A uniqueness theorem for semistar operations on 1-dimensional Prüfer domains

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Abstract

We study star and semistar operations on 1-dimensional Prüfer domains D .
We prove a uniqueness theorem for γ -representations of sigma operations on D .

§1 Introduction

Let $F(R)$ be the set of non-zero fractional ideals of an integral domain R , and let $\bar{F}(R)$ be the set of non-zero R -submodules of $q(R)$, where $q(R)$ is the quotient field of R . A mapping $I \mapsto I^*$ from $F(R)$ to $F(R)$ is called a star operation on R if it satisfies the following 5 conditions: For every $0 \neq x \in q(R)$ and every $I, J \in F(R)$, $R^* = R$; $(xI)^* = xI^*$; $I \subseteq I^*$; $(I^*)^* = I^*$; $I \subseteq J$ implies $I^* \subseteq J^*$. A mapping $I \mapsto I^*$ from $\bar{F}(R)$ to $\bar{F}(R)$ is called a semistar operation on R if it satisfies the following 4 conditions: For every $0 \neq x \in q(R)$ and every $I, J \in \bar{F}(R)$, $(xI)^* = xI^*$; $I \subseteq I^*$; $(I^*)^* = I^*$; $I \subseteq J$ implies $I^* \subseteq J^*$. The identity mapping from $F(R)$ to $F(R)$ is a star operation on R called the d-star operation on R . Similarly, we may define the d'-semistar operation on R . For every $I \in \bar{F}(R)$, set $I^{-1} := \{x \in q(R) \mid xI \subseteq R\}$, and set $I^\vee := (I^{-1})^{-1}$. Then the mapping $I \mapsto I^\vee$ from $F(R)$ to $F(R)$ is a star operation on R called the v-star operation on R . Similarly, we may define the v'-semistar operation on R . Let $\Sigma(R)$ (resp., $\Sigma'(R)$) be the set of star operations (resp., semistar operations) on R .

Let S be an overring of R . Let $\star$ be a semistar operation on S . If we set $I^{\delta(\star)} = (IS)^*$ for every $I \in \bar{F}(R)$, then $\delta(\star)$ is a semistar operation on R , called the descent of $\star$ to R . Reversely, for every $\star_1 \in \Sigma'(R)$, the semistar operation $\alpha(\star_1)$ on S is canonically defined, and is called the ascent of $\star_1$ to S . We define $\Sigma'_0(R)$ as $\{\star \in \Sigma'(R) \mid R^* = R\}$.

Let R be a domain, set $Y := Y(R) := \bar{F}(R) \setminus F(R)$. A mapping σ from Y to Y is called a sigma operation on R if it satisfies the following 4 conditions: $(xJ)^\sigma = xJ^\sigma$, $J \subseteq J^\sigma$, $(J^\sigma)^\sigma = J^\sigma$, and $I \subseteq J$ implies $I^\sigma \subseteq J^\sigma$ for every $I, J \in Y$ and $0 \neq x \in q(R)$. Let $\mathcal{S}(R)$ be the set of sigma operations on R .

The following (1.1)~(1.7) appear in [4].

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(1.1) Let $\{R_i \mid i \in \mathcal{I}\}$ be the set of overrings of a domain R with $R_i \neq R_j$ for every $i \neq j$. Then we have

- (1) $\Sigma'(R) = \bigcup_i \delta(\Sigma'_0(R_i))$ with disjoint.
- (2) $|\Sigma'(R)| = \sum_i |\Sigma'_0(R_i)|$.

Throughout the paper, let D be a 1-dimensional Prüfer domain with finitely many maximal ideals. Let $M_1, \dots, M_\nu$ be the maximal ideals of D . We may assume that $M_1, \dots, M_\mu$ are non-invertible ideals, and that $M_{\mu+1}, \dots, M_\nu$ are invertible ideals. Let $V_i := D_{M_i}$, let v_i be the valuation on $K := q(D)$ belonging to V_i , and let Γ_i be the value group of v_i for every i . For every subset $X \subseteq \{1, \dots, \mu\}$, there is a canonical mapping $\star_X$ from $F(D)$ to $F(D)$, and we have the following,

- (1.2)** (1) $\Sigma(D) = \{\star_X \mid X \text{ is a subset of } \{1, \dots, \mu\}\}$.
- (2) $|\Sigma(D)| = 2^\mu$.

