Liouville type theorem on conformal mapping for indefinite metrics associate with ultra-hyperbolic equations

KATSUNORI SHIMOMURA*

Dedicated to Professor Noriaki Suzuki on the occasion of his sixtieth birthday

Abstract

In [4], we proved that any transformation which preserves solutions of the wave equation is a similarity, an inversion, or a Bateman transformation. This paper gives a generalization of the results to ultra-hyperbolic type equations.

1. Introduction

The well-known Liouville's theorem states that every conformal mapping in the n -dimensional Euclidean space ($n \geq 3$) is a similarity or an inversion with respect to a sphere. In [4], we considered a Liouville type theorem associate with the wave equation, and proved that every conformal mapping for Lorentzian metric is one of the following: a similarity, an inversion composed with similarities, a Bateman transformation composed with similarities. We also determined the transformations which preserve solutions of the wave equation.

Let $\mathbb{R}^n$ be the n -dimensional Euclidean space ($n \geq 2$), and denote its point by $x = (x_1, \dots, x_n)$ and the norm of x by $|x|$. Let p and q be positive integers such that $p + q = n$. We consider a second order ultra-hyperbolic type equation

$$Uu = U_{p,q}u = \sum_{j=1}^p \frac{\partial^2 u}{\partial x_j^2} - \sum_{j=1}^q \frac{\partial^2 u}{\partial x_{p+j}^2} = 0. \quad (1.1)$$

We introduce an indefinite scalar product (non-degenerate indefinite symmetric bilinear form) of signature (p, q) on $\mathbb{R}^n$ by

$$\langle x, y \rangle = \langle x, y \rangle_{p,q} = \sum_{j=1}^p x_j y_j - \sum_{j=1}^q x_{p+j} y_{p+j}, \quad (1.2)$$

Received 16 September 2015; revised 23 October 2015

2010 *Mathematics Subject Classification.* 31B99, 35L99, 35A30

Key Words and Phrases. semi-euclidean, conformal mapping, ultra-hyperbolic equation, Bateman transformation

*Partially supported by Grant-in-aid for Scientific Research (C) No.25400126, 15K04934, Japan Society for the Promotion of Science.

*Ibaraki University, Mito, Ibaraki, 310-8512, Japan. (katsunori.shimomura.sci@vc.ibaraki.ac.jp)

associate with the ultra-hyperbolic type equation (1.1).

Put $J = J_{p,q} = \begin{pmatrix} I_p & O \\ O & -I_q \end{pmatrix}$, where I_k denotes the identity matrix of degree k .

Then the scalar product $\langle \cdot, \cdot \rangle$ is written by the Euclidean inner product $(\cdot, \cdot)$ as

$$\langle x, y \rangle = (Jx, y) = (x, Jy), \quad x, y \in \mathbb{R}^n. \quad (1.3)$$

The vector space $\mathbb{R}^n$ with this scalar product $\langle \cdot, \cdot \rangle_{p,q}$ is denoted by $\mathbb{R}^{p,q}$.

Definition 1.1. Let $D \subset \mathbb{R}^n$ be a domain, $f : D \rightarrow \mathbb{R}^n$ a C^2 -mapping, and φ a positive C^2 -function on D . A pair (f, φ) is called a transformation which preserves solutions of the ultra-hyperbolic equation, if f and φ satisfy the following conditions:

(1) $f'(x)$ is non-degenerate for all $x \in D$.

(2) For every solution u of the ultra-hyperbolic type equation (1.1) on $\mathbb{R}^n$, $\varphi \cdot (u \circ f)$ satisfies the ultra-hyperbolic type equation (1.1) on D .

In this note, we shall generalize the Liouville type theorem in [4] to $\mathbb{R}^{p,q}$ with $p + q \geq 3$, and determin all the transformations which preserve solutions of the ultra-hyperbolic equation.

Let $D \subset \mathbb{R}^n$ and $E \subset \mathbb{R}^k$ be domains and $f = (f_1, \dots, f_k) : D \rightarrow E$ a C^1 -mapping. The derivative of f is the linear mapping $f'(x) : \mathbb{R}^n \rightarrow \mathbb{R}^k$ such that for $u \in \mathbb{R}^n$

$$f'(x)u = \lim_{h \rightarrow 0} \frac{f(x + hu) - f(x)}{h}.$$

Definition 1.2. Let $D, E \subset \mathbb{R}^{p,q}$ be domains. A C^1 -mapping $f : D \rightarrow E$ is said to be conformal on D if there exists a function $\lambda(x) = \lambda_f(x) > 0$ such that

$$\langle f'(x)u, f'(x)v \rangle = \lambda(x)^2 \langle u, v \rangle \quad (1.4)$$

holds for all $x \in D$, and $u, v \in \mathbb{R}^{p,q}$.

Let $O(p, q) = \{R \in GL(n, \mathbb{R}); \langle Rx, Ry \rangle = \langle x, y \rangle \text{ for all } x, y \in \mathbb{R}^n\}$. Our Liouville type theorem is the following:

Theorem 1.1. Let $n \geq 3$ and f be a C^3 -mapping on a domain $D \subset \mathbb{R}^{p,q}$. If f is conformal, then f has one of the following forms.

$$f(x) = \frac{\alpha R(x - a)}{\langle x - a, x - a \rangle} + b, \quad \lambda_f(x) = \frac{\alpha}{|\langle x - a, x - a \rangle|}, \quad (a)$$

$$f(x) = \alpha R B_d(x - a) + b, \quad \lambda_f(x) = \frac{\alpha}{|\langle d, x - a \rangle|}, \quad (b)$$

or

$$f(x) = \alpha R x + b, \quad \lambda_f(x) = \alpha, \quad (c)$$

where $\alpha > 0$, $R \in O(p, q)$, $a, b, d \in \mathbb{R}^n$ with $\langle d, d \rangle = 0$, $|d| = 1$, and

$$B_d(x) = \frac{1}{\langle d, x \rangle} \left(\frac{\langle x, x \rangle}{2} d - Jd + Q_d x \right), \quad Q_d x = x - \langle x, Jd \rangle d - \langle x, d \rangle Jd.$$

In [6], we have defined the Bateman mapping $B_d(x)$ as a composition of inversions and translations. Thus the following Corollary follows immediately.

Corollary 1.1. *Let f be a conformal C^3 -mapping on a domain $D \subset \mathbb{R}^{p,q}$. Then f is obtained by composing similarities and inversions.*

We have also proved that the inversion is a composition of Bateman transformations and similarities in [6]. Thus the following Corollary also follows immediately.

Corollary 1.2. *Let f be a conformal C^3 -mapping on a domain $D \subset \mathbb{R}^{p,q}$. Then f is obtained by composing similarities and Bateman mappings.*

We shall also determine the transformations which preserve solutions of the ultra-hyperbolic equation. Theorem 1.1 gives the following theorem.

Theorem 1.2. *Let $n \geq 3$, $f = (f_1, \dots, f_n)$ be a C^3 -mapping, and φ a positive function on a domain $D \subset \mathbb{R}^n$. Assume (f, φ) be a transformation which preserves solutions of the ultra-hyperbolic equation.*

If $p \neq q$, then f is conformal, and f and φ have one of the following forms :

$$f(x) = \frac{\alpha R(x-a)}{\langle x-a, x-a \rangle} + b, \quad \varphi(x) = C|\langle x-a, x-a \rangle|^{\frac{2-n}{2}}, \quad (\text{a})$$

$$f(x) = \alpha R B_d(x-a) + b, \quad \varphi(x) = C|\langle d, x-a \rangle|^{\frac{2-n}{2}}, \quad (\text{b})$$

$$f(x) = \alpha R x + b, \quad \varphi(x) = C, \quad (\text{c})$$

where $\alpha > 0$, $R \in O(p, q)$, $a, b \in \mathbb{R}^n$, and $d \in \mathbb{R}^n$ with $\langle d, d \rangle = 0$ and $(d, d) = 1$.

