Note on spectral semistar operations, II

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Abstract

We study an infinite dimensional and non-local Prüfer domain D , and show that there are many non-spectral semistar operations on D .

0. Introduction

This is a continuation of [M3]. Let D be an integral domain with quotient field K . Let $\bar{F}(D)$ be the set of non-zero D -submodules of K , let $F(D)$ be the set of non-zero fractional ideals of D , i.e., $E \in F(D)$ if $E \in \bar{F}(D)$ and there is an element $d \in D \setminus \{0\}$ such that $dE \subseteq D$. A semistar operation on D is a mapping $\star : \bar{F}(D) \rightarrow \bar{F}(D)$, $E \mapsto E^\star$, such that for every $x \in K \setminus \{0\}$ and for every $E, F \in \bar{F}(D)$, the following conditions hold: $(xE)^\star = xE^\star$; $E \subseteq F$ implies $E^\star \subseteq F^\star$; $E \subseteq E^\star$, and $(E^\star)^\star = E^\star$. The identity mapping $E \mapsto E^d = E$ on $\bar{F}(D)$ is a semistar operation called the d-semistar operation on D . For every $\Delta \subseteq \text{Spec}(D)$, the mapping $E \mapsto \cap_{P \in \Delta} ED_P$ is a semistar operation on D , and is denoted by $\star_\Delta$. A semistar operation $\star$ on D is called spectral if there is a subset $\Delta \subseteq \text{Spec}(D)$ such that $\star = \star_\Delta$. G. Picozza [P, Corollary 4.13] proved the following:

Let D be a local domain, i.e., D has only one maximal ideal. Then every semistar operation on D is spectral if and only if D is a discrete valuation domain. In [M3, Theorem 1.1], we proved the following:

Let D be a finite dimensional domain. Then every semistar operation on D is spectral if and only if D is a discrete valuation domain.

A valuation domain V with value group Γ is called discrete if, for every pair H_1, H_2 of adjacent convex subgroups of Γ (say, $H_1 \subsetneq H_2$), the ordered factor group $\frac{H_2}{H_1}$ is order isomorphic with $\mathbf{Z}$. Next, for totally ordered abelian additive groups A and B , we introduce the lexicographic order on $A \oplus B$; If $x = a + b \in A \oplus B$ with $b > 0$, then $x > 0$.

We posed a question in [M3]. Thus, let A_1, A_2, B be non-zero totally ordered abelian additive groups. Let $v_0 : A_1 \oplus A_2 \oplus B \rightarrow A_1 \oplus B$ be the projection, let $w_0 : A_1 \oplus A_2 \oplus B \rightarrow A_2 \oplus B$ be the projection, and let $u_0 : A_1 \oplus A_2 \oplus B \rightarrow B$ be the projection. Let k be a field, let $k[X; A_1 \oplus A_2 \oplus B]$ be the group ring of $A_1 \oplus A_2 \oplus B$ over k , and let K be the quotient field of $k[X; A_1 \oplus A_2 \oplus B]$. Let $f = \sum a_i X^{\alpha_i} \in$

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$k[X; A_1 \oplus A_2 \oplus B]$ with $a_i \neq 0$ for every i , and $\alpha_i \neq \alpha_j$ for every $i \neq j$. We set $v(f) = \min_i v_0(\alpha_i)$, then v defines a valuation v on K with value group $A_1 \oplus B$. Similarly, we have a valuation w with value group $A_2 \oplus B$. Let V (resp., W) be the valuation domain of v (resp., w), and set $D := V \cap W$. Our question is the following:

[M3, Question 3.4]: Let A be the set of positive real numbers. For every $r \in A$ and for every $k \in \mathbf{Z}$, let e_r (resp., ϵ_r) be a symbol, let ke_r (resp., $k\epsilon_r$) be a copy of k , and let $\mathbf{Z}e_r$ (resp., $\mathbf{Z}\epsilon_r$) be a copy of $\mathbf{Z}$. Now set $A_1 := \bigoplus_{r \in A} \mathbf{Z}e_r$, set $A_2 := \bigoplus_{r \in A} \mathbf{Z}\epsilon_r$, and set $B := \mathbf{Z}e$, where e is another symbol. Then, is every semistar operation on D spectral? In this paper, we study the question and give a negative answer.

