point transformation for the classical and continuum formulation - a table from Oberlack and Waclawczyk (2006).</th>
</tr>
</thead>
<tbody>
<tr>
<td>i. $\Phi^* = \phi(\Phi, v_1, \ldots, v_n, t, \varepsilon)$</td>
<td>$\Phi^*$ = $\phi(\Phi, [v(x)], t, \varepsilon)$</td>
</tr>
<tr>
<td>ii. $v_{1}^* = \phi_{1}(\Phi, v_1, \ldots, v_n, t, \varepsilon)$</td>
<td>$v^*(x) = \phi_{1}(\Phi, [v(x)], x, t, \varepsilon)$, $x \in G$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>iii. $t^* = \phi_{t}(\Phi, v_1, \ldots, v_n, t, \varepsilon)$</td>
<td>$t^* = \phi_{t}(\Phi, [v(x)], t, \varepsilon)$</td>
</tr>
</tbody>
</table>

The transformed variables $\Phi^*, v^*(x), t^*$ become functionals in the continuum limit and instead of the finite set $v_1^* \ldots v_n^*$ in the continuum limit we defined $v^*(x) = \phi_x$, which is an explicit function of the variable $x$.

Analogously to the classical symmetry analysis, all variables of equation (120), i.e. the sets $t, v_1, \ldots, v_n$, as well as $\Phi$ and all its possible derivatives of any order, will be treated as independent variables. The differential operator defined for the ”classical” formulation in equation (49) has, for the configuration of variables considered here, the following form

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \Phi_{t} \frac{\partial}{\partial \Phi} + \Phi_{u_{i}} \frac{\partial}{\partial \Phi_{t}} + \cdots + \sum_{i=1}^{n} \Phi_{u_{i}} \frac{\partial}{\partial \Phi_{t}} + \cdots \ .$$

(130)

$$\frac{D}{Dv_i} = \frac{\partial}{\partial v_i} + \Phi_{v_i} \frac{\partial}{\partial \Phi} + \Phi_{u_{i}} \frac{\partial}{\partial \Phi_{v_i}} + \cdots + \sum_{i=1}^{n} \Phi_{u_{i}} \frac{\partial}{\partial \Phi_{v_i}} + \cdots \ .$$

(131)

Analogous definitions will apply in the continuum limit (121). The following differential operators, corresponding to (130) and (131) were derived in Oberlack and Waclawczyk (2006) for the ”continuum” limit

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \Phi_{t} \frac{\partial}{\partial \Phi} + \Phi_{u_{i}} \frac{\partial}{\partial \Phi_{t}} + \int G \, dx \, \Phi_{u_{i}(x)} \frac{\partial}{\partial \Phi_{t(x)}} + \cdots \ .$$

(132)
The derivatives of $\Phi$ in terms of the new differential operators have the following form

$$
\frac{D}{d\xi(x)} \Phi_j = \frac{D\Phi_j}{D\xi(x)} + \frac{D\Phi_j}{D\xi(x)} \frac{d\xi(x)}{d\xi} + \int_G d\xi' \frac{D\Phi_{j}\xi(x)}{D\xi'} \delta_{\xi(x)} + \cdots \cdot x \epsilon G .
$$

The derivatives of $\Phi$ are written in a Taylor series expansion about $\epsilon = 0$, cf. Eqs. (58), (59). A comparison between "classical" and "continuum" formulations were presented by Oberlack and Wacławczyk (2006) in form of table 2 where the terms of order $O(\epsilon^2)$ were neglected.

### Table 2 Comparison of infinitesimal transformations for the classical and continuum formulation - table from Oberlack and Wacławczyk (2006).

<table>
<thead>
<tr>
<th>Classical Formulation</th>
<th>Continuum Formulation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Phi^* = \Phi + \eta \Phi(x, v_1, \ldots, v_n, t) \epsilon$</td>
<td>$\Phi^* = \Phi + \eta \Phi(x, \epsilon(x), t) \epsilon$</td>
</tr>
<tr>
<td>$\Phi^<em>_x = \Phi^</em> + \xi(x, v_1, \ldots, v_n, t) \epsilon$</td>
<td>$\Phi^<em>_x = \Phi^</em> + \xi(x, \epsilon(x), t) \epsilon$</td>
</tr>
<tr>
<td>$\Phi^<em>_{x'} = \Phi^</em><em>{x'} + \xi</em>{x'}(x, v_1, \ldots, v_n, t) \epsilon$</td>
<td>$\Phi^<em>_{x'} = \Phi^</em><em>{x'} + \xi</em>{x'}(x, \epsilon(x), t) \epsilon$</td>
</tr>
<tr>
<td>$\Phi^<em>_{x''} = \Phi^</em><em>{x''} + \xi</em>{x''}(x, v_1, \ldots, v_n, t) \epsilon$</td>
<td>$\Phi^<em>_{x''} = \Phi^</em><em>{x''} + \xi</em>{x''}(x, \epsilon(x), t) \epsilon$</td>
</tr>
</tbody>
</table>

It should be noted that $\xi$, which denotes infinitesimals corresponding to $\epsilon$ is an explicit function of $x$. The same dependence holds true for infinitesimals corresponding to the functional derivatives of $\Phi$, such as $\zeta_{\epsilon(x)}$. In these cases the index of the set $[v(x)]$ has been given a different name such as $[v(x')]$ to avoid confusion with the parameter $x$. The remaining infinitesimals do not depend on $x$ explicitly.

We recall now that the key property of the Lie group method is that the finite transformations, given by the formulae (i)–(iii) in table 1, can be computed from their infinitesimal forms (iv–(vii)) by integrating the first order system of equations see also Eqs. (26) and (27)

$$
\frac{d\Phi^*}{d\epsilon} = \eta, \quad \frac{d\epsilon}{d\epsilon} = \xi, \quad \frac{d\xi(x)}{d\epsilon} = \xi^*, \quad x \epsilon G ,
$$

where the latter equations should be integrated with the initial condition

$$
\epsilon = 0 : \Phi^* = \Phi, \quad \epsilon = t, \quad \Phi^*(x) = v(x) .
$$

Now, to calculate the new, transformed variables $\Phi^*\epsilon(x), \Phi^*\epsilon'(x)$, $\epsilon'$ from equation (134) it is necessary to first find the infinitesimal forms $\eta, \xi, \xi^*$. In order to do this, the infinitesimals $\xi$ should be first expressed in terms of $\eta, \xi, \xi^*$ and independent variables $t, [v(x)], \Phi$. For this purpose, in the classical formulation differential operators (130) and (131) are applied to the transformed variable $\Phi^* = \psi(\Phi(x), v_1, \ldots, v_n, t, \epsilon)$, cf. equation (57). In the continuum counterpart this relation becomes

$$
\frac{D\Phi}{D\xi(x)dx} = \int_G \Phi_{x'}(x') \Phi(x)[v(x)], x', t, \epsilon dx' + \Phi_{x'} \Phi_{x}(x) \frac{D\Phi}{D\xi(x)dx} .
$$

When the infinitesimal forms (iv–(vii)), are introduced into equation (136) we obtain:

$$
\frac{D(\Phi + \eta \epsilon)}{D\xi(x)dx} = \int_G (\Phi_{x'} + \zeta_{x'}(x)\epsilon) \Phi(x)[v(x)] + \Phi_{x'} \Phi_{x}(x) \epsilon dx' + (\Phi_{x'} + \zeta_{x'}(x)\epsilon) \frac{D(\Phi_{x'})}{D\xi(x)dx} .
$$
Equation (137) can be further split into two equations, containing terms $O(1)$ and $O(\epsilon)$, respectively. The first of the two gives the identity
\[
\frac{D\Phi}{Dx} = \Phi_{,\alpha} (x) ,
\] (138)
from $O(\epsilon)$ we obtained a formula for the infinitesimal $\zeta_{,\alpha} (x)$
\[
\zeta_{,\alpha} (x) = \frac{D\eta}{Dx} = \int G \Phi_{,\alpha} (x) \frac{D\xi}{Dx} dx' - \Phi_{,\alpha} \frac{D\xi}{Dx} .
\] (139)
By analogy, a formula for the infinitesimal $\zeta$ was found in Wacławczyk and Oberlack (2013)
\[
\zeta = \frac{D\eta}{Dt} - \int G \Phi_{,\alpha} \frac{D\xi}{Dt} dx' - \Phi_{,\alpha} \frac{D\xi}{Dt} .
\] (140)
The infinitesimals of higher orders followed from the recursive formulæ:
\[
\frac{D\phi_{,\alpha (n-1)}}{Dx(x)} \frac{d\xi}{dx(x)} = \int G \Phi_{,\alpha (n-1)} \frac{D\phi}{Dx(x)} dx + \Phi_{,\alpha (n-1)} \frac{D\phi}{Dx(x)} ,
\] (141)
or
\[
\frac{D\phi_{,\alpha (n-1)}}{Dt} = \int G \Phi_{,\alpha (n-1)} \frac{D\phi}{Dt} dx + \Phi_{,\alpha (n-1)} \frac{D\phi}{Dt} .
\] (142)
Hence, up to this point, we recalled the derivation of formulæ for the infinitesimals $\zeta_1, \xi_{,\alpha}$ etc. in terms of $\eta, \xi, \xi_i, \zeta_0$. In the following section we will recall the procedure which finally led us to the determining equations, from the infinitesimal transformations $\eta, \xi, \xi_i$.

2.7.2. Generator X and its prolongations

Once all the necessary infinitesimal forms are obtained, they can be substituted into the equations (120) or (121) written in the transformed variables. In order to simplify notation we assume that Eqs. (120) and (121) contain derivatives up to second order only. The generalisation of the following relations to the case of higher order derivations is straightforward. After expansion of equation (129) in Taylor series about $\epsilon = 0$ in both classical and continuum formulation the expanded equation has the form:
\[
F + \epsilon X^{(p)} F + \frac{\epsilon^2}{2} \left[ X^{(p)} \right]^2 F + O(\epsilon^3) = 0 ,
\] (143)
where $X^{(p)}$ is called the prolongation of the generator $X$ of the $p^{th}$ order. In the classical formulation the generator $X$ is given by
\[
X = \eta \frac{\partial}{\partial \Phi} + \xi_i \frac{\partial}{\partial t} + \sum_{j=1}^n \xi_{,i} \frac{\partial}{\partial x_j} ,
\] (144)
see also equation (69). The following formulæ for the continuum limit holds
\[
X = \eta \frac{\partial}{\partial \Phi} + \xi_i \frac{\partial}{\partial t} + \int G dx \xi_i \frac{\delta}{\delta \Theta (x)}
\] (145)
and e.g. its second prolongation reads
\[
X^{(2)} = \eta \frac{\partial}{\partial \Phi} + \xi_i \frac{\partial}{\partial t} + \int G dx \xi_i \frac{\delta}{\delta \Theta (x)} + \xi_j \frac{\partial}{\partial \Phi} + \int G dx \xi_{,i} \frac{\delta}{\delta \Phi_{,i}} + \xi_{,j} \frac{\partial}{\partial \Phi} + \int G dx \xi_{,i} \frac{\delta}{\delta \Phi_{,j}} (x, \epsilon (x, \epsilon (x, \epsilon (x, \epsilon (x, \epsilon (x))) )) .
\] (146)
The first term in Eq. (143) equals zero, as follows from the equation (121). All the remaining terms $[X^{(p)}]^{n}$ in (143), representing a successive application of $X^{(2)}$ will be zero if the following relation holds
\[
X^{(2)} F = 0 .
\] (147)
Similar to (68), (143) leads to invariance condition
\[
\left[ X^{(p)} F \right]_{\epsilon = 0} = 0 ,
\] (148)
where, in the continuum formulation, the prolongation $X^{(p)}$ was expressed by formula (146) and the form of infinitesimals $\xi$ follow from the relations (139)–(142). The resulting condition constitutes an overdetermined system of linear differential equations. In the continuum limit we obtain a set of functional differential equations. This system can be finally solved for the infinitesimals $\eta, \xi_i$ and $\xi_{,\alpha}$.
2.7.3. Invariant solutions

The conditions for the invariant solutions of partial differential equations were discussed in Section 2.6. Analogous conditions apply to the case of functional differential equations. If equation (121) admits a symmetry given by the generator (145), then a solution $\Phi = \Theta(t, [u(x)])$ of this equation is called an invariant solution if it satisfies the relation

$$X [\Phi - \Theta(t, [u(x)])] = 0.$$  

(149)

After employing (145) and expanding the derivatives, the following, hyperbolic functional equation is obtained from (149)

$$x \frac{\partial \Theta}{\partial t} + \int_G x \frac{\delta \Theta}{\delta v(x)} dx = \eta.$$  

(150)

This equation can be solved by the method of characteristics. The corresponding system of equations reads

$$\frac{dt}{\xi_t} = \frac{d\Phi}{\eta} = \frac{\delta \Theta}{\delta v(x)} \text{ for each } x \in G.$$  

(151)

Above, $\Theta$ has been replaced by $\Phi$. Note that the last term in fact corresponds to an infinite set of equations for each point in $G$. The infinite set of constants, which is a solution of the above system, can be employed as new variables in Eq. (121).

2.7.4. Application to the Hopf formulation of the Burgers equation

In Waclawczyk and Oberlack (2013) we considered the following functional differential equation

$$\frac{\partial \Phi}{\partial t} = \int k^2 z(k) \frac{\delta^2 \Phi}{\delta z(k') \delta z(k'')} dk' dk'' - \mu \int k^2 z(k) \frac{\delta \Phi}{\delta z(k)} dk,$$  

(152)

where $\Phi([z(k)], t)$ is the characteristic functional which depends on the infinite set of variables $[z(k)]$. Equation (152) is a Hopf functional equation (cf. Hopf (1952)) derived from the Fourier transform of the viscous Burgers equation in the infinite domain

$$\frac{\partial \hat{v}(k, t)}{\partial t} = i \int k^2 z(k) \hat{v}(k', t) \hat{v}(k''', t) dk' dk'' \mu \hat{v}(k, t),$$  

(153)

where $\hat{v}(k, t)$ is the Fourier transform of the real variable $v(x, t)$. The variables $[z(k)]$ in equation (152) constitute a sample space of $\hat{v}(k, t)$ from equation (153). More information on the physical meaning of the Hopf equations will be given in Section 4.5. Here, we will only consider (152) as an example and show that new information can be delivered by the extended Lie group analysis of this equation. For this we presently search for symmetries of equation (152) where the prolongation is given by formula (146) and the invariance condition is given by (148). Doing so leads to the following system of differential equations determining the infinitesimals, $\xi_t, \xi_z, \eta$ and $\theta$

$$\Phi_{z(k)} \Phi_{z(k')}: \quad \frac{\delta \xi_t}{\delta \Phi} = 0,$$  

(154)

$$\Phi_{z(k)} \Phi_{z(k')}: \quad \frac{\delta \xi_z}{\delta \Phi} = 0, \text{ for each } k,$$  

(155)

$$\Phi_{z(k)} \Phi_{z(k')}: \quad \frac{\delta \xi_{z(k)}}{\delta \Phi} = 0, \text{ for each } k,$$  

(156)

$$\Phi_{z(k)} \Phi_{z(k')}: \quad \frac{\delta \eta}{\delta \Phi} = 0,$$  

(157)

$$\Phi_{z(k)} \Phi_{z(k')}: \quad -(k + k')z(k + k') \frac{\delta \xi_z}{\delta \phi(z(k))} + \int \left( k^2 z(k + k') \frac{\delta \xi_{z(k')}}{\delta \phi(z(k'))} \right) dk' = 0, \text{ for each } k \text{ and } k',$$  

(158)

$$\Phi_{z(k)} \Phi_{z(k')}: \quad \frac{\delta \xi_{z(k)}}{\delta \phi(z(k))} + \int k^2 z(k + k') \frac{\delta \xi_{z(k')}}{\delta \phi(z(k'))} dk' - \mu \int k^2 z(k) \frac{\delta \xi_{z(k)}}{\delta \phi(z(k))} dk' = 0, \text{ for each } k,$$  

(159)

$$\frac{\delta \eta}{\delta \phi(z(k))} - \int k^2 z(k) \frac{\delta \xi_{z(k)}}{\delta \phi(z(k))} dk' = 0.$$  

(160)

Although we were not able to find a general, complete solution of this system, a series of particular solutions have been derived. From all this only two sets of solutions have been identified as physically relevant and considered in order to find particular, invariant solutions of equation (152), namely

$$\xi_t = c_1 t, \quad \xi_z = c_2 z(k), \quad \eta = c_2 \Phi + c_3.$$  

(161)
for the inviscid case and
\[ \xi_i = c_1, \quad \xi_k = c_2 k(z(k)) + c_3 \eta = c_3 \Phi + c_4 \] (162)
for \( \mu \neq 0 \). From the infinitesimals (161), the following form of transformed variables can be derived based on equation (134)
\[ t' = t \exp(c_1 \epsilon) \quad z'(k) = z(k) \exp(c_1 \epsilon) \quad \Phi' = \Phi \] (163)
while the infinitesimals (162) correspond to
\[ t' = t + c_1 \epsilon \quad z'(k) = z(k) \exp(c_2 \epsilon) \quad \Phi' = \Phi. \] (164)
The infinitesimals (161) and (162) can be used to create the system of characteristic equations (151) in order to find invariant solutions of equation (152). In the case of functional equations the system (151) leads to an infinite number of integration constants. It was shown in Wacławczyk and Oberlack (2013) that the solution may be presented as a sum
\[ \Phi = 1 + C_1 + C_2 + \cdots + C_n + \ldots \] (165)
which, as will be shown later, is in fact a Taylor series expansion of \( \Phi \) around \( z = 0 \). Below, we will shortly recall this procedure for the inviscid case.

Using formula (151), we find that for each \( k \) the following relations are true
\[ \frac{dr}{t} = \frac{\delta(z(k))}{z(k)} \quad \delta \Phi = 0. \] (166)
Integrating the first equation of (166) leads to
\[ \ln t = \ln |c(k) z(k)| \] (167)
where \( c(k) \) is an integration constant for each \( k \). If we rewrite \( c(k) = f_1(k)/c_1(k) \) we obtain the solution
\[ c_1(k) = \frac{1}{t} f_1(k) z(k) \] (168)
where we assume that \( f_1(k) \) is an arbitrary function, which tends sufficiently fast to zero as \( |k| \to \infty \). The first constant is obtained by integrating equation (168) over all \( k \)
\[ C_1 = \frac{1}{t} \int f_1(k) z(k) dk. \] (169)
In order to find other integration constants, in fact, an infinite sequence of them, the following procedure may be employed. We first note that (166) is true for each \( k \), hence the following relations are satisfied for the "discrete" counterpart
\[ \frac{dr}{ds} = t \quad \frac{dz_k}{ds} = z_k^{\ep} \quad \frac{dz_{k-k'}}{ds} = z_{k-k'}^{\ep} \] (170)
where \( k \neq k' \). Next, the second equation of (170) was multiplied by \( z_k^{\ep} \), the third equation of (170) by \( z_{k-k'}^{\ep} \) and the two equations were added together leading to
\[ \frac{dr}{t} = \frac{\delta(z_k z_{k-k'})}{2 z_k z_{k-k'}}. \] (171)
After integrating the above formula and going to the continuum limit we obtained
\[ c_2(k, k') = \frac{1}{t^2} f_2(k, k') z(k') z(k-k') \] (172)
With this, the second integration constant reads
\[ C_2 = \int f_2(k, k') dk' = \frac{1}{t^2} \int f_2(k, k') z(k') z(k-k') dk' \] (173)
Analogously, an infinite set of integration constants with convolutions
\[ \int \ldots f_{n+1}(k, k', \ldots, k^{(n)}) z(k^{(n)}) \ldots z(k') z(k-k' - \cdots - k^{(n)}) dk \ldots dk^{(n)} \] where \( n = 3, 4, \ldots \) was created
\[ C_3 = \frac{1}{t^2} \int f_3(k, k', k'') z(k'') z(k-k' - k'') dk' dk'' \] (174)
have the momentum and the continuity equation for a Newtonian fluid under constant density and viscosity conditions. In Cartesian tensor notation we write the system of functional equations determining infinitesimals (154–160). Although a general solution to this system was not known, the starting point of the analysis to follow is the three dimensional Navier-Stokes equations for an incompressible fluid assuming Newtonian fluid behavior under constant density.

\[ C_4 = \frac{1}{\rho} \int \int \int f_4(k',k'',k''')z(k'')z(k'')z(k'\cdot(k-k''-k'''))dk'dk''dk''' , \]  
\[ C_{n+1} = \frac{1}{\rho} \int \cdots \int f_{n+1}(k,\ldots,k^n)z(k')z(k'\cdot-k^n)\cdots dk\ldots dk^n . \]  

The constants \( C_1, C_2, \ldots, C_n \) were substituted to equation (165). Here, we can recognize that the constants \( C_n \) can be regarded as terms in a Taylor series expansion of the functional \( \Phi \) where the kernels \( 1/t^n f_n \) are related to functional derivatives of \( \Phi \) at the origin. We will address this issue in some more detail in Section 4.5.

Analogously, the following constants for the viscous case (164) were derived in Waclawczyk and Oberlack (2013)

\[ C_1 = \int_0^\infty C(k)dk = \int_0^\infty f_1(k)z(k)\exp\left(-\frac{1}{c}kr\right)dk , \]  
\[ C_2 = \int \int_0^\infty \exp\left(-\frac{1}{c}kr\right)f_2(k,k')z(k-k')dk'dk' , \]  
\[ C_{n+1} = \int \cdots \int \int_0^\infty \exp\left(-\frac{1}{c}kr\right)f_{n+1}(k,\ldots,k^n)z(k')\cdots z(k\cdot-k^n)dk\ldots dk^n . \]  

It was shown in Waclawczyk and Oberlack (2013) that the Taylor-series form of solution (165) allows to calculate \( n \)-th order statistics of the Burgers equation (153). This leads to special solutions representing the time decay of moments observed in works of other authors, e.g. the temporal decay \( \tilde{v}(k, t) \sim 1/t \) for incompressible case reported by Bec and Khanin (2007) or the relation \( \tilde{v}(k, t) \sim \exp(-\mu kt) \) for large wave numbers for the viscous case, cf. Rosenblatt (1967).

To sum up, it was shown that the extended Lie group analysis applied to a functional equation (152) leads to the system of functional equations determining infinitesimals (154–160). Although a general solution to this system was not found, we derived special solutions (161) and (162), which lead to special solutions of the Hopf-Burgers equation (152) of the Taylor-series form (165).

3. Classical symmetries of Euler and Navier-Stokes equations

3.1. Navier-Stokes equations

The starting point of the analysis to follow is the three dimensional Navier-Stokes equations for an incompressible fluid assuming Newtonian fluid behavior under constant density and viscosity conditions. In Cartesian tensor notation we have the momentum and the continuity equation

\[ \mathcal{M}_i(x) = \frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} + \frac{\partial P}{\partial x_i} - \nu \frac{\partial^2 U_i}{\partial x_i \partial x_j} = 0 , \quad i = 1, 2, 3 , \quad \frac{\partial U_i}{\partial x_i} = 0 \]  

where \( t \in \mathbb{R}^*, x \in \mathbb{R}^3, U = U(x, t) \) and \( P = P(x, t) \) represent time, position vector, instantaneous velocity vector and pressure respectively, while pressure has been normalized by a constant density.