If $p = q$, then either f is conformal and (f, φ) satisfies one of the above (a), (b), and (c), or Sf is conformal and (f, φ) satisfies one of the following:

$$f(x) = \frac{\alpha S R(x-a)}{\langle x-a, x-a \rangle} + b, \quad \varphi(x) = C|\langle x-a, x-a \rangle|^{\frac{2-n}{2}}, \quad (\text{a}')$$

$$f(x) = \alpha S R B_d(x-a) + b, \quad \varphi(x) = C|\langle d, x-a \rangle|^{\frac{2-n}{2}}, \quad (\text{b}')$$

$$f(x) = \alpha S R x + b, \quad \varphi(x) = C, \quad (\text{c}')$$

where α, R, a, b, d are the same as above, and $S = \begin{pmatrix} O & I_p \\ I_p & O \end{pmatrix}$.

Remark 1. In the case of $p = q$, (f, φ) satisfies one of (a), (b), and (c), if $\langle \nabla f_j, \nabla f_j \rangle > 0$ ($1 \leq j \leq p$). If $\langle \nabla f_j, \nabla f_j \rangle < 0$ ($1 \leq j \leq p$), then (f, φ) satisfies one of (a'), (b'), and (c').

Corollary 1.3. *Let $n \geq 3$. If $p \neq q$, every transformation which preserves solutions of the ultra-hyperbolic equation is obtained by composing similarities and inversions. If $p = q$, every transformation which preserves solutions of the ultra-hyperbolic equation is obtained by composing the linear map S , and similarities and inversions.*

Corollary 1.4. *Let $n \geq 3$. If $p \neq q$, every transformation which preserves solutions of the ultra-hyperbolic equation is obtained by composing similarities and Bateman transformations. If $p = q$, every transformation which preserves solutions of the ultra-hyperbolic equation is obtained by composing S and similarities and Bateman transformations.*

2. Conformal mapping

In the previous section, we introduced the scalar product $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{p,q}$ and the associated space $\mathbb{R}^{p,q}$. A vector $u \in \mathbb{R}^{p,q}$ is called null if $\langle u, u \rangle = 0$ and $u \neq 0$. For a null vector u , we define a conjugate of u by $u^* = |u|^{-2}Ju$, so that $\langle u, u^* \rangle = 1$ and $\langle u^*, u^* \rangle = 0$. Clearly, u and u^* are linearly independent. If a null vector u satisfies $|u| = 1$, then $u^* = Ju$.

Recall that $O(p, q) = \{R \in \text{GL}(n, \mathbb{R}); \langle Rx, Ry \rangle = \langle x, y \rangle \text{ for all } x, y \in \mathbb{R}^{p,q}\}$. $O(p, q)$ is the group of all matrices which preserve the scalar product $\langle \cdot, \cdot \rangle$. It is clear that $R \in O(p, q)$ if and only if ${}^tRJR = J$. Note that $R \in O(p, q)$ if and only if $J = RJ{}^tR$.

For a subspace X of $\mathbb{R}^{p,q}$, we define a subspace $X^{\perp_J}$ of $\mathbb{R}^{p,q}$ by

$$X^{\perp_J} = \{x \in \mathbb{R}^{p,q}; \langle x, y \rangle = 0 \text{ for all } y \in X\},$$

while we denote by $X^\perp$ the Euclidean orthogonal complement of X in $\mathbb{R}^n$. The following lemma will be used in section 3.

Lemma 2.1. *Let $\alpha \neq 0$, $R \in O(p, q)$, and X be a subspace of $\mathbb{R}^{p,q}$. Then*

$$X^{\perp_J} = (JX)^\perp = J(X^\perp), \quad (X^{\perp_J})^{\perp_J} = X, \quad (2.1)$$

$$\alpha R(X^{\perp_J}) = (\alpha RX)^{\perp_J}. \quad (2.2)$$

Proof. Since $\langle x, y \rangle = (x, Jy)$, $x \in X^{\perp_J}$ is equivalent to $x \in (JX)^\perp$. Since $\langle x, y \rangle = 0$ is equivalent to $(Jx, y) = 0$ and $J^2 = I$, $x \in X^{\perp_J}$ is equivalent to $x \in J(X^\perp)$. We have the first equality. The second equality follows from $(X^{\perp_J})^{\perp_J} = J(((JX)^\perp)^\perp) = J^2X = X$. $x \in \alpha R(X^{\perp_J})$ if and only if $\langle \alpha^{-1}R^{-1}x, y \rangle = 0$ for all $y \in X$. Then if $x \in \alpha R(X^{\perp_J})$

$$\langle \alpha \alpha^{-1} R R^{-1} x, \alpha R y \rangle = \langle x, \alpha R y \rangle = 0$$

for all $y \in X$, and hence $x \in (\alpha RX)^{\perp_J}$, and vice versa. $\square$

We need the assumption $n \geq 3$ for the following lemma.

Lemma 2.2. (1) *Let $n \geq 3$, $u \in \mathbb{R}^{p,q}$ and $V = \{v; \langle u, v \rangle = 0\}$. Then $\{v \in V; \langle v, v \rangle \neq 0\}$ is dense in V .*

(2) *Let $n \geq 3$. For any $u, v \in \mathbb{R}^{p,q}$ with $\langle u, v \rangle = 0$, there exist sequences $\{u_n\}, \{v_n\}$ in $\mathbb{R}^{p,q}$ such that $|u - u_n|, |v - v_n| \rightarrow 0$, $\langle u_n, v_n \rangle = 0$, $\langle u_n, u_n \rangle \neq 0$, and $\langle v_n, v_n \rangle \neq 0$.*

Proof. (1) $n \geq 3$ implies $p \geq 2$ or $q \geq 2$. Assume $p \geq 2$. Then there exists $w_+ \in \mathbb{R}^p$ such that $w_+ \perp u_+$ where $u_+ = (u_1, \dots, u_p)$ in $\mathbb{R}^p$. Put $w = (w_+, 0, \dots, 0) \in \mathbb{R}^{p,q}$.

For $v \in V$ with $\langle v, v \rangle = 0$, put $v_t = v + tw$ ($t \in \mathbb{R}$). Then $v_t \in V$ for all $t \in \mathbb{R}$ because $\langle v_t, u \rangle = \langle v, u \rangle + t\langle w, u \rangle = 0 + t(w_+, u_+) = 0$. On the other hand, $\langle v_t, v_t \rangle = 2t\langle v, w \rangle + t^2\langle w, w \rangle \neq 0$ except for at most 2 values of t because $\langle w, w \rangle = |w_+|^2 \neq 0$. Since $\lim_{t \rightarrow 0} v_t = v$, the assertion holds in the case $p \geq 2$. In the case $q \geq 2$, the assertion follows similarly.

(2) If $\langle u, u \rangle \neq 0$, put $u_n = u$ for all n . Then by (1), there exists a sequence $\{v_n\}$ such that $\langle u_n, v_n \rangle = \langle u, v_n \rangle = 0$, $\langle v_n, v_n \rangle \neq 0$, and $|v - v_n| \rightarrow 0$.

Let $\langle u, u \rangle = 0$. Then there exists a sequence $\{u_n\}$ such that $|u - u_n| \rightarrow 0$ and $\langle u_n, u_n \rangle \neq 0$. Consider the hyperplanes $V_n = \{v : \langle u_n, v \rangle = 0\}$ and let P_n be the Euclidean orthogonal projection to V_n . For each n , (1) implies that there exists a vector v_n such that $v_n \in V_n$, $|v_n - P_n v| < 1/n$, $\langle v_n, v_n \rangle \neq 0$. Since $u_n \rightarrow u$, $P_n v \rightarrow v$ and hence $|v - v_n| \rightarrow 0$. $\square$

Lemma 2.3. (1) Let $\max(p, q) \geq 3$. Then for any $u, v \in \mathbb{R}^{p,q}$ with $\langle u, v \rangle = 0$, there exists $w \in \mathbb{R}^{p,q}$ with $\langle w, w \rangle \neq 0$ such that $\langle u, w \rangle = \langle v, w \rangle = 0$.