Let Z_i be the D -submodule of K generated by the set $\{\frac{1}{p} \mid p \in M_i \setminus \bigcup_{j \neq i} M_j\}$ for every i . We set $Y := Y(D) := \bar{F}(D) \setminus F(D)$. Let $\emptyset \neq \alpha := \{i_1, \dots, i_n\} \subseteq \{1, \dots, \nu\}$, and let $\beta := \{j_1, \dots, j_m\} \subseteq \{1, \dots, \mu\}$ with $\alpha \cap \beta = \emptyset$. Then we set $y_\alpha^\beta := y_{i_1 \dots i_n}^{j_1 \dots j_m} := y_{i_1, \dots, i_n}^{j_1, \dots, j_m} := Z_{i_1} \cdots Z_{i_n} M_{j_1} \cdots M_{j_m}$. If $\beta = \emptyset$, we set also $y_\alpha := y_\alpha^\beta$.

(1.3) (1) Let $\mathcal{S}$ be a subset of $Y := Y(D)$ with $\mathcal{S} \ni K$, where $K := q(D)$. Assume that, if $A \in \mathcal{S}$ and $0 \neq x \in K$ then $xA \in \mathcal{S}$. Then the mapping $J \mapsto \bigcap \{A \mid \mathcal{S} \ni A \supseteq J\}$ from Y to Y is a sigma operation on D .

(2) Let σ be a sigma operation on D . Then the subset $\{J^\sigma \mid J \in Y\} \subseteq Y$ defines σ .

Let $A_1, A_2 \in \bar{F}(D)$. If there is $0 \neq x \in K$ such that $A_2 = xA_1$, then A_1 and A_2 are said similar, and is denoted by $A_1 \sim A_2$.

Let $\{A_1, \dots, A_l\} \ni K$ be a finite subset of Y . Then, by (1.3), the set $\{xA_1, \dots, xA_l \mid 0 \neq x \in K\}$ defines a sigma operation on D , and is denoted by $\sigma(A_1, \dots, A_l)$. The representation $\sigma(A_1, \dots, A_l)$ is called reduced if $A_i \not\sim A_j$ for every $i \neq j$.

In the following, α, β, γ denote subsets of $\{1, \dots, \nu\}$. Let σ be a sigma operation on D . If σ has an expression of the form $\sigma(y_{\alpha_1}^{\beta_1}, \dots, y_{\alpha_l}^{\beta_l})$, then the expression is called a y-representation of σ .

Let $\sigma(y_{\alpha_1}^{\beta_1}, \dots, y_{\alpha_l}^{\beta_l})$ be a reduced y-representation of σ . Assume that, if $\sigma = \sigma(y_{\alpha_1}^{\beta_1}, \dots, y_{\alpha_i}^{\beta_i}, y_\alpha^\beta)$ for some α, β , then there is i such that $y_\alpha^\beta \sim y_{\alpha_i}^{\beta_i}$. Then it is called a maximal y-representation.

Let $\sigma(y_{\alpha_1}^{\beta_1}, \dots, y_{\alpha_l}^{\beta_l})$ be a reduced y-representation of σ . Assume that, for every $y_{\alpha_i}^{\beta_i} \neq K$, we have $\sigma(y_{\alpha_1}^{\beta_1}, \dots, y_{\alpha_{i-1}}^{\beta_{i-1}}, y_{\alpha_{i+1}}^{\beta_{i+1}}, \dots, y_{\alpha_l}^{\beta_l}) \neq \sigma$. Then it is called a minimal y-representation.

(1.4) Every sigma operation σ on D can be expressed as $\sigma(y_{\alpha_1}^{\beta_1}, \dots, y_{\alpha_l}^{\beta_l}, K)$, where $\emptyset \subsetneq \alpha_i \subseteq \{1, \dots, \nu\}$ and $\emptyset \subseteq \beta_j \subseteq \{1, \dots, \mu\}$ with $\alpha_i \cap \beta_i = \emptyset$. That is, every sigma operation has a y-representation.