1. A Theorem of E. Houston

Let D be a domain with quotient field K . A star operation on D is a mapping $\star : F(D) \rightarrow F(D)$, $E \mapsto E^\star$, such that for every $x \in K \setminus \{0\}$ and for every $E, F \in F(D)$, the following conditions hold: $D^\star = D$; $(xE)^\star = xE^\star$; $E \subseteq F$ implies $E^\star \subseteq F^\star$; $E \subseteq E^\star$, and $(E^\star)^\star = E^\star$.

For a Prüfer domain D , call two maximal ideals M, N of D dependent if $M \cap N$ contains a non-zero prime ideal. This defines an equivalence relation on the set of maximal ideals of D , and we denote by $\{A_\lambda \mid \lambda \in \Lambda\}$ the corresponding partition. For each $\lambda \in \Lambda$, denote by P_λ the largest prime ideal of D contained in $\bigcap_{M \in A_\lambda} M$, and let $S_\lambda = \bigcap_{M \in A_\lambda} D_M$. Recently, E. Houston proved the following,

Theorem 1 ([H]). An integrally closed domain D admits only finitely many star operations if and only if D is a Prüfer domain which satisfies the following conditions: (1) Each non-zero element of D is contained in only finitely many maximal ideals, (2) $|A_\lambda| = 1$ for almost all $\lambda \in \Lambda$, (3) $|\text{Spec}(\frac{D}{P_\lambda})| < \infty$ for each $\lambda \in \Lambda$, and (4) D has only finitely many non-invertible maximal ideals.

Moreover, under these conditions, if we denote by $\text{Star}(D)$ the set of star operations on D , then $|\text{Star}(D)| = \prod_{\lambda \in \Lambda} |\text{Star}(S_\lambda)|$.

For semistar operations, we have the following,

Theorem 2 ([M2, Section 5, (5.2), Theorem]). An integrally closed domain D admits only finitely many semistar operations if and only if D is a finite dimensional Prüfer domain which has only finitely many maximal ideals.

2. An Answer to Question

Throughout the section, as in §0, let A be the set of positive real numbers, for every $r \in A$ and for every $k \in \mathbf{Z}$, let e_r (resp., ϵ_r) be a symbol, let ke_r (resp., $k\epsilon_r$) be a copy of k , let $\mathbf{Z}e_r$ (resp., $\mathbf{Z}\epsilon_r$) be a copy of $\mathbf{Z}$, let $A_1 := \bigoplus_{r \in A} \mathbf{Z}e_r$, let $A_2 := \bigoplus_{r \in A} \mathbf{Z}\epsilon_r$, and let $B := \mathbf{Z}e$, where e is another symbol. Let $v_0 : A_1 \oplus A_2 \oplus B \rightarrow A_1 \oplus B$ be the projection, let $w_0 : A_1 \oplus A_2 \oplus B \rightarrow A_2 \oplus B$ be the projection, and let $u_0 : A_1 \oplus A_2 \oplus B \rightarrow B$ be the projection. Let k be a field, let $k[X; A_1 \oplus A_2 \oplus B]$ be the group ring of $A_1 \oplus A_2 \oplus B$ over k , and let K be the quotient field of $k[X; A_1 \oplus A_2 \oplus B]$. For $f = \sum a_i X^{\alpha_i} \in k[X; A_1 \oplus A_2 \oplus B]$ with $a_i \neq 0$ for every i , and $\alpha_i \neq \alpha_j$ for every

$i \neq j$, we set $v(f) := \min_i v_0(\alpha_i)$, then v defines a valuation v on K with value group $A_1 \oplus B$. Similarly, we have valuations w and u on K ; $v : K \rightarrow (A_1 \oplus B) \cup \{\infty\}$, $w : K \rightarrow (A_2 \oplus B) \cup \{\infty\}$, $u : K \rightarrow B \cup \{\infty\}$. Let V (resp., W, U) be the valuation domain of v (resp., w, u), and let M' (resp., N', P'_0) be the maximal ideal of V (resp., W, U), set $D := V \cap W$, and set $M := M' \cap D$, $N := N' \cap D$, and $P_0 := D \cap P'_0$. Then $M' = MV$, $N' = NW$ and $P'_0 = P_0U$. Moreover, D is a Prüfer domain with exactly two maximal ideals M and N . Clearly, P_0 is a height one prime ideal of D with $P_0 \subseteq M \cap N$.

Lemma 1 ([M1, Lemmas 2 and 4]). (1) D is a Bezout domain.