(2) Let $n = 3$ or $p = q = 2$. Then for any $u, v \in \mathbb{R}^{p,q}$ with $\langle u, u \rangle \neq 0$, $\langle v, v \rangle \neq 0$, and $\langle u, v \rangle = 0$, there exists $w \in \mathbb{R}^{p,q}$ with $\langle w, w \rangle \neq 0$ such that $\langle u, w \rangle = \langle v, w \rangle = 0$.

Proof. (1) Let $p \geq 3$, and $u, v \in \mathbb{R}^{p,q}$ with $\langle u, v \rangle = 0$. There exists $(w_1, \dots, w_p) \neq 0$ such that $(w_1, \dots, w_p) \perp (u_1, \dots, u_p)$ and $(w_1, \dots, w_p) \perp (v_1, \dots, v_p)$. Then the vector $w = (w_1, \dots, w_p, 0, \dots, 0) \in \mathbb{R}^{p,q}$ satisfies the conditions. The case $q \geq 3$ follows similarly.

(2) Let $n = 3$, $u, v \in \mathbb{R}^{p,q}$, and assume $\langle u, u \rangle \neq 0$, $\langle v, v \rangle \neq 0$, and $\langle u, v \rangle = 0$. Then u and v are linearly independent. Since $\dim(\mathbb{R}u + \mathbb{R}v)^{\perp_J} = 1$, there exists a non-zero vector $w \in (\mathbb{R}u + \mathbb{R}v)^{\perp_J}$. The vectors u, v, w are linearly independent. In fact, if $au + bv + cw = 0$, $a = 0$ and $b = 0$ follow by taking the scalar product with u and v , respectively. Hence $c = 0$ because $w \neq 0$. Then the linear mapping $Cx = \begin{pmatrix} \langle u, x \rangle \\ \langle v, x \rangle \\ \langle w, x \rangle \end{pmatrix}$ is

injective. Therefore $Cw = \begin{pmatrix} \langle u, w \rangle \\ \langle v, w \rangle \\ \langle w, w \rangle \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \langle w, w \rangle \end{pmatrix} \neq 0$ and hence $\langle w, w \rangle \neq 0$.

Finally, let $p = q = 2$, $u, v \in \mathbb{R}^{p,q}$, $\langle u, u \rangle \neq 0$, $\langle v, v \rangle \neq 0$, and $\langle u, v \rangle = 0$. Then u and v are linearly independent. Since $\dim(\mathbb{R}u + \mathbb{R}v)^{\perp_J} = n - 2 = 2$, there exists a pair of linearly independent vectors $x, y \in (\mathbb{R}u + \mathbb{R}v)^{\perp_J}$. The vectors u, v, x, y are linearly independent. In fact, if $au + bv + cx + dy = 0$, $a = 0$ and $b = 0$ follow by taking the scalar product with u and v , respectively. Then $c = d = 0$ follows from the linear independence of x and y . If $\langle x, x \rangle \neq 0$ or $\langle y, y \rangle \neq 0$ holds, we can choose $w = x$ or $w = y$, respectively. Assume that $\langle x, x \rangle = \langle y, y \rangle = 0$ hold. Since the linear

mapping $Cz = \begin{pmatrix} \langle u, z \rangle \\ \langle v, z \rangle \\ \langle x, z \rangle \\ \langle y, z \rangle \end{pmatrix}$ is injective and $y \neq 0$, $Cy = \begin{pmatrix} \langle u, y \rangle \\ \langle v, y \rangle \\ \langle x, y \rangle \\ \langle y, y \rangle \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \langle x, y \rangle \\ 0 \end{pmatrix} \neq 0$.

Then $w = x + y$ satisfies $\langle w, w \rangle = 2\langle x, y \rangle \neq 0$, $\langle u, w \rangle = 0$, and $\langle v, w \rangle = 0$. $\square$

The following proposition follows easily from the chain rule.

Proposition 2.1. (1) If $f : D \rightarrow E$ and $g : E \rightarrow F$ are conformal mappings, then the composition mapping $g \circ f : D \rightarrow F$ is also a conformal mapping, and

$$\lambda_{g \circ f}(x) = \lambda_g(f(x))\lambda_f(x), \quad x \in D. \quad (2.3)$$

(2) If $f : D \rightarrow E$ is a conformal mapping, then f has a local inverse f^{-1} at each point of $f(D)$, f^{-1} is also conformal, and

$$\lambda_{f^{-1}}(f(x)) = \frac{1}{\lambda_f(x)}. \quad (2.4)$$

In the following, we list fundamental conformal mappings.

Example 2.1 (Similarity). The mapping

$$f(x) = \alpha Rx + a \quad (\alpha \in \mathbb{R}, \alpha > 0, R \in O(p, q), a \in \mathbb{R}^n)$$

is a conformal mapping defined on $\mathbb{R}^{p,q}$ satisfying $\lambda_f(x) = \alpha$. We call such mapping similarity (mapping).

We introduce the notation $T_a(x) = x - a$ for later use. $\lambda_{T_a}(x) = 1$.

Example 2.2 (Inversion). The mapping

$$K(x) = \frac{x}{\langle x, x \rangle}$$

is a conformal mapping defined on each connected component of $\{x; \langle x, x \rangle \neq 0\}$. We call K the inversion (mapping). By simple calculation, we have $K^{-1} = K$ and

$$\lambda_K(x) = \frac{1}{|\langle x, x \rangle|}. \quad (2.5)$$

Example 2.3 (Bateman mapping [6]). For each null vector d , the mapping defined on each connected component of $\{x; \langle d, x \rangle \neq 0\}$

$$B_d(x) = \frac{1}{\langle d, x \rangle} \left(\frac{\langle x, x \rangle}{2} d - d^* + Q_d x \right),$$

where $Q_d x = x - \langle x, d^* \rangle d - \langle x, d \rangle d^*$, is a conformal mapping because

$$B_d = T_{d^*} \circ K \circ T_{-\frac{1}{2}d} \circ K \circ T_{d^*}.$$

We call B_d the Bateman mapping. As we have shown in [6],

$$B_d^{-1} = B_{-d}. \quad (2.6)$$

By the chain rule and (2.5),

$$\lambda_{B_d}(x) = \frac{1}{|\langle d, x \rangle|}. \quad (2.7)$$

In fact, since $\langle T_{-\frac{1}{2}d}(K(y)), T_{-\frac{1}{2}d}(K(y)) \rangle = \frac{1 + \langle d, y \rangle}{\langle y, y \rangle}$, we have

$$\langle (T_{-\frac{1}{2}d} \circ K \circ T_{d^*})(x), (T_{-\frac{1}{2}d} \circ K \circ T_{d^*})(x) \rangle = \frac{1 + \langle d, x \rangle - 1}{\langle T_{d^*}(x), T_{d^*}(x) \rangle} = \frac{\langle d, x \rangle}{\langle T_{d^*}(x), T_{d^*}(x) \rangle}.$$

By (2.3), $\lambda_{B_d}(x) = 1 \cdot \lambda_K(T_{-\frac{1}{2}d} \circ K \circ T_{d^*}(x)) \cdot 1 \cdot \lambda_K(T_{d^*}(x)) \cdot 1$ and hence

$$\lambda_{B_d}(x) = \frac{1}{|\langle (T_{-\frac{1}{2}d} \circ K \circ T_{d^*})(x), (T_{-\frac{1}{2}d} \circ K \circ T_{d^*})(x) \rangle|} \frac{1}{|\langle T_{d^*}(x), T_{d^*}(x) \rangle|} = \frac{1}{|\langle d, x \rangle|}.$$

3. Liouville type theorem for conformal mapping

In this section, we shall prove Theorem 1.1.