(1.5) Let $\sigma := \sigma(y_{\alpha_1}^{\beta_1}, \dots, y_{\alpha_l}^{\beta_l}, K)$ be a reduced y-representation. Then the following conditions (1) and (2) are equivalent:

- (1) The y-representation is maximal.
- (2) (i) If $\beta \subseteq \beta_i$ for some i , then $y_{\alpha_i}^\beta \in \{y_{\alpha_1}^{\beta_1}, \dots, y_{\alpha_l}^{\beta_l}\}$.
- (ii) If $\emptyset \neq \alpha_i \cap \alpha_j$ for some i, j , then $y_{\alpha_i \cap \alpha_j}^{\beta_i \cup \beta_j} \in \{y_{\alpha_1}^{\beta_1}, \dots, y_{\alpha_l}^{\beta_l}\}$.

(1.6) Let σ be a sigma operation on D . Let $\sigma(y_{\alpha_1}^{\beta_1}, \dots, y_{\alpha_l}^{\beta_l}, K) = \sigma(y_{A_1}^{B_1}, \dots, y_{A_n}^{B_n}, K)$ be maximal y-representations of σ . Then $l = n$ and $\{y_{\alpha_1}^{\beta_1}, \dots, y_{\alpha_l}^{\beta_l}\} = \{y_{A_1}^{B_1}, \dots, y_{A_n}^{B_n}\}$.

(1.7) Let σ be a sigma operation on D . Let $\sigma(y_{\alpha_1}, \dots, y_{\alpha_l}, K) = \sigma(y_{A_1}, \dots, y_{A_n}, K)$ be minimal y-representations of σ . Then $l = n$ and $\{\alpha_1, \dots, \alpha_l\} = \{A_1, \dots, A_n\}$.

In (2.6) of §2 of this paper, we prove a uniqueness theorem for y-representations of sigma operations. In §3 (resp., §4), we study the domains D with $\mu = 0$ (resp., $\mu = 1$) and $\nu = 3$, and calculate $|\mathcal{S}(D)|$ and $|\Sigma'(D)|$.

§2 The minimal y-representations

In this section, we prove a uniqueness theorem (2.6) for minimal y-representations.

Following (2.1) ~ (2.4) appear in [4]. Let σ be a sigma operation on D , and let $A \in Y$. If $A^\sigma = A$, then A is said to be contained in σ , and is denoted by $A \in \sigma$.

(2.1) Let $\sigma := \sigma(y_{\alpha_1}^{\beta_1}, \dots, y_{\alpha_l}^{\beta_l}, K)$ be reduced. Let $y_\alpha^\beta \neq K$. Then $y_\alpha^\beta \in \sigma$ if and only if there are $i_1, \dots, i_k$ such that $\alpha = \alpha_{i_1} \cap \dots \cap \alpha_{i_k}$ with $\beta \subseteq \beta_{i_1} \cup \dots \cup \beta_{i_k}$.

(2.2) Let $\sigma := \sigma(y_\alpha^{\beta_1}, \dots, y_\alpha^{\beta_l}, y_{A_1}^{B_1}, \dots, y_{A_n}^{B_n}, K)$, and set $\beta := \beta_1 \cup \dots \cup \beta_l$. Then we have $\sigma = \sigma(y_\alpha^\beta, y_{A_1}^{B_1}, \dots, y_{A_n}^{B_n}, K)$. Further, if $\sigma(y_\alpha^{\beta_1}, \dots, y_\alpha^{\beta_l}, y_{A_1}^{B_1}, \dots, y_{A_n}^{B_n}, K)$ is of minimal y-representation, then so is $\sigma(y_\alpha^\beta, y_{A_1}^{B_1}, \dots, y_{A_n}^{B_n}, K)$.

(2.3) Let $\sigma := \sigma(y_{\alpha_1}^{\beta_1}, \dots, y_{\alpha_l}^{\beta_l}, K)$, and let $y^\sigma = y$, where $y \in Y$. Then $\sigma = \sigma(y_{\alpha_1}^{\beta_1}, \dots, y_{\alpha_l}^{\beta_l}, K, y)$.

