Generalizations of Bateman's transformation for general indefinite metrics

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Abstract

Bateman's transformation is associated with the Lorentzian metric and preserves solutions of the wave equation. We generalize Bateman's transformation for general indefinite semi-euclidean metrics. Then we show that the generalized transformation preserves solutions of the equation associated with given indefinite metric.

1. Introduction

Let x, y, z, t be the coordinate of $\mathbb{R}^4$. In [1], Bateman introduced his transformation defined on $\{z - ct \neq 0\} \subset \mathbb{R}^4$, which transforms a function $w(x, y, z, t)$ to

$$\frac{1}{z - ct} w(B(x, y, z, t)) := \frac{1}{z - ct} w\left(\frac{x}{z - ct}, \frac{y}{z - ct}, \frac{r^2 - 1}{2(z - ct)}, \frac{r^2 + 1}{2c(z - ct)}\right), \quad (1.1)$$

where $r^2 = x^2 + y^2 + z^2 - c^2 t^2$. Bateman's transformation preserves the solutions of the wave equation

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 w}{\partial t^2} = 0.$$

Note that the above wave operator is associated with the indefinite metric

$$\langle (x, y, z, t), (x, y, z, t) \rangle := x^2 + y^2 + z^2 - c^2 t^2.$$

In our previous paper [3], we determined the transformations which preserve the solution of the wave equation on $\mathbb{R}^n$ ($n \geq 3$). There we introduced a transformation which is a modification of (1.1). We called the transformation Bateman transformation, which played crucial roles in that paper. In [4], we gave some relations between the Lorentzian inversion and the Bateman transformation. The relations enable us to generalize the Bateman's transformation (1.1) naturally to general metrics.

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The key was the following relation. Let $K(x, y, z, t) = (\frac{x}{r^2}, \frac{y}{r^2}, \frac{z}{r^2}, \frac{t}{r^2})$ be the Lorentzian inversion, and put $S(x, y, z, t) = (x, y, z - \frac{1}{2}, t + \frac{1}{2c})$ and $T(x, y, z, t) = (x, y, z + \frac{1}{2}, t + \frac{1}{2c})$. Then B, K, S , and T satisfy

$$B = SKTKS. \quad (1.2)$$

The purpose of this note is to generalize the Bateman's transformation to general indefinite semi-euclidean metrics by making use of a relation corresponds to (1.2) and to show that the generalized transformation preserve the solutions of the equation associated with the given indefinite metric.

2. U_γ -morphism

Let $\mathbb{R}^n$ be the n -dimensional Euclidean space ($n \geq 2$), and denote its point by $x = (x_1, \dots, x_n)$. Let $\gamma = (\gamma_{ij})$ be a non-degenerate symmetric (n, n) -matrix of signature (p, q) where $p + q = n$ and $1 \leq p, q \leq n - 1$. We define an indefinite scalar product on $\mathbb{R}^n$ by

$$\langle u, v \rangle = \langle u, v \rangle_\gamma = \sum_{i,j=1}^n \gamma_{ij} u_i v_j.$$

We consider the second order partial differential operator associated with γ on $\mathbb{R}^n$

$$U_\gamma w = \sum_{i,j=1}^n \gamma^{ij} \frac{\partial^2 w}{\partial x_i \partial x_j}, \quad (2.1)$$

where (γ^{ij}) is the inverse matrix of (γ_{ij}) .

Definition 2.1. Let $D \subset \mathbb{R}^n$ be an open set, $f : D \rightarrow \mathbb{R}^n$ a C^2 -mapping, and φ a non-vanishing C^2 -function on D . A pair (f, φ) is called a U_γ -morphism, if f and φ satisfy the following conditions:

- (1) $f'(x)$ is non-degenerate for all $x \in D$.
- (2) For every solution w of the equation $U_\gamma w = 0$ on $f(D)$, $\varphi \cdot (w \circ f)$ satisfies the equation $U_\gamma(\varphi \cdot (w \circ f)) = 0$ on D .

Here, $f'(x) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is the first derivative of f at $x \in D$ defined by $f'(x)v = \lim_{h \rightarrow 0} \frac{f(x + hv) - f(x)}{h}$.