The first derivative of a C^1 -mapping $f : \mathbb{R}^n \rightarrow \mathbb{R}^k$ is the linear mapping $f'(x) : \mathbb{R}^n \rightarrow \mathbb{R}^k$

$$f'(x)u = \lim_{h \rightarrow 0} \frac{f(x + hu) - f(x)}{h} \quad (u \in \mathbb{R}^n).$$

The second derivative of a C^2 -mapping $f : \mathbb{R}^n \rightarrow \mathbb{R}^k$ is the symmetric bilinear mapping $f''(x) : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^k$ such that

$$f''(x)[u, v] = \lim_{h \rightarrow 0} \frac{1}{h} (f'(x + hu)v - f'(x)v) \quad (u, v \in \mathbb{R}^n).$$

The third derivative of a C^3 -mapping $f : \mathbb{R}^n \rightarrow \mathbb{R}^k$ is the symmetric trilinear mapping $f'''(x) : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^k$ such that

$$f'''(x)[u, v, w] = \lim_{h \rightarrow 0} \frac{1}{h} (f''(x + hu)[v, w] - f''(x)[v, w]) \quad (u, v, w \in \mathbb{R}^n).$$

Lemma 3.1. *If f is a conformal C^2 -mapping on a domain $D \subset \mathbb{R}^{p,q}$, then for each $u, v, w \in \mathbb{R}^{p,q}$,*

$$\langle f''(x)[u, v], f'(x)w \rangle + \langle f''(x)[u, w], f'(x)v \rangle = 2\lambda(x)\langle v, w \rangle \lambda'(x)u. \quad (3.1)$$

Note that $\lambda'(x)$ is a linear map from $\mathbb{R}^n$ to $\mathbb{R}$.

Proof. Taking the derivative in u direction of the conformality condition (1.4)

$$\langle f'(x)v, f'(x)w \rangle = \lambda(x)^2 \langle v, w \rangle,$$

we have (3.1). □

Lemma 3.2. *Let f be a conformal C^2 -mapping on a domain $D \subset \mathbb{R}^{p,q}$. If $u, v, w \in \mathbb{R}^{p,q}$ and $\langle u, v \rangle = \langle v, w \rangle = \langle w, u \rangle = 0$, then*

$$\langle f''(x)[u, v], f'(x)w \rangle = 0.$$

Proof. We put $A[u, v, w] = \langle f''(x)[u, v], f'(x)w \rangle$ for $u, v, w \in \mathbb{R}^{p,q}$. Then A satisfies

$$A[v, u, w] = A[u, v, w], \quad (3.2)$$

$$A[u, v, w] = -A[u, w, v] \text{ if } \langle v, w \rangle = 0. \quad (3.3)$$

In fact (3.2) is clear from the symmetry of the bilinear map $f''(x)$, and (3.3) immediately follows from Lemma 3.1.

If $u, v, w \in \mathbb{R}^n$ and $\langle u, v \rangle = \langle v, w \rangle = \langle w, u \rangle = 0$, then we have $A[u, v, w] = A[v, u, w]$ by (3.2), $A[v, u, w] = -A[v, w, u]$ by (3.3) and $\langle u, w \rangle = 0$, $-A[v, w, u] = -A[w, v, u]$ by (3.2), $-A[w, v, u] = A[w, u, v]$ by (3.3) and $\langle u, v \rangle = 0$, $A[w, u, v] = A[u, w, v]$ by (3.2), $A[u, w, v] = -A[u, v, w]$ by (3.3) and $\langle v, w \rangle = 0$. Therefore $A[u, v, w] = 0$. This proves the lemma. $\square$

Lemma 3.3. *Let f be a conformal C^2 -mapping on a domain $D \subset \mathbb{R}^{p,q}$. If λ is equal to a constant α on D , then f is a similarity $f(x) = \alpha Rx + b$, with $R \in O(p, q)$ and $b \in \mathbb{R}^n$.*

Proof. Since λ is constant, (3.1) implies $\langle f''(x)[u, v], f'(x)w \rangle = 0$ for all $u, v, w \in \mathbb{R}^{p,q}$. Then $A[u, v, w] = \langle f''(x)[u, v], f'(x)w \rangle$ satisfies (3.2) and

$$A[u, v, w] = -A[u, w, v] \quad (u, v, w \in \mathbb{R}^{p,q}). \quad (3.4)$$

From the same argument as in the proof of Lemma 3.2, $A[u, v, w] = 0$ follows for all $u, v, w \in \mathbb{R}^n$. Since $f'(x)$ is surjective, $f''(x)[u, v] = 0$ for all $u, v \in \mathbb{R}^n$, and hence $f''(x) = 0$. Then $\alpha^{-1}f'(x)$ is equal to a constant matrix R satisfying $\langle Ru, Rv \rangle = \langle u, v \rangle$. Therefore $R \in O(p, q)$ and $f(x) = \alpha Rx + b$, where $b \in \mathbb{R}^n$. $\square$

Lemma 3.4. *Let f be a conformal C^2 -mapping on a domain $D \subset \mathbb{R}^{p,q}$. If*

$$\lambda_f(x) = \frac{\alpha}{|\langle x, x \rangle|}$$

with some constant $\alpha > 0$ on D , then there exist $R \in O(p, q)$ and $b \in \mathbb{R}^n$ such that

$$f(x) = \alpha RK(x) + b.$$

Proof. By the assumption and (2.5),

$$\lambda_{f \circ K}(x) = \lambda_f(K(x)) \cdot \lambda_K(x) = \frac{\alpha}{|\langle K(x), K(x) \rangle|} \cdot \frac{1}{|\langle x, x \rangle|} = \frac{\alpha |\langle x, x \rangle|^2}{|\langle x, x \rangle| |\langle x, x \rangle|} = \alpha.$$

Then by Lemma 3.3, $(f \circ K)(x) = \alpha Rx + b$, with $R \in O(p, q)$ and $b \in \mathbb{R}^n$. Hence the lemma follows from $K^{-1}(x) = K(x)$. $\square$

Lemma 3.5. *Let f be a conformal C^2 -mapping on a domain $D \subset \mathbb{R}^{p,q}$. If*

$$\lambda_f(x) = \frac{\alpha}{|\langle d, x \rangle|}$$

with some null vector d and a constant $\alpha > 0$ on D , then

$$f(x) = \alpha RB_d(x) + b,$$

with some $R \in O(p, q)$ and $b \in \mathbb{R}^n$.

Proof. By (2.6), (2.7), and the assumption,

$$\lambda_{f \circ B_d^{-1}}(x) = \lambda_f(B_{-d}(x)) \cdot \lambda_{B_{-d}}(x) = \frac{\alpha}{|\langle d, B_{-d}(x) \rangle|} \cdot \frac{1}{|\langle d, x \rangle|}.$$

Since $(-d)^* = -d^*$ and $\langle Q_{-d}x, d \rangle = 0$,

$$\langle d, B_{-d}(x) \rangle = \frac{1}{\langle -d, x \rangle} \langle d, -\frac{\langle x, x \rangle}{2}d + d^* + Q_{-d}x \rangle = \frac{\langle d, d^* \rangle}{\langle -d, x \rangle} = \frac{-1}{\langle d, x \rangle},$$

and hence

$$\lambda_{f \circ B_d^{-1}}(x) = \alpha.$$

Therefore $(f \circ B_d^{-1})(x) = \alpha R x + b$ with $R \in O(p, q)$ and $b \in \mathbb{R}^n$. The lemma follows. $\square$

By Lemmas 3.3, 3.4 and 3.5, to prove Theorem 1.1, it suffices to prove the following:

Theorem 3.1. *Let $n \geq 3$ and f be a conformal C^3 -mapping on a domain $D \subset \mathbb{R}^{p,q}$. Then λ_f has one of the following forms:*

$$\lambda_f(x) = \frac{\alpha}{|\langle x - a, x - a \rangle|}, \quad (3.5)$$

$$\lambda_f(x) = \frac{\alpha}{|\langle d, x - a \rangle|}, \quad (3.6)$$

$$\lambda_f(x) = \alpha, \quad (3.7)$$

where $\alpha > 0$ is a constant, $a \in \mathbb{R}^n$, and $d \in \mathbb{R}^n$ is a null vector.

In the following, we prepare several lemmas to prove Theorem 3.1.