(2) $\bar{F}(D) \setminus F(D) = \{K\}$.

(3) Every star operation $\star$ on D can be uniquely extended to a semistar operation on D .

Lemma 2 ([M4, Example 1]). Every non-zero convex subgroup of A_1 is generated by a subset of $\{e_r \mid r \in A\}$. And, for every $r \in A$, the subgroups $L_r = \{k_1 e_{r_1} + \cdots + k_n e_{r_n} \mid r_1 < \cdots < r_n \leq r, \text{ every } k_i \in \mathbf{Z}\}$ and $L'_r = \{k_1 e_{r_1} + \cdots + k_n e_{r_n} \mid r_1 < \cdots < r_n < r, \text{ every } k_i \in \mathbf{Z}\}$ are convex subgroups of A_1 . The set $\{L_r, L'_r \mid r \in A\} \cup \{\{0\}, A_1\}$ is the set of all convex subgroups of A_1 .

Lemma 3. (1) For every $r \in A$, set $H_r = \{k_1 e_{r_1} + \cdots + k_n e_{r_n} \mid r_1 < \cdots < r_n \leq r, \text{ every } k_i \in \mathbf{Z}\}$, and set $H'_r = \{k_1 e_{r_1} + \cdots + k_n e_{r_n} \mid r_1 < \cdots < r_n < r, \text{ every } k_i \in \mathbf{Z}\}$. Then the set $\{H_r, H'_r \mid r \in A\} \cup \{A_1\} \cup \{\{0\}, A_1 \oplus B\}$ is the set of all convex subgroups of $A_1 \oplus B$.

(2) If Q is a non-zero prime ideal of D with $Q \subseteq M \cap N$, then $Q = P_0$.

Proof. (1) The proof follows from Lemma 2.

(2) Suppose that $Q \not\supseteq P_0$. Take an element $d \in Q \setminus P_0$. Set $v(d) = \sum a_i e_i + ce$ with the $a_i, c \in \mathbf{Z}$. Since $d \notin P_0$, we have $c = 0$. Set $x := X^{\sum a_i e_i}$. Then $v(d) = v(x) = \sum a_i e_i$. Since $w(x) = 0$, we have $x \in Q \setminus N$, hence $Q \not\subseteq N$; a contradiction.

Lemma 4. D has an infinite number of semistar operations $\star$ with $D^\star = D$.

Proof. By Lemma 3 (1), we have $\dim(D) = \infty$. By Lemma 3 (2), P_0 is the largest prime ideal of D contained in $M \cap N$. Since $|\text{Spec}(\frac{D}{P_0})| = \infty$, D has an infinite number of star operations by Theorem 1, that is, $|\text{Star}(D)| = \infty$. Every member $\star \in \text{Star}(D)$ can be uniquely extended to a semistar operation $\star'$ on D (Lemma 1 (3)). Since $D^{\star'} = D$, the proof is complete.

Lemma 5. (1) $\star_{\{M, N\}} = \text{id}$.

(2) Let P, Q be incomparable prime ideals of D . Set $\star = \star_{\{P, Q\}}$, and set $T = D_P \cap D_Q$. Then $E^\star = ET$ for every $E \in F(D)$.

Proof. (1) Let $E \in F(D)$ and let $x \in E^\star = ED_M \cap ED_N$. We have $x = \frac{e_1}{s} = \frac{e_2}{t}$, where $e_1, e_2 \in E$, $s \in D \setminus M$ and $t \in D \setminus N$. Then $(s, t) = D$, hence $1 = sd_1 + td_2$ with $d_1, d_2 \in D$. Then $x = sxd_1 + txd_2 = e_1d_1 + e_2d_2 \in E$, hence $E^\star = E$, and hence $\star =$

d.

(2) Set $\Delta := \{P, Q\}$. We may assume that $P \subseteq M$ and $Q \subseteq N$. Set $PD_P \cap T = M_1, QD_Q \cap T = N_1$. T is a Prüfer domain with exactly two maximal ideals M_1, N_1 . Further $T_{M_1} = D_P, T_{N_1} = D_Q$. Obviously $D^* = T$. Let J be a non-zero ideal of T . As in (1), $J = JT_{M_1} \cap JT_{N_1} = JD_P \cap JD_Q$. Let I be a non-zero ideal of D . Then $I^* = (ID)^* = (IT)^* = IT$, i.e., $I^* = IT$.