By definition, it is easy to see that if (f, φ_f) and (g, φ_g) are U_γ -morphisms such that the image of f is contained in the domain of g , then $(g \circ f, (\varphi_g \circ f) \cdot \varphi_f)$ is also a U_γ -morphism. We call this transformation $(g \circ f, (\varphi_g \circ f) \cdot \varphi_f)$ the composition of (f, φ_f) and (g, φ_g) .

We list some examples of U_γ -morphism. Let $O_\gamma(n) = \{R \in \text{GL}(n, \mathbb{R}); R\gamma^{-1}R = \gamma^{-1}\}$.

Example 2.1 (similarity). Let $\alpha, C \in \mathbb{R}$, $\alpha > 0$, $C \neq 0$, $R \in O_\gamma(n)$, $a \in \mathbb{R}^n$,

$$f(x) = \alpha R x + a, \quad \varphi(x) = C.$$

Then the pair (f, φ) is a U_γ -morphism on $\mathbb{R}^n$. We call this transformation similarity.

Example 2.2 (inversion). Let $D = \{x; \langle x, x \rangle \neq 0\}$ and let

$$K(x) = \frac{1}{\langle x, x \rangle} x, \quad \varphi_K(x) = |\langle x, x \rangle|^{\frac{2-n}{2}}.$$

Then the pair (K, φ_K) is a U_γ -morphism defined on D . We call this transformation inversion.

In fact, K satisfies $K^2 = I$ and this implies that $K'(x)$ is non degenerate on D . Next, by the chain rule,

$$\begin{aligned} U_\gamma(\varphi \cdot (w \circ K)) &= \sum_{i,j=1}^n \gamma^{ij} \left(\frac{\partial^2 \varphi}{\partial x_i \partial x_j} \cdot (w \circ K) + 2 \frac{\partial \varphi}{\partial x_i} \frac{\partial}{\partial x_j} (w \circ K) + \varphi \cdot \frac{\partial^2}{\partial x_i \partial x_j} (w \circ K) \right) \\ &= (U_\gamma \varphi) \cdot (w \circ K) + \sum_{k=1}^n \left(2 \sum_{i,j=1}^n \gamma^{ij} \frac{\partial \varphi}{\partial x_i} \frac{\partial K_k}{\partial x_j} + \varphi \sum_{i,j=1}^n \gamma^{ij} \frac{\partial^2 K_k}{\partial x_i \partial x_j} \right) \cdot \left(\frac{\partial w}{\partial y_k} \circ K \right) \\ &\quad + \varphi \sum_{k,l=1}^n \left(\sum_{i,j=1}^n \gamma^{ij} \frac{\partial K_k}{\partial x_j} \frac{\partial K_l}{\partial x_i} \right) \cdot \left(\frac{\partial^2 w}{\partial y_l \partial y_k} \circ K \right) \end{aligned}$$

holds. Since

$$\begin{aligned} \sum_{i,j=1}^n \gamma^{ij} \frac{\partial^2 \varphi}{\partial x_i \partial x_j} &= 0, \quad \sum_{i,j=1}^n \gamma^{ij} \frac{\partial \varphi}{\partial x_i} \frac{\partial K_k}{\partial x_j} = -(2-n) \frac{x_k}{\langle x, x \rangle^2} \varphi, \\ \sum_{i,j=1}^n \gamma^{ij} \frac{\partial^2 K_k}{\partial x_i \partial x_j} &= 2(2-n) \frac{x_k}{\langle x, x \rangle^2}, \quad \sum_{i,j=1}^n \gamma^{ij} \frac{\partial K_l}{\partial x_i} \frac{\partial K_k}{\partial x_j} = \frac{\gamma^{lk}}{\langle x, x \rangle^2}, \end{aligned}$$

we have

$$U_\gamma(\varphi \cdot (w \circ K)) = \varphi \sum_{k,l=1}^n \left(\frac{\partial^2 w}{\partial y_l \partial y_k} \circ K \right) \frac{\gamma^{kl}}{\langle x, x \rangle^2} = \frac{\varphi}{\langle x, x \rangle^2} ((U_\gamma w) \circ K),$$

which shows that (K, φ) is a U_γ -morphism.