The following lemma is well-known in linear algebra.

Lemma 3.6. *Let $l(u, v)$ is a symmetric bilinear form on $\mathbb{R}^n$. If $l(u, v) = 0$ for all $u, v \in \mathbb{R}^n$ with $\langle u, v \rangle = 0$, then there exists a constant $c \in \mathbb{R}$ such that*

$$l(u, v) = c \langle u, v \rangle \quad (u, v \in \mathbb{R}^{p,q}).$$

Lemma 3.7. *Let f be a conformal C^2 -mapping on D . If $\langle u, v \rangle = 0$, then for each $x \in D$,*

$$f''(x)[u, v] \in \mathbb{R}f'(x)u + \mathbb{R}f'(x)v.$$

Proof. By Lemma 3.2, if $\langle u, v \rangle = 0$ and $w \in (\mathbb{R}u + \mathbb{R}v)^{\perp_J}$,

$$\langle f''(x)[u, v], f'(x)w \rangle = 0.$$

Then $f''(x)[u, v] \in \left(f'(x)(\mathbb{R}u + \mathbb{R}v)^{\perp_J}\right)^{\perp_J}$ holds, and (2.2) implies $\left(f'(x)(\mathbb{R}u + \mathbb{R}v)^{\perp_J}\right)^{\perp_J} = f'(x)((\mathbb{R}u + \mathbb{R}v)^{\perp_J})^{\perp_J} = f'(x)(\mathbb{R}u + \mathbb{R}v) = \mathbb{R}f'(x)u + \mathbb{R}f'(x)v$. $\square$

Lemma 3.8. *Let $n \geq 3$. Let f be a conformal C^2 -mapping on $D \subset \mathbb{R}^{p,q}$. If $\langle u, v \rangle = 0$, then for each $x \in D$,*

$$f''(x)[u, v] = \lambda(x)^{-1} \lambda'(x) v f'(x) u + \lambda(x)^{-1} \lambda'(x) u f'(x) v. \quad (3.8)$$

Proof. Let $u, v \in \mathbb{R}^{p,q}$. Applying Lemma 3.1 with $w = v$, we have

$$\langle f''(x)[u, v], f'(x)v \rangle = \lambda(x) \langle v, v \rangle \lambda'(x) u. \quad (3.9)$$

Fix arbitrary $u \in \mathbb{R}^{p,q}$. If $v \in \mathbb{R}^{p,q}$ satisfies $\langle u, v \rangle = 0$, then

$$f''(x)[u, v] = a f'(x) u + b f'(x) v \quad (3.10)$$

with some $a = a(x, u, v)$ and $b = b(x, u, v)$ by Lemma 3.7. From (3.9), we have

$$a \langle f'(x) u, f'(x) v \rangle + b \langle f'(x) v, f'(x) v \rangle = \lambda(x) \langle v, v \rangle \lambda'(x) u.$$

Since $\langle u, v \rangle = 0$ and f is conformal,

$$b \lambda(x)^2 \langle v, v \rangle = \lambda(x) \langle v, v \rangle \lambda'(x) u.$$

Similarly we have

$$a \lambda(x)^2 \langle u, u \rangle = \lambda(x) \langle u, u \rangle \lambda'(x) v.$$

Thus we have

$$a = \lambda(x)^{-1} \lambda'(x) v, \quad b = \lambda(x)^{-1} \lambda'(x) u \quad (3.11)$$

for all u, v with $\langle u, v \rangle = 0$, $\langle u, u \rangle \neq 0$, and $\langle v, v \rangle \neq 0$, because $\lambda(x) > 0$.

By Lemma 2.2 and continuity, we can drop $\langle u, u \rangle \neq 0$ and $\langle v, v \rangle \neq 0$. $\square$

Lemma 3.9. *Let $n \geq 3$. If f is a conformal C^3 -mapping, then $\rho(x) = \frac{1}{\lambda(x)}$ is C^∞ and*

$$\rho''(x)[u, v] = c \langle u, v \rangle \quad (3.12)$$

where c is a constant.

Proof. Since f is C^3 , $\rho = 1/\lambda$ is C^2 and $\rho' = -\lambda'/\lambda^2$. Then by (3.8), if $\langle u, v \rangle = 0$,

$$\rho(x) f''(x)[u, v] + \rho'(x) v f'(x) u + \rho'(x) u f'(x) v = 0.$$

Taking the derivative in direction w , we have

$$\begin{aligned} & \rho'(x) w f''(x)[u, v] + \rho(x) f'''(x)[u, v, w] + \rho''(x)[v, w] f'(x) u \\ & + \rho'(x) v f''(x)[u, w] + \rho''(x)[u, w] f'(x) v + \rho'(x) u f''(x)[v, w] = 0. \end{aligned} \quad (3.13)$$

Assume $\langle u, v \rangle = 0$ and interchange u and w in the equation. Then

$$\begin{aligned} & \rho'(x) u f''(x)[w, v] + \rho(x) f'''(x)[w, v, u] + \rho''(x)[v, u] f'(x) w \\ & + \rho'(x) v f''(x)[w, u] + \rho''(x)[w, u] f'(x) v + \rho'(x) w f''(x)[v, u] = 0. \end{aligned} \quad (3.14)$$

Subtract (3.14) from (3.13). Then

$$\rho''(x)[v, w]f'(x)u - \rho''(x)[v, u]f'(x)w = 0$$

holds for all $u, v, w \in \mathbb{R}^{p,q}$ satisfying $\langle u, v \rangle = \langle w, v \rangle = 0$. Taking a scalar product with $f'(x)w$, we have

$$\langle \rho''(x)[v, w]f'(x)u - \rho''(x)[v, u]f'(x)w, f'(x)w \rangle = 0,$$

and hence if $\langle u, v \rangle = \langle w, v \rangle = 0$,

$$\rho''(x)[v, w]\lambda(x)^2\langle u, w \rangle - \rho''(x)[v, u]\lambda(x)^2\langle w, w \rangle = 0.$$

By Lemma 2.3, we can choose w so that $\langle u, w \rangle = \langle w, v \rangle = 0$ and $\langle w, w \rangle \neq 0$, if $\langle u, u \rangle \neq 0$ and $\langle v, v \rangle \neq 0$. Then we obtain

$$\rho''(x)[u, v] = 0.$$

By continuity and Lemma 2.2, we can drop $\langle u, u \rangle \neq 0$ and $\langle v, v \rangle \neq 0$. By Lemma 3.6,

$$\rho''(x)[u, v] = c(x)\langle u, v \rangle \quad (3.15)$$

holds for all $u, v \in \mathbb{R}^{p,q}$ with some $c(x) \in \mathbb{R}$. Then for $j = 1, \dots, n$,

$$\frac{\partial^2 \rho}{\partial x_j^2}(x) = \rho''(x)[e_j, e_j] = c(x)\varepsilon_j,$$

where $\varepsilon_j = 1$ ($1 \leq j \leq p$), $\varepsilon_j = -1$ ($p+1 \leq j \leq n$). Hence ρ is C^∞ because ρ satisfies an elliptic equation

$$\sum_{j=1}^p q \frac{\partial^2 \rho}{\partial x_j^2}(x) + \sum_{j=1}^q p \frac{\partial^2 \rho}{\partial x_{p+j}^2}(x) = 0.$$