3.2. Symmetries of the Euler and Navier-Stokes equations

The Euler equations, i.e. equation (180) with \( \nu = 0 \), admit a ten-parameter symmetry group,

\[ T_1 : \ t' = t + k_1 , \quad x' = x , \quad U^* = U , \quad P^* = P , \]  
\[ T_2 : \ t' = t , \quad x' = e^{k_2}x , \quad U^* = e^{k_2}U , \quad P^* = e^{2k_2}P , \]  
\[ T_3 : \ t' = e^{k_3}t , \quad x' = e^{k_3}x , \quad U^* = e^{2k_3}U , \quad P^* = e^{2k_3}P , \]  
\[ T_4 - T_6 : \ t' = t , \quad x' = a \cdot x , \quad U^* = a \cdot U , \quad P^* = P , \]  
\[ T_7 - T_9 : \ t' = t , \quad x' = x + f(t) , \quad U^* = U + \frac{df}{dt} , \]  
\[ P^* = P - x \cdot \frac{df}{dt} , \]  
\[ T_{10} : \ t' = t , \quad x' = x , \quad U^* = U , \quad P^* = P + f_3(t) , \]  

where \( k_1-k_3 \) are independent group-parameters, \( a \) denotes a constant rotation matrix with the properties \( a \cdot a^T = a^T \cdot a = I \) and \( |a| = 1 \). Moreover \( f(t) = (f_1(t), f_2(t), f_3(t))^T \) with twice differentiable functions \( f_1-f_3 \) and \( f_3(t) \) may have arbitrary time dependence.
According to (25) the corresponding infinitesimal generators have the form

\[
X_1 = \frac{\partial}{\partial t},
\]

\[
X_2 = \left[ x_i \frac{\partial}{\partial x_i} + U_i \frac{\partial}{\partial U_j} + 2P \frac{\partial}{\partial P} + 2\nu \frac{\partial}{\partial \nu} \right],
\]

\[
X_3 = \left[ \frac{\partial}{\partial t} - U_i \frac{\partial}{\partial U_i} - 2P \frac{\partial}{\partial P} - \nu \frac{\partial}{\partial \nu} \right],
\]

\[
X_4 = \left[ -x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2} - U_2 \frac{\partial}{\partial U_1} + U_1 \frac{\partial}{\partial U_2} \right],
\]

\[
X_5 = \left[ -x_3 \frac{\partial}{\partial x_2} + x_2 \frac{\partial}{\partial x_3} - U_2 \frac{\partial}{\partial U_2} + U_3 \frac{\partial}{\partial U_3} \right],
\]

\[
X_6 = \left[ -x_3 \frac{\partial}{\partial x_3} + x_1 \frac{\partial}{\partial x_3} - U_3 \frac{\partial}{\partial U_1} + U_1 \frac{\partial}{\partial U_3} \right],
\]

\[
X_7 = f_1(t) \frac{\partial}{\partial x_1} + \frac{df_1(t)}{dt} \frac{\partial}{\partial U_1} - x_1 \frac{d^2 f_1(t)}{dt^2} \frac{\partial}{\partial P},
\]

\[
X_8 = f_2(t) \frac{\partial}{\partial x_2} + \frac{df_2(t)}{dt} \frac{\partial}{\partial U_2} - x_2 \frac{d^2 f_2(t)}{dt^2} \frac{\partial}{\partial P},
\]

\[
X_9 = f_3(t) \frac{\partial}{\partial x_3} + \frac{df_3(t)}{dt} \frac{\partial}{\partial U_3} - x_3 \frac{d^2 f_3(t)}{dt^2} \frac{\partial}{\partial P},
\]

\[
X_{10} = f_4(t) \frac{\partial}{\partial P}.
\]

(182)

Each of the symmetries have a distinct physical meaning. \( T_1 \) means time translation i.e. any physical experiment is independent of the actual starting point. \( T_4-T_6 \) designate rotation invariance which refers to the possibility to let an experiment undergo a fixed rotation without changing physics. Note, that this does not mean moving into a rotating system since this does significantly change physics and hence is not a symmetry. The symmetries \( T_7-T_9 \) comprise translational invariance in space for constant \( f_1-f_3 \) as well as the classical Galilei group if \( f_1-f_3 \) are linear in time. These are key properties of classical mechanics referring to the fact that physics is independent of the location or if moved at a constant speed. In its rather general form \( T_7-T_9 \) and \( T_{10} \) are direct consequences of an incompressible fluid and do not have a counterpart in the case of compressible fluids. The complete record of all point-symmetries (181) was first published by Pukhnachev (1972).

Invoking a formal transfer from Euler to the Navier-Stokes equations symmetry properties change and a recombination of the two scaling symmetries \( T_2 \) and \( T_3 \) in (181) is observed

\[
T_{NS} : \ t' = e^{2k_3 t}, \ x' = e^{k_3 x}, \ U' = e^{-k_3} U, \ P' = e^{-2k_3} P,
\]

(183)

with \( k_3 = 2k_{NS} \) and \( k_2 = k_{NS} \), while the remaining groups stay unaltered. The corresponding generator reads

\[
X_{NS} = 2t \frac{\partial}{\partial t} + x_i \frac{\partial}{\partial x_i} - U_j \frac{\partial}{\partial U_j} - 2P \frac{\partial}{\partial P}.
\]

(184)

It should be noted that additional symmetries exist for dimensional restricted cases such as plane or axisymmetric flows (see Andreev and Rodionov, 1988; Cantwell, 1978).

4. Introduction to statistical descriptions of turbulence

4.1. Statistical averaging

In the following we define the classical Reynolds averaging. The quantity \( Z \) represents an arbitrary statistical variable, i.e. \( U \) and \( P \), which in the following we also denote as instantaneous value. According to the classic definition by Reynolds, all instantaneous quantities are decomposed into their mean and their fluctuation value

\[
Z = \overline{Z} + z.
\]

(185)

Here, the overbar denotes a statistically averaged quantity whereas the lower-case \( z \) denotes the fluctuation value of \( Z \). In an engineering context the most general definition of a statistically averaged quantity is given by an ensemble average
4.2. Reynolds averaged transport equations

After $U$ and $P$ are decomposed according to the Reynolds decomposition, i.e. $U = \overline{U} + u$ and $P = \overline{p} + p$, we gain an averaged versions of the momentum and continuity equations

$$\frac{\partial \overline{U}_i}{\partial t} + \overline{U}_j \frac{\partial \overline{U}_i}{\partial x_j} = - \frac{\partial \overline{p}}{\partial x_i} + \nu \frac{\partial^2 \overline{U}_i}{\partial x_k \partial x_k}, \quad i = 1, 2, 3, \quad \frac{\partial \overline{U}_k}{\partial x_k} = 0 , \quad \frac{\partial \overline{D}_k}{\partial x_k} = 0 , \quad \frac{\partial D_k}{\partial x_k} = 0 ,$$

with $\partial \overline{D}_k / \partial t + \overline{U}_k \partial p / \partial x_k$.

At this point we observe the well-known closure problem of turbulence since, compared to the original set of equations, the unknown Reynolds stress tensor $\overline{u}_i u_k$ appeared. However, rather different from the classical approach we will not proceed with deriving the Reynolds stress tensor transport equation, which contains additional four unclosed tensors. Instead, the multi-point correlation approach is put forward the reason being twofold.

First, if the infinite set of correlation equations is considered, the closure problem is somewhat bypassed. Second, the multi-point correlation delivers additional information on the turbulence statistics such as length scale information which may not be gained from the Reynolds stress tensor, which is a single-point quantity.

For this we need the equations for the fluctuating quantities $u$ and $p$, which are derived by taking the differences between the averaged and the non-averaged equations, i.e. (180) and (187). The resulting fluctuation equations read

$$N_i(x) = \frac{D u_i}{D t} + u_k \frac{\partial U_i}{\partial x_k} = \frac{\partial \overline{u}_i u_k}{\partial x_k} + \frac{\partial \overline{u}_k u_i}{\partial x_k}, \quad i = 1, 2, 3 , \quad \frac{\partial \overline{u}_k}{\partial x_k} = 0 , \quad \frac{\partial \overline{D}_k}{\partial x_k} = 0 .$$

4.3. Multi-point correlation equations

It is considered that the idea of two- and multi-point equations in turbulence was first established in Keller and Friedmann (1924). At the time it was assumed that all correlation equations of orders higher than two may be neglected. Theoretical considerations showed that all higher order correlations should be accounted for. Consequently, all multi-point correlation equations have been considered in the symmetry analysis that follows below.

Two different sets of multi-point correlation (MPC) equations will be derived below. The first is based on the instantaneous values of $U$ and $P$ while the second follows the classical notation based on the fluctuating quantities $u$ and $p$.

4.3.1. MPC equations: instantaneous approach

In order to write the MPC equations in a very compact form, we introduce the following notation. The multi-point velocity correlation tensor of order $n + 1$ is defined as follows:

$$H_{(i_0 + 1)} = H_{(i_0 i_1 \ldots i_n)} = \overline{U}_{i_0}(x_{i_0}, t) \cdots \overline{U}_{i_n}(x_{i_n}, t) ,$$

where the index $i$ of the farthermost left quantity refers to its tensor character, while its superscript in curly brackets denotes the tensor order. The central term exemplifies this since a list of $n + 1$ tensor indices is given, where the index in parenthesis is a counter for the tensor order. It is important to mention that the index counter starts with 0 which is an advantage when introducing a new coordinate system based on the Euclidean distance of two or more space points with (189). The mean velocity is given by the first order tensor as $H_{(0)} = H_{(0)} = \overline{U}_i$.

In some cases the list of indices is interrupted by one or more other indices, which is pointed out by attaching the replaced value in square brackets to the index $H_{(i_0 + 1)[i_0 \cdots i_n]} = \overline{U}_{i_0}(x_{i_0}, t) \cdots \overline{U}_{i_n}(x_{i_n}, t) \overline{U}_{i_0}(x_{i_0}, t) \overline{U}_{i_1}(x_{i_1}, t) \cdots \overline{U}_{i_n}(x_{i_n}, t) . \quad (190)$

This is further extended by

$$H_{(i_0 + 1)[i_1 \cdots i_n]}(x_{i_0}) \Rightarrow x_{i_0} = \overline{U}_{i_0}(x_{i_0}, t) \cdots \overline{U}_{i_n}(x_{i_n}, t) \overline{U}_{i_0}(x_{i_0}, t) \overline{U}_{i_1}(x_{i_1}, t) \cdots \overline{U}_{i_n}(x_{i_n}, t) , \quad (191)$$

where not only that index $i_0 + 1$ is replaced by $k_0$, but also that the independent variable $x_{i_0 + 1}$ is replaced by $x_{i_0}$. If indices are missing e.g. between $i_{(t-1)}$ and $i_{(t+1)}$ we define

$$H_{(i_0 \cdots i_0)} = \overline{U}_{i_0}(x_{i_0}, t) \cdots \overline{U}_{i_0}(x_{i_0}, t) \overline{U}_{i_0}(x_{i_0}, t) \cdots \overline{U}_{i_0}(x_{i_0}, t) . \quad (192)$$
Finally, if pressure is involved we write
\[
I_{i[m]} = U_{i[0]}(x_{i[m]}, t) \cdot \ldots \cdot U_{i[l-1]}(x_{i[l-1]}, t)P(x_{i[l]}, t)U_{i[l]}(x_{i[l]}, t) \cdot \ldots \cdot U_{i[m]}(x_{i[m]}, t),
\]
which is, considering all the above definitions, sufficient to derive the MPC equations from the equations of instantaneous velocity and pressure i.e. equations (180).

Applying the Reynolds averaging operator according to the sum below
\[
S_{l_{i[l]}(x_{i[0]}, \ldots, x_{i[m]}, t)} = M_{i[0]}(x_{i[0]}, t)U_{i[1]}(x_{i[1]}, t) \cdot \ldots \cdot U_{i[l]}(x_{i[l]}, t)
+ U_{i[0]}(x_{i[0]}, t)M_{i[1]}(x_{i[1]}, t)U_{i[2]}(x_{i[2]}, t) \cdot \ldots \cdot U_{i[l]}(x_{i[l]}, t)
+ \ldots
+ U_{i[0]}(x_{i[0]}, t) \cdot \ldots \cdot U_{i[l-2]}(x_{i[l-2]}, t)M_{i[l-1]}(x_{i[l-1]}, t)U_{i[l]}(x_{i[l]}, t)
+ U_{i[0]}(x_{i[0]}, t) \cdot \ldots \cdot U_{i[l-2]}(x_{i[l-2]}, t)M_{i[l-1]}(x_{i[l-1]}, t)U_{i[l]}(x_{i[l]}, t),
\]
we obtain the S-equation which writes
\[
S_{l_{i[l]}} = \frac{\partial H_{i[l]}}{\partial t} + \sum_{l=0}^{n} \left[ \frac{\partial H_{i[l]}(x_{i[l]}, t \rightarrow x_{m})}{\partial x_{k_{0}}} \right] + \frac{\partial I_{i[l]}(0)}{\partial x_{k_{0}}} = 0 \quad \text{for} \quad n = 1, \ldots, \infty .
\]

In general, equation (195) implies the full multi-point statistical information of the Navier-Stokes equations at the expense of dealing with an infinite dimensional chain of differential equations starting with order 2 i.e. \( n = 1 \). It is a remarkable fact that (195) is a linear equation which considerably simplifies the finding of Lie symmetries to be pointed out below.

From the second equation of (180) a continuity equation for \( H_{i[0]} \) and \( I_{i[0]} \) can be derived. This leads to
\[
\frac{\partial H_{i[l]}(0 \rightarrow k_{0})}{\partial x_{k_{0}}} = 0 \quad \text{for} \quad l = 0, \ldots, n , \quad \frac{\partial I_{i[l]}(k_{0} \rightarrow m_{0})}{\partial x_{k_{0}}} = 0 \quad \text{for} \quad k, l = 0, \ldots, n \quad \text{and} \quad k \neq l .
\]

At this point we adopt the classic notation of distance vectors. Accordingly the usual position vector \( x \) is employed and the remaining independent spatial variables are expressed as the difference of two position vectors \( x_{i[0]} \) and \( x_{i[m]} \). The coordinate transformation are
\[
x = x_{i[0]} , \quad r_{i[0]} = x_{i[0]} - x_{i[m]} \quad \text{with} \quad l = 1, \ldots, n \quad \text{and} \quad \frac{\partial}{\partial x_{k_{0}}} = \frac{\partial}{\partial x_{l}} - \sum_{l=1}^{n} \frac{\partial}{\partial r_{k_{0}}}, \quad \frac{\partial}{\partial x_{k_{0}}} = \frac{\partial}{\partial r_{k_{0}}} \quad \text{for} \quad l \geq 1 .
\]

For consistency the first index \( i[0] \) is replaced by \( i \). Thus, the indices of the tensor \( H_{i[l]} \) are \( H_{i[l];i_{2}, \ldots, i_{l-1};i_{0}} \). Using the rules of transformation (197) the S-equation leads to
\[
S_{i[l]} = \frac{\partial H_{i[l]}}{\partial t} + \frac{\partial H_{i[l]}(x_{i[l]} \rightarrow x)}{\partial x_{l}} - \sum_{l=1}^{n} \left[ \frac{\partial H_{i[l]}(x_{i[l]} \rightarrow x_{m_{0}})}{\partial r_{k_{0}}} \right] - \frac{\partial H_{i[l]}(x_{i[l]} \rightarrow x_{m_{0}})}{\partial x_{k_{0}}} - \frac{\partial^{2} H_{i[l]}}{\partial x_{l} \partial x_{k_{0}}} + \sum_{l=1}^{n} \left[ \frac{\partial^{2} H_{i[l]}}{\partial x_{l} \partial r_{k_{0}}} + \frac{\partial^{2} H_{i[l]}}{\partial x_{k_{0}} \partial r_{l}} \right] = 0
\]
for \( n = 1, \ldots, \infty \),

and the two continuity equations become
\[
\frac{\partial H_{i[l]}(0 \rightarrow k_{0})}{\partial x_{k_{0}}} = 0, \quad \frac{\partial H_{i[l]}(0 \rightarrow k_{0})}{\partial r_{k_{0}}} = 0 \quad \text{for} \quad l = 1, \ldots, n
\]
and
\[
\frac{\partial I_{i[l]}(k_{0} \rightarrow m_{0})}{\partial x_{m_{0}}} = 0 \quad \text{for} \quad k = 1, \ldots, n , \quad \frac{\partial I_{i[l]}(k_{0} \rightarrow m_{0})}{\partial r_{m_{0}}} = 0 \quad \text{for} \quad k = 0, \ldots, n, l = 1, \ldots, n, k \neq l .
\]
4.3.2. MPC equations: fluctuation approach In the present subsection we adopt the classical approach i.e. all correlation functions are based on the fluctuating quantities \( u \) and \( p \) as introduced by Reynolds and not on the full instantaneous quantities \( U \) and \( P \) as in the previous subsection. Hence, similar to (189) we have the multi-point correlation for the fluctuation velocity

\[
R_{(u_{i_1} \ldots u_{i_n})} = R_{(u_{i_1}, \ldots, u_{i_n})} = u_{i_1}(x_{i_1}) \cdot \ldots \cdot u_{i_n}(x_{i_n}) .
\]  

(201)

All other correlations defined in sub-section 4.3.1 are defined accordingly i.e. equivalent to the definitions (190)-(193) we respectively define \( R_{(u_{i_1}u_{i_2} \ldots u_{i_n})} \), \( R_{(u_{i_1}u_{i_2} \ldots u_{i_n})} \) and \( P_{u_{i_1}u_{i_2} \ldots u_{i_n}} \).

Finally, we define the correlation equation in analogy to (194) where \( M_i \) is replaced by the equation for the fluctuations (188) denoted by \( N_i \) and \( U_i \) and \( P \) are substituted by \( u_i \) and \( p \). The resulting equation is denoted by \( T_{(u_{i_1}u_{i_2} \ldots u_{i_n})} \).

\[
T_{(u_{i_1}u_{i_2} \ldots u_{i_n})} = \frac{\partial R_{(u_{i_1}u_{i_2} \ldots u_{i_n})}}{\partial t} + \sum_{l=0}^n \left[ U_{i_1}^* u_{i_2} \frac{\partial R_{(u_{i_1}u_{i_2} \ldots u_{i_n})}}{\partial x_{i_2}} + R_{(u_{i_1}u_{i_2} \ldots u_{i_n})} \frac{\partial U_{i_1}^*}{\partial x_{i_2}} + \frac{\partial R_{(u_{i_1}u_{i_2} \ldots u_{i_n})}}{\partial x_{i_2}} \frac{\partial x_{i_2}}{\partial x_{i_2}} \right] = 0, \quad \text{for } n = 1, \ldots, \infty .
\]  

(202)

From the second of the equations (188) a continuity equation for \( R_{(u_{i_1}u_{i_2} \ldots u_{i_n})} \) and \( P_{u_{i_1}u_{i_2} \ldots u_{i_n}} \) can be derived which have identical form to (196) for \( H_{(u_{i_1}u_{i_2} \ldots u_{i_n})} \) and \( I_{(u_{i_1}u_{i_2} \ldots u_{i_n})} \) complete the system.

The first tensor equation of the infinite chain (202) propagates \( R_{(u_{i_2})} \) which has a close link to the Reynolds stress tensor, i.e.

\[
\overline{u_{i_1}u_{i_2}}(x_{i_2}) = \lim_{x_{i_2} \to x_{i_2}} R_{(u_{i_1}u_{i_2})} = \lim_{x_{i_2} \to x_{i_2}} R_{(u_{i_1}u_{i_2})} \text{ for } k \neq l ,
\]  

(203)

which is the key unclosed quantity in the Reynolds stress transport equation (187). Here \( x_{i_2} \) and \( x_{i_2} \) can be arbitrary vectors out of \( x_{i_2}, \ldots, x_{i_2} \).

Similar to (195), also equation (202) implies all statistical information of the Navier-Stokes equations. However, apart from the latter simple relation to the Reynolds stress tensor (203) it possesses the key disadvantage of being a non-linear infinite dimensional system of differential equations, which has a strong cross-coupling between tensors as first and the second moments are present in all higher order moment equations. This makes the extraction of Lie symmetries from this equation rather cumbersome and essentially impossible. In contrast, in equation (195) each moment of the order \( n \) is only coupled to the tensor of the order \( n + 1 \), which simplifies the analysis considerably.

Of course, there is a unique relation between the instantaneous \((H, I)\) and the fluctuation approach \((R, P)\) though the actual crossover to discuss in particular with increasing tensor order because they may only be given in recursive form. Since needed later we give the first relations

\[
\begin{align*}
H_0 &= \overline{u}, \\
H_{(u_{i_1}k_{i_1})} &= \overline{u_{i_1}u_{i_1}} + R_{(u_{i_1}k_{i_1})}, \\
H_{(u_{i_1}u_{i_2}k_{i_1}k_{i_2})} &= \overline{u_{i_1}u_{i_2}u_{i_1}} + R_{(u_{i_1}k_{i_1})u_{i_2}} + R_{(u_{i_1}k_{i_1})u_{i_2}} + R_{(u_{i_1}k_{i_1})u_{i_2}}, \ldots
\end{align*}
\]

(204) - (206)

where the indices also refer to the spatial points as indicated.

4.4. Lundgren-Monin-Novikov hierarchy for PDF functions

As already stated in the introductory section 1, the LMN approach is one of the three known methods, which provide full statistical description of turbulence. In this approach one assumes that for the velocity field there exist a probability density function (PDF), which describe the joint probabilities of measuring contemporarily sets of velocities at multiple points in space. When calculating mean values thereof, these PDFs play the role of the weighting measure.

For this, we subsequently briefly outline the approach presented in the seminal paper by Lundgren (1967). Let us therefore take into consideration an ensemble of incompressible flows occupying the entire infinite space \( \mathbb{R}^3 \) and having identical physical properties but different initial conditions. Let the velocity field of each member of the ensemble be denoted by \( U \) in agreement with the notation of the preceding sections. It is assumed that \( U \) satisfies Navier-Stokes and continuity equations and that the statistical distribution of \( U \) over the ensemble at the initial time \( t_0 \) is given. The main
goal is the statistical distribution of the velocity field as it evolves with time. In order to do so, let us define multi-point PDFs in the usual way: the 1-point PDF \( f_1(x_0, u_0, t) \) is such that \( f_1(x_0, u_0, t) \, du_0 \) expresses the probability to measure a velocity in an infinitesimal interval \( du_0 \) around \( u_0 \) at position \( x_0 \) (or, equivalently, the fraction of systems in the ensemble such that the given condition is satisfied). The 1-point PDF can be written as:

\[
  f_1(x_0; u_0, t) = \delta(U(x_0, t) - u_0) .
\]  

(207)

where \( \delta(U(x_0, t) - u_0) \) is the Dirac delta function and can be understood as a velocity distribution function for a single element of the ensemble, whereas \( f_1 \) is the average distribution function. Analogously to (207), the 2-point PDF, which denotes the joint probability to measure two given velocities \( u_0 \) and \( u_1 \) at two defined points \( x_0 \) and \( x_1 \) in space at the same time \( t \), can be expressed as:

\[
  f_2(u_0, u_1; x_0, x_1, t) = \delta(U(x_0, t) - u_0) \delta(U(x_1, t) - u_1) .
\]  

(208)

This can be continued so forth to the definition of the \( n \) point PDF

\[
  f_{n+1}(u_0, \ldots, u_n; x_0, \ldots, x_n, t) = \delta(U(x_0, t) - u_0) \cdots \delta(U(x_n, t) - u_n) .
\]  

(209)

Sometimes the following abbreviation will be used in this paper, cf. Lundgren (1967):

\[
  f_{n+1} \equiv f_{n+1}(0, \ldots, n) \equiv f_{n+1}(u_0, \ldots, u_n; x_0, \ldots, x_n, t) .
\]

In the LMN language it is the moments associated to the PDFs that correspond to the multi-point moments (189) of the MPC approach, i.e. to the components of the tensor \( H \):

\[
  H_{i_{n+1}} \equiv U_{i_{n+1}}(x_0, t) \cdots U_{i_0}(x_n, t) \equiv \int f_{n+1} \, u_{i_0} \cdots u_{i_n} \, du_0 \cdots du_n .
\]

(210)

4.4.1. Properties of the PDFs

In order for the previously defined PDFs to be well defined from a physical point of view, they are required to satisfy four conditions Lundgren (1967):

I. The reduction or normalization property imposed by the concept of probability:

\[
  \int f_1(u_0; x_0, t) \, du_0 = 1;
\]

\[
  \int f_2(u_0, u_1; x_0, x_1, t) \, du_0 \, du_1 = f_1(u_0; x_0, t); \quad \vdots
\]

(211)

II. An infinite number of “continuity” conditions dictated by the incompressibility of the fluid:

\[
  \partial_{x_{i_{n+1}}} \int f_{n+1} \, du_i = 0, \quad \forall i \in [0, \ldots, n];
\]

(212)

III. The “coincidence” property, required by the condition for the velocity field to be well defined:

\[
  \lim_{|x_0 - x_1| \to 0} f_2(0, 1) = f_1(0) \delta(u_1 - u_0); \quad \lim_{|x_0 - x_1| \to 0} f_3(0, 1, 2) = f_2(0, 1) \delta(u_2 - u_0); \quad \vdots
\]

(213)

IV. The “separation” property (here shown only for the 2-point PDF) which expresses the fact that the velocities of two fluid elements tends to become independent if the two points are set far apart from each other:

\[
  \lim_{|x_0 - x_1| \to \infty} f_2(0, 1) = f_1(0)f_1(1).
\]

(214)
4.4.2. The LMN hierarchy  Analogously to the MPCE hierarchy in Section 4.3 an infinite chain of equations for the multipoint PDF’s can be derived based on the Navier-Stokes equations. In the paper of Lundgren (1967) the continuity equation was first used in order to eliminate the pressure term from the Navier-Stokes equations and present it in the following form

\[
\frac{\partial U_{0m}}{\partial t} + U_{0m} \frac{\partial U_{0m}}{\partial x_{0m}} = \frac{1}{4\pi} \frac{\partial}{\partial x_{0m}} \int \frac{1}{|x_1(x_0) - x_{0m}|} \left[ U_{1m}(x_1, t) \frac{\partial U_{1m}}{\partial x_{0m}} \right] \, dx_{1m} + \nu \frac{\partial^2}{\partial x_{0m} \partial x_{0m}} U_{0m}(x_0, t).
\]  

In order to derive the first equation in the hierarchy, the time derivative of formula (207) should be calculated, leading to

\[
\frac{\partial f_1}{\partial t} = \frac{\partial}{\partial t} \delta(U(x_0, t) - u_0) = -\frac{\partial U_{0m}}{\partial t} \frac{\partial}{\partial u_{0m}} \delta(U(x_0, t) - u_0).
\]  

Next, the Navier-Stokes equations (215) was substituted for the time derivative $\frac{\partial U}{\partial t}$ in equation (216) leading, after additional transformations have been implemented, to the following first equation of the LMN hierarchy (for details see Lundgren (1967))

\[
\frac{\partial f_1}{\partial t} + v_{0m} \frac{\partial f_1}{\partial x_{0m}} = \frac{1}{4\pi} \frac{\partial}{\partial x_{0m}} \int \left( \frac{\partial}{\partial x_{0m}} \left( \frac{1}{|x_1(x_0) - x_{0m}|} \right) \right) v_{1m} \frac{\partial}{\partial x_{1m}} f_2(0, 1) \, dx_{1m} + \frac{\partial^2}{\partial v_{0m} \partial v_{0m}} \left[ \lim_{|x_1(x_0) - x_{0m}| \to 0} \frac{\partial^2}{\partial x_{1m} \partial x_{1m}} \int v_{1m} f_2(0, 1) \, dx_{1m} \right].
\]  

In a similar way equation for the $n = 1$ PDF function may be derived, starting from the time derivative of equation (209), substituting (215) and leading to

\[
\frac{\partial f_{n+1}}{\partial t} + \sum_{k=0}^{n} v_{k} \frac{\partial f_{n+1}}{\partial x_{k}} = \frac{1}{4\pi} \frac{\partial}{\partial x_{k}} \int \left( \frac{\partial}{\partial x_{k}} \left( \frac{1}{|x_{k+1}(x_{k}) - x_{k}|} \right) \right) v_{k+1} \frac{\partial}{\partial x_{k+1}} f_{n+2} \, dx_{k+1} + \sum_{k=0}^{n} \frac{\partial^2}{\partial v_{k} \partial v_{k}} \left[ \lim_{|x_{k+1}(x_{k}) - x_{k}| \to 0} \frac{\partial^2}{\partial x_{k+1} \partial x_{k+1}} \int v_{k+1} f_{n+2} \, dx_{k+1} \right].
\]  

Hence, the LMN hierarchy constitutes an infinite chain of equations where, on the $n + 1$ level, the unknown $n + 2$-point PDF is needed. As it was pointed in Friedrich et al. (2012), the chain can be formally truncated at the $n + 1$ level by replacing the terms with $f_{n+2}$ by conditional averages. For the one-point PDF equations and in the case of homogeneous, isotropic turbulence these conditionally averaged quantities where estimated based on the DNS data in Wilczek et al. (2011a,b) in order to study the deviations of the PDF from Gaussianity.