(2.4) Let $\sigma := \sigma(y_{\alpha_1}^{\beta_1}, \dots, y_{\alpha_l}^{\beta_l}, K)$ be a reduced y-representation. The following conditions are equivalent:

- (1) The y-representation is minimal.
- (2) If $\alpha_{i_1} \cap \dots \cap \alpha_{i_{k-1}} = \alpha_{i_k}$ with $i_1, \dots, i_k$ distinct, then $\beta_{i_1} \cup \dots \cup \beta_{i_{k-1}} \not\subseteq \beta_{i_k}$.

Let X be a set, and let $\{A_1, \dots, A_n\}$ be a finite set of subsets of X . Then $\max\{A_1, \dots, A_n\}$ denotes the set $\{A_i \mid A_i \not\subseteq A_j \text{ for every } A_j \neq A_i\}$.

(2.5) **Example** Let σ be a sigma operation on D . Let $\sigma(y_{\alpha_1}^{\beta_1}, y_{\alpha_2}^{\beta_2}, K) = \sigma(y_{A_1}^{B_1}, \dots, y_{A_n}^{B_n}, K)$ both be minimal y-representations of σ . Then $\{\alpha_1, \alpha_2\} = \{A_1, \dots, A_n\}$.

Proof Obviously, we have $n \geq 1$. By (2.2), we may assume that α_1, α_2 are mutually distinct, and that $A_1, \dots, A_n$ are mutually distinct.

First, assume that $\{\alpha_1\} = \max\{\alpha_1, \alpha_2\}$. We confer (2.1). For every i , we have $\alpha_i \subseteq A_j$ for some j , and $A_j \subseteq \alpha_k$ for some k . It follows that $\alpha_1 = A_i$ for some i , say, $\alpha_1 = A_1$. Since $y_{\alpha_2}^{\beta_2} \in \sigma(y_{A_1}^{B_1}, \dots, y_{A_n}^{B_n}, K)$, we have $n \geq 2$. Since $y_{A_2}^{B_2} \in \sigma$, we may assume that $\alpha_2 = A_2$. If $n > 2$, then $y_{A_n}^{B_n} \in \sigma$ implies a contradiction.

Assume that $\{\alpha_1, \alpha_2\} = \max\{\alpha_1, \alpha_2\}$. We may assume that $\{A_1, A_2\} = \max\{A_1, \dots, A_n\}$, $\alpha_1 = A_1$ and $\alpha_2 = A_2$. Since $y_{\alpha_1}^{\beta_1} \in \sigma$, we have $\beta_1 \subseteq B_1$. Since $y_{A_1}^{B_1} \in \sigma$, we have $B_1 \subseteq \beta_1$. Hence $\beta_1 = B_1$. Similarly, $\beta_2 = B_2$. Hence, $\sigma(y_{A_1}^{B_1}, y_{A_2}^{B_2}, K) = \sigma = \sigma(y_{A_1}^{B_1}, y_{A_2}^{B_2}, \dots, y_{A_n}^{B_n}, K)$. By the minimality, we have $n = 2$.

(2.6) Theorem Let σ be a sigma operation on D , and let $\sigma = \sigma(y_{\alpha_1}^{\beta_1}, \dots, y_{\alpha_l}^{\beta_l}, K)$ be minimal y -representations of σ . Then we have $\{\alpha_1, \dots, \alpha_l\} = \{A_1, \dots, A_n\}$.