To show $c(x)$ is constant, take the derivative of (3.15) in direction w . We have

$$\rho'''(x)[u, v, w] = c'(x)w\langle u, v \rangle,$$

and then

$$\begin{aligned} \langle (c'(x)w)u - (c'(x)u)w, v \rangle &= c'(x)w\langle u, v \rangle - c'(x)u\langle w, v \rangle \\ &= \rho'''(x)[u, v, w] - \rho'''(x)[w, v, u] = 0 \end{aligned}$$

for all $u, v, w \in \mathbb{R}^{p,q}$. That implies

$$(c'(x)w)u - (c'(x)u)w = 0,$$

and hence

$$c'(x)u = c'(x)w = 0$$

for each linearly independent pair of u and w . Therefore $c'(x) = 0$. $\square$

Lemma 3.10. *Let $n \geq 3$. If f is conformal and C^3 on D , then $\lambda = \lambda_f$ satisfies one of the following on D with $c, b, l \in \mathbb{R}$, $c \neq 0$, $l > 0$, $a, d \in \mathbb{R}^n$, $d \neq 0$:*

$$1/\lambda_f(x) = c\langle x - a, x - a \rangle + b, \quad (3.16)$$

$$1/\lambda_f(x) = \langle d, x - a \rangle, \quad (3.17)$$

$$1/\lambda_f(x) = l. \quad (3.18)$$

Proof. By Lemma 3.9, $\rho(x) = \lambda(x)^{-1}$ is equal to a polynomial of degree at most 2. Integrating the equation (3.12), we have the lemma. $\square$

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. By Lemma 3.10, λ_f satisfies (3.16), (3.17), or (3.18). Fix arbitrary $x \in D$. Since f is conformal, f has a conformal local inverse f^{-1} defined in a neighborhood W of x . Then Lemma 3.10 implies that f^{-1} satisfies one of the following in W with $c', b', l' \in \mathbb{R}$, $c' \neq 0$, $l' > 0$, $a', d' \in \mathbb{R}^n$, $d' \neq 0$:

$$\lambda_{f^{-1}}(x)^{-1} = c'\langle x - a', x - a' \rangle + b',$$

$$\lambda_{f^{-1}}(x)^{-1} = \langle d', x - a' \rangle,$$

$$\lambda_{f^{-1}}(x)^{-1} = l'.$$

By (2.4), we have the following possibilities:

$$(c\langle x - a, x - a \rangle + b)(c'\langle f(x) - a', f(x) - a' \rangle + b') = 1, \quad (3.19)$$

$$(c\langle x - a, x - a \rangle + b)\langle d', f(x) - a' \rangle = 1, \quad (3.20)$$

$$\langle d, x - a \rangle(c'\langle f(x) - a', f(x) - a' \rangle + b') = 1, \quad (3.21)$$

$$\langle d, x - a \rangle\langle d', f(x) - a' \rangle = 1, \quad (3.22)$$

$$\lambda_f(x) = l, \quad (3.23)$$

because λ_f is constant if and only if $\lambda_{f^{-1}}$ is constant.

We begin with the case (3.19). Assume $b \neq 0$. We can choose a point $z \in D$ so that $v = z - a$ satisfies $\langle v, v \rangle \neq 0$. The line

$$\ell : v(t) = a + tv \quad (t \in \mathbb{R}) \quad (3.24)$$

is orthogonal to every hypersurface $S_\eta = \{x; c\langle x - a, x - a \rangle + b = \eta\}$ for $\eta \in \mathbb{R}$ and $\eta \neq b$ with respect to the indefinite metric. Since f is conformal, the image curve $f(\ell)$ is orthogonal to every image hypersurface $f(S_\eta)$. By (3.19), $f(S_\eta)$ is contained in $S'_{1/\eta} = \{y; c'\langle y - a', y - a' \rangle + b' = 1/\eta\}$. Then $f(\ell)$ is orthogonal to every S'_η , and hence $f(\ell) = \{f(v(t)); t \in \mathbb{R}\}$ must be in a line passing through a' . Therefore,

$$f(v(t)) = a' + \beta(t)v' \quad (3.25)$$

where $v' \in \mathbb{R}^n$ and $\beta(t)$ is a real valued C^3 -function. Differentiating (3.25), we have

$$f'(v(t))v = \beta'(t)v'.$$

The conformality of f implies that

$$\beta'(t)^2 \langle v', v' \rangle = \langle f'(v(t))v, f'(v(t))v \rangle = \lambda_f(v(t))^2 \langle v, v \rangle = \frac{\langle v, v \rangle}{(c\langle v, v \rangle t^2 + b)^2}.$$

Since $b \neq 0$,

$$\beta(t) = \begin{cases} p_1 \arctan p_2 t & \text{if } \frac{b}{c\langle v, v \rangle} > 0 \\ \frac{1}{2} p_1 \log \frac{p_2 t - 1}{p_2 t + 1} & \text{if } \frac{b}{c\langle v, v \rangle} < 0 \end{cases} \quad (3.26)$$

with some $p_1, p_2 \in \mathbb{R}$. Especially, $\beta(t)$ is a transcendental function.

On the other hand, substituting (3.24) and (3.25) into (3.19), we have

$$c' \beta(t)^2 \langle v', v' \rangle + b' = \frac{1}{t^2 + b},$$

which means $\beta(t)^2$ is a rational function. This is a contradiction. Therefore $b = 0$ and thus we have (3.5) by putting $\alpha = 1/|c|$.

Next we show that the case of (3.20) does not occur. Let v , ℓ , and S_η be as above. Equation (3.20) shows that $f(S_\eta)$ ($\eta \in \mathbb{R}, \eta \neq 0$) is contained in the hyperplane $P_{1/\eta} = \{y; \langle d', y - a' \rangle = 1/\eta\}$ with $d' \neq 0$. Since f is conformal, $f(\ell)$ is orthogonal to every $f(S_\eta)$ and hence to every hyperplane $P_{1/\eta}$. Therefore, $f(\ell)$ is a line of direction d' , and there exist $w \in \mathbb{R}^n$ and a real valued function $\beta(t)$ such that

$$f(v(t)) = w + \beta(t)d'. \quad (3.27)$$

Differentiating the both sides, we have

$$f'(v(t))v = \beta'(t)d'.$$

The conformality of f implies that

$$\beta'(t)^2 \langle d', d' \rangle = \langle f'(v(t))v, f'(v(t))v \rangle = \lambda_f(v(t))^2 \langle v, v \rangle = \frac{\langle v, v \rangle}{(c\langle v, v \rangle t^2 + b)^2}.$$

Then $\langle d', d' \rangle \neq 0$ because $c^{-1} \neq 0$.

Assume $b \neq 0$. Then

$$\beta(t) = \begin{cases} p_1 \arctan p_2 t & \text{if } \frac{b}{c\langle v, v \rangle} > 0 \\ \frac{1}{2} p_1 \log \frac{p_2 t - 1}{p_2 t + 1} & \text{if } \frac{b}{c\langle v, v \rangle} < 0 \end{cases} \quad (3.28)$$

with some $p_1, p_2 \in \mathbb{R}$. Especially, $\beta(t)$ is a transcendental function.

On the other hand, substituting (3.27) into (3.20), we have

$$(c\langle v, v \rangle t^2 + b)(\langle d', d' \rangle \beta(t) + \langle d', w - a' \rangle) = 1,$$

and hence $\beta(t)$ is a rational function. This is a contradiction. Therefore $b = 0$. Then $\beta'(t)^2 = \frac{\langle d', d' \rangle^{-1}}{c^2 \langle v, v \rangle t^4}$ and $\beta(t) + b' = \frac{\langle d', d' \rangle^{-1}}{c \langle v, v \rangle t^2}$, where $b' = \frac{\langle d', w - a' \rangle}{\langle d', d' \rangle}$, which is also a contradiction. Thus the case (3.20) does not occur.

Interchanging the role of f and f^{-1} , the same argument shows the case (3.21) does not occur.

Finally, we treat the case of (3.22). For each hyperplane $P_\eta = \{x; \langle d, x - a \rangle = \eta\}$, $f(P_\eta)$ is contained in the hyperplane $P'_{1/\eta} = \{y; \langle d', y - a' \rangle = 1/\eta\}$ by (3.22). For each $w \in \mathbb{R}^n$, the line

$$\ell : v(t) = w + td \quad (t \in \mathbb{R}) \quad (3.29)$$

is orthogonal to all P_η . Since f is conformal, the image curve $f(\ell)$ is orthogonal to every $P'_{1/\eta}$. Hence $f(\ell)$ has direction d' . Therefore, there exist $w' \in \mathbb{R}^n$ and a C^3 -function $\beta(t)$ such that

$$f(v(t)) = w' + \beta(t)d'. \quad (3.30)$$

Differentiating by t , we have

$$f'(v(t))d = \beta'(t)d'.$$

Substituting (3.29) and (3.30) into (3.22), we have

$$(\langle d, d' \rangle t + \langle d, w - a \rangle)(\langle d', d' \rangle \beta(t) + \langle d', w' - a' \rangle) = 1,$$

and hence $\beta(t)$ is a rational function.