3. Generalization of Bateman's transformation

First we prepare the involution associated with the metric. Let

$$\gamma = |\gamma|J, \quad (|\gamma| = \sqrt{t\gamma\gamma}, \quad J \in O(n))$$

be the polar decomposition of γ . Since γ is real symmetric, $|\gamma| = \sqrt{\gamma^2}$, $|\gamma|$ is also real symmetric, and hence $\gamma|\gamma| = |\gamma|\gamma$ holds. Therefore the matrices $\gamma, \gamma^{-1}, |\gamma|, |\gamma|^{-1}, J, J^{-1}$ are mutually commutative. Hence

$$J^2 = I, \quad J \in O_\gamma(n)$$

holds because $J\gamma^{-1t}J = \gamma^{-1}J^2 = \gamma^{-1}$. Since $\gamma J = \gamma^2|\gamma|^{-1} = |\gamma|$ is positive definite,

$$\langle x, Jx \rangle_\gamma = \langle x, x \rangle_{|\gamma|} > 0 \quad (x \neq 0).$$

Let $u \in \mathbb{R}^n$ satisfy $u \neq 0$ and $\langle u, u \rangle = 0$. We define a conjugate of u by

$$u^* = \frac{Ju}{\langle u, Ju \rangle}.$$

Then

$$(u^*)^* = u, \quad \langle u^*, u^* \rangle = 0$$

because $J^2 = I$. Since $\langle u, u^* \rangle = 1$, u and u^* are linearly independent. Note that

$$(\alpha u)^* = \frac{1}{\alpha} u^* \quad (\alpha > 0), \quad (3.1)$$

$$\langle \alpha u + \beta u^*, \alpha u + \beta u^* \rangle = 2\alpha\beta \quad (\alpha, \beta \in \mathbb{R}) \quad (3.2)$$

For $a \in \mathbb{R}^n$, we define the translation T_a by $T_a(x) = x - a$. Clearly $T_a(Ax) = AT_{A^{-1}a}(x)$ holds for any $A \in \text{GL}(n, \mathbb{R})$.

Now we generalize Bateman's mapping and transformation. Let $u \in \mathbb{R}^n$ satisfy $u \neq 0$ and $\langle u, u \rangle = 0$. We define the (γ, u) -Bateman mapping B_u on $\{x; \langle x, u \rangle \neq 0\}$ by

$$B_u(x) = T_{u^*} K T_{-\frac{1}{2}u} K T_{u^*}(x).$$

Next we put

$$\varphi_{B_u}(x) = |\langle x, u \rangle|^{\frac{2-n}{2}}.$$

We call the pair (B_u, φ_{B_u}) (γ, u) -Bateman transformation.

This (γ, u) -Bateman transformation is a generalization of the original Bateman's transformation (see Remark 1 below).

In our previous papers [3] and [4], we called the mapping $B = B_u J$ Bateman mapping, where $\gamma = \text{diag}(-1, 1, \dots, 1)$ and $u = (1, 1, 0, \dots, 0)$.

4. Results

Theorem 4.1. *Let $u \in \mathbb{R}^n$ with $u \neq 0$ and $\langle u, u \rangle = 0$. Then B_u and φ_{B_u} satisfy the following.*

(1)

$$B_u(x) = \frac{1}{\langle x, u \rangle} \left(\frac{\langle x, x \rangle}{2} u - u^* + Q_u x \right), \quad (4.1)$$

where $Q_u x = x - \langle x, u^* \rangle u - \langle x, u \rangle u^*$ is a linear mapping.

(2) $B_u^{-1} = B_{-u}$.

(3) (B_u, φ_{B_u}) is a U_γ -morphism.

Proof. We first prove (1). Since $\langle u, u \rangle = 0$, $\langle u^*, u^* \rangle = 0$, and $\langle u, u^* \rangle = 1$,

$$(K T_{-\frac{1}{2}u} K)(y) = \frac{y + \frac{1}{2}\langle y, y \rangle u}{1 + \langle y, u \rangle + \frac{1}{4}\langle y, y \rangle \langle u, u \rangle} = \frac{y + \frac{1}{2}\langle y, y \rangle u}{1 + \langle y, u \rangle} = \frac{y + \frac{1}{2}\langle y, y \rangle u}{\langle u^*, u \rangle + \langle y, u \rangle}.$$