4.5. Hopf functional approach  E. Hopf (1952) introduced another very general approach to the description of turbulence. He considered the case where the number of points in the PDF goes to infinity, so that the probability density function becomes a probability density functional $F([\rho(x)], t)$, where, instead of the vector of sample space variables $v_0, \ldots, v_n$ at points $x_0, \ldots, x_n$, one deals with a continuous set of sample space variables $\rho(x)$. Further, instead of dealing with the probability density functional itself it was more convenient to consider its functional Fourier transform, called the characteristic functional

\[
\Phi([\rho(x)], t) = \int e^{ivy} F([\rho(x)], t) \, dx = e^{ivy}.
\]  

where the integration is performed with respect to the probability measure $F([\rho(x)], t) \, dx$ and $(y, v) = \int \rho \, u_{\alpha} y_{\alpha} \, dx$ is a scalar product of two vector fields. We recall here that in the probability theory the $n + 1$-point characteristic function is defined as the following $n + 1$-dimensional inverse Fourier transform of the $n + 1$-point PDF $f_{n+1}$

\[
\Phi_{n+1} = \int e^{ivy_{0}} \ldots e^{ivy_{n+1}} f_{n+1}(v_0, \ldots, v_n, x_0, \ldots, x_n, t) \, dv_0 \ldots dv_n.
\]  

Hence, $\Phi$ may be treated as a functional analogue of the characteristic function for $n \to \infty$. The functional embodies the statistical properties of the fluid flow in a more concise form than the infinite set of functions $\Phi_n$.

Solutions of $\Phi$ are admitted only, if at any time $t$ the following conditions are fulfilled $\Phi^*(y, t) = \Phi(-y, t)$, $\Phi(0, t) = 1$ and $|\Phi(y, t)| \leq 1$. These conditions follow from the properties of probability density functional, which is strictly positive $F([\rho(x)], t) \geq 0$ and its integral over entire sample space equals 1.
An introduction to the functional approach and definitions of functional derivatives have already been presented in Section 2.7, cf. equation (118), where also the extension of Lie group analysis towards such equations was detailed. Here we note that with the definition (219) moments of the velocity can be calculated as the functional derivatives of the characteristic functional at the origin \( y = 0 \). The first functional derivative of \( \Phi \) is given by

\[
\frac{\partial \Phi(y(x),t)}{\partial y_{l_0}(x(0))} = \int_{\Omega} \frac{i}{2} \eta_{l_0}(x(0)) \exp^{i(\theta y(\theta))} F([e(x)], t) \delta x(x) = iU_{l_0}(x(0), t) \exp^{i(\theta y(\theta))}
\]

hence, at \( y = 0 \)

\[
\frac{\partial \Phi(y(x),t)}{\partial y_{l_0}(x(0))} \bigg|_{y=0} = iU_{l_0}(x(0), t).
\]

The \( n + 1 \) order derivative of \( \Phi \) give

\[
\frac{\partial^{n+1} \Phi(y(x),t)}{\partial y_{l_0}(x(0)) \partial y_{l_1}(x(1)) \cdots \partial y_{l_n}(x(n))} \bigg|_{y=0} = \frac{n^{n+1} U_{l_0}(x(0), t) U_{l_1}(x(1), t) \cdots U_{l_n}(x(n), t)}{n!}.
\]

Based on the Navier-Stokes equations, E. Hopf derived an evolution equation for the characteristic functional. It is only one equation (not a hierarchy) and all turbulence statistics can formally be calculated from the solution of the Hopf equation. We do not present here the derivation of this equation, and an interested reader is referred to the work of Hopf (1952), Monin and Yaglom (1971). The Hopf equation for velocity in the physical space reads

\[
\frac{\partial \Phi}{\partial t} = \int_R y_h(x) \left[ t \frac{\partial}{\partial x_l} \delta y_l(x) + v \nabla^2 \delta y_l(x) - \frac{\partial \Pi}{\partial x_l} \right] dx
\]

where, in order to eliminate pressure functional \( \Pi \) from the equation, vector field \( \vec{y} \) such that \( y(x) = \vec{y}(x) + \nabla \phi \) was introduced by Hopf (1952). The scalar \( \phi \) is chosen such that \( \vec{y} = \vec{0} \) at the boundary \( B \) and the continuity equation is satisfied \( \nabla \cdot \vec{y} = 0 \). With this, the Hopf functional equation reads

\[
\frac{\partial \Phi}{\partial t} = \int_R y_h(x) \left[ t \frac{\partial}{\partial x_l} \delta y_l(x) + v \nabla^2 \delta y_l(x) \right] dx.
\]

The first RHS term describes convection of velocity and the second appears due to the presence of the viscosity. The Hopf equation may also be written for the velocity in Fourier space. In such a case it has the following form

\[
\frac{\partial \Phi}{\partial t} = \int_{\mathbb{R}} \frac{\delta}{\delta \vec{z}(k)} \left[ \int_{\mathbb{R}} \frac{\partial \Phi}{\partial \vec{z}(k')} \delta k' \delta \vec{z}(k') \right] \delta k' - v \int_{\mathbb{R}} \left| \vec{k} \right|^2 \frac{\delta \Phi}{\delta \vec{z}(k)} \delta \vec{z}(k) \delta k,
\]

where

\[
\vec{z} = z - \frac{k \cdot \vec{z}(k)}{|k|^2} k.
\]

Again, the first term on the RHS of (226) represents the contribution of the triadic interactions due to the convection term and the second, contribution of viscosity.

The compactness of the form is an advantage of the Hopf approach, however making this equation of practical use for applications or finding meaningful solutions is challenging. Recently, a link between the functional approach and simulation of turbulence in terms of the many-particle dynamics has been established Hosokawa (2006). Solutions of the Hopf equation for the statistically stationary case allowing to calculate higher-order and multipoint statistics from the known first-order moments have been proposed in Shen and Wray (1991). General forms of solutions to equation (225) were discussed by Vishik (1976) and in the paper of Hopf (1952). Such solution was presented as a regular Taylor expansion

\[
\Phi = 1 + C_1 + C_2 + \cdots,
\]

where \( C_{n+1} \) is a polynomial functional of degree \( n + 1 \) in \( y(x) \) of the form

\[
C_{n+1} = \int \cdots \int K_{l_0 \cdots l_{n+1}}(x(0), \ldots x(n), t) y_{l_0}(x(0)) \cdots y_{l_n}(x(n)) \ dx(0) \ldots dx(n)
\]

and the kernel function \( K \) is defined by

\[
K_{l_0 \cdots l_{n+1}}(x(0), \ldots x(n), t) = \frac{1}{(n+1)!} \frac{\partial^{n+1} \Phi}{\partial y_{l_0}(x(0)) \cdots y_{l_n}(x(n))} \bigg|_{y=0}.
\]
This, according to the relation (223) is related to the $n + 1$-point velocity correlation function. Hence, the subsequent terms in the proposed form of solution (228) represent the contributions of different velocity statistics, up to infinite order.

Generally, considering the Hopf functional equation is difficult due to the lack of analytical methods. Here, Lie group analysis is an ideal candidate to at least partly fill this gap. Lie group analysis allows to derive the invariant, i.e. symmetry-based solutions of the equation. These solutions often describe certain regularities or attractors, like e.g., in the case of turbulence, scaling laws for ensemble-averaged statistics. The extension of Lie group analysis towards functional equations was presented in Section 2.7, based on the previous contributions of Oberlack and Waclawczyk (2006) and Waclawczyk and Oberlack (2013). The Lie group analysis of the Hopf formulation of the Burgers equation described in Section 2.7.4 led to invariant solutions which have the Taylor-series form, analogous to (228) cf. (165). Application of the method to Hopf equation for turbulence (225) or (226) is a subject of currently on-going research.

5. Symmetries of statistical equations of turbulence

5.1. Symmetries of MPC equations

5.1.1. Symmetries of the MPC equations implied by Euler and Navier-Stokes symmetries

Adopting the classical Reynolds notation first, where the instantaneous quantities are split into mean and fluctuating values, we may directly derive from (181)

\[
T_1^* : t^* = t + k_1, \quad x^* = x, \quad r_{(i)}^* = r_{(i)}, \quad \overline{U} = \overline{U},
\]

\[
\overline{U}^* = \overline{U}, \quad R_{(n)}^* = R_{(n)}, \quad P_{(n)}^* = P_{(n)},
\]

\[
T_2^* : t^* = t, \quad x^* = e^{k_2}x, \quad r_{(i)}^* = e^{k_2}r_{(i)}, \quad \overline{U}^* = e^{k_2}\overline{U},
\]

\[
\overline{U}^* = e^{2k_2}\overline{U}, \quad R_{(n)}^* = e^{nk_2}R_{(n)}, \quad P_{(n)}^* = e^{(n+2)k_2}P_{(n)},
\]

\[
T_3^* : t^* = e^{k_3}t, \quad x^* = x, \quad r_{(i)}^* = r_{(i)}, \quad \overline{U}^* = e^{-k_3}\overline{U},
\]

\[
\overline{U}^* = e^{-2k_3}\overline{U}, \quad R_{(n)}^* = e^{-nk_2}R_{(n)}, \quad P_{(n)}^* = e^{-(n+2)k_2}P_{(n)},
\]

\[
T_4^* - T_6^* : t^* = t, \quad x^* = a \cdot x, \quad r_{(i)}^* = r_{(i)}, \quad \overline{U}^* = a \cdot \overline{U}, \quad \overline{U}^* = \overline{U},
\]

\[
R_{(n)}^* = A_{(n)} \otimes R_{(n)}, \quad P_{(n)}^* = A_{(n)} \otimes P_{(n)},
\]

\[
T_7^* - T_9^* : t^* = t, \quad x^* = x + f(t), \quad r_{(i)}^* = r_{(i)}, \quad \overline{U}^* = \overline{U} + \frac{df}{dt},
\]

\[
\overline{U}^* = \overline{U} - x \frac{d^2f}{dt^2}, \quad R_{(n)}^* = R_{(n)}, \quad P_{(n)}^* = P_{(n)},
\]

\[
T_{10}^* : t^* = t, \quad x^* = x, \quad r_{(i)}^* = r_{(i)}, \quad \overline{U}^* = \overline{U},
\]

\[
\overline{U}^* = \overline{U} + f_2(t), \quad R_{(n)}^* = R_{(n)}, \quad P_{(n)}^* = P_{(n)}.
\]

where all function and parameter definitions are adopted from 3.2 and A is a concatenation of rotation matrices as

\[
A_{(k_1\cdots k_n, h_1\cdots h_n)} = a_{(k_1\cdots k_n,l_1\cdots l_n)}.
\]

The latter symmetries may also be transformed into the H-notation according to equation (195). This will be omitted for briefness and also because the turbulent scaling laws to be derived and discussed below are rarely considered in this notation.

5.1.2. Statistical symmetries of the MPC equations

The concept of an extended set of symmetries for the MPC equation in the form (195) or (202) may e.g. be taken from Oberlack (2000) and Khujadze and Oberlack (2004). Its importance was not observed therein - rather it was stated that they may be mathematical artifacts of the averaging process and probably physically irrelevant. The set of new symmetries was first presented and its key importance for turbulence recognised in Oberlack and Rosteck (2010) and later extended in Rosteck and Oberlack (2011).

The actual finding of symmetries of the non-rotating MPC is rather difficult since an infinite system of equations has to be analyzed. For this task, however, it is considerably easier to investigate the linear H-I-system (195)-(196) rather than the non-linear R-P-system (202). However, since the latter formulation is more common, the symmetries will finally be re-written in this notation.

This new set of symmetries for the H-I-system (195)-(196) can be separated in three distinct sets of symmetries

\[
T_{11}^* : t^* = t, \quad x^* = x, \quad r_{(i)}^* = r_{(i)} + k_{(i)}, \quad H_{(n)}^* = H_{(n)} , \quad I_{(n)}^* = I_{(n)}.
\]

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Below, we shortly recall the findings. In the translation of the relative coordinates (232) \( k_{ij} \) represents the related set of group parameters. Note that this group is not related to the classical translation group in usual \( x \)-space (here \( T_7 = T_9 \) in equation (181) with \( f = \text{const.} \)).

The second set of statistical symmetries (233) was in fact already partially identified in Oberlack (2000), however, mistakenly taken for the Galilean group. In the above general form it was identified in Oberlack and Rosteck (2010), where \( C_{[n]} \) and \( D_{[n]} \) refer to group parameters and further extended in Rosteck and Oberlack (2011), so that \( C_{[n]} \) is a function of time \( C_{[n]}(t) \) and then a temporal derivative of \( C_{[n]} \) appears also for the transformation of \( I_{[n]}^* \).

It is considered that Kraichnan (1965) recognised this first. He observed the first element of the infinite set of symmetries in (233), which is in fact valid for the mean velocity, \( \overline{U_t} = H_{[1]} \) i.e. \( \overline{U} = \overline{U} + C_{[1]} \) and which he named random Galilean invariance.

It is essentially this latter statistical group that is one of the key ingredients for the logarithmic law of the wall which in fact constitutes a solution of the infinite set of MPC equations to be shown below.

The third statistical group (234) that has been identified denotes simple scaling of all MPC tensors with the same factor \( e^\zeta \).

Furthermore there exists at least one more symmetry, which consists of a combination of multi-point velocity and of pressure-velocity correlations (see Rosteck and Oberlack, 2011). Its concrete form is omitted at this point because it is not needed for the further considerations.

It should finally be added that due to the linearity of the MPC equation (195) another generic symmetry is admitted. This is in fact featured by all linear differential equations (see Bluman et al., 2010). It merely reflects the super-position principle of linear differential equations though usually cannot directly be adopted for the practical derivation of group invariant solutions.

We may transform (232)-(234) into classical notation to obtain

\[
\begin{align*}
T^e_{2} : & \quad t' = t, \quad x' = x, \quad r_{[0]}^{*} = r_{[0]} + k_{[0]}, \quad \overline{U}^* = \overline{U}, \\
T^{e}_{2} : & \quad t' = t, \quad x' = x, \quad r_{[0]}^{*} = r_{[0]}, \quad P_{[n]} = P_{[n]}, \\
T^{e}_{2} : & \quad t' = t, \quad x' = x, \quad r_{[0]}^{*} = r_{[0]}, \quad \overline{U}_{[n]}^{*} = \overline{U}_{[n]} + C_{[n]} , \nonumber
\end{align*}
\]

(235)

(236)

(237)

(238)

where for the translation symmetry (233) only \( n = 1 \) and \( n = 2 \) are presented in (236) and (237). Despite of the fact that each of these groups appear to be almost trivial, since they are simple translational groups in the dependent coordinates, they exhibit an increasing complexity with increasing tensor order if written in the \((\overline{U}, R)\) formulation.

5.2. Symmetries of the LMN hierarchy

5.2.1. Symmetries of the LMN hierarchy implied by Euler and Navier-Stokes symmetries

Similarly as the MPC hierarchy, the LMN equations are invariant under the classical symmetries of Navier-Stokes equations. This issue was investigated by Wacławczyk et al. (2014), where the form of classical symmetries written in terms of PDF’s was derived. Below, we shortly recall the findings.

The invariance under time and space translations, cf. (231), can be very easily inspected. Equation (218) for \( \nu = 0 \) is invariant under two scaling groups:

\[
\begin{align*}
T^{*}_{2} : & \quad t' = t, \quad x_{[0]}^{*} = e^{k_2} x_{[0]}, \quad v_{[0]}^{*} = e^{k_2} v_{[0]}, \quad f_{[r+1]}^{*} = e^{-3(k_2+1)k_2} f_{[r+1]}, \quad f_{[r+2]}^{*} = e^{-3(k_2+2)k_2} f_{[r+2]} , \\
T^{*}_{3} : & \quad t' = e^{k_3} t, \quad x_{[0]}^{*} = x_{[0]}, \quad v_{[0]}^{*} = e^{k_2} v_{[0]}, \quad f_{[r+1]}^{*} = e^{3(k_2+1)k_3} f_{[r+1]}, \quad f_{[r+2]}^{*} = e^{3(k_2+2)k_3} f_{[r+2]}.
\end{align*}
\]

(239)

(240)
which can be compared to the analogous symmetry of the MPCE, cf. (231). The scaling of the PDFs $f_{n+1}$ and $f_{n+2}$ assures that the normalization property (211) is satisfied for both transformed functions $f^*_{n+1}$ and $f^*_{n+2}$. In the viscous case the two above symmetries reduce to one scaling group with $k_3 = 2k_2$.

It was shown in Wacławczyk et al. (2014) that the LMN equations are invariant with respect to the Galilean transformations $t^* = t, x_j^* = x_j + v_0 t, v_j^* = v_j + \gamma_0$, where $v_0$ is a constant vector. However, the extended Galilean transformations $T^* - T_0$ in (181), which read $t^* = t, x^* = x + f(t)$ and $U^* = U + f'(t)$, where the vector $f$ is time-dependent, was broken due to the formulation of the pressure term chosen for constructing the LMN hierarchy. The reason was that the instantaneous pressure must undergo the transformation $p^* = p - \gamma \cdot f''(t)$ in order for the NS equation to be invariant. However, the latter transformation, being not bounded in $\mathbb{R}^3$, is not compatible with the integral representation of the pressure-gradient term that is present in the LMN hierarchy.

### 5.2.2. Statistical symmetries of the LMN hierarchy

The statistical symmetries of the LMN hierarchy were first identified in Wacławczyk et al. (2014) and their derivation will be shortly recalled below. We discuss here a set of transformations of the PDFs under which the LMN equations (218) turn out to be invariant and which corresponds to the set of statistical symmetries (233) and (234) found for the MPCE, where the moments $H$ are, respectively shifted by a constant and scaled. For simplicity of notation, it is convenient to first derive symmetries of equation (217), where only the one- and two-point PDF are present. Finally, a generalisation to the $n$-point PDF will be given.

#### Scaling symmetry

As it was shown in Wacławczyk et al. (2014) scaling of moments according to (234) transforms a PDF of a turbulent signal into a non-continuous PDF with delta function at the origin $x = 0$. Such function would correspond to a PDF of a turbulent signal interrupted by intervals where $U = 0$. In Wacławczyk et al. (2014) we first presented a one-point PDF as a Fourier transform of the characteristic function $\Phi_1$ (see Pope, 2000)

$$f_i(t; x, t) = \frac{1}{(2\pi)^3} \int e^{-i \omega y} \Phi_1(y; x, t) \, dy,$$  
(241)

where the index $(0)$ has been skipped. The one-point velocity statistics can be calculated as the $n$-th order derivative of $\Phi_1$ at the origin

$$\frac{\partial \Phi_1}{\partial y_{i0}} \bigg|_{y=0} = i U_{i0}(x, t),$$

$$\frac{\partial^2 \Phi_1}{\partial y_{i0} \partial y_{j0}} \bigg|_{y=0} = (i^2) U_{i0}(x, t) U_{j0}(x, t),$$

$$\frac{\partial^3 \Phi_1}{\partial y_{i0} \partial y_{j0} \partial y_{k0}} \bigg|_{y=0} = (i^3) U_{i0}(x, t) U_{j0}(x, t) U_{k0}(x, t), \ldots$$

(242)

Hence, the Taylor-series expansion of the characteristic function can be written as

$$\Phi_1 = 1 + i U_{i0}(x, t) y_{i0} - \frac{1}{2!} U_{i0}(x, t) U_{j0}(x, t) y_{i0} y_{j0} + \cdots.$$  
(243)

with summation over repeating indices $i_{(0)}, j_{(0)}, k_{(0)}, \ldots = 1, \ldots, 3$.

If we substitute the transformation of moments (234) into (243) we obtain

$$\Phi_1 = 1 + i e^{ik} U_{i0}(x, t) y_{i0} - \frac{1}{2!} e^{ik} U_{i0}(x, t) U_{j0}(x, t) y_{i0} y_{j0} + \cdots.$$  
(244)

We note that the symmetry (243) transforms moments of the velocity, starting from the first-order moment, whereas the first term in the above Taylor series expansion is in fact the normalisation of the PDF ($\Phi_1(0) = 1$, hence $\int f_{n+1} \, dy_0 \cdots dy_{(n)} = 1$). This term cannot be scaled, in order not to violate the properties of the PDF. Substituting (244) into equation (241) we obtain the transformed PDF, which can be written in the following form

$$f^*_n(t; x, t) = \delta(u) + e^{ik}(f_1 - \delta(u_{(0)})).$$  
(245)

As it was shown in Wacławczyk et al. (2014), the symmetry (245) may be extended to the $n + 1$-point PDF according to

$$f^*_{n+1} = \delta(u_{(0)}) \cdots \delta(u_{(0)}) + e^{ik} [f_{n+1} - \delta(u_{(0)}) \cdots \delta(u_{(0)})].$$  
(246)

Being a PDF, the function $f^*_{n+1}$ must satisfy all properties of a PDF. Hence, we note that
I. \[
\delta(v_0) \cdots \delta(v_n) + e^k (f_{n+1} - \delta(v_0) \cdots \delta(v_n)) \geq 0, \quad \forall v \in \mathbb{R}^3, \forall x \in \mathbb{R}^3,
\] (247)

which, for a continuous function \(f_{n+1}\) implies that \(e^k \leq 1\), hence \(k \leq 0\). Apparently, such restrictions for the scaling parameter \(k\) means that the group axioms given in section 2.2 are not satisfied by the transformation (246).