On the other hand, the conformality of f implies that

$$\beta'(t)^2 \langle d', d' \rangle = \lambda_f(v(t))^2 \langle d, d \rangle = \frac{\langle d, d \rangle}{(\langle d, d \rangle t + b)^2},$$

where $b = \langle d, w - a \rangle$. If $\langle d, d \rangle \neq 0$, then $\langle d', d' \rangle \neq 0$ and $\beta(t)$ is a logarithm function. This is a contradiction. Therefore $\langle d, d \rangle = 0$ and thus we have (3.6) by putting $\alpha = 1/|d|$ and normalizing d . $\square$

4. Transformation which preserves solutions of the ultra-hyperbolic equation

In this section, we shall give the proof of Theorem 1.2.

By definition, it is easy to see that if (f, φ_f) and (g, φ_g) are transformations which preserves solutions of the ultra-hyperbolic equation such that the image of f is contained in the domain of g , then $(g \circ f, \varphi_f \cdot (\varphi_g \circ f))$ is also a transformation which preserves solutions of the ultra-hyperbolic equation. We call this transformation $(g \circ f, \varphi_f \cdot (\varphi_g \circ f))$ the composition of (f, φ_f) and (g, φ_g) . First, we remark that all the transformations in Theorem 1.2 are the composition of the following fundamental transformations Examples 4.1 – 4.4.

Example 4.1 (similarity). Let $\alpha, C \in \mathbb{R}$, $\alpha, C > 0$, $R \in O(p, q)$, $a \in \mathbb{R}^n$. The pair of a similarity $f(x) = \alpha Rx + a$ and a positive constant function $\varphi(x) = C$ is a transformation which preserves solutions of the ultra-hyperbolic equation.

Example 4.2 (inversion [6]). The pair of the inversion mapping K and the function $\varphi_K(x) = |\langle x, x \rangle|^{\frac{2-n}{2}}$ is a transformation which preserves solutions of the ultra-hyperbolic equation on each connected component of $\{x \in \mathbb{R}^{p,q}; \langle x, x \rangle \neq 0\}$.

Example 4.3 (Bateman transformation [6]). Let d be a null vector. Then the pair of the Bateman mapping B_d and the function $\varphi_{B_d}(x) = |\langle d, x \rangle|^{\frac{2-n}{2}}$ is a transformation which preserves solutions of the ultra-hyperbolic equation on each connected component half space of $\{x \in \mathbb{R}^{p,q}; \langle d, x \rangle \neq 0\}$.

We also remark that all the mappings of the above three transformations are conformal. In the case $p = q$, we have a transformation of which the mapping is not conformal.

Example 4.4. Let $p = q$ and $S = \begin{pmatrix} O & I_p \\ I_p & O \end{pmatrix}$. The pair of the mapping $f(x) = Sx$ and the constant function $\varphi(x) = 1$ is a transformation which preserves solutions of the ultra-hyperbolic equation on $\mathbb{R}^{p,q}$.

The following theorem gives a characterization of the transformations which preserves solutions of the ultra-hyperbolic equation.

Theorem 4.1. *The pair (f, φ) is a transformation which preserves solutions of the ultra-hyperbolic equation, if and only if $f = (f_1, f_2, \dots, f_n)$ and φ satisfy the following equations on D :*

$$U\varphi = 0, \quad (4.1)$$

$$\varphi Uf_j + 2\langle \nabla \varphi, \nabla f_j \rangle = 0 \quad (j = 1, 2, \dots, n) \quad (4.2)$$

$$\langle \nabla f_i, \nabla f_j \rangle = 0 \quad (i, j = 1, 2, \dots, n, i \neq j) \quad (4.3)$$

$$\langle \nabla f_i, \nabla f_i \rangle = -\langle \nabla f_j, \nabla f_j \rangle \neq 0 \quad (1 \leq i \leq p, p+1 \leq j \leq n), \quad (4.4)$$

where $\nabla f_j = (\frac{\partial f_j}{\partial x_1}, \frac{\partial f_j}{\partial x_2}, \dots, \frac{\partial f_j}{\partial x_n})$, $j = 1, 2, \dots, n$.

Proof. By the chain rule, the following holds for any C^2 -function u :

$$\begin{aligned} U(\varphi \cdot (u \circ f)) &= \sum_{j=1}^n \varepsilon_j \frac{\partial^2}{\partial x_j^2} (\varphi \cdot (u \circ f)) \\ &= (U\varphi) \cdot (u \circ f) \\ &\quad + \sum_{k=1}^n (\varphi \cdot Uf_k + 2\langle \nabla \varphi, \nabla f_k \rangle) \cdot \left(\frac{\partial u}{\partial y_k} \circ f \right) + \sum_{k,l=1}^n \varphi \cdot \langle \nabla f_k, \nabla f_l \rangle \cdot \left(\frac{\partial^2 u}{\partial y_k \partial y_l} \circ f \right). \end{aligned} \quad (4.5)$$

Assume that (f, φ) is a transformation which preserves solutions of the ultra-hyperbolic equation.

First we take $u = 1$. Since $\frac{\partial u}{\partial y_k} = \frac{\partial^2 u}{\partial y_k \partial y_l} = 0$ ($1 \leq k, l \leq n$), we have (4.1): $U\varphi = 0$.

Take $u(y) = y_j$ ($1 \leq j \leq n$). Since $\frac{\partial u}{\partial y_k} = \delta_{jk}$, $\frac{\partial^2 u}{\partial y_k \partial y_l} = 0$ ($1 \leq k, l \leq n$), we have (4.2): $\varphi Uf_j + 2\langle \nabla \varphi, \nabla f_j \rangle = 0$.

Take $u(y) = y_i y_j$ ($1 \leq i, j \leq n$, $i \neq j$). Since $\frac{\partial^2 u}{\partial y_k \partial y_l} = \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}$ ($1 \leq k, l \leq n$), we have (4.3): $\langle \nabla f_i, \nabla f_j \rangle = 0$ ($1 \leq i, j \leq n$, $i \neq j$).

Finally, take $u(y) = y_i^2 + y_j^2$ ($1 \leq i \leq p$, $p+1 \leq j \leq n$). Since $\frac{\partial^2 u}{\partial y_i^2} = \frac{\partial^2 u}{\partial y_j^2} = 2$, $\frac{\partial^2 u}{\partial y_k^2} = 0$ ($k \neq i, j$), we have $\langle \nabla f_i, \nabla f_i \rangle + \langle \nabla f_j, \nabla f_j \rangle = 0$ ($1 \leq i \leq p$, $p+1 \leq j \leq n$). Together with (4.3), we have $f' J^t f' = \langle \nabla f_1, \nabla f_1 \rangle J$. Since f is non-degenerate, the left hand side is a regular matrix. Therefore, $\langle \nabla f_1, \nabla f_1 \rangle \neq 0$.