Putting $x = y + u^* = T_{u^*}^{-1}(y)$, the right hand side equals to

$$\begin{aligned} \frac{x - u^* + \frac{1}{2}\langle x - u^*, x - u^* \rangle u}{\langle x, u \rangle} &= \frac{1}{\langle x, u \rangle} \left(x - u^* + \frac{\langle x, x \rangle}{2} u - \langle x, u^* \rangle u \right) \\ &= \frac{1}{\langle x, u \rangle} \left(\frac{\langle x, x \rangle}{2} u - u^* + Q_u x \right) + u^*. \end{aligned}$$

Therefore $B_u(x) = T_{u^*} K T_{-\frac{1}{2}u} K T_{u^*}(x) = \frac{1}{\langle x, u \rangle} \left(\frac{\langle x, x \rangle}{2} u - u^* + Q_u x \right)$ holds.

Next, since $(-u)^* = -u^*$ and $K^{-1} = K$,

$$B_u^{-1} = T_{-u^*} K^{-1} T_{\frac{1}{2}u} K^{-1} T_{-u^*} = T_{(-u)^*} K T_{-\frac{1}{2}(-u)} K T_{(-u)^*} = B_{-u}.$$

Finally, to show (3), it suffices to show that (B_u, φ_{B_u}) is the composition of U_γ -morphisms $(T_{u^*}, 1)$, (K, φ_K) , $(T_{-\frac{u}{2}}, 1)$, (K, φ_K) , and $(T_{u^*}, 1)$. In fact, since

$$\begin{aligned} \langle T_{-\frac{1}{2}u} K(y), T_{-\frac{1}{2}u} K(y) \rangle \langle y, y \rangle &= \left\langle \frac{y}{\langle y, y \rangle} + \frac{u}{2}, \frac{y}{\langle y, y \rangle} + \frac{u}{2} \right\rangle \langle y, y \rangle \\ &= 1 + \langle u, y \rangle = \langle u, u^* + y \rangle, \end{aligned}$$

by substituting $y = T_{u^*}(x) = x - u^*$ into above, we have

$$\langle T_{-\frac{1}{2}u} K T_{u^*}(x), T_{-\frac{1}{2}u} K T_{u^*}(x) \rangle \langle T_{u^*}(x), T_{u^*}(x) \rangle = \langle u, x \rangle.$$

Then

$$\begin{aligned} \varphi_{T_{u^*} K T_{-\frac{1}{2}u} K T_{u^*}}(x) &= 1 \cdot \varphi_K(T_{-\frac{1}{2}u} K T_{u^*}(x)) \cdot 1 \cdot \varphi_K(T_{u^*}(x)) \cdot 1 \\ &= \varphi_K(T_{-\frac{1}{2}u} K T_{u^*}(x)) \varphi_K(T_{u^*}(x)) \\ &= |\langle u, x \rangle|^{\frac{2-n}{2}} = \varphi_{B_u}(x). \end{aligned}$$

Therefore, (B_u, φ_{B_u}) is a U_γ -morphism. $\square$

Remark 1. The above (B_u, φ_{B_u}) is actually a generalization of the original Bateman's transformation. If $n = 4$, $\gamma = \text{diag}(1, 1, 1, -c^2)$ and $u = (0, 0, 1, \frac{1}{c})$, then $J = \text{diag}(1, 1, 1, -1)$, $u^* = (0, 0, \frac{1}{2}, -\frac{1}{2c})$, $\langle x, u \rangle = x_3 - cx_4$, and $Q_u x = (x_1, x_2, 0, 0)$. By (4.1), $B_u(x) = \frac{1}{x_3 - cx_4} (x_1, x_2, \frac{\langle x, x \rangle - 1}{2}, \frac{\langle x, x \rangle + 1}{2c})$ on $D = \{x \in \mathbb{R}^4; x_3 - cx_4 > 0\}$ and (γ, u) -Bateman transformation is

$$\frac{1}{x_3 - cx_4} w\left(\frac{x_1}{x_3 - cx_4}, \frac{x_2}{x_3 - cx_4}, \frac{\langle x, x \rangle - 1}{2(x_3 - cx_4)}, \frac{\langle x, x \rangle + 1}{2c(x_3 - cx_4)}\right),$$

which is equal to Bateman's transformation on D .