II. the normalization condition (211) is satisfied

III. the coincidence property (213) is satisfied as \(\delta(v_0) \cdots \delta(v_n) = \delta(v_0)\delta(v_0 - v_{(1)}) \cdots \delta(v_{(n-1)} - v_{(n)}))\).

IV. the divergence or continuity condition (212) is satisfied, since \(\delta(v_0) \cdots \delta(v_n)\) does not depend on the space variable;

V. The separation property (214) is not satisfied unless the PDF \(f_{n+1}\) itself is a delta function.

If we note that \(\delta(v_0) \cdots \delta(v_n)\) does not depend on time and space variables, we see that the LMN equations (218) are in fact invariant under the transformation (246). The moments calculated from the transformed PDFs read

\[
\frac{\partial}{\partial x} U_{x_1}(x_1, t) \cdots U_{x_n}(x_n, t) = \int [f_{n+1} e^k + (1 - e^k) \delta(v_0) \cdots \delta(v_n)] v_{(n)} \cdots v_{(1)} dv_{(0)} =
\]

\[
= e^k U_{x_1}(x_1, t) \cdots U_{x_n}(x_n, t) = e^k \mathcal{H}_{[n+1]},
\] (248)

which is identical to equation (234).

**Translation symmetry of moments**

If we substitute the translation symmetry of moments (233) into the Taylor-series expansion (243), we obtain

\[
\Phi_i(s; x, t) = \Phi_1(s; x, t) + \frac{1}{2!} C_{x_1 y_{x_1}} - \frac{1}{3!} C_{x_1 y_{x_1} y_{x_2}} y_{x_2} y_{x_3} y_{x_3} y_{x_4} \cdots
\]

The underbraced sum is the Taylor series expansion of a function \(\phi(y)\) which equals 0 at the origin and its derivatives at the origin equals, respectively \(i C_{x_1 y_{x_1}}, -C_{x_1 y_{x_1} y_{x_2}}, -i C_{x_1 y_{x_1} y_{x_2} y_{x_3}}\) etc. If equation (249) is substituted into equation (241), the transformed PDF reads

\[
f'_i(v; x, t) = f(v; x, t) + \frac{1}{(2\pi)^3} \int e^{-2\pi y} \psi(y) dy = f_1(v; x, t) + \psi(v)
\] (250)

where \(\psi(v)\) is an inverse Fourier transform of \(\phi(y)\) and

\[
\int \psi(v) dv = 0.
\] (251)

which follows from the fact that \(\phi(0) = 0\). Note that neither \(\phi(y)\) nor \(\psi(y)\) depend on \(x\) or time \(t\).

In Wacławczyk et al. (2014) the considerations were also generalised to the case of the \(n + 1\)-point PDF leading to the followig form of the transformed PDF

\[
f^*_{n+1} = f_{n+1} + \psi(v_0) \delta(v_0 - v_{(1)}) \cdots \delta(v_{(n)} - v_{(n)}).
\] (252)

It can be readily verified that the transformations (252) correspond to the statistical symmetry (233) where the MPC \(\mathbf{H}\) tensors are translated. Indeed, under the transformations (252), the moments of the PDFs are mapped to:

\[
\frac{\partial}{\partial x} U_{x_1}(x_1, t) \cdots U_{x_n}(x_n, t) = \int [f_{n+1} v_{(n)} \cdots v_{(1)} + \psi(v_0) \delta(v_0 - v_{(1)}) \cdots \delta(v_{(n)} - v_{(n)})] v_{(n)} \cdots v_{(1)} dv_{(0)} =
\]

\[
= U_{x_1}(x_1, t) \cdots U_{x_n}(x_n, t) + \int \psi(v_0) v_{(n)} \cdots v_{(1)} dv_{(0)} = \mathcal{H}_{[n+1]} + C_{[n+1]}.
\] (253)

In Wacławczyk et al. (2014) the conditions, which the above transformed PDF functions have to satisfy, were examined:

I. the coincidence property (213) is satisfied by the transformed PDFs (252)

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II. because of the probabilistic interpretation of the PDFs, it is locally required that:

\[ f_{n+1} + \psi(v_0)\delta(v_0) \delta(v_1) \ldots \delta(v_n) \geq 0, \quad \forall v \in \mathbb{R}^3, \forall x \in \mathbb{R}^3, \]  

(254)

while the normalization condition globally imposes:

\[ \int \psi(v) \, dv = 0; \]  

(255)

which agrees with equation (251).

III. the divergence or continuity condition (212) is satisfied, since \( \psi \) does not depend on the space variable;

IV. the separation property (214) is not satisfied, as the transformation of the PDF is independent of the spatial variable and thus cannot satisfy this limiting behavior. On the other hand, let us note that, while it is reasonable to require this property, this is never used in the derivation of the equations of the LMN hierarchy. Moreover, this property is not satisfied by the corresponding symmetries of the MPC equations, either.

It was noted in Waclawczyk et al. (2014) that the function \( f_{n+1} \) transformed according to (252) is always a solution of equation (218). However, such \( f_{n+1} \) may not be a PDF any more, as it may have negative values. Then, (252) may not necessary be a Lie group but still is a symmetry of the LMN equation equation (218). This implies that the constants \( C_{n+1} \) obtained in equation (253) are not arbitrary, but due to the condition (254) on the functions \( f_{n+1} \) and \( \psi \) we expect that they might be limited to a certain range. We conclude that considering the symmetries of the LMN hierarchy provides additional restrictions on the group parameters, which were not observed in the MPC approach.

In order to derive these restrictions for the translation symmetry (252) we attempted in Waclawczyk et al. (2014) to find its physical interpretation. We started with the observation that the \( n+1 \)-point PDF of a laminar field constant in space and time reads

\[ f_L(v_0, \ldots, v_n; x_0, \ldots, x_n, t) = \delta(v_0 - U^{(l)}_L) \cdots \delta(v_n - U^{(l)}_L) = \]  

(256)

\[ = \delta(v_0 - U^{(l)}_L) \delta(v_1 - v_0) \cdots \delta(v_n - v_0) = f(v_0) \delta(v_1 - v_0) \cdots \delta(v_n - v_0), \]

where \( U^{(l)}_L \) is the laminar velocity and \( \omega \) is an element from the probability space (i.e. \( U^{(l)}_L \) can be different in different flow realisations). If both laminar and turbulent solutions are elements of the ensemble, then the PDF can be written as a sum of a turbulent and laminar part \( f = g_T + g_L \), where

\[ g_L = g(v_0) \delta(v_1 - v_0) \cdots \delta(v_n - v_0), \]  

(257)

\[ \int g_L \, dv_0 g_L \leq 1. \]

Let us compare equation (257) with the shape symmetry (252). Due to the condition (251) the translation function in (252) has, at a certain range of \( v_0 \), negative and infinite values at the diagonal \( v_1 = v_0, v_2 = v_0 \) etc. Hence, we come to the conclusion that for a non-zero function \( \psi \) the transformed PDF \( f_{n+1}^{(T)} \) could be non-negative only if the function \( f_{n+1} \) would contain a laminar part (257) which is also infinite at \( v_1 = v_0, v_2 = v_0 \) for arbitrary separations \( x_1 - x_0, x_2 - x_0 \) etc. Equation (252) would then transform the laminar part of the PDF such that we would obtain \( f^{(T)} = g_T + g_L^{(T)} = g_T + g_L + \psi \), where \( g_L^{(T)} \geq 0 \). We argue that the restrictions on the constants \( C_{n+1} \) will follow from the range of laminar solutions for velocity realisable in the given flow configuration. We address here the particular case of the fully developed plane Poiseuille channel flow. Due to the presence of the non-moving boundaries the laminar part of the PDF takes the form

\[ f_L(v_0, \ldots, v_n; x_0, \ldots, x_n, t) = \delta(v_0 - U^{(l)}_L) \left( 1 - \frac{y^2}{y^2_{20}} \right) \cdots \delta(v_n - U^{(l)}_L) \left( 1 - \frac{y^2}{y^2_{20}} \right) \]  

where \( U_L = [U_L, 0, 0] \) is the streamwise velocity in the centerline and \( y(k) = x_{20} \) is the wall-normal coordinate. Using properties of the delta function the above equation can be rewritten as

\[ f_L(v_0, \ldots, v_n; x_0, \ldots, x_n, t) = \]  

(258)

\[ = F(y_0, \ldots, y_n) \delta(v'_0 - U^{(l)}_L) \delta(v'_1 - v'_0) \cdots \delta(v'_n - v'_0) = \]  

(259)
where \( f'(v_{(0)}) = \delta(v_{(0)} - U_{(0)}^L) \) is, according to the definition (207) a PDF of the variable \( v_{(0)}' \).

\[
F(y_{(0)}, \ldots, y_{(n)}) = \left( 1 - \frac{y_{(0)}^2}{H^2} \right)^{-1} \cdots \left( 1 - \frac{y_{(n)}^2}{H^2} \right)^{-1}
\]

and for each \( k \)

\[
v_{1n}' = v_{1n} \left( 1 - \frac{y_{(0)}^2}{H^2} \right)^{-1}, \quad v_{2n}' = v_{2n}, \quad v_{3n}' = v_{3n}.
\]

If we compare equation (258) with Eqs. (256) and (252) it follows that the translation symmetry in the channel flow has the following form

\[
f_{n+1}^L = f_{n+1} + F(y_{(0)}, \ldots, y_{(n)}) \psi(v_{(0)}') \delta(v_{(1)}' - v_{(0)}') \cdots \delta(v_{(n)}' - v_{(0)}').
\]

We will now take into considerations both the scaling (246) and the translation symmetry for the channel flow (262). First, the scaling symmetry transforms a PDF \( f_{n+1} \) of a turbulent signal into the PDF with delta function at \( x = 0 \)

\[
f_{n+1}^L = \epsilon^k f_{n+1} + (1 - \epsilon^k)F(y_{(0)}, \ldots, y_{(n)}) \delta(v_{(0)}' - U_L) \delta(v_{(1)}' - v_{(0)}') \cdots \delta(v_{(n)}' - v_{(0)}').
\]

Next, the translation symmetry with the function \( \psi \) defined as \( \psi = (1 - \epsilon^k)[\delta(v_{(0)}' - U_L) - \delta(v_{(0)}')] \) transforms equation (263) into

\[
f_{n+1}^L = \epsilon^k f_{n+1} + (1 - \epsilon^k)F(y_{(0)}, \ldots, y_{(n)}) \delta(v_{(0)}' - U_L) \delta(v_{(1)}' - v_{(0)}') \cdots \delta(v_{(n)}' - v_{(0)}').
\]

To sum up, both symmetries, scaling and translation transform a PDF of a turbulent signal into the PDF of an intermittent laminar-turbulent flow, see Fig. 1. This would correspond to a situation where the flow in a channel is induced by a certain pressure difference \( \Delta P \), such that the resulting Reynolds number \( Re = U_L H/\nu \) where \( U_L \) is the bulk velocity, \( H \) is the channel half-width and \( \nu \) is the kinematic viscosity is close to the critical value \( Re_{cr} \). For certain range of \( Re \) both, laminar or turbulent solutions are possible with certain probability, leading to the PDF of the form given in equation (264).

The PDF (264) gives rise to the following translations of the moments

\[
\mathbb{U}_1(x_{(0)}, t) \cdots \mathbb{U}_1(x_{(0)}, t) = \epsilon^k \mathbb{U}_1(x_{(0)}, t) \cdots \mathbb{U}_1(x_{(0)}, t) + (1 - \epsilon^k)C_{1,1} \left( 1 - \frac{y_{(0)}^2}{H^2} \right)^{-1} \cdots \left( 1 - \frac{y_{(0)}^2}{H^2} \right).
\]

Such considerations may allow to determine values of the coefficients \( C_{1,1} \) in transformation (265). The second RHS term in equation (265) can be identified as the laminar contribution, hence \( C_1 = U_L, C_{11} = U_L^2 \) etc. where \( U_L \) is the velocity at the centerline for a laminar flow at a given \( Re \) number. We note here that laminar solutions are observed only within a certain range of \( Re \) numbers which sets limits for the coefficients \( C_{1,1} \), e.g. \(-U_{Lmax} \leq C_1 \leq U_{Lmax} \) where \( U_{Lmax} \) is the maximal laminar velocity at the centerline observed for the channel flow.
5.3. Symmetries of the Hopf equation

5.3.1. Symmetries of the Hopf equation implied by Euler and Navier-Stokes symmetries  

In order to verify the invariance of the Hopf functional equation under the scaling groups $T_2$ and $T_3$ in (181) and $T_{NS}$ in (183) we first consider transformations of $n + 1$-point characteristic functions. From the relation

$$ \Phi_{n+1}^* = \int e^{\Phi_0} y_m^* \cdots e^{\Phi_0} f_{n+1}^* \, dx_0^* \cdots dx_n^* $$

(266)

we find, that the scaling symmetries (239) and (240) will hold, if $y_0^* = e^{-k} y_0$ and $y_i^* = e^{k} y_i$ for each $i$, as in such a case the exponent $v_0^* y_0^* = v_0 y_0$ remain unchanged and using (239) we obtain

$$ \Phi_{n+1}^* = \int e^{3(n+1)k^2} y_m^* \cdots e^{n} f_{n+1} e^{3(n+1)k^2} \, dx_0^* \cdots dx_n^* = \Phi_{n+1}. $$

(267)

The same holds for the second scaling group (240), i.e. the $n + 1$-point characteristic function (262) transforms as $\Phi_{n+1}^* = \Phi_{n+1}$. We expect that the same should hold in the limit $n \to \infty$ i.e. for the characteristic functional.

In this case the first, space scaling group might be problematic at the first sight as in the definition of the characteristic functional

$$ \Phi^* = \phi(U(x), y(x)dx), $$

(268)

the integral over space $x$ is present in the exponent and as we remember from formula (239), the space scales as $x^* = xe^{y^*}$. The answer is that in the continuum case instead of the discrete $k$-th variable $y_0$, we deal with $y(x)dx$. The sums are replaced by integrals in the continuum limit and hence $y(x)dx$ should scale as $y(x)dx$ in the discrete case, i.e. $y_i dx^* = e^{-k} y_i dx$. Because $dx$ scales as $dx^* = dx e^{y^*}$ it follows that $y_i^* = e^{y_i} y_i$.

To sum up, it can be shown that the Hopf functional equation (225) for $\mu = 0$ is invariant under the following scaling transformation of variables

$$ T_2 : \Phi^* = \Phi, \quad x^* = e^{y^*} x, \quad t^* = t, \quad y_i^* dx^* = e^{-y} y_i dx, \quad y_i^* = e^{y} y_i. $$

(269)

$$ T_3 : \Phi^* = \Phi, \quad x^* = x, \quad t^* = e^{y} t, \quad y_i^* dx^* = e^{y} y_i dx, \quad y_i^* = e^{y} y_i. $$

(270)

For non-vanishing viscosity $\mu \neq 0$, instead of two scaling groups we obtain one scaling according to (183).

Generalised Galilean invariance $T_7 - T_9$ in (181), takes the form $t^* = t, x^* = x + f(t) t, u^* = u + df(t)/dt$, where $f(t) = f_1(t), f_2(t), f_3(t)$. Subsequently we brieﬂy prove Galilean invariance of equation (224) with the assumption that $df(t)/dt = U_0$ is a vector with constant components. Hence, instead of generalised Galilean invariance we consider only classical Galilean transformations, since in this case it is not necessary to transform the pressure functional in equation (224). The functional $\Phi$ transforms under the Galilean invariance as follows

$$ \Phi^* = \phi(U(x), y(x)dx), \quad \Phi^* = \phi(U(x), y(x)dx), $$

(271)

Hence, the transformation of $\Phi$ may be written as

$$ \Phi^* = C(y(x)) \Phi, $$

(272)

where $C(y(x))$ is a functional

$$ C(y(x)), t = \phi(y(x)) U_0 dx $$

(273)

such that $C(0) = 1$ and derivatives

$$ \frac{\delta^2 C(y(x)), t}{\delta y_{i_0}(x) \cdots \delta y_{i_n}(x)} = t^i U_{i_0} \cdots U_{i_n} e^i(y(x)) U_0 dx $$

(274)

are not explicit functions of $x$. The $n$-th derivative of the transformed functional $\Phi^*$ at $y = 0$ gives

$$ \frac{\delta^n \Phi^*}{\delta y_{i_0}(x) \cdots \delta y_{i_n}(x)} |_{y = 0} = t^n (U_{i_0}(x, t) + U_{i_0}(x, t) \cdots (U_{i_n}(x, t) + U_{i_n}(x, t)), $$

(275)

as expected for Galilean invariance.

The space derivatives equation (224) transform as $\partial/\partial x_i = \partial/\partial x_i$ and the integral over infinite space $\int dx^* = \int dx$. As the variables $y^* = y$, also the functional derivative remains unchanged $\delta/\delta y_i(x)$. In order to transform
the time derivative in equation (224) we first follow Klauder (2011) and we present the variable \( y(x) \) with the use of the test sequence \( \{ y_n \} \)

\[
y(x) = \sum_{n=1}^{\infty} y_n h_n(x).
\]  

(276)

where \( \{ h_n(x) \} \) is a set of orthogonal functions. The derivative \( \partial / \partial t \) can be presented as

\[
\frac{\partial}{\partial t} \frac{\partial}{\partial t} + \frac{\partial}{\partial t} \frac{\partial}{\partial x_i} + \sum_{n=1}^{\infty} \int dx^2 \frac{\partial y_n}{\partial t} \frac{\partial}{\partial y_k}.
\]

(277)

Hence, all new variables \( r^*, x^* \) and the infinite set \( \{ y_n \} \) are differentiated with respect to \( t \). The last term in the above formula can be rewritten in terms of derivatives of \( \{ y(x) \}^* \) as follows

\[
\frac{\partial (y(x))^*}{\partial t} = \sum_{n=1}^{\infty} \frac{\partial y_n}{\partial t} h_n(x^*) + \sum_{n=1}^{\infty} y_n \frac{\partial h_n}{\partial t}.
\]

(278)

It follows that

\[
\sum_{n=1}^{\infty} \frac{\partial y_n}{\partial t} h_n(x^*) = \frac{\partial (y(x))^*}{\partial t} - \frac{\partial y(x)^*}{\partial x_i} \frac{\partial x_i}{\partial t}.
\]

(279)

Finally, the time derivative for the Galilei invariance transforms as

\[
\frac{\partial}{\partial t^*} = \frac{\partial}{\partial t} - U_{00} \frac{\partial}{\partial x_l} + U_{0l} \int dx \frac{\partial y_l}{\partial x_l} \frac{\partial}{\partial y_k}.
\]

(280)

The transformed functional equation (224) can be rewritten as

\[
\frac{\partial C([y(x)])}{\partial t^*} = \int y_k(x) \left[ i \frac{\partial}{\partial x_l} \delta C([y(x)]) + v \nabla^2 \frac{\partial C([y(x)])}{\partial y_k} \right] \frac{\partial C([y(x)])}{\partial x_k} dx.
\]

With (280) the LHS of equation (281) reads

\[
\frac{\partial C([y(x)])}{\partial t^*} = C([y(x)]) \frac{\partial C([y(x)])}{\partial t} - U_{00} \int y_k(x) \frac{\partial C([y(x)])}{\partial y_k} dx = \frac{\partial C([y(x)])}{\partial t} - U_{00} C([y(x)]) \int y_k(x) \frac{\partial C([y(x)])}{\partial y_k} dx,
\]

(282)

where the second equality follows from the fact that \( \delta C/\delta y_k = iU_{00}C \) does not depend explicitly on \( x \), moreover, as the functional \( C([y(x)]) \) does not depend on time, it was taken out of the time derivative.

The functional derivative of \( \Phi^* \) in (281) reads

\[
C([y(x)]) \frac{\partial \Phi}{\partial y_k(x)} + \Phi \frac{\delta C([y(x)])}{\delta y_k(x)}
\]

(283)

and we note that Laplacian \( \nabla^2 \) of the second term is zero as this term is not a function of \( x \). Hence, the last RHS term of equation (281) inside the integral reads \( C([y(x)])v \nabla^2 \frac{\partial \Phi}{\partial y_k(x)} \). Further, the second functional derivative of \( \Phi^* \) reads

\[
\frac{\delta^2 \Phi}{\delta y_k(x) \delta y_k(x)} + \frac{\delta \Phi}{\delta y_k(x) \delta y_k(x)} \frac{\delta C}{\delta y_k(x)} + \frac{\delta \Phi}{\delta y_k(x) \delta y_k(x)} \frac{\delta^2 C}{\delta y_k(x) \delta y_k(x)}.
\]

(284)

Again, the derivative \( \partial / \partial x_i \) of the last term is zero, as it does not depend on \( x \). In addition we also have

\[
\frac{\partial}{\partial x_i} \left[ \frac{\delta \Phi}{\delta y_k(x) \delta y_k(x)} \right] = \frac{\delta C}{\delta y_k(x) \delta x_i} \frac{\delta \Phi}{\delta y_k(x)} = 0,
\]

(285)

where the first equality follows from the fact that the derivative of \( C \) does not depend explicitly on \( x \) and the second, from the continuity condition. It can be seen from the definition of the Hopf functional (219) that its derivative with respect to \( y(x) \) reads \( IU(x, t) \exp [iU(x, t) \cdot y(x) dx] \), hence differentiating once again with respect to \( x_i \) is zero as \( \partial U_i / \partial x_i = 0 \). The second term in equation (284), introduced into equation (281), leads to

\[
\int y_k(x) \left[ \frac{\partial}{\partial x_i} \frac{\delta \Phi}{\delta y_k(x)} \right] dx,
\]

(286)

which cancels with the term on the LHS (see 282). With this, the Hopf equation can be reformulated to the form (225) with the variable \( \tilde{y} \) and finally, \( C([y(x)]) \) can be taken out of integral on the RHS. This finally proves the Galilean invariance of equation (225).
5.3.2. Statistical symmetries of the Hopf equation

As it was discussed in Section 4.5, E. Hopf proposed to present the solution of the Hopf functional as the infinite series expansion

\[ \Phi = 1 + C_1 + C_2 + \cdots, \tag{287} \]

where

\[ C_{n=1} = \int K_{\alpha_1-\alpha_0}(x(0), \ldots, x(n), t) y_{a_0}(x(0)) \cdots y_{a_0}(x(n)) dx(0) \cdots dx(n) \tag{288} \]

with the kernel function given by equation (229)

\[ K_{\alpha_1-\alpha_0}(x(0), \ldots, x(n), t) = \frac{1}{(n+1)!} \frac{\partial^{n+1} \Phi}{\partial y_{a_0}(x(0)) \cdots \partial y_{a_0}(x(n))} \bigg|_{y=0} = \frac{\partial^n}{(n+1)!} U_{a_0}(x(0), t) \cdots U_{a_0}(x(n), t). \tag{289} \]

If we substitute the statistical symmetries of moments from Section 5.1.2 into the above equations we find that the scaling symmetry (234) transforms the kernel functions as \( K^{\ast}_{\alpha_1-\alpha_0} = e^{\tilde{e}} K_{\alpha_1-\alpha_0} \), and hence, equation (287) reads

\[ \Phi^{\ast} = 1 + e^{\tilde{e}} (\Phi^1 + \Phi^2 + \cdots), \tag{290} \]

or

\[ \Phi^{\ast} = 1 + e^{\tilde{e}} (\Phi - 1). \tag{291} \]

Similarly, the statistical symmetry (233) translates the kernel functions (229) by a constant which leads to the following translation of the \( n + 1 \)th term in the Taylor series expansion

\[ \Phi_{n+1}^\ast = \Phi_{n+1} + \int C_{\alpha_1-\alpha_0} y_{a_0}(x(0)) \cdots y_{a_0}(x(n)) dx(0) \cdots dx(n). \tag{292} \]

Hence, the translation symmetry of the characteristic functional \( \Phi \), corresponding to (233) can be written in the following form

\[ \Phi^\ast = \Phi + \Psi([y(x)]), \tag{293} \]

where \( \Psi \) is a functional such that its \( n \)th functional derivative at the origin equals \( C_{\alpha_1-\alpha_0} \).