Conversely, assume that f and φ satisfy (4.1), (4.2), (4.3), and (4.4). Substituting (4.1), (4.2) and (4.3) into (4.5), we have

$$U(\varphi \cdot (u \circ f)) = \varphi \sum_{j=1}^n \langle \nabla f_j, \nabla f_j \rangle \cdot \left(\frac{\partial^2 u}{\partial y_j^2} \circ f \right).$$

Putting $\lambda_1 = \langle \nabla f_1, \nabla f_1 \rangle$, (4.4) gives

$$U(\varphi \cdot (u \circ f))(x) = \varphi(x) \lambda_1(x) ((Uu) \circ f)(x).$$

This shows that $Uu = 0$ implies $U(\varphi \cdot (u \circ f)) = 0$. Thus we have the theorem. $\square$

Theorem 4.2. *Let (f, φ) be a transformation which preserves solutions of the ultra-hyperbolic equation defined on $D \subset \mathbb{R}^{p,q}$. If $p \neq q$, then f is conformal on D such that $\lambda_f(x)^2 = \langle \nabla f_1, \nabla f_1 \rangle$. If $p = q$, then either f is conformal on D such that $\lambda_f^2 = \langle \nabla f_1, \nabla f_1 \rangle$, or $Sf(x)$ is conformal on D such that $\lambda_{Sf}^2 = -\langle \nabla f_1, \nabla f_1 \rangle$.*

Proof. Put $\lambda_1 = \langle \nabla f_1, \nabla f_1 \rangle$ and let $x \in D$ be fixed. (4.3) and (4.4) is equivalent to

$$f'(x) J^t f'(x) = \lambda_1(x) J.$$

Since f is non-degenerate, $\lambda_1(x) \neq 0$ and hence

$${}^t f'(x)^{-1} J f'(x)^{-1} = \lambda_1(x)^{-1} J.$$

Therefore ${}^t f'(x) J f'(x) = \lambda_1(x) J$ and hence $\langle f'(x)u, f'(x)v \rangle = \lambda_1(x) \langle u, v \rangle$ for all $u, v \in \mathbb{R}^{p,q}$. If $\lambda_1(x) > 0$ for all $x \in D$, then f is conformal and $\lambda_f(x)^2 = \lambda_1(x)$.

Next, assume $\lambda_1(x) < 0$. Put $e_1 = (1, 0, \dots, 0)$, $e_2 = (0, 1, \dots, 0)$, $\dots$, $e_n = (0, \dots, 0, 1)$. We shall show that the set of $2p$ vectors $\{e_1, \dots, e_p, \nabla f_1(x), \dots, \nabla f_p(x)\}$ and the set of $2q$ vectors $\{e_{p+1}, \dots, e_{p+q}, \nabla f_{p+1}(x), \dots, \nabla f_{p+q}(x)\}$ are both linearly independent. Assume $\alpha_1 e_1 + \dots + \alpha_p e_p + \alpha_{p+1} \nabla f_1(x) + \dots + \alpha_{2p} \nabla f_p(x) = 0$ and $\beta_1 e_{p+1} + \dots + \beta_q e_{p+q} + \beta_{q+1} \nabla f_{p+1}(x) + \dots + \beta_{2q} \nabla f_{p+q}(x) = 0$. Put $u = \alpha_1 e_1 + \dots + \alpha_p e_p = -\alpha_{p+1} \nabla f_1(x) - \dots - \alpha_{2p} \nabla f_p(x)$ and $v = \beta_{p+1} e_{p+1} + \dots + \beta_q e_{p+q} = -\beta_{q+1} \nabla f_{p+1}(x) - \dots - \beta_{2q} \nabla f_{p+q}(x)$. Then by (4.3), (4.4) and $\langle \nabla f_1(x), \nabla f_1(x) \rangle = \lambda_1(x) < 0$,

$$\begin{aligned} 0 &\leq \sum_{j=1}^p \alpha_j^2 = \langle u, u \rangle = \lambda_1(x) \sum_{j=1}^p \alpha_{p+j}^2 \leq 0, \\ 0 &\geq -\sum_{j=1}^q \beta_j^2 = \langle v, v \rangle = -\lambda_1(x) \sum_{j=1}^q \beta_{q+j}^2 \geq 0, \end{aligned}$$

that imply $\alpha_1 = \dots = \alpha_{2p} = 0$ and $\beta_1 = \dots = \beta_{2q} = 0$. Therefore $2p \leq n$ and $2q \leq n$, which imply $p = q$ because $p + q = n$. Consequently, $p \neq q$ implies $\lambda_1 > 0$. Moreover,

$$(Sf)'(x)J {}^t(Sf)'(x) = Sf'(x)J {}^t f'(x) {}^t S = \lambda_1(x)SJS = -\lambda_1(x)J.$$

Thus Sf is conformal on D and $\lambda_{Sf}^2 = -\langle \nabla f_1, \nabla f_1 \rangle$. $\square$

Next lemma shows that φ is uniquely determined by f except a constant multiple.

Lemma 4.1. *If (f, φ_1) and (f, φ_2) are two transformations which preserve solutions of the ultra-hyperbolic equation with same mapping f on a domain. Then φ_2 is a constant multiple of φ_1 .*

Proof. By (4.2), we have

$$Uf_j = -2\langle \nabla \log \varphi_1, \nabla f_j \rangle, \quad Uf_j = -2\langle \nabla \log \varphi_2, \nabla f_j \rangle, \quad (j = 0, 1, \dots, n).$$

and then

$$\langle \nabla f_j, \nabla \log \frac{\varphi_2}{\varphi_1} \rangle = \langle \nabla f_j, \nabla \log \varphi_2 - \nabla \log \varphi_1 \rangle = 0, \quad (j = 0, 1, \dots, n).$$

Since the matrix $f'(x)$ is non-degenerate, $\nabla \log \frac{\varphi_2(x)}{\varphi_1(x)} = 0$ for all $x \in D$. Therefore, φ_2 is a constant multiple of φ_1 . $\square$

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. Let $p \neq q$, or $p = q$ and $\langle \nabla f_1, \nabla f_1 \rangle > 0$. Then by Corollary 4.2, f is conformal on D . Then by Theorem 1.1, f has the form (a), (b), or (c).

In the case (a) : $f(x) = \frac{\alpha R(x-a)}{\langle x-a, x-a \rangle} + b$, put $h(x) = \alpha Rx + b$ and $g(x) = x - a$. Then $f(x) = (h \circ K \circ g)(x)$. Since $(h, 1)$ and $(g, 1)$ are transformations which preserve solutions of the ultra-hyperbolic equation, Lemma 4.1 implies φ is a constant multiple of

$$1 \cdot \varphi_K(g(x)) \cdot 1 = |\langle x-a, x-a \rangle|^{\frac{2-n}{2}}.$$

Therefore $\varphi(x) = C|\langle x-a, x-a \rangle|^{\frac{2-n}{2}}$ with $C > 0$.

In the case (b) : $f(x) = \alpha RB_d(x-a) + b$, put $h(x) = \alpha Rx + b$ and $g(x) = x - a$. Then $f(x) = (h \circ B_d \circ g)(x)$. Since $(h, 1)$ and $(g, 1)$ are transformations which preserve solutions of the ultra-hyperbolic equation, φ is a constant multiple of

$$1 \cdot \varphi_{B_d}(g(x)) \cdot 1 = |\langle d, x-a \rangle|^{\frac{2-n}{2}}$$

by Lemma 4.1. Therefore $\varphi(x) = C|\langle d, x-a \rangle|^{\frac{2-n}{2}}$ with $C > 0$.

In the case (c) : $f(x) = \alpha Rx + b$ is the similarity. So φ is a positive constant.

If $p = q$ and $\langle \nabla f_1, \nabla f_1 \rangle < 0$, then Sf is conformal by Corollary 4.2. Since $S(Sf) = S^2 f = f$, we have (a'), (b'), and (c'). $\square$

References

- [1] H. Bateman, The conformal transformations of a space of four dimensions and their applications to geometrical optics, *Proc. London Math. Soc.*, **7** (1909) 70 – 89.
- [2] M. Berger, *Geometry*, Springer-Verlag, 1986.
- [3] B. O’Neill, *Semi-Riemannian geometry*, Academic Press, 1983.
- [4] K. Shimomura, Liouville type theorem associate with the wave equation, *Math. J. Ibaraki Univ.*, **43** (2011), 51 – 64.
- [5] K. Shimomura, A relation between the Lorentzian inversion and the Bateman transformation, *Math. J. Ibaraki Univ.*, **44** (2012), 1 – 5.
- [6] K. Shimomura, Generalizations of Bateman’s transformation for general indefinite metrics, *Math. J. Ibaraki Univ.*, **45** (2013), 7 – 13.