Proposition 4.1. Let u satisfy $u \neq 0$ and $\langle u, u \rangle = 0$ and let $\alpha \in \mathbb{R} \setminus \{0\}$. Then the linear map Q_u has the following properties;

$$\begin{aligned} Q_u u &= Q_u u^* = 0, \quad Q_u^2 = Q_u, \quad \langle Q_u x, u \rangle = \langle Q_u x, u^* \rangle = 0, \\ Q_{\alpha u} &= Q_u = Q_{u^*} = Q_{Ju}, \quad \langle Q_u x, y \rangle = \langle x, Q_u y \rangle. \end{aligned}$$

Proof. First, $Q_u u = Q_u u^* = 0$ and $\langle Q_u x, u \rangle = \langle Q_u x, u^* \rangle = 0$ follows from $\langle u, u \rangle = \langle u^*, u^* \rangle = 0$ and $\langle u, u^* \rangle = 1$. Then $Q_u u = Q_u u^* = 0$ implies $Q_u^2 = Q_u$. Since $(u^*)^* = u$ and $(\alpha u)^* = \alpha^{-1} u^*$, $Q_{\alpha u} = Q_u = Q_{u^*} = Q_{Ju}$ holds. Finally, $\langle Q_u x, y \rangle = \langle x, Q_u y \rangle$ follows by interchanging x and y in $\langle Q_u x, y \rangle = \langle x, y \rangle - \langle x, u^* \rangle \langle u, y \rangle - \langle x, u \rangle \langle u^*, y \rangle$. $\square$

Remark 2. The above proposition suggests that every $x \in \mathbb{R}^n$ can be written as $x = \langle x, u^* \rangle u + \langle x, u \rangle u^* + Q_u x$ and the vectors $u, u^*, Q_u x$ are mutually orthogonal with respect to γ . Conversely, if $x \in \mathbb{R}^n$ can be written as $x = \alpha u + \beta u^* + z$ with $\alpha, \beta \in \mathbb{R}$ and $\langle u, z \rangle = \langle u^*, z \rangle = 0$, then $z = Q_u x$. Therefore we can regard Q_u as a “ γ -orthogonal projection to the complement of the subspace spanned by u and its adjoint u^* ”.

By definition, (γ, u) -Bateman transformation is a composition of inversion and translations. Conversely, inversion is a composition of (γ, u) -Bateman transformation and similarities. We introduce a linear map $S_v x := x - \frac{2\langle x, v \rangle}{\langle v, v \rangle} v$ for v with $\langle v, v \rangle \neq 0$.

Proposition 4.2. For v with $\langle v, v \rangle \neq 0$ and $\alpha \in \mathbb{R} \setminus \{0\}$, S_v satisfies

$$S_{\alpha v} = S_v, \quad S_v \in O_\gamma(n), \quad S_v^2 = I, \quad \left\langle \frac{x + S_v x}{2}, v \right\rangle = 0.$$

Proof. $S_{\alpha v} = x - \frac{2\langle x, \alpha v \rangle}{\langle \alpha v, \alpha v \rangle} \alpha v = x - \frac{2\langle x, v \rangle}{\langle v, v \rangle} v = S_v$ by definition. $S_v \in O_\gamma(n)$ because

$$\langle S_v x, S_v x \rangle = \langle x, x \rangle - 4 \frac{\langle x, v \rangle}{\langle v, v \rangle} \langle x, v \rangle + 4 \frac{\langle x, v \rangle^2}{\langle v, v \rangle^2} \langle v, v \rangle = \langle x, x \rangle.$$

Since $\langle S_v x, v \rangle = \langle x, v \rangle - 2 \frac{\langle x, v \rangle}{\langle v, v \rangle} \langle v, v \rangle = -\langle x, v \rangle$, $\left\langle \frac{x + S_v x}{2}, v \right\rangle = 0$ and

$$S_v^2 x = S_v x - 2 \frac{\langle S_v x, v \rangle}{\langle v, v \rangle} v = S_v x + 2 \frac{\langle x, v \rangle}{\langle v, v \rangle} v = x. \quad \square$$

Remark 3. The above proposition suggests that S_v can be regarded as a “reflection with respect to the hyperplane which is γ -orthogonal to v ”.