To sum up the content of the preceding sections, it was shown that all methods for the full statistical description of turbulence, namely MPC hierarchy, LMN hierarchy for PDF’s and the Hopf characteristic functional equations are invariant under classical symmetries of Navier-Stokes equations and additionally under the set of statistical symmetries: translation of moments and scaling. Through the analysis of the PDF equations the statistical translation and scaling symmetries (246) and (252) were identified as closely connected to intermittency. Both symmetries acting simultaneously transform the PDF e.g. from turbulent to intermittent (laminar/turbulent) or laminar PDF. Hence, the statistical symmetries indicate the fact that solutions of Navier-Stokes equations may have physically very different character. Such transformation could only be observed in the statistical approach, hence the statistical symmetries were not found in the Lie group analysis of the Navier-Stokes equations in its classical form (180).

In principle, the Lie group analysis of infinite hierarchies of equations, such as LMN or MPC cannot be performed with the use of common computer algebra systems. Indeed, the statistical symmetries (232–234) of MPC equations first were rather guessed than calculated and their set may not be complete. Still in an ingenious proof Rosteck (2013) showed that the infinitesimals of the infinite number of MPC all only depend on a finite number of variables. With this he made the system accessible to computer algebra system using some successive procedure.

Another option is the possibility to apply Lie group method to Hopf’s functional equation in order to find possibly new statistical symmetries and, in a next step derive corresponding transformations in the LMN and MPC hierarchy. For this purpose, the extended Lie group method, introduced in Section 2.7 could be used. This issue is the subject of the current research.

6. Turbulent scaling laws

In this final section we intend to show that based on the symmetry machinery we derive turbulent scaling laws for various canonical flows. This content is based mainly on the PhD thesis Rosteck (2013) and papers where results of DNS calculations were compared with the theoretical findings i.e. Mehdizadeh and Oberlack (2010), Avsarkisov et al. (2014), Oberlack and Rosteck (2010).
The rather classical idea of a turbulent scaling law usually refers to two distinct facts:

(i) Introducing a certain set of dimensional parameters such as the wall-friction velocity $u_f$ or boundary layer thickness $\Theta$ to non-dimensionless statistical turbulence variables, for instance the mean velocity, and which in turn leads to a collapse of data if one external parameter is varied such as the Reynolds number.

(ii) An explicit mathematical function is given for statistical turbulence variables such as the mean velocity, Reynolds stresses, etc.

Presently we primarily contemplate with the second definition while the normalisation according to (i) will be introduced on dimensional reasons as well as employing classical arguments. In order to rigorously derive such laws directly from the MPC equations we employ the idea of a group invariant solution detailed in Section 2.6.

It appears to be the driving mechanism for quantities of statistical turbulence which have the strong tendency to establish invariant solutions of the MPC equations while at the same time maximising the number of symmetries involved being limited by the boundary condition.

In the remaining two subsections we adopt the latter condition for the derivation of the accordant invariant solution alternatively later also named turbulent scaling laws, which is the usual phrase in the turbulence literature.

### 6.1. Classical near-wall turbulent shear flows

Due to its practical importance wall bounded shear flows are by far the most intensively investigated turbulent flow and have been so for more than a century. These studies employ a vast number of numerical, experimental and modelling approaches. From all the theoretical approaches the universal law of the wall is the most widely cited and also accepted approach with its essential ingredient being the logarithmic law of the wall. Though a variety of different approaches have been put forward for its derivation neither of them have employed the full multi-point equations, which are the basis for statistical turbulence. In the following we demonstrate that the log-law is an invariant solution of the infinite set of multi-point equations and, further, it is shown that it essentially relies on new statistical symmetry groups (233) and (234) or rewritten in classical notation according to (236) and (234), respectively, according to (238).

Within this subsection we exclusively examine wall-parallel turbulent flows only depending on the wall-normal coordinate $x_2$. Further, we only explicitly write the two-point correlation $R_{ij}$, though all results are also valid for all higher order correlations. This finally yields

$$\bar{U}_1(x) = \bar{U}_1(x_2), \quad \bar{U}_2(x) = 0, \quad \bar{U}_3(x) = 0, \quad \bar{P}(x) = \bar{P}(x_1, x_2), \quad R_{ij} = R_{ij}(x_2, r), \quad \ldots$$

(294)

In the limit $r \to 0$ from the two-point correlation tensor $R_{ij}$ the Reynolds stress tensor $\bar{u}_i\bar{u}_j$, subsequently denoted by $\bar{R}_{ij}(x_2)$ is obtained.

In the following, we start by deducing the governing equations which represent a reduced version of the Navier-Stokes and MPC equations in comparison to Eqs. (187) and (198). From these equations the set of symmetries can be established on dimensional reasons as well as employing classical arguments. In order to rigorously derive such laws directly from the MPC equations we employ the idea of a group invariant solution detailed in Section 2.6.

Beginning with the simplified averaged Navier-Stokes equations

$$\begin{align*}
\frac{\partial \bar{U}_1}{\partial t} + \frac{\partial H_{12}[U^{(1)} \to 0]}{\partial x_2} + \frac{\partial \bar{P}}{\partial x_1} - \nu \frac{\partial^2 \bar{U}_1}{\partial x_2^2} &= 0 \\
\frac{\partial H_{22}[U^{(1)} \to 0]}{\partial x_2} + \frac{\partial \bar{P}}{\partial x_2} &= 0, \quad \frac{\partial H_{22}[U^{(1)} \to 0]}{\partial x_2} = 0
\end{align*}$$

(295)

(296)

we may further consider the transport equations for the MPCs in turbulent plane shear flows

$$0 = \frac{\partial H_{10}[U^{(1)} \to 0]}{\partial t} + \frac{\partial H_{10}[U^{(1)} \to 0]}{\partial x_2} + \frac{\partial H_{10}[U^{(1)} \to 0]}{\partial x_2} - \frac{\partial^2 H_{10}}{\partial x_2^2}$$

$$+ \sum_{k=1}^{n-1} \left[ \frac{\partial H_{10}[U^{(1)} \to 0]}{\partial r_i^{(1)}} - \frac{\partial H_{10}[U^{(1)} \to 0]}{\partial r_i^{(1)}} + \frac{\partial H_{10}[U^{(1)} \to 0]}{\partial r_i^{(1)}} \right]$$

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\[
- \frac{\partial H_{i=1}}{\partial r_{i=1}^{(j)}} + \nu \left( 2 \frac{\partial H_{i=1}}{\partial x_2 \partial r_{i=1}^{(j)}} - \frac{\partial H_{i=1}}{\partial r_{i=1}^{(j)}} \frac{\partial r_{i=1}^{(j)}}{\partial x_2} - \sum_{m=1}^{n-1} \frac{\partial H_{i=1}}{\partial r_{i=1}^{(m)}} \frac{\partial r_{i=1}^{(m)}}{\partial x_2} \right)
\]

(297)

and, additionally, the continuity equations

\[
\frac{\partial H_{i=1}[m=0+1]}{\partial x_m} - \sum_{j=1}^{n} \frac{\partial H_{i=1}[m=0+1]}{\partial r_{i=1}^{(j)}} = 0
\]

\[
= 0
\]

\[
\frac{\partial I_{i=1}[k=0+n]}{\partial x_n} - \sum_{j=1}^{n} \frac{\partial I_{i=1}[k=0+n]}{\partial r_{i=1}^{(j)}} = 0
\]

\[
= 0
\]

\[
\frac{\partial I_{i=1}[k=m+n]}{\partial r_{i=1}^{(j)}} = 0
\]

\[
= 0
\]

have to be obeyed.

It can be shown that all scaling symmetries and the translation symmetries in \( x_2 \), \( U_i \) and \( R_{ij} \), cf. (231) and (232) remain in the simplified system (295–297). Additionally, a linear function in \( x_3 \) can be added to the mean velocity and also to the two-point correlations \( H_{11} \), \( H_{12} \), \( H_{22} \) and \( H_{33} \) forming an extended set of statistical symmetries for the case of turbulent shear flow:

\[
Z_{c12} : \quad x_i^j = x_i \quad \bar{P} = \bar{P} + k_{12} \quad I_{i=1}[q]_i = I_{i=1}[q]_i
\]

\[
H_{i=1}^{12} = H_{i=1}^{12} + k_{12} \quad \text{for } i_{[n]} \neq 12, 21
\]

\[
Z_{c11} : \quad x_i^j = x_i \quad \bar{P} = \bar{P} \quad I_{i=1}[q]_i = I_{i=1}[q]_i
\]

\[
H_{i=1}^{11} = H_{i=1}^{11} + k_{11} \quad \text{for } i_{[n]} \neq 11
\]

\[
Z_{c22} : \quad x_i^j = x_i \quad \bar{P} = \bar{P} \quad I_{i=1}[q]_i = I_{i=1}[q]_i
\]

\[
H_{i=1}^{22} = H_{i=1}^{22} + k_{22} \quad \text{for } i_{[n]} \neq 22, 12, 21
\]

\[
Z_{c33} : \quad x_i^j = x_i \quad \bar{P} = \bar{P} \quad I_{i=1}[q]_i = I_{i=1}[q]_i
\]

\[
H_{i=1}^{33} = H_{i=1}^{33} + k_{33} \quad \text{for } i_{[n]} \neq 33
\]

Merging of the occurring classical (231), statistical (232) and additional symmetries (298) leads to the characteristic system, by analogy to equation (108). For the later comparison with the DNS data we want to apply these symmetries in the fluctuation approach (i.e. for the components of the tensor \( R_{ij} \), not \( H_{ij} \)), hence we transform them according to (204–206) as it was done for other symmetries, going from the instantaneous \( \mathbf{H} \) to the fluctuation \( \mathbf{R} \) approach cf. equation (231).

We state only terms up to the two-point correlations \( R_{ij} \). The characteristic system with the invariants corresponding to the considered symmetries reads

\[
\frac{d x_2}{k_{NS} x_2 + k_{G,2}} = \frac{d r_1^{(j)}}{k_{NS} r_1^{(j)}} = \frac{d U_i}{k_{NS} U_i} = \frac{d \bar{P}}{\bar{P} - 2 k_{NS} + k_{ij} \bar{P} + k_{i1}^2}
\]

\[
\frac{d R_{11}}{(-k_{NS} + k_J) U_1 + k_{r,1} + k_{11}} = \frac{d R_{12}}{(-k_{NS} + k_J) U_2 + k_{r,2} + k_{12}}
\]

\[
\frac{d R_{13}}{(-k_{NS} + k_J) U_3 + k_{r,3} + k_{13}} = \frac{d R_{22}}{(-k_{NS} + k_J) U_2 + k_{r,2} + k_{22}}
\]

\[
\frac{d R_{23}}{(-k_{NS} + k_J) U_3 + k_{r,3} + k_{23}} = \frac{d R_{33}}{(-k_{NS} + k_J) U_3 + k_{r,3} + k_{33}}
\]

(302)
were each constant corresponds to one of the previously mentioned symmetries:

\[
\begin{align*}
  k_{\text{NS}} & : \text{Navier-Stokes scaling} & k_i & : \text{new scaling symmetry} \\
  k_{G,2} & : \text{translation in } x_2 & k_{\text{tr},1} & : \text{moment translation in } \bar{U}_1 (\text{H-approach}) \\
  k_{\text{tr},i} & : \text{moment translation in } R_{ij} & k_{i} & : \text{additional symmetry } i.
\end{align*}
\]

Solving the hyperbolic system of differential equations given through the characteristic system (302) different results can be deducted depending on the group parameters \( k_i \). If special relations hold between these multipliers, particular results emerge such as logarithmic and exponential scaling laws. For the case of shear flows, the invariant solutions of the system (302) can only be calculated up to the two-point correlations as every higher moment will be considerably more complicated to derive. In order to save the space and avoid writing lengthy formulas we will only present results of calculations after taking the limit \( r \to 0 \) in two-point correlations. The formulas for mean velocity \( \bar{U}_1 \) and Reynolds-stresses \( \bar{\sigma}_{ij} \) can be later verified by substituting to the averaged Navier-Stokes equations and later compared with the DNS data. Additionally, by inserting the derived solutions to the averaged Navier-Stokes equations we can determine the averaged pressure and obtain conditions for integration constants. This procedure will be performed in the following paragraphs for the algebraic, exponential and logarithmic solutions of the characteristic system (302).

**Navier-Stokes - algebraic solution:** With the conditions \( k_t \neq ak_{\text{NS}} \), \( a \in \mathbb{N}, a \geq 1, k_{\text{NS}} \neq 0 \) from the characteristic system (302) we obtain the relation for the distance vector

\[
\bar{r}_i^{(j)} = k_{G,2} + k_{\text{NS}} x_2 \frac{r_i^{(j)}}{k_{i}},
\]

and the following algebraic scaling laws for the mean velocity and Reynolds stresses in the limit \( r \to 0 \)

\[
\bar{U}_1(x_2) = C_{1,1} \left[ k_{G,2} + k_{\text{NS}} x_2 \right] \frac{k_{\text{NS}}}{k_i} - \frac{k_{\text{tr},1} + k_{i}}{k_i - k_{\text{NS}}} \]

\[
(303)
\]

and

\[
\begin{align*}
\bar{R}_{12} &= C_{1,2} [k_{G,2} + k_{\text{NS}} x_2] \frac{k_{\text{NS}}}{k_i} - k_{1,2} + k_{G,2} + x_2 (k_t - 2k_{\text{NS}}) \\
&\quad - k_{1,2} (k_t - 2k_{\text{NS}} + 6k_t^2), \\
(304)
\end{align*}
\]

\[
\begin{align*}
\bar{R}_{22} &= C_{1,2} [k_{G,2} + k_{\text{NS}} x_2] \frac{k_{\text{NS}}}{k_i} - k_{2,2} + k_{G,2} + x_2 (k_t - 2k_{\text{NS}}) \\
&\quad - k_{2,2} (k_t - 2k_{\text{NS}} + 6k_t^2), \\
(305)
\end{align*}
\]

\[
\begin{align*}
\bar{R}_{33} &= C_{1,3} [k_{G,2} + k_{\text{NS}} x_2] \frac{k_{\text{NS}}}{k_i} - k_{3,3} + k_{G,2} + x_2 (k_t - 2k_{\text{NS}}) \\
&\quad - k_{3,3} (k_t - 2k_{\text{NS}} + 6k_t^2), \\
(306)
\end{align*}
\]

\[
\begin{align*}
\bar{R}_{11} &= C_{1,1} \left[ k_{G,2} + k_{\text{NS}} x_2 \right] \frac{k_{\text{NS}}}{k_i} - k_{1,1} + 2k_{\text{tr},1} (k_t - 2k_{\text{NS}}) \\
&\quad - C_{1,1} \left[ k_{G,2} + k_{\text{NS}} x_2 \right] \frac{k_{\text{NS}}}{k_i} - k_{1,1} + 2k_{\text{tr},1} (k_t - 2k_{\text{NS}}) \\
&\quad - C_{1,1} \left[ k_{G,2} + k_{\text{NS}} x_2 \right] \frac{k_{\text{NS}}}{k_i} - k_{1,1} + 2k_{\text{tr},1} (k_t - 2k_{\text{NS}}) \\
&\quad - C_{1,1} \left[ k_{G,2} + k_{\text{NS}} x_2 \right] \frac{k_{\text{NS}}}{k_i} - k_{1,1} + 2k_{\text{tr},1} (k_t - 2k_{\text{NS}}) \\
&\quad - C_{1,1} \left[ k_{G,2} + k_{\text{NS}} x_2 \right] \frac{k_{\text{NS}}}{k_i} - k_{1,1} + 2k_{\text{tr},1} (k_t - 2k_{\text{NS}}) \\
(307)
\end{align*}
\]

where \( C_{1,1}, C_{1,11}, C_{1,12}, C_{1,22}, C_{1,33} \) are the integration constants.

In order to check if an invariant solution of the first order exists, the general mean velocity (303) and the MPCs (304) and (305) are inserted into the averaged Navier-Stokes equations (295) which leads to formulas for the pressure derivatives \( \partial \bar{P} / \partial x_1 \) and \( \partial \bar{P} / \partial x_2 \) (not shown here). Next, the condition that the derivative of \( \bar{P} \) with respect to \( x_1 \) should not depend on \( x_2 \) leads to the following relation between the integration constants

\[
C_{1,12} = \nu C_{1,1} k_{\text{NS}} \left( \frac{k_{\text{NS},n}}{k_{\text{NS},m}} - 1 \right). 
\]

(308)

With this, the arising pressure is

\[
\bar{P} = C_{1,22} \left[ k_{G,2} + k_{\text{NS}} x_2 \right] \frac{k_{\text{NS}}}{k_i} - k_{1,2} + k_{G,2} + x_2 (k_t - 2k_{\text{NS}}) \\
- k_{2,2} (k_t - 2k_{\text{NS}} + 6k_t^2) \frac{x_1 + C_{1,1} + 2k_{G,2} + x_2 (k_t - 2k_{\text{NS}}) + k_{G,2} - 2k_{\text{NS}}}{k_{G,2} - 5k_{\text{NS}} + 6k_{\text{NS}}}.
\]

**Navier-Stokes - exponential solution:** If scaling of the Navier-Stokes equations is excluded, \( k_{\text{NS}} = 0 \), but \( k_t \neq 0 \) and \( k_{G,2} \neq 0 \), the characteristic system (302) does not provide an algebraic solution any more. Instead, we obtain

\[
\bar{p}^{(j)}_i = r_i^{(j)}.
\]
which further leads to an exponential solution for the averaged velocity
\[ \overline{U}_1(x_2) = C_{I,1} e^{\frac{k_{r,1} + k_{c,1}}{k_s}} \] (309)

and the Reynolds-stress components
\[ \dot{R}_{12} = C_{I,12} e^{\frac{k_{r,1} + k_{c,1}}{k_s} x_2} - \frac{k_{r,1} + k_{c,1}}{k_s} \] (310)
\[ \dot{R}_{22} = C_{I,22} e^{\frac{k_{r,1} + k_{c,1}}{k_s} x_2} - \frac{k_{r,1} + k_{c,1}}{k_s} \] (311)
\[ \dot{R}_{33} = C_{I,33} e^{\frac{k_{r,1} + k_{c,1}}{k_s} x_2} - \frac{k_{r,1} + k_{c,1}}{k_s} \] (312)
\[ \dot{R}_{11} = -C_{I,1} e^{\frac{k_{r,1} + k_{c,1}}{k_s} x_2} + e^{\frac{k_{r,1} + k_{c,1}}{k_s} x_2} \left( C_{I,11} + 2C_{I,1} x_2 \frac{2k_{r,1} + k_{c,1}}{k_{G,2}} \right) + \frac{3k_{r,1} + 4k_{r,1}k_{c,1} + k_{c,1}^2}{k_s^2} \left( 1 - \frac{k_{r,1}}{k_s} - 2\frac{k_{r,1}}{k_s} \right). \] (313)

To check if the solutions above define an invariant solution of the first order we have to implement these terms into the averaged Navier-Stokes equations (295) and use the condition that \( \overline{P} \) can be only linear in \( x_1 \). This leads to the following relation between the integration constants
\[ C_{I,12} = \nu C_{I,1} \frac{k_{r,1}}{k_{G,2}}, \] (314)
and the following algebraic relation for the pressure
\[ \overline{P} = -C_{I,22} e^{\frac{k_{r,1} + k_{c,1}}{k_s} x_1} + 2\frac{k_{r,1}}{k_s} x_1 + C_{I,4} x_2 \] (315)

**Navier-Stokes - logarithmic solution:**

A logarithmic scaling law for the average velocity is observed if \( k_s = k_{NS} \) and \( k_s \neq 0 \). We present below the results for \( k_{G,2} = 0 \), although a more general solution with \( k_{G,2} \neq 0 \) can also be derived (see Rosteck (2013)).

\[ \overline{U}_1 = C_{I,1} + \frac{k_{r,1} + k_{c,1}}{k_{NS}} \ln |x_2|, \] (316)
\[ \dot{R}_{12} = \frac{C_{I,12}}{x_2} + \frac{k_{r,1} + k_{c,1}}{k_{NS}} \] (317)
\[ \dot{R}_{22} = \frac{C_{I,22}}{x_2} + \frac{k_{r,1} + k_{c,1}}{k_{NS}} + 2 \frac{k_{r,1}}{k_s} x_2, \] (318)
\[ \dot{R}_{33} = \frac{C_{I,33}}{x_2} + \frac{k_{r,1} + k_{c,1}}{k_{NS}} + 2 \frac{k_{r,1}}{k_s} x_2, \] (319)
\[ \dot{R}_{11} = -C_{I,1} + C_{I,11} x_2 + k_{r,1} x_2 \frac{k_{r,1} + k_{c,1}}{k_s} - \frac{4k_{r,1} + 6k_{r,1}k_{c,1} + k_{c,1}^2}{k_{NS}^2} \ln |x_2| \] (320)
\[ - \ln |x_2| - \frac{k_{r,1} x_2 - 6k_{r,1} x_2 - 2k_{r,1}^2 C_{I,11} k_{NS} - 2k_{c,1} C_{I,11} k_{NS}}{k_{NS}} \]
\[ + \frac{k_{r,1} x_2 - 6k_{r,1} x_2 - 2k_{r,1}^2 C_{I,11} k_{NS} + 2k_{c,1} C_{I,11}}{k_{NS}}. \]

Again we have to validate that these solutions fulfill the averaged Navier-Stokes equations (295). The derivative \( \partial \overline{P} / \partial x_1 \) has to be independent of \( x_2 \) which can be obtained for
\[ C_{I,12} = \nu C_{I,1} \frac{k_{r,1}}{k_{NS}}, \] (321)

Then the averaged pressure is
\[ P^* = -\frac{k_{c,2}}{2} x_1 - k_{222} x_2 - \frac{C_{I,22}}{x_2} + C_{I,1} P. \]
6.1.1. Poiseuille Flow The first application of the previously derived solutions shall be the well-known channel flow, see Figure 2. We are especially interested in the near-wall region, where our logarithmic solution can be applied. Further, the center region will be studied where an algebraic law can fit the data very well.

For this we assume a constant mean pressure gradient $\partial P/\partial x_1$. The mean flow shall only depend on the wall-normal direction $x_2$. Further, two parallel plane walls shall be given, described through $x_2 = 0$ and $x_2 = 2h$. To be consistent with classical notation we introduce the non-dimensional velocity $U_i/\bar{U}_i$, pressure $\bar{P}/\bar{U}_i^2$ and wall-normal coordinate $x_2^* = x_2 u_\tau/\nu$, where $u_\tau$ is the friction velocity defined as $u_\tau = \sqrt{\tau_w/\rho}$ and

$$\tau_w = \mu \left. \frac{\partial \bar{U}_1}{\partial x_2} \right|_{x_2=0}$$

is the shear stress at the wall. With this, the averaged Navier-Stokes equation reads

$$\frac{\partial R^*_2}{\partial x_2^*} [r^{(1)} \rightarrow 0] + \frac{\partial \bar{P}^*}{\partial x_1^*} + \frac{\partial \bar{U}_1^*}{\partial x_2^*} = 0 .$$

In the classical universal law-of-the-wall it is assumed that very close to the wall the turbulent stress $\bar{R}_2$ is negligible and the mean velocity can be described by the linear function $\bar{U}_1^* = x_2^*/\kappa$. Further away from the wall, another characteristic behaviour of the averaged velocity can be observed, i.e. the so-called logarithmic boundary layer, cf. Pope (2000)

$$\bar{U}_1^* = \frac{1}{\kappa} \ln x_2^* + B .$$

The constant $\kappa$ is called Kármán’s constant. There is a great amount of data for this type of flow employed to determine the values for $\kappa$ and $B$. So far, it seems that $\kappa \approx 0.4$ and $B \approx 5.1$ can fit most of the data.

For the center of a turbulent channel flow no generally accepted law can be found in the literature. One of the oldest approximations is the empirical algebraic law

$$\frac{U_0 - \bar{U}(x_2)}{U_r} = 5.08 \left(1 - \frac{x_2}{h}\right)^{3/2}$$

derived by Darcy (1858), where this formula is written as a deficit law. $U_0$ is the averaged velocity in the center of the channel. Considerably later von Kármán (1930), proposed an equation combining an algebraic law with a logarithm:

$$\frac{U_0 - \bar{U}(x_2)}{U_r} = -\frac{1}{\kappa} \left(1 - \frac{x_2}{h}\right)^{1/2} + \ln \left(1 - \left(1 - \frac{x_2}{h}\right)^{1/2}\right) .$$

This equation can be written in a more general form

$$\frac{U_0 - \bar{U}(x_2)}{U_r} = -\frac{1}{\kappa} \left(1 - \frac{x_2}{h}\right)^{1/2} + C \ln \left(\frac{C - \left(1 - \frac{x_2}{h}\right)^{1/2}}{C}\right)$$

while $C$ is constant of order one (see Hunt (1954)).

