A relation between the Lorentzian inversion and the Bateman transformation

KATSUNORI SHIMOMURA*

Abstract

In our previous paper [1], we proved that every transformation which preserves the wave equation is a similarity or a Lorentzian inversion composed with similarities or a Bateman transformation composed with similarities. In this paper, we give several relations between Bateman transformation and Lorentzian inversion. We also prove that only Lorentzian inversion or Bateman transformation is enough to generate the set of all transformations which preserve the wave equation.

1. Introduction and known results

In [1], we considered Liouville type theorem associate with the wave equation, and proved that every conformal mapping for the Lorentzian metric is one of the following: a similarity, a inversion composed with similarities, a Bateman transformation composed with similarities. We also determined the transformations which preserve the solutions of the wave equation. The purpose of this note is to discuss relations among these transformations and to give generators of the group of all transformations which preserve the wave equation (see Theorems 3.1 and 3.2 below).

Let $\mathbb{R}^{n+1}$ be the $(n+1)$ -dimensional Euclidean space ($n \geq 2$), and denote its point by $x = (x_0, x_1, \dots, x_n)$. We introduce Lorentzian scalar product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^{n+1}$ defined by

$$\langle x, y \rangle = -x_0y_0 + x_1y_1 + \dots + x_ny_n.$$

The vector space $\mathbb{R}^{n+1}$ with this Lorentzian scalar product is called Lorentzian $(n+1)$ -space, and is denoted by $\mathbb{R}^{1,n}$.

Let $D, E \subset \mathbb{R}^{1,n}$ be domains and $f : D \rightarrow E$ a C^1 -mapping. The first derivative $f'(x)$ of f at $x \in D$ is a linear mapping $\mathbb{R}^{1,n} \rightarrow \mathbb{R}^{1,n}$ such that

$$f'(x)u = \lim_{h \rightarrow 0} \frac{f(x+hu) - f(x)}{h}, \quad u \in \mathbb{R}^{1,n}.$$

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*Ibaraki University, Mito, Ibaraki, 310-8512, Japan. (shimomur@mx.ibaraki.ac.jp)

Definition 1.1. A C^1 -mapping $f : D \rightarrow E$ is said to be Lorentzian conformal if there exists a function $\lambda(x) = \lambda_f(x) > 0$ defined on D such that

$$\langle f'(x)u, f'(x)v \rangle = \lambda(x)^2 \langle u, v \rangle \quad (1.1)$$

holds for all $x \in D$ and all $u, v \in \mathbb{R}^{1,n}$.

Note that a composition of two Lorentzian conformal mapping is also Lorentzian conformal.

Next, we consider the wave equation

$$Wu := \frac{\partial^2 u}{\partial x_0^2} - \sum_{j=1}^n \frac{\partial^2 u}{\partial x_j^2} = 0$$

on $\mathbb{R}^{1,n}$ and transformations which preserve the solutions of the wave equation.

Definition 1.2. Let $D \subset \mathbb{R}^{n+1}$ be a domain, $f : D \rightarrow \mathbb{R}^{1,n}$ a C^2 -mapping, and φ a positive C^2 -function on D . A pair (f, φ) is called a transformation which preserves the wave equation, if f and φ satisfy the following conditions:

(1) $f'(x)$ is non-degenerate for all $x \in D$.

(2) For every solution u of the wave equation on $\mathbb{R}^{1,n}$, $\varphi \cdot (u \circ f)$ satisfies the wave equation on D .

Note that if (f, φ_f) and (g, φ_g) are transformations which preserve the wave equation such that the image of f is contained in the domain of g , then the composition $(g \circ f, (\varphi_g \circ f) \cdot \varphi_f)$ is also a transformation which preserves the wave equation.

Let

$$O(1, n) := \{R \in \text{GL}(n+1, \mathbb{R}); \langle Rx, Ry \rangle = \langle x, y \rangle \text{ for all } x, y \in \mathbb{R}^{1,n}\}.$$

In the following, we list fundamental Lorentzian conformal mappings and corresponding transformations which preserve the wave equation.