Theorem 4.2. Let $u \in \mathbb{R}^n$ satisfy $u \neq 0$ and $\langle u, u \rangle = 0$. Put $v = u + Ju$. Then

$$K = \frac{\langle u, Ju \rangle}{2} S_v B_u B_{Ju} = \frac{\langle u, Ju \rangle}{2} S_v B_u J B_u J \quad (4.2)$$

holds. This shows that the inversion is a composition of Bateman transformations and similarities.

Proof. First we shall show $J B_u J = B_{Ju}$. Since $J \in O_\gamma(n)$ and $J^2 = I$,

$$\begin{aligned} J B_u J(x) &= \frac{1}{\langle Jx, u \rangle} \left(\frac{\langle Jx, Jx \rangle}{2} Ju - Ju^* + J(Jx - \langle Jx, u^* \rangle u - \langle Jx, u \rangle u^*) \right) \\ &= \frac{1}{\langle x, Ju \rangle} \left(\frac{\langle x, x \rangle}{2} Ju - (Ju)^* + x - \langle x, (Ju)^* \rangle Ju - \langle x, Ju \rangle (Ju)^* \right) \\ &= B_{Ju}. \end{aligned}$$

Therefore, it suffices to show the first equality in (4.2). Since $\langle (Ju)^*, u \rangle = 0$, $Q_{Ju} = Q_u$ and $\langle Q_u x, u \rangle = 0$,

$$\langle B_{Ju}(x), u \rangle = \frac{1}{\langle x, Ju \rangle} \frac{\langle x, x \rangle}{2} \langle Ju, u \rangle.$$

Since $\langle Q_u x, Q_u x \rangle = \langle Q_u x, x \rangle$ and $\langle Q_u x - x, x \rangle = -2\langle x, u \rangle \langle u^*, x \rangle$,

$$\langle B_{Ju}(x), B_{Ju}(x) \rangle = \frac{1}{\langle x, Ju \rangle^2} (\langle Q_u x, Q_u x \rangle - \langle x, x \rangle) = -\frac{2\langle x, u \rangle}{\langle x, Ju \rangle \langle u, Ju \rangle}.$$

$Q_u^2 = Q_u$ and $Q_u u = Q_u u^* = 0$ imply

$$Q_u B_{Ju}(x) = \frac{1}{\langle x, Ju \rangle} Q_u x.$$

Note that $\langle v, v \rangle = \langle u + Ju, u + Ju \rangle = 2\langle u, Ju \rangle > 0$. Therefore

$$\begin{aligned} B_u B_{Ju}(x) &= \frac{1}{\langle B_{Ju}(x), u \rangle} \left(\frac{\langle B_{Ju}(x), B_{Ju}(x) \rangle}{2} u - u^* + Q_u B_{Ju}(x) \right) \\ &= \frac{2\langle x, Ju \rangle}{\langle x, x \rangle \langle Ju, u \rangle} \left(-\frac{\langle x, u \rangle u}{\langle x, Ju \rangle \langle u, Ju \rangle} - \frac{Ju}{\langle u, Ju \rangle} + \frac{Q_u x}{\langle x, Ju \rangle} \right) \\ &= \frac{2}{\langle x, x \rangle \langle Ju, u \rangle} \left(-\frac{\langle x, u \rangle u}{\langle u, Ju \rangle} - \frac{\langle x, Ju \rangle Ju}{\langle u, Ju \rangle} + x - \frac{\langle x, Ju \rangle u}{\langle u, Ju \rangle} - \frac{\langle x, u \rangle Ju}{\langle u, Ju \rangle} \right) \\ &= \frac{2}{\langle x, x \rangle \langle Ju, u \rangle} \left(x - \frac{\langle x, u + Ju \rangle (u + Ju)}{\langle v, v \rangle / 2} \right) \\ &= \frac{2}{\langle Ju, u \rangle} \left(\frac{x}{\langle x, x \rangle} - \frac{2\langle \frac{x}{\langle x, x \rangle}, v \rangle v}{\langle v, v \rangle} \right) = \frac{2}{\langle Ju, u \rangle} S_v K(x). \end{aligned}$$

Since $S_v^2 = I$, $K(x) = \frac{\langle Ju, u \rangle}{2} S_v B_u B_{Ju}(x)$. □

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