In the following we will show that an algebraic law, derived from Lie group analysis provides a very good fit to the core region data, while the classical logarithmic law is also a Lie symmetry induced solution of the MPCE.

The scaling laws we calculated in the last subsection using Lie symmetries will be compared to DNS data of Jiménez and Hoyas (2008) and Hoyas and Jiménez (2008) at a Reynolds number of $Re= 2003$.

**Logarithmic layer**
In order to simplify the appearing complex coefficients we introduce new constants allowing us to write the scaling laws in a compact form. Considering the averaged velocity (316), the prefactor and the constant will be replaced by
\[
\kappa = \frac{k_{NS}}{k_{r,1} + k_{s}}, \quad B = C_{I,1}.
\]
The scaling laws of the Reynolds stress tensor (317–320) will be further simplified by defining
\[
\alpha_{ij} = \frac{k_{ij}}{k_{NS}} \quad \text{for } ij = 12, 22, 33, \quad \alpha_{11} = -C_{I,1}^2 + \frac{k_{ij} k_{NS} - 4k_{r,1}^2 - 6k_{r,1} k_{s} - 2k_{s}^2 + 4k_{r,1} C_{I,1} k_{NS} + 2k_{s} C_{I,1}}{k_{NS}^2}.
\]
and the linear terms can be simplified employing
\[
\beta_{12} = \frac{k_{12}}{2}, \quad \beta_{22} = k_{32}, \quad \beta_{11} = \frac{k_{11}}{k_{s}}, \quad \beta_{33} = k_{33}.
\]
The factor \( \kappa \) and the coefficient of \( \ln(x_2^1) \) in the formula (320) for \( \bar{R}_{11} \) are linear independent so that there is an independent parameter \( \gamma \)
\[
\gamma = \frac{4k_{r,1}^2 + 6k_{r,1} k_{s} + 2k_{s}^2 - 2k_{r,1} C_{I,1} k_{NS} - 2k_{s} C_{I,1} k_{NS}}{k_{NS}^2}.
\]
With this, the averaged velocity (316) simplifies to
\[
\bar{U}_1^x = \frac{1}{k} \ln \left| x_2^x \right| + B \tag{323}
\]
and the Reynolds stress tensor (317–320) can be represented through
\[
\bar{R}_{ij}^* = \frac{C_{I,1}}{x_2^x} + \alpha_{ij} + \beta_{ij} x_2^x \quad \text{for } ij = 12, 22, 33, \quad \bar{R}_{11}^* = \frac{C_{I,1}}{x_2^x} + \alpha_{11} + \beta_{11} x_2^x + \frac{1}{k^2} \ln^2 \left| x_2^x \right| - \gamma \ln \left| x_2^x \right|. \tag{324}
\]
Moreover, due to the condition (321) a formula for the parameter \( C_{I,12} \) reads
\[
C_{I,12} = \frac{1}{k}. \tag{325}
\]
As the formulas for the Reynolds-stresses (324) are written in non-dimensional form, the viscosity \( \nu \) is not present in the above equation.

As mentioned above, (323) and (324) will be validated against the data of Jiménez and Hoyas (2008); Hoyas and Jiménez (2008). In Figure 3 the result of the fitting in the range \( 55 \leq x_2^x \leq 325 \) is considered. A convincing fit for the averaged velocity can be found for the parameters \( \kappa = 0.405 \) and \( B = 5.07 \). In Figure 3a one can recognise the linear region on the very left, then the Buffer layer and on the very right the log region is observed. In the logarithmic region we also plot the Reynolds stress tensor which can be found in the right hand of Figure 3 and in Figure 4. We are able to show that our predictions (324) can describe the flow in this region very well. The fit of all parameters can be found in table 3. It should be mentioned that the parameter \( C_{I,12} \) is determined by equation (325) with \( \kappa \approx 0, 405 \). Moreover, in the 11-component, the leading order term in formula (324) is the logarithm squared with the factor \( 1/k^2 \). It is important to note that exactly this term is fixed through the parameter \( \kappa \) determined by the mean velocity. If \( \kappa \) would be smaller, we would not be able to fit the 11-component with the DNS data.

Centre region:

Presently it is our aim to show that the mean velocity in the center region of a channel flow is nicely mimiced by an algebraic law. The parameters of the averaged velocity (303) for \( k_{G,2} = 0 \) are abbreviated as follows
\[
\gamma = \frac{k_{s} - k_{NS}}{k_{NS}}, \quad B = -\frac{k_{r,1} + k_{s}}{k_{s} - k_{NS}}.
\]
The constants in the Reynolds stresses (304–307) for \( k_{G,2} = 0 \) may be summarised to
\[
\alpha_{ij} = -\frac{k_{ij}}{k_{s} - 2k_{NS}} \quad \text{for } ij = 12, 22, 33,
\]
Fig. 3 The DNS data of Jiménez and Hoyas (2008); Hoyas and Jiménez (2008) are compared with the derived scaling laws. Here, the velocity (♦) and the components $\tilde{R}_{11}^+$ (□), $\tilde{R}_{22}^+$ (△) and $\tilde{R}_{33}^+$ (+) of the DNS data are presented. The solid lines indicate the corresponding fits.

Table 3 List of parameters in formulas (324) fitted to the DNS data of Jiménez and Hoyas (2008); Hoyas and Jiménez (2008).

<table>
<thead>
<tr>
<th>$\tilde{R}_{ij}$</th>
<th>$a_{ij}$</th>
<th>$b_{ij}$</th>
<th>$c_{ij}$</th>
<th>$\gamma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{R}_{11}$</td>
<td>954</td>
<td>-163</td>
<td>0.035</td>
<td>61.66</td>
</tr>
<tr>
<td>$\tilde{R}_{22}$</td>
<td>-12.14</td>
<td>1.42</td>
<td>-0.0008</td>
<td></td>
</tr>
<tr>
<td>$\tilde{R}_{33}$</td>
<td>23.64</td>
<td>2.13</td>
<td>-0.0017</td>
<td></td>
</tr>
<tr>
<td>$\tilde{R}_{12}$</td>
<td>1/κ</td>
<td>-1.00</td>
<td>0.0005</td>
<td></td>
</tr>
</tbody>
</table>

Fig. 4 The Reynolds stress component $\tilde{R}_{12}^+$ from the data of Jiménez and Hoyas (2008); Hoyas and Jiménez (2008) is fitted through the scaling laws for $\tilde{R}_{12}^+$ (solid line)
\[ \alpha_{11} = \frac{k_1^2 k_s}{(k_s - 2k_{NS})(k_s - k_{NS})^2} + \frac{k_2^2(3k_s - 2k_{NS})}{(k_s - 2k_{NS})(k_s - k_{NS})^2} - \frac{k_{ij}}{(k_s - 2k_{NS})} + \frac{2k_{R1}k_{C1}(2k_s - k_{NS})}{(k_s - 2k_{NS})(k_s - k_{NS})^2}, \]

and the terms linear in \( x_2 \) can be simplified using the definitions

\[ \beta_{ij} = -2k_{ij} \frac{k_s - 2k_{NS}}{k_s^2 - 5k_s k_{NS} + 6k_s^2} \quad \text{for } ij = 11, 22, 33, \quad \beta_{12} = -k_{12} \frac{k_s - 2k_{NS}}{k_s^2 - 5k_s k_{NS} + 6k_s^2}. \]

Finally, the coefficient of an additional term in the 11-component (307) can be replaced by

\[ \sigma = -\frac{2k_{11}k_s C_{l1}}{k_{NS}(k_s - k_{NS})} - \frac{2k_{R1}(2k_s - k_{NS}) C_{l1}}{k_{NS}(k_s - k_{NS})}. \]

Inserting these definitions into the algebraic scaling laws, the averaged velocity (303) becomes

\[ U^+_1 = C_{l1} \left| \frac{x_2}{h} \right|^{\gamma} + \hat{B}, \]

As it is common to rewrite the centre region using a deficit law, we obtain

\[ \overline{U}^{def+}_1 = U^{+\ast}_1 - \overline{U}_1 \left( \frac{x_2}{h} \right) = -C_{l1} \left| \frac{x_2}{h} \right|^\gamma + B, \] (326)

where \( U^{+\ast}_1 \) represents the normalized velocity of the centreline. Then, the quantity \( B = \overline{U}_1 - \hat{B} \) is introduced. The components of Reynolds stress tensor (304–306) can be simplified accordingly

\[ \tilde{R}^+_i = C_{l1i} \left( \frac{x_2}{h} \right)^{\gamma - 1} + \alpha_{ij} \frac{x_2}{h} + \beta_{ij} \frac{x^2}{h} \] (327)

for \( ij = 12, 22, 33 \). Finally, the 11-component is considered, where the long expression (307) may be reduced to

\[ \tilde{R}^+_{11} = -C_{l11} \left| \frac{x_2}{h} \right|^{\gamma} + C_{l11} \left| \frac{x_2}{h} \right|^{\gamma - 1} + C_{l11} \left| \frac{x_2}{h} \right|^\gamma + \alpha_{11} + \beta_{11} \frac{x_2}{h}, \] (328)

Moreover, the condition (308) written for the non-dimensionalised Reynolds-stress tensor components (327) leads to the formula

\[ C_{l12} = \frac{C_{l12}}{Re}. \] (329)

![Fig. 5](https://example.com/fig5.png) The velocity \( \Phi \), the components \( \tilde{R}^+_1(\square), \tilde{R}^+_1(\circ), \tilde{R}^+_2(\Delta) \) and \( \tilde{R}^+_3(\ast) \) are compared with fits (solid lines) of the derived scaling laws.
The deduced scaling laws are fitted and compared to the data of Jiménez and Hoyas (2008); Hoyas and Jiménez (2008), see Figure 5. The algebraic law for the mean velocity mimics approximately 60 % of the half-channel, see Figure 5, where parameters have been approximated to $C_{1,1} = -6.1, B = -0.0090$ and $\gamma = 1.87$. Further, we may also compare the components of the Reynolds stress tensor to the scaling laws (327) and (328). The condition (329) leads to the coefficient $C_{1,12} = 0.0057$ of the 12 component. In figure 5 all components are plotted, representing a very good agreement to the DNS data. The parameters of the graphs in figure 5 are summarized in table 4. The leading order term of $\tilde{R}_{11}$ in estimation (328) is the term containing $C_{1,1}^2$, which is fixed through the parameter of the averaged velocity. Also $C_{1,2}$ is given by the previous fits, nevertheless, a very good agreement with the data can be observed.

<table>
<thead>
<tr>
<th>Table 4</th>
<th>List of parameters in formulas (327) and (328) fitted to the data of Jiménez and Hoyas (2008); Hoyas and Jiménez (2008).</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_{ij}$</td>
<td>$\beta_{ij}$</td>
</tr>
<tr>
<td>$R_{11}$</td>
<td>0.88</td>
</tr>
<tr>
<td>$R_{12}$</td>
<td>0</td>
</tr>
<tr>
<td>$R_{22}$</td>
<td>-3.01</td>
</tr>
<tr>
<td>$R_{33}$</td>
<td>-4.56</td>
</tr>
</tbody>
</table>

6.1.2. Boundary Layer | Let us consider a fundamental boundary layer flow, with the wall located at $x_2 = 0$. Classically, the outer part is approximated by a defect law

$$\frac{U_\infty - \bar{U}_{1}(x_2)}{U_\tau} = f\left(\frac{x_2}{\delta}\right),$$

where $U_\infty$ is the outer velocity and $\delta$ is the boundary layer thickness. As in the channel flow, a logarithmic sublayer can also be observed in the near-wall region. A more elaborate description of the flow behaviour was suggested by Coles (1956) who proposed a scaling law for the whole boundary layer,

$$\frac{\bar{U}(x_2)}{U_\tau} = f\left(\frac{x_2}{\nu}\right) + C\omega\left(\frac{x_2}{\delta}\right),$$

(see Coles (1956)). Coles determined the function $\omega$ through experimental data and approximated it by

$$\omega = 1 + \sin\left(\frac{2\pi}{\delta} - 1\right).$$

In the following, we intend to describe statistical boundary layer data using the scaling laws derived from the Lie symmetry theory. Hereby, two regions in the boundary layer shall be distinguished.

The traditional theory suggests a logarithmic law in the near-wall region of a boundary layer flow. However, in George (2006) alternatively an algebraic law is suggested. These two approaches will be compared applying the symmetry induced scaling laws. Thereby, additionally to the mean velocity the Reynolds stress tensor will be considered.

The second analysis on the wake region is inspired by the assumption in Oberlack (2001), stating an exponential law in this region. We will compare it to an algebraic law, which fits the data in this region also very well.

Following classical boundary layer theory some special characteristic scales and Reynolds numbers shall be defined. Beside the friction velocity defined in (322) we have a viscous length scale $l_\tau = \nu/u_\tau$, and the outer velocity $U_\infty = U(x_2 \rightarrow$
shall be introduced. In order to describe the thickness of the boundary layer, various definitions are known. Presently we adopt the momentum thickness 
\[
\theta = \int_0^\infty \frac{U(x)}{U_\infty} \left( 1 - \frac{U(x)}{U_\infty} \right) dx 
\]
leading to the Reynolds number 
\[
Re_\theta = \frac{\theta U_\infty}{v},
\]
which will be used to describe the following turbulent flows.

**Logarithmic layer**

In the logarithmic region, the characteristic velocity and length scale are given by \( u_* \) and \( l^* \), so that we adopt the classical definitions of the plus-variables given in Section 6.1.1. The scaling laws in the logarithmic subregion will be compared to the data of Simens et al. (2009), Jiménez et al. (2010), who created their DNS data at \( Re_\theta = 1968 \).

![Fig. 7 The velocity \( \bar{U}_1 \) is compared with fits (solid lines) of the derived scaling laws. The logarithmic fit can be found on the left-hand and the algebraic on the right-hand side.](image)

Both logarithmic and an algebraic law will be fitted to the DNS data, while both approaches were already simplified in the previous subsection. The logarithmic scaling law (323)-(325) was given by

\[
\bar{U}_1 = \frac{1}{k} \ln(x_2^+) + B,
\]
\[
\tilde{R}_{ij}^* = \frac{C_{ij}}{x_2^{\gamma_i}} + \alpha_{ij} + \beta_{ij} x_2 \quad \text{for } i,j = 12, 22, 33, \quad \tilde{R}_{11}^* = \frac{C_{11}}{x_2^{\gamma_1}} + \alpha_{11} + \beta_{11} x_2 - \frac{1}{k^2} \ln^2(x_2^+) - \gamma \ln(x_2^+) \quad (330)
\]

while the condition

\[
C_{1,12} = \frac{1}{k} \quad (331)
\]

has to be considered for the 12-component. Let us also recall the algebraic scaling law, (326)-(329),

\[
U_1^* (x_2) = C_{i,1} x_2^{\gamma_i} + B,
\]
\[
\tilde{R}_{ij}^* = C_{ij} x_2^{\gamma_i} + \alpha_{ij} + \beta_{ij} x_2 \quad i,j = 12, 22, 33, \quad \tilde{R}_{11}^* = -C_{1,1} x_2^{2\gamma} + C_{1,11} x_2^{\gamma_1 - 1} + \sigma x_2^{\gamma} + \alpha_{11} + \beta_{11} x_2 \quad (332)
\]

extended by the condition

\[
C_{i,12} = C_{i,1} \gamma. \quad (333)
\]
Hence, these scaling laws shall be compared to the data of Simens et al. (2009); Jiménez et al. (2010), while all fits are generated in the region $35 \leq x_2^+ \leq 100$, in order to have the possibility to assess the results of both approaches.

The mean velocity is fitted in Figure 7, where the logarithmic fit can be found on the left-hand and the algebraic on the right-hand side. The parameters for the logarithmic approach are $\kappa = 0.418$, $B = 5.31$ and for the algebraic approach $C_{1,1} = 11.07$, $\gamma = 0.128$, $B = -3.38$. Both approximate the data equally well.

Similarly, a very good agreement was obtained for the Reynolds stresses for both, logarithmic (330) and algebraic (333) cases. To save space we will only present results for the logarithmic case, the interested reader is referred to Rosteck (2013) for more data on the algebraic case. Due to condition (331) the parameter $C_{1,12} = 0.00122$ is fixed. Then the remaining parameters are derived by fitting the scaling laws to the data, see table 5.

Comparing parameters it is noticed that $\beta_{ij}$ for $\tilde{R}_{11}^{+}$ is much higher than for the other Reynolds stress components. This parameter represents the Lie-point symmetry (298) adding a linear term in $x_2$ to the Reynolds stress component. In $\tilde{R}_{12}^{+}$ and $\tilde{R}_{13}^{+}$, the corresponding symmetries can be neglected and a good agreement with the data can be observed. Especially, the parameter for $\tilde{R}_{12}^{+}$ is extremely small $\beta_{12} = 7 \cdot 10^{-6}$, which will be explained in the following.

Since the data are created for a zero-pressure gradient (ZPG), we may assume $\partial P / \partial x_1 = 0$, so that the mean pressure does not change in streamwise direction. Inserting this assumption into the averaged Navier-Stokes equations (295) and substituting the scaling laws (330) this implies that $\beta_{12} = 0$. Hence, a ZPG assumption leads directly to $\beta_{12} = 0$. The graphs of the fits are given in Figure 8. The dashed line for the component $\tilde{R}_{12}^{+}$ describes the fit if $\beta_{12} = 0$ is claimed. This assumption leads to a slightly worse fit as can be seen in the figure.

Deficit region

Presently we consider the region between the near-wall log-region and the outer constant flow. The dimensionless velocity $\tilde{U}_1 = \bar{U}_1 / \bar{u}_r$ is defined as before, while the characteristic length scale in the deficit region is $\delta_{99}$, describing the distance from the wall, where 99% of the outer flow velocity is reached. Hence, the dimensionless length is $x_2 / \delta_{99}$. Introducing the corresponding Reynolds number we have

$$Re = \frac{\bar{U}_1 \delta_{99}}{\nu}. \quad (334)$$
As shown in Oberlack (2001) an exponential scaling law can describe the mean velocity in the deficit region. Here, we intend to verify if this also holds true for the Reynolds stress components. The exponential scaling law for the averaged velocity (309) and the Reynolds stress tensor (310–313) may be written in a more elegant form, by introducing the parameters

\[ \gamma = \frac{k_s}{k_{G,2}}, \quad \tilde{B} = -\frac{k_{r,1} + k_{13}}{k_s}. \]  

(335)

For the Reynolds stress tensor (310–313) we additionally have

\[ \alpha_{ij} = \frac{k_{22}k_s + 2k_{22}k_{G,2}}{k_s^2} \quad \text{for } i = 22, 33, \quad \alpha_{12} = \frac{k_{12}k_s + k_{13}k_{G,2}}{k_s^2} \]  

\[ \alpha_{11} = \frac{3k_{22}^2 + 4k_{r,1}k_{22} + k_{22}^2}{k_s^2} - \frac{k_{11} - 2k_{11}k_{G,2}}{k_s}. \]  

and

\[ \beta_{ij} = -2\frac{k_{r,ij}}{k_s} \quad \text{for } i = 11, 22, 33, \quad \beta_{12} = -\frac{k_{12}x_2}{k_s}, \quad \sigma = 2C_{i,1} \frac{2k_{r,1} + k_{13}}{k_{G,2}}. \]  

(337)

It can be shown that all above parameters are independent, so that the scaling laws can be transferred into a more compact representation. With this the averaged velocity (309) reads

\[ U_1^+ = C_{i,1}e^{\gamma x_2/\delta y} + B. \]  

(338)

As the data are presented in the deficit law formulation, it holds

\[ U_1^{+ \text{erf}} = U_1^{+ \infty} - \frac{x_2}{2} \left( \frac{\partial x_2}{\partial y_9} \right) = -C_{i,1}e^{\gamma x_2/\delta y} + B, \]  

(339)

where \( B = -\tilde{B} + U_1^{+ \infty} \). Further, the Reynolds stress tensor (310–313) reduces to

\[ \tilde{R}_{ij} = C_{i,1}e^{\gamma x_2/\delta y} + \alpha_{ij} + \beta_{ij} \left( \frac{x_2}{\delta y_9} \right) \quad \text{for } i = 12, 22, 33, \]  

\[ \tilde{R}_{11} = -C_{i,1}e^{2\gamma x_2/\delta y_{10}} + C_{i,1}e^{\gamma x_2/\delta y} + \sigma \left( \frac{x_2}{\delta y_9} \right) e^{\gamma x_2/\delta y} + \alpha_{11} + \beta_{11} \left( \frac{x_2}{\delta y_9} \right). \]  

(340)

From the additional condition (314) we obtain \( C_{i,12} = C_{i,1}/Re \).

The previously derived scaling laws shall be compared to the DNS data of Schlatter and Örlü (2010). The comparison of DNS and theory for the mean velocity in equation (339) can be found in Figure 9, where the deficit law is presented. In Figure 10, DNS data and calculated scaling laws of the Reynolds stress tensor (340) are illustrated. All data are fitted in the region \( 0.3 \leq x_2/\delta y_9 \leq 0.8 \), where both mean velocity and Reynolds stress tensor are fitted very well. Noticeable it is that \( \tilde{R}_{12} \) is the only component which does not provide an excellent fit. This is in contrast to the case of the channel flow in the last subsection, where also \( \tilde{R}_{12} \) was fitted very well.

The parameter fits of the DNS data to (339) and (340) are given in table 6.

We notice that ratios between various parameters are almost constant for all Reynolds numbers. More precisely, we observe

\[ \frac{\alpha}{C_{i,1}} \approx 0.33, \quad \frac{\alpha_{22}}{\beta_{22}} \approx -1.3, \quad \frac{\alpha_{33}}{\beta_{33}} \approx -1.2. \]  

(341)

For Reynolds numbers high enough there also holds

\[ \frac{\alpha_{22}}{C_{i,1}} \approx -1.5, \quad \frac{\alpha_{33}}{C_{i,1}} \approx -1.5. \]  

(342)

If we use these conditions as assumptions of the scaling laws less parameters have to be fitted. Since the ratios seem to be constant we can suppose that the suggested scaling laws present a good description for the deficit region.

Apart from the exponential approach, we want to compare the DNS data of Schlatter and Örlü (2010) to an algebraic description in the deficit region. In order to gain simplified scaling laws the parameters in the formula of the averaged velocity (303) are replaced by

\[ C_{i,1} = -C_{i,1}k_{NS}^{-1}, \quad A = k_{G,2}/k_{NS}, \quad \gamma = k_s/k_{NS} - 1, \quad \tilde{\alpha}_{i,1} = -k_{i,1} + k_{13}/k_s - k_{NS}, \quad \tilde{\alpha}_{i,1} = U_1^{+ \infty} - \tilde{\alpha}_{i,1}. \]  

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With this we substitute the constant part of (304–307) by

\[ \alpha_{11} = \frac{k_{11}^2 k_x + k_{11} k_x (3 k_x - 2 k_{NS}) + 2 k_{11} k_{11} (2 k_x - k_{NS})}{(k_x - 2 k_{NS}) (k_x - k_{NS})^2}, \]

\[ \alpha_{12} = -\frac{k_{12}}{k_x - 2 k_{NS}} - \frac{k_{12} k_{22}}{k_x^2 - 2 k_{NS} k_{NS} + 6 k_{NS}^2}, \]

\[ \alpha_{ij} = -\frac{k_{ij}}{k_x - 2 k_{NS}} - \frac{2 k_{ij} k_{22}}{k_x^2 - 5 k_{NS} k_{NS} + 6 k_{NS}^2} \] for \( i, j = 22, 33 \).

The linear parts are simplified using

\[ \beta_{12} = -\frac{k_{12}}{k_x - 3 k_{NS}}, \beta_{ij} = -\frac{2 k_{ij}}{k_x - 3 k_{NS}} \] for \( i, j = 11, 22, 33 \).

For the integration factor we use

\[ C_{ij} = C_{ij} \frac{k_{NS}^2}{k_{NS}} \] for \( i, j = 12, 11, 22, 33 \)

and the additional term in \( \bar{R}_{11} \) is abbreviated as follows

\[ \sigma = -\frac{2 k_{11} k_{11} C_{11} + 2 k_{11} (2 k_x - k_{NS}) C_{11}}{k_{NS} (k_x - k_{NS})} \frac{k_{NS}}{k_{NS}^2} \] for \( i, j = 12 \).