Example 1.1 (Lorentzian similarity). The mapping $f(x) = \alpha Rx + a$, where $\alpha \in \mathbb{R}$, $\alpha > 0$, $R \in O(1, n)$, and $a \in \mathbb{R}^{1,n}$, is a Lorentzian conformal mapping defined on $\mathbb{R}^{1,n}$. The pair $(f, 1)$ is a transformation which preserves the wave equation.

Example 1.2 (Lorentzian inversion). The mapping $K(x) = \frac{x}{\langle x, x \rangle}$ is a Lorentzian conformal mapping defined on each component of $\{x \in \mathbb{R}^{1,n}; \langle x, x \rangle \neq 0\}$. K satisfies $K^{-1} = K$. The pair $(K, |\langle x, x \rangle|^{\frac{1-n}{2}})$ is a transformation which preserves the wave equation.

Example 1.3 (Bateman transformation). The mapping

$$B(x) = \left(\frac{\langle x, x \rangle + 1}{2(x_0 + x_1)}, \frac{\langle x, x \rangle - 1}{2(x_0 + x_1)}, \frac{x_2}{x_1 + x_0}, \dots, \frac{x_n}{x_1 + x_0} \right)$$

is a Lorentzian conformal mapping defined on each component of $\{x \in \mathbb{R}^{1,n}; x_0 + x_1 \neq 0\}$. The pair $(B, |x_0 + x_1|^{\frac{1-n}{2}})$ is a transformation which preserves the wave equation.

In [1], we proved the following two theorems:

Theorem A. *Let $n \geq 2$ and f be a Lorentzian conformal C^4 -mapping defined on a domain $D \subset \mathbb{R}^{1,n}$. Then f has one of the following forms:*

$$f(x) = \alpha RK(x - a) + b, \quad (a)$$

$$f(x) = \alpha RB(S(x - a)) + b, \quad (b)$$

$$f(x) = \alpha Rx + b, \quad (c)$$

where $\alpha > 0$, $R, S \in O(1, n)$, $a, b \in \mathbb{R}^{1,n}$.

Theorem B. *Let $n \geq 2$, f be a C^4 -mapping, and φ a positive function defined on a domain $D \subset \mathbb{R}^{1,n}$. If (f, φ) be a transformation which preserve the wave equation, then f and φ have one of the following forms:*

$$f(x) = \alpha RK(x - a) + b, \quad \varphi(x) = C|\langle x - a, x - a \rangle|^{\frac{1-n}{2}}, \quad (a)$$

$$f(x) = \alpha RB(S(x - a)) + b, \quad \varphi(x) = C|\langle v, S(x - a) \rangle|^{n-1}, \quad (b)$$

$$f(x) = \alpha Rx + b, \quad \varphi(x) = C, \quad (c)$$

where $\alpha > 0$, $R, S \in O(1, n)$, $a, b \in \mathbb{R}^{1,n}$, and $v = (-1, 1, 0, \dots, 0) \in \mathbb{R}^{1,n}$.

2. Relations between Bateman transformation and Lorentzian inversion

In the last section, we introduced Lorentzian scalar product. Let J be the $(n+1) \times (n+1)$ diagonal matrix

$$J = \text{diag}(-1, 1, \dots, 1).$$

Then the Lorentzian scalar product $\langle \cdot, \cdot \rangle$ is written by the Euclidean scalar product $(\cdot, \cdot)$ as

$$\langle x, y \rangle = (Jx, y) = (x, Jy), \quad x, y \in \mathbb{R}^{1,n}.$$

Recall that $O(1, n) = \{R; {}^tRJR = J\} = \{R; RJ^tR = J\}$. $O(1, n)$ is the group of all matrices which preserve the Lorentzian scalar product.