Lemma 6. Let $\Delta = \{P_\lambda \mid \lambda \in \Lambda\}$ with every $P_\lambda \subseteq M$, let $\star = \star_\Delta$, and set $P := \cup_\lambda P_\lambda$.

(1) We have $D^* = \cap_\lambda D_{P_\lambda} = D_P$.

(2) If $P = M$, then, for every prime ideal Q of D with $Q \subsetneq M$, we have $Q \subseteq P_\lambda$ for some λ .

Proof. (1) Let $x \notin D_P$, and set $m := \frac{1}{x}$. Since D_P is a valuation domain, we have $m \in PD_P$, and $sm \in P$ for some $s \in D \setminus P$. Then $sm \in P_\lambda$ for some λ . Since $1 = mx = sm \frac{x}{s} \in P_\lambda \frac{x}{s}$, we have $\frac{x}{s} \notin D_{P_\lambda}$, hence $x \notin D_{P_\lambda}$. Therefore $\cap_\lambda D_{P_\lambda} = D_P$.

(2) Take an element $d \in M \setminus Q$. Then we have $\frac{d}{1} \in P_\lambda D_M$ for some λ and $\frac{d}{1} \notin QD_M$. It follows that $P_\lambda D_M \supseteq QD_M$. Hence $P_\lambda \supseteq Q$.

Lemma 7. Let $\Delta_1 = \{P_\lambda \mid \lambda \in \Lambda\}$ with every $P_\lambda \subseteq M$, let $\Delta_2 = \{Q_\sigma \mid \sigma \in \Sigma\}$ with every $Q_\sigma \subseteq N$, set $\Delta = \Delta_1 \cup \Delta_2$, and let $\star = \star_\Delta$. Set $P = \cup_\lambda P_\lambda$, and set $Q = \cup_\sigma Q_\sigma$.

(1) $D^* = \cap_\lambda D_{P_\lambda} \cap_\sigma D_{Q_\sigma} = D_P \cap D_Q$.

(2) $D^* = D$ if and only if $P = M$ and $Q = N$.

Proof. (1) follows from Lemma 6 (1).

(2) $T := D_P \cap D_Q$ is a Prüfer domain with maximal ideals $PD_P \cap T$ and $QD_Q \cap T$. Assume that $D^* = D$. Then $T = D$ by (1). Hence $PD_P \cap D = M$ and $QD_Q \cap D = N$. Hence $P = M$ and $Q = N$.

Lemma 8. Let X_0 be the set of spectral semistar operations $\star$ with $D^* = D$. Then $|X_0| = 4$.

Proof. We confer Lemmas 5, 6 and 7. Let $\star \in X_0$, and let $\star = \star_\Delta$. By Lemma 6 (1), we have $\Delta = \Delta_1 \cup \Delta_2$, $\Delta_1 = \{P_\lambda \mid \lambda \in \Lambda\}$ with every $P_\lambda \subseteq M$ and $\Delta_2 = \{Q_\sigma \mid \sigma \in \Sigma\}$ with every $Q_\sigma \subseteq N$. Set $P := \cup_\lambda P_\lambda$ and $Q := \cup_\sigma Q_\sigma$. By Lemma 7, we have $P = M$ and $Q = N$. We have the following four cases: (1) $\Delta_1 \ni M$ and $\Delta_2 \ni N$, (2) $\Delta_1 \not\ni M$ and $\Delta_2 \ni N$, (3) $\Delta_1 \ni M$ and $\Delta_2 \not\ni N$, (4) $\Delta_1 \not\ni M$ and $\Delta_2 \not\ni N$.

The case (1): $\star = d$.

The case (2): We may assume that Δ_1 is the set of all prime ideals P' of D with $P' \subsetneq M$. Then $M^* = D$ and $N^* = N$.

The case (3): We may assume that Δ_2 is the set of all prime ideals Q' of D with $Q' \subsetneq N$. Then $M^* = M$ and $N^* = D$.

The case (4): We may assume that Δ_1 (resp., Δ_2) is the set of all prime ideals

of D with $P' \subsetneq M$ (resp., $Q' \subsetneq N$). Then $M^\star = D$ and $N^\star = D$.

Lemmas 4 and 8 imply the following,

There is an infinite number of non-spectral semistar operations $\star$ on D with $D^\star = D$.

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