The exponential result above shall be compared to an algebraic scaling law, the basic formulation of which is given by

\[ \bar{U}_{1}^{def+} = C_{\bar{U}} \left( \frac{x_2}{\delta_{yy}} + A \right)^{\gamma} + C_{\bar{U}} \] (343)

\[ \bar{R}_{ij} = C_{ij} \left( \frac{x_2}{\delta_{yy}} + A \right)^{\gamma} + \alpha_{ij} + \beta_{ij} x_2 \] (344)

\[ \bar{R}_{11} = -C_{\bar{R}} (\frac{x_2}{\delta_{yy}} + A)^{2\gamma} + C_{\bar{R}} \left( \frac{x_2}{\delta_{yy}} + A \right)^{\gamma} + \sigma \left( \frac{x_2}{\delta_{yy}} + A \right)^{\gamma} + \alpha_{11} + \beta_{11} \frac{x_2}{\delta_{yy}} \] (345)

extended by the condition \( C_{11,12} = -C_{\bar{R}} \gamma / Re \).

The fitting region \( 0.2 \leq x_2 / \delta_{yy} \leq 0.7 \) is translated by 0, 1 compared to the exponential description. The reason is simple, as we have chosen for both approaches the optimal region to apply the corresponding fit. In Figure 9 the fit of the mean velocity of both approaches is compared. We notice that both exponential (339) and the algebraic description (343) show a good agreement to the DNS data.
Fig. 10 DNS data of Schlatter and Örlü (2010) for the Reynolds stress tensor are presented at different Reynolds numbers: $Re_\theta = 760$ (■), $Re_\theta = 1410$ (▲), $Re_\theta = 2540$ (□), $Re_\theta = 4060$ (△). The corresponding fits of the exponential approach in the region 0, 3-0, 8 given by the formulas (340) are represented through solid lines.

Table 6 List of parameters in formulas (340) fitted to the data of Schlatter and Örlü (2010).

<table>
<thead>
<tr>
<th>Re</th>
<th>760</th>
<th>1410</th>
<th>2540</th>
<th>3270</th>
<th>4060</th>
</tr>
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<tr>
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</tr>
<tr>
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<td>331.2</td>
<td>337.3</td>
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<tr>
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<td>-147.1</td>
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<tr>
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<td>-107.7</td>
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<tr>
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<td>-1.10</td>
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<tr>
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<td>-3.49</td>
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<td>524.0</td>
<td>591.9</td>
<td>609.2</td>
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<tr>
<td>$\alpha_{12}$</td>
<td>-2.24</td>
<td>-2.44</td>
<td>-2.44</td>
<td>-2.44</td>
<td>-2.44</td>
</tr>
<tr>
<td>$\sigma_{13}$</td>
<td>-3.54</td>
<td>-1.95</td>
<td>-2.80</td>
<td>-2.74</td>
<td>-3.00</td>
</tr>
<tr>
<td>$\beta_{33}$</td>
<td>-1.24</td>
<td>-1.24</td>
<td>-1.24</td>
<td>-1.23</td>
<td>-1.24</td>
</tr>
<tr>
<td>$\gamma_{33}$</td>
<td>1.24</td>
<td>1.30</td>
<td>1.18</td>
<td>1.16</td>
<td>1.17</td>
</tr>
<tr>
<td>$\beta_{23}$</td>
<td>1.88</td>
<td>3.39</td>
<td>4.24</td>
<td>4.19</td>
<td>4.45</td>
</tr>
<tr>
<td>$\sigma_{33}$</td>
<td>-0.54</td>
<td>-1.95</td>
<td>-2.80</td>
<td>-2.74</td>
<td>-3.00</td>
</tr>
<tr>
<td>$\alpha_{33}$</td>
<td>3.25</td>
<td>6.23</td>
<td>7.36</td>
<td>6.74</td>
<td>7.02</td>
</tr>
<tr>
<td>$\sigma_{33}$</td>
<td>-3.34</td>
<td>-5.09</td>
<td>-5.80</td>
<td>-5.42</td>
<td>-5.62</td>
</tr>
<tr>
<td>$\beta_{33}$</td>
<td>-1.15</td>
<td>-4.21</td>
<td>-5.44</td>
<td>-4.74</td>
<td>-5.06</td>
</tr>
<tr>
<td>$\gamma_{33}$</td>
<td>-1.15</td>
<td>-4.21</td>
<td>-5.44</td>
<td>-4.74</td>
<td>-5.06</td>
</tr>
</tbody>
</table>

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In Figure 11 the Reynolds stress tensor and the corresponding fits of the algebraic approach (344) and (345) are displayed. In the algebraic approach all Reynolds stress components can be described very well except for $R_{12}$, which shows small deviations.

The parameters of Figures 11 are given in table 7. Especially for $Re\geq 1410$, some ratios of parameters seem to be constant. Hence, we can assume

$$\frac{\alpha}{\beta} \approx 1.04, \quad \frac{\alpha_{22}}{\beta_{22}} \approx 1.23, \quad \frac{\alpha_{33}}{\beta_{33}} \approx 1.18.$$  

Some parameters appear to be almost independent of Reynolds number, such as the exponential $\gamma$ and the two parameters $\alpha_{12}$ and $\beta_{12}$ of the Reynolds stress $R_{12}$. Since both approaches provide good agreements with the DNS data, it is difficult to decide which description is more appropriate.

| Table 7 | List of parameters in formulas (344) fitted to the DNS data of Schlatter and Örlü (2010). |
|-----------------|---------------------------------|-----------------|-----------------|-----------------|
| Re  | 760 | 1410 | 2540 | 3270 | 4060 |
| $\alpha$ | 12.78 | 31.93 | 31.19 | 32.59 | 33.72 |
| $C$ | -12.97 | -30.57 | -30.03 | -31.39 | -32.33 |
| $\gamma$ | 0.42 | 0.23 | 0.24 | 0.23 | 0.24 |
| $\sigma$ | 0.042 | 0.26 | 0.22 | 0.022 | 0.024 |
| $\alpha_{11}$ | -54.22 | -699.8 | -592.0 | -650.0 | -720.0 |
| $\beta_{11}$ | 86.91 | 95.27 | 115.4 | 116.8 | 127.0 |
| $C_{11}$ | 8.51 | 29.25 | 23.28 | 22.42 | 30.35 |
| $\sigma$ | 130.7 | 1536 | 1382 | 1523 | 1639 |
| $\alpha_{12}$ | -1.16 | -1.16 | -1.16 | -1.15 | -1.16 |
| $\beta_{12}$ | 1.12 | 1.03 | 1.02 | 1.01 | 1.02 |
| $\alpha_{22}$ | -1.99 | -2.34 | -2.39 | -2.45 | -2.54 |
| $\beta_{22}$ | -0.39 | -0.64 | -0.60 | -0.62 | -0.73 |
| $\alpha_{33}$ | 3.17 | 4.21 | 4.19 | 4.39 | 4.45 |
| $\beta_{33}$ | -3.05 | -3.62 | -3.56 | -3.69 | -3.70 |
| $C_{33}$ | -0.40 | -0.92 | -0.85 | -0.93 | -0.99 |

### 6.2. Channel flow with constant wall-transpiration

In this subsection, the channel flow configuration is slightly changed since we assume a constant cross flow normal to the wall due to wall transpiration. Consequently, one convection term in the averaged Navier-Stokes equations remains and a different set of governing equations is formed. Therefore, the scaling symmetries, as stated in the previous sections can only be obtained in modified way. In the following, we will calculate different scaling laws and compare them to the recent data of Avsarkisov et al. (2014). Older DNS data for this flow are available from Sumitani and Kasagi (1995), who studied this case involving heat transfer. They investigated the averaged velocities and the Reynolds stress tensor and observed that the flow behaviour is very different compared to the simple channel flow. Other DNS simulations are done in Nikitin and Pavelev (1998) as well as Chung and Sung (2001); Chung et al. (2002).

As in all previous approaches the transpiration velocity $U_T$ is assumed to be constant which was validated in all DNS studies. The dependencies of the flow field and the pressure are given by

$$\overline{U}_1 = \overline{U}_1(x_2), \quad \overline{U}_2 = U_T, \quad \overline{U}_3 = 0, \quad \overline{P} = \overline{P}(x_2, x_1).$$

With these assumptions the set of governing equations is determined. The averaged Navier-Stokes equations in the fluctuation $R$ approach read

$$U^* \frac{\partial U^*}{\partial x_2} = \frac{\partial P^*}{\partial x_1} + \nu \frac{\partial^2}{\partial x_2^2} - \frac{\partial \overline{P}^*}{\partial x_2},$$

$$0 = - \frac{\partial P^*}{\partial x_2} - \frac{\partial \overline{U}^*}{\partial x_2},$$

and in the instantaneous $H$ approach we obtain

$$0 = - \frac{\partial \overline{P}^*}{\partial x_2} + \nu \frac{\partial^2}{\partial x_2^2} - \frac{\partial \overline{H}_{22}}{\partial x_2}$$

$$0 = - \frac{\partial \overline{P}^*}{\partial x_2} - \frac{\partial \overline{H}_{22}}{\partial x_2},$$

$$0 = - \frac{\partial \overline{H}_{22}}{\partial x_2}.\quad (348)$$
Fig. 11 DNS data of Schlatter and Örlü (2010) for the Reynolds stress tensor are presented for different Reynolds numbers: $Re_\theta = 760$ (■), $Re_\theta = 1410$ (▲), $Re_\theta = 2540$ (□), $Re_\theta = 4060$ (△). The corresponding fits of the algebraic approach (344) and (345) for the region 0,3-0,8 are represented through solid lines.
The MPC equations will be presented below only in the $\mathbf{H}$ approach. In the considered case they reduce to

$$
S_{ij(n)} = \frac{\partial H_{ij(n)}^{[m]}[r^{(i)} \rightarrow 0]}{\partial x_2} + \frac{\partial H_{ij(n)}^{[m]}[r^{(j)} \rightarrow 0]}{\partial x_2} - \frac{\partial^2 H_{ij(n)}^{[m]}[r^{(i)} \rightarrow 0]}{\partial x_2 \partial x_2} + \sum_{k=1}^{n-1} \left[ \frac{\partial H_{ij(n)}^{[m]}[r^{(i)} \rightarrow 0]}{\partial r_k^{(j)}} - \frac{\partial H_{ij(n)}^{[m]}[r^{(j)} \rightarrow 0]}{\partial r_k^{(j)}} \right] + \frac{\partial H_{ij(n)}^{[m]}[r^{(i)} \rightarrow 0]}{\partial x_2} \frac{\partial r_k^{(j)}}{\partial x_2} \right]
$$

(349)

Further, also the continuity equations (199), (200) have to be considered.

Due to the presence of the transpiration velocity $U_T$, some of the symmetries of the MPC equations may be broken, hence it is necessary to investigate the invariance of Eqs. (348–349) under transformations of variables. Transformations in the instantaneous ($\mathbf{H}$) approach can be transferred back to the fluctuations ($\mathbf{R}$) approach to finally calculate invariant solutions for the Reynolds stresses. The classical scaling symmetry, arising for the Navier-Stokes equations is

$$
Z_{NS} : \quad x^* = e^{k_\nu x} x , \quad r^{(j)*} = e^{k_\nu} r^{(j)} , \quad \bar{U}_1 = e^{-k_\nu} \bar{U}_1 , \quad \bar{P} = e^{-2k_\nu} \bar{P} , \quad H^*_{ij} = e^{-nu} H_{ij} , \quad \Gamma_{ij(n)}^{*p} = e^{-(nu+1)k_\nu} \Gamma_{ij(n)}^{*p} ,
$$

(350)

while the new statistical scaling symmetry in the $\mathbf{H}$ formulation reads

$$
Z_h : \quad x^* = x , \quad r^{(j)*} = r^{(j)} , \quad \bar{U}_1 = \bar{U}_1 + k_{21} , \quad \bar{P} = \bar{P} , \quad H^*_{ij} = e^{k_\nu} H_{ij} , \quad \Gamma_{ij(n)}^{*p} = e^{k_\nu} \Gamma_{ij(n)}^{*p} ,
$$

(351)

Moreover, due to the reduction of the general averaged Navier-Stokes equations and MPC equations to the above system (348), (349) additional symmetries can be verified. The first of them represents a translation of the mean velocity

$$
\bar{r}^* = \bar{r}^* + k_{21} x , \quad \bar{r}^{(j)*} = r^{(j)} , \quad \bar{U}_1 = \bar{U}_1 + k_{21} , \quad \bar{P} = \bar{P} , \quad R^*_{ij} = R_{ij} , \quad R^*_{ij(n)}^{*p} = R_{ij(n)}^{*p} .
$$

(352)

In addition, further statistical Lie-point symmetry exist which corresponds to linear terms in $x_2$, respectively $2x_2 + r_2$, that can be added to each two-point correlation. These transformations yield

$$
Z_{c11} : \quad x_1^* = x_1 , \quad r^{(j)*} = r^{(j)} , \quad \bar{P} = \bar{P} , \quad \Gamma_{ij(n)}^{*p} = \Gamma_{ij(n)}^{*p} ,
$$

(353)

$$
Z_{c12} : \quad x_1^* = x_1 , \quad r^{(j)*} = r^{(j)} \quad \bar{P} = \bar{P} , \quad \Gamma_{ij(n)}^{*p} = \Gamma_{ij(n)}^{*p} ,
$$

(353)

$$
Z_{c22} : \quad x_1^* = x_1 , \quad r^{(j)*} = r^{(j)} \quad \bar{P} = \bar{P} , \quad \Gamma_{ij(n)}^{*p} = \Gamma_{ij(n)}^{*p} ,
$$

(353)

$$
Z_{c33} : \quad x_1^* = x_1 , \quad r^{(j)*} = r^{(j)} \quad \bar{P} = \bar{P} , \quad \Gamma_{ij(n)}^{*p} = \Gamma_{ij(n)}^{*p} ,
$$

(353)

From the classical symmetries, translation in space with respect to $x_2$ is retained. While from the statistical symmetries the moment translation symmetry in all higher-order correlations can be verified for the $\mathbf{H}$ approach. After rewriting the symmetry transformation in the fluctuation ($\mathbf{R}$) approach and deriving the corresponding infinitesimals, the following characteristic system is to be solved

$$
\frac{dx_2}{k_{NS} x_2 + k_{G2}} = \frac{dr^{(j)*}}{k_{NS} r^{(j)*}} = \frac{d\bar{U}_1}{(k_4 - k_{NS}) \bar{U}_1 + k_{21}}
$$
corresponds to one of the occurring symmetries:

Subsequently both an algebraic and a logarithmic scaling law for the mean velocity will be deduced. Each constant above corresponds to one of the occurring symmetries:

\( k_i \): new statistical scaling symmetry in the transition case
\( k_{NS} \): Navier-Stokes symmetry in the transition case
\( k_{G,2} \): translation symmetry in \( x_2 \)
\( k_{r,1} \): translation in \( \overline{U}_1 \) (H approach)
\( k_{3} \): additional translation in \( \overline{U}_1 \) (R approach)
\( k_{ij} \): translation in \( H_{ij} \)
\( k_{ij} \): additional symmetry \( ij \).

The characteristic systems (354) will be solved for different conditions on the parameters \( k_i \) and \( k_{NS} \). For the later comparison to the DNS data, only the algebraic and the logarithmic solutions are deduced.

**Algebraic solution:**

This first case represents the most general one since all symmetry parameters are considered to obey \( k_{NS} \neq n \cdot k_i \ (n \in \mathbb{Z}), k_{NS} \neq 0 \). Solving the first equality in the characteristic system (354) we obtain the invariant

\[
\rho_i^{(j)} = \frac{r_i^{(j)}}{k_{G,2} + k_{NS} x_2}.
\]

The second equality can be employed to calculate the averaged velocity

\[
\overline{U}_1 = \frac{k_{i1}(k_{G,2} + k_{NS} x_2)_{NS}^{\frac{k_{i1}}{k_{NS} - k_i}}}{k_{NS} - k_i} + C_{I1}(k_{G,2} + k_{NS} x_2)_{NS}^{\frac{k_{i1}}{k_{NS} - k_i}} - 1,
\]

where we have to assume \( k_{G,2} + k_{NS} x_2 > 0 \). In principle, the characteristic system for the two-point correlations can be solved. However, presently we consider the limit \( r \to 0 \), which corresponds to the Reynolds stress tensor components

\[
\begin{align*}
\hat{R}_{11} &= -C_{I1}(k_{G,2} + k_{NS} x_2)_{NS}^{\frac{k_{i1}}{k_{NS} - k_i}} - k_{NS}(k_{NS} - k_i)C_{I11}(k_{G,2} + k_{NS} x_2)_{NS}^{\frac{k_{i1}}{k_{NS} - k_i}} - k_{NS} k_{r,1} C_{I1}(k_{G,2} + 2k_{NS} x_2)_{NS}^{\frac{k_{i1}}{k_{NS} - k_i}} - k_{NS} k_{r,1} C_{I1}(k_{G,2} + k_{NS} x_2)_{NS}^{\frac{k_{i1}}{k_{NS} - k_i}} - \frac{k_{i1}(2k_{NS} k_{r,1} + k_i k_{r,1} - 2k_{NS} k_{r,1})}{k_{NS} k_{r,1} (k_{NS} - k_i) (k_{NS} - k_i)^2 - 2k_{NS} (k_{NS} + 2k_{NS})}, \\
\hat{R}_{12} &= C_{I1}(k_{G,2} + k_{NS} x_2)_{NS}^{\frac{k_{i1}}{k_{NS} - k_i}} - C_{I1}(k_{G,2} + k_{NS} x_2)_{NS}^{\frac{k_{i1}}{k_{NS} - k_i}} - k_{NS} k_{r,1} C_{I1}(k_{G,2} + k_{NS} x_2)_{NS}^{\frac{k_{i1}}{k_{NS} - k_i}} - k_{NS} k_{r,1} C_{I1}(k_{G,2} + 2k_{NS} x_2)_{NS}^{\frac{k_{i1}}{k_{NS} - k_i}} - \frac{k_{i1}(2k_{NS} k_{r,1} + k_i k_{r,1} - 2k_{NS} k_{r,1})}{k_{NS} k_{r,1} (k_{NS} - k_i) (k_{NS} - k_i)^2 - 2k_{NS} (k_{NS} + 2k_{NS})}, \\
\hat{R}_{22} &= \frac{k_{i1} (k_{G,2} + k_{NS} x_2)_{NS}^{\frac{k_{i1}}{k_{NS} - k_i}}}{k_{NS} - k_i} + C_{I1}(k_{G,2} + k_{NS} x_2)_{NS}^{\frac{k_{i1}}{k_{NS} - k_i}} - 1, \\
\hat{R}_{23} &= -\frac{k_{i1} (k_{G,2} + k_{NS} x_2)_{NS}^{\frac{k_{i1}}{k_{NS} - k_i}}}{k_{NS} - k_i} + C_{I1}(k_{G,2} + k_{NS} x_2)_{NS}^{\frac{k_{i1}}{k_{NS} - k_i}} - 2k_{NS} (k_{NS} + 2k_{NS}) - \frac{k_{i1}(2k_{NS} k_{r,1} + k_i k_{r,1} - 2k_{NS} k_{r,1})}{k_{NS} k_{r,1} (k_{NS} - k_i) (k_{NS} - k_i)^2 - 2k_{NS} (k_{NS} + 2k_{NS})}, \\
\hat{R}_{33} &= -\frac{k_{i1} (k_{G,2} + k_{NS} x_2)_{NS}^{\frac{k_{i1}}{k_{NS} - k_i}}}{k_{NS} - k_i} + C_{I1}(k_{G,2} + k_{NS} x_2)_{NS}^{\frac{k_{i1}}{k_{NS} - k_i}} - 2k_{NS} (k_{NS} + 2k_{NS}) - \frac{k_{i1}(2k_{NS} k_{r,1} + k_i k_{r,1} - 2k_{NS} k_{r,1})}{k_{NS} k_{r,1} (k_{NS} - k_i) (k_{NS} - k_i)^2 - 2k_{NS} (k_{NS} + 2k_{NS})}.
\end{align*}
\]
Inserting these solutions into the first averaged Navier-Stokes equation, \(348\), and postulating that \(\partial \bar{T}/\partial x_1\) should not depend on \(x_2\) we obtain the condition

\[
C_{I,12} = \nu C_{I,1}k_{NS} \left( \frac{k_x}{k_{NS}} - 1 \right) - C_{I,1}U_T k_{G,2} \tag{361}
\]

and for the mean pressure

\[
\bar{p} = \frac{k_{G,2}}{k_x - 3k_{NS}} x_1 + \frac{2k_{G,2}}{k_x - 3k_{NS}} x_2 - C_{I,22} (k_{G,2} + k_{NS} x_2) \left( \frac{\dot{\nu}}{k_x} \right)^2.
\]

**Logarithmic solution:**

Here the restriction \(k_x = k_{NS}\) is claimed which leads to a logarithmic type of solution. Considering \(354\) we find the invariant

\[
\hat{\rho}^{(i)}_1 = \frac{r^{(j)}_1}{k_{G,2} + k_{NS} x_2}
\]

and for the mean velocity

\[
\bar{U}_1 = C_{I,1} + \frac{k_{G,2}}{k_{NS}} \ln(k_{G,2} + k_{NS} x_2).
\]

Since the logarithm is only defined for positive values we have to claim \(k_{G,2} + k_{NS} x_2 > 0\). Still these functions fulfill the differential equations as this was already explained for the channel flow case. With this the two-point correlations and Reynolds stresses in the limit \(r \to 0\) are calculated

\[
\bar{r}_{12} = \frac{C_{I,12} + (k_{G,2} + k_{NS} U_T^2 x_2) - k_{G,2} + k_{NS} x_2}{k_{G,2} + k_{NS} x_2} - \frac{k_{G,2} + k_{NS} x_2}{2(k_{G,2} + k_{NS} x_2)^2},
\]

\[
\bar{r}_{22} = \frac{C_{I,22} + (k_{G,2} + k_{NS} U_T^2 x_2) + k_{G,2} + k_{NS} x_2}{k_{G,2} + k_{NS} x_2},
\]

\[
\bar{r}_{33} = \frac{C_{I,33} + (k_{G,2} + k_{NS} x_2) + k_{G,2} + k_{NS} x_2}{k_{G,2} + k_{NS} x_2},
\]

and

\[
\bar{r}_{11} = \frac{k_{G,2}}{k_{NS}} \ln^2(k_{G,2} + k_{NS} x_2) - \frac{k_{G,2} + k_{NS} x_2}{k_{NS}} \frac{k_{G,2} + k_{NS} x_2}{k_{NS}} C_{I,1} \ln(k_{G,2} + k_{NS} x_2)
\]

\[
+ k_{G,2} + k_{NS} x_2 \ln^2(k_{G,2} + k_{NS} x_2) \frac{k_{G,2} + k_{NS} x_2}{k_{NS}} + \frac{k_{G,2} + k_{NS} x_2}{k_{NS}} C_{I,1}.
\]

These solutions are to be inserted into the averaged Navier-Stokes equations, \(346\) leading to the condition

\[
C_{I,12} = \nu k_{G,2} \frac{k_{G,2} k_{I,12}}{2k_{NS}^2} + \left( \frac{k_{G,2} k_{I,12}}{k_{NS}} - \frac{k_{G,2} k_{I,12}}{k_{NS}} U_T U_T + U_T U_T C_{I,1} \right) k_{G,2} \tag{365}
\]

and a relation for the mean pressure

\[
\bar{T} = -\frac{k_{G,2} x_1}{2k_{NS}} + \frac{C_{I,22} + (k_{G,2} + k_{NS} U_T^2 x_2) + k_{G,2} x_2}{k_{G,2} + k_{NS} x_2} + C_{I,1} \bar{T}.
\]

**6.2.1. Comparison to DNS Data**

Our aim in this subsection, is to analyse both the logarithmic region near the wall as well as the core region, where a logarithmic and an algebraic scaling laws are taken into consideration. For both approaches the mean velocity and all components of the Reynolds stress tensor will be compared to the DNS data. For this we introduce the distinct friction velocities

\[
u = \sqrt{u_{\text{rb}}^2 + u_{\text{rb}}^2},
\]

\[u_{\text{rb}} = \sqrt{\frac{1}{\nu} \frac{\partial \bar{U}_1}{\partial x_2}},
\]

\[u_{x_2} = \sqrt{\frac{1}{\nu} \frac{\partial \bar{U}_2}{\partial x_2}}.
\]
while the final expression may be simplified to

\[ x^+ = \frac{x}{u_\tau}, \quad \text{so that we gain} \]


where the wall-friction velocity based Reynolds number is defined as

\[ \text{Re}_\tau = \frac{hu_\tau}{\nu}. \]

For the near-wall log-law we may further define the non-dimensional spatial variables

\[ x^+_i = \text{Re}_\tau x_i, \quad \text{and} \quad \tau_i^+(j) + = \text{Re}_\tau \tau_i^+(j). \]

**Near-wall logarithmic sublayer**

We observed already for the classical channel flow that a Lie symmetry induced logarithmic sublayer can be identified near the wall. The same can be found for the channel flow with transpiration, to be shown below. In the following we want to check that the calculated scaling law from the previous subsection can describe the behaviour of the flow.