For $a \in \mathbb{R}^{1,n}$, we define the translation operator T_a by $T_a(x) = x - a$. Clearly $T_a(Ax) = AT_{A^{-1}a}(x)$ holds for any $A \in GL(n+1, \mathbb{R})$.

Proposition 2.1. *Let $u \in \mathbb{R}^{1,n}$ such that $\langle u, u \rangle \neq 0$. Then*

$$KT_uK = \frac{1}{\langle u, u \rangle} (T_u S_u K T_{\hat{u}}),$$

where $\hat{u} = \frac{u}{\langle u, u \rangle} = K(u)$ and $S_u(x) = x - 2\frac{\langle x, u \rangle}{\langle u, u \rangle}u$.

Proof. First,

$$(KT_uK)(x) = \frac{\langle x, x \rangle^{-1}x - u}{\langle \langle x, x \rangle^{-1}x - u, \langle x, x \rangle^{-1}x - u \rangle} = \frac{x - \langle x, x \rangle u}{1 - 2\langle x, u \rangle + \langle x, x \rangle \langle u, u \rangle}.$$

Since $\langle u, u \rangle \neq 0$ and $\langle \hat{u}, \hat{u} \rangle = \frac{1}{\langle u, u \rangle}$, the right hand side equals to

$$\begin{aligned} \frac{x - \langle x, x \rangle u}{\langle u, u \rangle (\langle \hat{u}, \hat{u} \rangle - 2\langle x, \hat{u} \rangle + \langle x, x \rangle)} &= \frac{1}{\langle u, u \rangle} \left(\frac{x - \hat{u} - 2\langle x - \hat{u}, \hat{u} \rangle u}{\langle x - \hat{u}, x - \hat{u} \rangle} - u \right) \\ &= \frac{1}{\langle u, u \rangle} T_u \left(K(T_{\hat{u}}x) - 2 \frac{\langle K(T_{\hat{u}}x), u \rangle}{\langle u, u \rangle} u \right) = \frac{1}{\langle u, u \rangle} (T_u S_u K T_{\hat{u}})(x). \end{aligned}$$

We remark that $S_u \in O(1, n)$. $\square$

Let $u \in \mathbb{R}^{1,n}$ satisfy $u \neq 0$ and $\langle u, u \rangle = 0$. Then $\langle u, Ju \rangle = (u, u) \neq 0$. We define the mapping B_u on $\{x; \langle x, u \rangle \neq 0\}$ by

$$B_u(x) = \frac{1}{\langle x, u \rangle} \left(\frac{\langle x, x \rangle}{2} u - \frac{1}{\langle u, Ju \rangle} Ju + x - \frac{\langle x, Ju \rangle}{\langle u, Ju \rangle} u - \frac{\langle x, u \rangle}{\langle u, Ju \rangle} Ju \right).$$

Put $u_0 = (1, 1, 0, \dots, 0)$. Then the Bateman mapping B can be obtained by B_{u_0} as

$$B = B_{u_0} J = J B_{Ju_0}. \quad (2.1)$$

Proposition 2.2. *Let $u \in \mathbb{R}^{1,n}$ satisfy $u \neq 0$ and $\langle u, u \rangle = 0$. Then B_u is Lorentzian conformal. We also have $B_u^{-1} = B_{-u}$ and*

$$B_u = T_{u^*} K T_{-\frac{1}{2}u} K T_{u^*}, \quad (2.2)$$

where $u^* = \frac{Ju}{\langle u, Ju \rangle}$.