The first step is to rewrite the logarithmic scaling laws (362)–(364) into a compact form. The available DNS data suggest \( k_{G,2} = 0 \), similar to the classical log-law. The parameters for the mean velocity (362) are replaced by

\[ B_1 = C_{l,1}, \quad \kappa = \frac{k}{k_{c,1}}, \]

so that we gain

\[ \overline{U}_1^+ = B_1 + \frac{1}{k} \ln(x_2^+). \]

The Reynolds stress components (363–364) significantly compress using

\[ B_{ij} = \frac{k_{i,j}}{k_{c,1}}, \quad B_{11} = -2 \frac{k_{i,1}^2}{k_{c,1}^2} + 2 \frac{k_{i,j} k_{j,1}}{k_{c,1}^2} + 2 \frac{k_{j,1} C_{i,1}}{k_{c,1}} + \frac{k_{i,1}}{k_{c,1}} C_{j,1} - C_{i,j}^2, \]

\[ B_{12} = \frac{k_{1,2} - k_{1,1} U_T + k_{2,1} U_T - U_T C_{1,1}}{k_{c,1}}, \quad B_{22} = \frac{k_{2,2} - k_{c,1} U_T^2}{k_{c,1}}, \quad B_{33} = \frac{k_{3,3}}{k_{c,1}}; \]

\[ C_{ij} = \frac{C_{i,j}}{k_{c,1}}. \]

Equivalently, the terms linear in \( x_2 \) in (363–364) can be expressed by

\[ D_{11} = \frac{k_{c,1}}{k_{c,1}}, \quad D_{12} = \frac{k_{c,2}}{k_{c,1}}, \quad D_{22} = \frac{k_{c,2}}{k_{c,1}}, \quad D_{33} = \frac{k_{c,3}}{k_{c,1}}, \]

while the final expression may be simplified to

\[ E = -2 \frac{k_{c,1}}{k_{c,1}} (k_{1,1} - k_{3,1} + k_{c,1} C_{1,1}), \]

which is independent of the other ones because of the appearance of \( k_{1,1} \). With this, the Reynolds stress tensor reads

\[ \tilde{R}_{11}^+ = \frac{1}{\kappa} \ln^2(x_2) + E \ln(x_2) + B_{11} + D_{11} x_2 + C_{11}, \]

\[ \tilde{R}_{12}^+ = \frac{C_{12}}{x_2} + B_{12} + D_{12} x_2 - \frac{U_T}{\kappa} \ln(x_2), \]

\[ \tilde{R}_{22}^+ = \frac{C_{22}}{x_2} + B_{22} + D_{22} x_2, \]

\[ \tilde{R}_{33}^+ = \frac{C_{33}}{x_2} + B_{33} + D_{33} x_2. \]
The parameter $C_{12}$ is fixed through the condition (365) written for $k_{G,2} = 0$ which, for the non-dimensionalized quantities, transforms to

$$C_{12} = \frac{1}{\kappa}$$

The mean velocity (366) and the Reynolds stresses (371–374) are compared to DNS data of A vsarkisov et al. (2014) shown in Figures 12 and 13. Though not shown, very similar results are obtained for other Reynolds numbers and transpiration velocities.

The parameters of the mean velocity (366) are given by $\kappa = 0.395$, $B_1 = -4.657$, i.e. the von Kármán constant is almost the same as for channel flow without transpiration, though the additive constant is very different.

A list of parameters for the scaling laws (371–374) is given in table 8.

We realize that $D_{ij}$ is relatively small for $R_{12}^+$, $R_{22}^+$ and $R_{33}^+$. This refers to the fact that we may neglect the additional symmetries (353), where a term linear in $x_2$ was added to the two-point correlations. On the other hand this linear symmetry is crucial to describe $R_{11}^+$ correctly.

**Centre region**

In the centre region we intend to compare different scaling laws, i.e. a second logarithmic scaling law and an algebraic...
Furthermore, we introduce a generalized integration constant

From (365), the condition

\[ B_{11} = \frac{k_{i} k_{NS}}{k_{NS}^{3}} + A E \ln(k_{NS}) - \frac{A}{\kappa} A \ln(k_{w,a}) , \]

so that (362) reduces to

\[ \overline{U}_1 = B_1 + T \ln(\xi_2) . \]  

The Reynolds stress tensor contains additional parameters, which can be written in a condensed form as

\[ R_{11} = \frac{k_{NS}^{2} C_{i,j} - k_{G,2} k_{NS}^{2}}{k_{NS}^{3}} + A E \ln(k_{NS}) - \frac{A}{\kappa} \ln(k_{w,a}) , \]

\[ B_{12} = \frac{C_{i,j}}{k_{NS}} - \frac{U_{T}}{k} \ln(k_{NS}) , \]

\[ C_{11} = \frac{1}{k_{NS}^{2}} (-2 k_{i}^{2} + 2 k_{i1} k_{NS} + 2 k_{NS} k_{i1} + k_{11} k_{NS} - 2 k_{NS} k_{i1} C_{i,j} - k_{NS}^{2} C_{i,j}^{2} + E \ln(k_{w,a}) - \frac{1}{\kappa} A \ln(k_{w,a}) , \]

\[ C_{12} = \frac{1}{k_{NS}^{2}}(k_{i1} k_{NS} - k_{i1} U_{T} + k_{i} U_{T} - k_{NS} U_{T} C_{i,j} - \frac{U_{T}}{k} \ln(k_{NS}) , \]

\[ D_{11} = \frac{k_{NS}}{k_{NS}^{2}} D_{12} = \frac{k_{NS}}{k_{NS}^{2}} D_{22} = \frac{k_{NS}}{k_{NS}^{2}} D_{33} = \frac{k_{NS}}{k_{NS}^{2}} , \]

and

\[ E = -2 \frac{k_{i1}}{k_{NS}}(k_{i1} - k_{i} + k_{NS} C_{i,j}) , \]

using the above expressions in (363) and (364) results in the following Reynolds stress tensor components

\[ \hat{R}_{11}^{+} = -\xi_{2}^{2} \ln(\xi_{2}) + E \ln(\xi_{2}) + \frac{B_{11} + \xi_{2} C_{11} + \xi_{2}^{2} D_{11}}{\xi_{2}} , \]

\[ \hat{R}_{12}^{+} = \frac{B_{12} + \xi_{2} C_{12} + \xi_{2}^{2} D_{12}}{\xi_{2}} - \xi_{2} U_{T} \ln(\xi_{2}) , \]

\[ \hat{R}_{22}^{+} = \frac{B_{22} + \xi_{2} C_{22} + \xi_{2}^{2} D_{22}}{\xi_{2}} , \]

\[ \hat{R}_{33}^{+} = \frac{B_{33} + \xi_{2} C_{33} + \xi_{2}^{2} D_{33}}{\xi_{2}} . \]

From (365), the condition \( B_{12} = \nu T \), has to be fulfilled.

Alternatively, considering the algebraic solution (355) and (356–360), we may define parameters for the mean velocity

\[ B_{i} = \frac{k_{i1}}{k_{NS} - k_{i}} , \]

\[ C_{i} = C_{i,j} x_{2}^{2} x_{2}^{n-1} , \]

\[ A = \frac{k_{G,2}}{k_{NS}} , \]

\[ \gamma = \frac{k_{i}}{k_{NS}} - 1 \]

which transforms the equation (355) into

\[ \overline{U}_{1}^{+} = B_{1} + C_{1} (\xi_{2} + A)^{y} . \]

Furthermore, we introduce a generalized integration constant

\[ C_{ij} = C_{i,j} k_{NS}^{2} , \]

\[ C_{12} = C_{12} + C_{1} U_{T} A \]

Table 8  List of parameters in formulas (371–374) fitted to the DNS data of A vsarkisov et al. (2014).

<table>
<thead>
<tr>
<th></th>
<th>( B_{ji} )</th>
<th>( C_{ji} )</th>
<th>( D_{ji} )</th>
<th>( E )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R_{11} )</td>
<td>-69</td>
<td>125.8</td>
<td>0.143</td>
<td>4.40</td>
</tr>
<tr>
<td>( R_{12} )</td>
<td>0.664</td>
<td>1/\kappa</td>
<td>0.004</td>
<td></td>
</tr>
<tr>
<td>( R_{13} )</td>
<td>12.05</td>
<td>-157.9</td>
<td>-0.006</td>
<td></td>
</tr>
<tr>
<td>( R_{23} )</td>
<td>14.8</td>
<td>-111.8</td>
<td>-0.016</td>
<td></td>
</tr>
</tbody>
</table>
Fig. 14 Logarithmic solution (375) and (378) (solid lines, left-hand side plots) and algebraic solution (380) and (384) (solid lines, right-hand side plots) compared to the DNS data of Avsarkisov et al. (2014): ♦: mean velocity, □: $\tilde{R}_{11}^+$, △: $\tilde{R}_{22}^+$, ×: $\tilde{R}_{33}^+$, ◦: $\tilde{R}_{12}^+$. 
and alternative parameter definitions

\[
B_{11} = \frac{k_{11}}{k_s - 2k_{NS}} - \frac{k_{11}(2k_{NS}k_{tr,1} + k_s(-2k_{tr,1} + k_{11}))}{(k_s - 2k_{NS})(k_s - k_{NS})^2} - \frac{2k_{c11}k_{G,2}}{k_s^2 - 5k_sk_{NS} + 6k_{NS}^2},
\]

\[
B_{12} = \frac{k_{12}(k_{NS} - k_s) + k_{tr,1}U_T(k_s - k_{NS}) - k_{NS}k_{tr,1}U_T}{k_s^2 - 3k_sk_{NS} + 2k_{NS}^2} - \frac{2k_{c12}k_{G,2}}{k_s^2 - 5k_sk_{NS} + 6k_{NS}^2},
\]

\[
B_{22} = \frac{k_{2,22} + (k_s - 2k_{NS})U_T^2}{k_s - 2k_{NS}} - \frac{2k_{c22}k_{G,2}}{k_s^2 - 5k_sk_{NS} + 6k_{NS}^2},
\]

\[
B_{33} = \frac{k_{33}}{k_s - 2k_{NS}} - \frac{2k_{c33}k_{G,2}}{k_s^2 - 5k_sk_{NS} + 6k_{NS}^2},
\]

\[
D_{11} = \frac{2k_{11}}{k_s - 3k_{NS}}, \quad D_{12} = \frac{k_{12}}{k_s - 3k_{NS}}, \quad D_{22} = \frac{k_{22}}{k_s - 3k_{NS}}, \quad D_{33} = \frac{2k_{33}}{k_s - 3k_{NS}},
\]

and

\[
E = \frac{k_{NS}k_{tr,1} + k_s(k_{11} - k_{tr,1})}{(k_s - k_{NS})k_{NS}} C_{1,1} k_{NS}^{\frac{1}{\gamma - 1}},
\]

which occur in the Reynolds stress tensor and can be generalized corresponding to the given formulas (356–360), so that finally the concise expressions

\[
\hat{R}_{11}^* = -C_{1,1}^2(A + \tilde{x}_2)^{\gamma - 1} + C_{1,1}(A + \tilde{x}_2)^{\gamma - 1} + E(A + 2\tilde{x}_2)(A + \tilde{x}_2)^{\gamma - 1} + B_{11} + D_{11}\tilde{x}_2,
\]

\[
\hat{R}_{12}^* = C_{1,2}^2(\tilde{x}_2 + A)^{\gamma - 1} - C_{1,1}U_T(\tilde{x}_2 + A)^{\gamma - 1} + B_{12} + D_{12}\tilde{x}_2,
\]

\[
\hat{R}_{22}^* = C_{2,2}^2(\tilde{x}_2 + A)^{\gamma - 1} + B_{22} + D_{22}\tilde{x}_2,
\]

\[
\hat{R}_{33}^* = C_{3,3}^2(\tilde{x}_2 + A)^{\gamma - 1} + B_{33} + D_{33}\tilde{x}_3,
\]

emerge. The condition (361) in the new parameters is given by

\[
C_{1,2} = \gamma C_{1} - C_{1}A U_T = \frac{\gamma C_{1}}{Re} - C_{1}A U_T.
\]

In the following we compare both the algebraic and logarithmic approaches to the DNS data of Avsarkisov et al. (2014) at the same Reynolds number and transpiration velocity. In Figure 14 the logarithm solution (375) and (378) (left-hand side graphs) and the algebraic solution (380) and (384) (right-hand side graphs) are fitted. From these figures we observe that the algebraic approach can fit all quantities in a somewhat broader region than the logarithmic approach. Both approaches work similarly well also for different Reynolds numbers and transpiration velocities, see Figures 15-18.

In order to distinguish which of the both approaches is to be preferred we have listed in table 9 and 10 the values of the respective formulas (375), (378) and (380), (384) fitted to the DNS data of Avsarkisov et al. (2014).

Then we want to study the behaviour of the parameters if an algebraic solution is assumed as can be seen in Figure 17 and 18. The crucial parameter is the mean velocity where we recognize that $\gamma$ in table 9 has essentially the same value.
Due to this extended universality we believe that the log-law is to be preferred.

Surely also $D_1$ seems to be almost constant. This would mean, that $R_{12}^*$ has only parameters which are fixed, since also $C_{12}$ is determined through condition (365), respectively (361).

### 6.3. Rotating channel flow

The final application which we want to consider is the rotating channel flow, where two different rotational axes are examined. Here we want to only briefly outline this case, without going into such details as in the previous sections. The interested reader is referred to Ref. (Rosteck, 2013).

Using our symmetry analysis to gain scaling laws, the calculated symmetries have to be transformed into the coordinate system of a rotating frame (see e.g. Rosteck, 2013). In the second step the invariant system can be developed, though for each rotational axis the symmetries must be determined separately.

This leads to a rather complex and involved form of the characteristic equation, so details have to be omitted and only results for the mean flow will be presented. In the present sub-section we consider two cases of a rotating channel i.e. rotation about the $x_2$- and the $x_3$-axis.

We first assume that the rotational axis lies along the $x_3$ direction i.e. only $\Omega_3$ is non-zero. Applying Lie symmetry analysis the classical symmetries i.e. scaling in space and the Galilei invariance are used and they are extended by the action of the new scaling symmetry (238) and the translation of the velocities (236). Solving the resulting system for the invariant solution, we gain an exponential type of solution which in normalised form reads

$$\frac{\overline{U}_1(x_2) - \overline{U}_{cl}}{\Omega_3 h} = A(\Omega_3) \left\{ e^{\gamma(\Omega_3)x_3/h} - 1 \right\} ,$$

where $A(\Omega_3)$ and $\gamma(\Omega_3)$ are unknown functions of the rotation number $\Omega_3 = \frac{2\Omega_3 h}{U_{cl}}$, though its functional form has not been determined from symmetry theory yet. $U_3$ and $\overline{U}_{cl}$ are respectively bulk and center line velocity. $\gamma(\Omega_3)$ converges to zero for increasing $\Omega_3$ while $A(\Omega_3)$ tends to a constant in this limit. Carrying out the latter limit $\Omega_3 \rightarrow \infty$ we obtain the well-known scaling law for a rotating channel about the $x_3$-axis (see Oberlack, 2001)

$$\overline{U}_1(x_2) = A_{x_2} \Omega_3 x_2 + \overline{U}_{cl} .$$

A clear validation of (385) and (386) is given in Figure 19 for various $\Omega_3$ taken from the DNS of Kristoffersen and Andersson (1993). Interesting enough the value for $A_{x_2}$ appears to be very close to 2.

As the stresses are not available from the literature but only the corresponding kinetic energy (formulas may be taken from Rosteck (2013)) this is shown in Figure 20.

The final case to be considered is the one of channel flow rotating about $x_3$, where two velocity components $\overline{U}_1$ and $\overline{U}_3$ have to be taken into consideration since the Coriolis force induces a cross flow. Again, both mean velocities may only depend on $x_2$. Different to the previous case is that one additional symmetry appears, namely translation in time i.e. $T^{\ast'}$.

---

Table 10 List of parameters in formulas 380 and (384) fitted to the data of Avsarkisov et al. (2014).

<table>
<thead>
<tr>
<th>Re</th>
<th>$U_T$</th>
<th>250</th>
<th>0.05</th>
<th>250</th>
<th>0.1</th>
<th>250</th>
<th>0.16</th>
<th>250</th>
<th>0.26</th>
<th>250</th>
<th>0.1</th>
<th>250</th>
<th>0.16</th>
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<td>fit region</td>
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<td>0.5-1.6</td>
<td>0.5-1.6</td>
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<td>0.5-1.6</td>
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<td></td>
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<td></td>
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</tr>
<tr>
<td>$A$</td>
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<tr>
<td>$B_1$</td>
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<td>17.91</td>
<td>12.72</td>
<td>28.25</td>
<td>18.34</td>
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<td>$C_1$</td>
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<td>$\gamma$</td>
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<td>-1.33</td>
<td>-0.69</td>
<td>-1.05</td>
<td>-0.28</td>
<td>-0.73</td>
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<tr>
<td>$B_{21}$</td>
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<td>$C_{21}$</td>
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<td>$D_{21}$</td>
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<td>-5.51</td>
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<td>$E$</td>
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<td>31.72</td>
<td>33.01</td>
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<td>44.11</td>
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<tr>
<td>$D_{22}$</td>
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<td>-2.99</td>
<td>-3.24</td>
<td>-3.12</td>
<td>-2.99</td>
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<td>$D_{33}$</td>
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<td>8.11</td>
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</table>

Table 10
Fig. 15 Logarithmic solution (375) and (378) (solid lines) compared to the DNS data of Avsarkisov et al. (2014) at different Reynolds numbers and transition velocities: ♦: mean velocity, □: $\tilde{R}_{11}^+$, △: $\tilde{R}_{22}^+$, ×: $\tilde{R}_{33}^+$, ◦: $\tilde{R}_{12}^+$. 
Fig. 16 Logarithmic solution (375) and (378) (solid lines) compared to the DNS data of Avsarkisov et al. (2014) at different Reynolds numbers and transition velocities: ♦: mean velocity, □: $\tilde{R}_{11}$, △: $\tilde{R}_{12}$, ×: $\tilde{R}_{33}$, ◦: $\tilde{R}_{12}$. 
Fig. 17  Algebraic solution (380) and (384) (solid lines) compared to the DNS data of Avsarkisov et al. (2014) at different Reynolds numbers and transition velocities: ♦: mean velocity, □: $\tilde{R}^{+}_{11}$, △: $\tilde{R}^{+}_{22}$, ×: $\tilde{R}^{+}_{33}$, ◦: $\tilde{R}^{+}_{12}$. 

$Re= 250, \ U_T = 0.05$

$Re= 250, \ U_T = 0.26$
Fig. 18  Algebraic solution (380) and (384) (solid lines) compared to the DNS data of Avsarkisov et al. (2014) at different Reynolds numbers and transition velocities: ♦: mean velocity, □: $\tilde{R}_{11}^+$, △: $\tilde{R}_{22}^+$, ×: $\tilde{R}_{33}^+$, ◦: $\tilde{R}_{12}^+$. 

Re = 450, $U_T = 0.1$

Re = 450, $U_T = 0.16$
Fig. 19 Comparison of the mean velocity scaling law (---) in (385)/(386) with the DNS data (· · ·) of Kristoffersen and Andersson (1993) at rotation rates $\omega_0 = 0.1, 0.15, 0.2, 0.3$ and $Re_\tau = 194$.

From this we derive the new $\Omega_2$ depending scaling laws

$$
\frac{\bar{U}_1}{u_\tau} = \left(\frac{y}{h}\right)^b \left[ a_1 \cos \left(c\omega_0 \cdot \ln \frac{y}{h}\right) + a_2 \sin \left(c\omega_0 \cdot \ln \frac{y}{h}\right) \right] + d_1(\omega_0),
$$

$$
\frac{\bar{U}_3}{u_\tau} = \left(\frac{y}{h}\right)^b \left[ a_1 \sin \left(c\omega_0 \cdot \ln \frac{y}{h}\right) - a_2 \cos \left(c\omega_0 \cdot \ln \frac{y}{h}\right) \right] + d_2(\omega_0).\tag{387}
$$

with $\omega_0 = 2\Omega_2 h / u_{\tau 0}$.

For the rotating case $\omega_0 \neq 0$ we compare the DNS data of Mehdizadeh and Oberlack (2010) at $Re_\tau = 360$ with the scaling law (387). Results are depicted for two different rotation numbers $\omega_0$ (defined below equation 387) in the Figure 21 exhibiting an excellent fit in the center of the channel for all cases. $u_{\tau 0}$ refers to the friction velocity of the non-rotating case. It is to note from all the DNS data sets in Mehdizadeh and Oberlack (2010) we find that with an increasing $\Omega_2$ the magnitude of $\bar{U}_1$ and $\bar{U}_3$ switches position since with increasing rotation rates $\bar{U}_1$ is suppressed while $\bar{U}_3$ increases up to a certain point and decreases again though to a smaller rate compared to $\bar{U}_1$. This behavior is exactly described by the scaling law (387). Because of its complexity the corresponding stresses have been omitted.

7. Summary and outlook

Within the present overview article it was shown that the admitted symmetry groups of the infinite set of multi-point correlation equations are considerable extended by three classes of groups compared to those originally stemming from the Euler and the Navier-Stokes equations. In fact, it was demonstrated that it is exactly these symmetries which are essentially needed to validate certain classical scaling laws such as the log-law from first principles and also to derive a large set of new scaling laws. Moreover, it was argued that the corresponding extended set of symmetries can be found also for the LMN equations for velocity PDFs and the Hopf functional approach.

Implicitly, symmetries have been used in turbulence modeling for several decades since essentially all symmetries of Euler and the Navier-Stokes equations have been made part of modern turbulence models. Still, this is only partially true for the new statistical symmetries. In fact, some of them have been employed even in very early turbulence models since many of them where calibrated against the log-law. Many other symmetries, however, have never been made use of and in fact, it might be even impossible to make turbulence models consistent with some of the symmetries such as the new scaling symmetry (234).

Even with these new symmetry groups at hand, which give a much deeper understanding on turbulence statistics, there are still many open questions to be answered. (i) So far completeness of all admitted symmetries of the MPC equation has not been shown. This appears to be necessary not only from a theoretical point of view but rather essential...
Fig. 20  Comparison of the kinetic energy scaling (−) with the DNS data (···) of Kristoffersen and Andersson (1993) at rotation rates $Ro_3 = 0.1$, $0.15$, $0.2$, $0.5$ and $Re_τ = 194$.

to generate scaling for all higher moments. (ii) From turbulence data it is apparent that the appearing group parameters do have certain decisive values which are to be determined. In some very rare cases such as the classical decaying turbulence case values such as the decay exponent may be determined from integral invariants. Still, a general scheme is unknown. (iii) Finally, we clearly observe that certain scaling laws such as the log law only cover certain regions of a turbulent flow and are usually embedded within other layers of turbulence. Hence, the matching of turbulent scaling laws remains an open question.

An apparent extension of the presented machinery to turbulent heat transfer problems can be achieved by adding the scalar heat transfer equation to the Navier-Stokes equations (180). This might be either in the most simple case of a passive temperature equation or, if buoyancy is considered, the temperature equation again, while the momentum equation (180) is extended by the Boussinesq approximation. In both cases, the multi-point system (195) is extended by an additional tensor composed of the temperature correlated with $m$ velocities, where $m = 1, \ldots, \infty$. Note, that because of the linearity of the temperature equation and the linearity of the temperature in the momentum equation due the Boussinesq approximation, it is not necessary to derive correlations and equations which contain the temperature more than once.

Other extension such as to Navier-Stokes equations for compressible fluids, i.e. the gas dynamics equations prolonged by viscous terms, are less straightforward. The reason is that the corresponding equation may only contain terms, where density and one of the velocities are at the same point.

Finally, depending on the Mach number $Ma$, the gas-dynamics equations may change type and hence, for $Ma > 1$ correlations may only be uniquely defined within the Mach cone. This, apparently, posses an additionally non-trivial constraint on the problem which is unsolved to date.

References


Fig. 21  Comparison of the scaling law (−) in (387) with the DNS data (···) of Mehdizadeh and Oberlack (2010) at a) $R_o = 0.011$, b) $R_o = 0.18$.