Proof. We first prove (2.2). Since $\langle u, u \rangle = 0$ and $\langle u, u^* \rangle = 1$,

$$(K T_{-\frac{1}{2}u} K)(x) = \frac{x + \frac{1}{2}\langle x, x \rangle u}{1 + \langle x, u \rangle + \frac{1}{4}\langle x, x \rangle \langle u, u \rangle} = \frac{x + \frac{1}{2}\langle x, x \rangle u}{1 + \langle x, u \rangle} = \frac{x + \frac{1}{2}\langle x, x \rangle u}{\langle u^*, u \rangle + \langle x, u \rangle}.$$

Note that $\langle u^*, u^* \rangle = 0$. Putting $y = x + u^* = T_{u^*}^{-1}(x)$, the right hand side equals to

$$\begin{aligned} \frac{y - u^* + \frac{1}{2}\langle y - u^*, y - u^* \rangle u}{\langle y, u \rangle} &= \frac{1}{\langle y, u \rangle} \left(y - \frac{Ju}{\langle u, Ju \rangle} + \frac{\langle y, y \rangle}{2} u - \langle y, \frac{Ju}{\langle u, Ju \rangle} \rangle u \right) \\ &= \frac{1}{\langle y, u \rangle} \left(\frac{\langle y, y \rangle}{2} u - \frac{Ju}{\langle u, Ju \rangle} + y - \frac{\langle y, Ju \rangle}{\langle u, Ju \rangle} u - \frac{\langle y, u \rangle}{\langle u, Ju \rangle} Ju \right) + \frac{Ju}{\langle u, Ju \rangle} \\ &= B_u(y) + u^* = (T_{u^*}^{-1} B_u T_{u^*}^{-1})(x). \end{aligned}$$

Therefore $B_u = T_{u^*} K T_{-\frac{1}{2}u} K T_{u^*}$ holds.

Next, the conformality of B_u follows immediately from this equation.

Finally, to show $B_u^{-1} = B_{-u}$, note that $(-u)^* = -u^*$ holds by definition. Then

$$B_u^{-1} = T_{-u^*} K^{-1} T_{\frac{1}{2}u} K^{-1} T_{-u^*} = T_{(-u)^*} K T_{-\frac{1}{2}(-u)} K T_{(-u)^*} = B_{-u}$$

because $K^{-1} = K$. $\square$

Proposition 2.3. *We have the following relation:*

$$B^2 = J_1 K = K J_1, \quad (2.3)$$

where $J_1 = \text{diag}(1, -1, 1, \dots, 1) \in O(1, n)$.

Proof. Put $y = B(x)$. Then

$$\frac{1}{y_0 + y_1} = \frac{x_0 + x_1}{\langle x, x \rangle}, \quad \langle y, y \rangle = \frac{x_0 - x_1}{x_0 + x_1}, \quad \frac{\langle y, y \rangle + 1}{2} = \frac{x_0}{x_0 + x_1}, \quad \frac{\langle y, y \rangle - 1}{2} = \frac{-x_1}{x_0 + x_1}.$$

Hence

$$\begin{aligned} B^2(x) &= B(y) = \frac{1}{y_0 + y_1} \left(\frac{\langle y, y \rangle + 1}{2}, \frac{\langle y, y \rangle - 1}{2}, y_2, \dots, y_n \right) \\ &= \left(\frac{x_0}{\langle x, x \rangle}, \frac{-x_1}{\langle x, x \rangle}, \frac{x_2}{\langle x, x \rangle}, \dots, \frac{x_n}{\langle x, x \rangle} \right) = (J_1 K)(x) = (K J_1)(x). \end{aligned}$$

□

3. Results

In this way, we obtain the following theorems from Theorems A and B and relations (2.1), (2.2), and (2.3).

Theorem 3.1. *Let $n \geq 2$. Then the set of all Lorentzian conformal C^4 -mappings can be generated by K and Lorentzian similarities. And the set of all C^4 -transformations which preserve the wave equation can be generated by Lorentzian inversion K and Lorentzian similarities.*

Theorem 3.2. *Let $n \geq 2$. Then the set of all Lorentzian conformal C^4 -mapping can be generated by B and Lorentzian similarities. And the set of all C^4 -transformations which preserve the wave equation can be generated by Bateman transformation B and Lorentzian similarities.*

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References

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