Proof Assume the contrary. By (2.2), we may assume that $\alpha_1, \dots, \alpha_l$ are mutually distinct, and that $A_1, \dots, A_n$ are mutually distinct. We may assume that $\alpha_1 = A_1, \dots, \alpha_a = A_a$, and that $\alpha_{a+1}, \dots, \alpha_l, A_{a+1}, \dots, A_n$ are mutually distinct. We have $a > 1$ by (2.1). By the assumption, we have either $a < l$ or $a < n$. Let $1 \leq b \leq a$. Since $y_{A_b}^{B_b} \in \sigma$, we have $\sigma = \sigma(y_{\alpha_1}^{\beta_1}, \dots, y_{\alpha_l}^{\beta_l}, K, y_{A_b}^{B_b})$ by (2.3). By (2.2), we have $\sigma = \sigma(y_{\alpha_1}^{\beta_1}, \dots, y_{\alpha_b}^{\beta_b \cup B_b}, \dots, y_{\alpha_l}^{\beta_l}, K)$. Similarly, we have $\sigma = \sigma(y_{A_1}^{B_1}, \dots, y_{A_b}^{B_b \cup \beta_b}, \dots, y_{A_n}^{B_n}, K)$. Therefore, we may assume that $\beta_1 = B_1, \dots, \beta_a = B_a$. Let, say, A_n be a maximal element in $\{\alpha_{a+1}, \dots, \alpha_l, A_{a+1}, \dots, A_n\}$. Note that, for any $X \in \{\alpha_{a+1}, \dots, \alpha_l, A_{a+1}, \dots, A_n\} \setminus \{A_n\}$, we have $X \not\supseteq A_n$. Since $y_{A_n}^{B_n} \in \sigma(y_{\alpha_1}^{\beta_1}, \dots, y_{\alpha_l}^{\beta_l}, K)$, by (2.1), there is a subset $\{i_1, \dots, i_c\} \subseteq \{1, \dots, a\}$ such that $A_n = \alpha_{i_1} \cap \dots \cap \alpha_{i_c}$ and that $B_n \subseteq \beta_{i_1} \cup \dots \cup \beta_{i_c}$. Hence, $A_n = A_{i_1} \cap \dots \cap A_{i_c}$ and $B_n \subseteq B_{i_1} \cup \dots \cup B_{i_c}$. Confering (2.4), this is a contradiction to the minimality of the y -representation $\sigma(y_{A_1}^{B_1}, \dots, y_{A_n}^{B_n}, K)$.

In (2.6), $\{y_{\alpha_1}^{\beta_1}, \dots, y_{\alpha_l}^{\beta_l}\} = \{y_{A_1}^{B_1}, \dots, y_{A_n}^{B_n}\}$ need not hold (cf., [4, §4, (1.5)]).

§3 Domains with $\mu = 0$ and $\nu = 3$

As in the introduction, let D be a 1-dimensional Prüfer domain with finitely many maximal ideals. Let $M_1, \dots, M_\nu$ be the maximal ideals of D , let $M_1, \dots, M_\mu$ be non-invertible, and let $M_{\mu+1}, \dots, M_\nu$ be invertible. Let $\star$ be a mapping from $\bar{F}(D)$ to $\bar{F}(D)$. If $(M_i Z_j)^\star = (M_i^\star Z_j)^\star$ for every $i \neq j$, then $\star$ is said to satisfy the condition (C). Every semistar operation satisfies the condition (C). Let $s \in \Sigma(D)$, and let $\sigma \in \mathcal{S}(D)$. Then the pair (s, σ) canonically defines a mapping from $\bar{F}(D)$ to $\bar{F}(D)$. If (s, σ) satisfies the condition (C), we say that s and σ are compatible with each other.

In this section, we study domains D with $\mu = 0$ and $\nu = 3$. We calculate $|\mathcal{S}(D)|$ and $|\Sigma'(D)|$. The following (3.1) ~ (3.3) appear in [4].

(3.1) Let $s \in \Sigma(D)$, and let $\sigma \in \mathcal{S}(D)$. Then $\star := (s, \sigma)$ is a semistar operation

on D if and only if it satisfies the condition (C).

(3.2) Let $s \in \Sigma(D)$, let $\sigma \in \mathcal{S}(D)$, and set $\star := (s, \sigma)$. Then the following conditions (1) and (2) are equivalent:

- (1) $\star$ satisfies the condition (C).
- (2) Let $M_j^s = D$ for some j . Then we have $y_\alpha^{jj_1 \cdots j_k} \notin \sigma$ for any α and $\{j_1, \dots, j_k\} \subseteq \{1, \dots, \mu\}$ with $j, j_1, \dots, j_k$ distinct with $k \geq 0$ such that $\alpha \cap \{j, j_1, \dots, j_k\} = \emptyset$.

(3.3) Every semistar operation $\star$ on D with $D^\star = D$ is uniquely of the form (s, σ) , where s (resp., σ) is a star (resp., sigma) operation on D with compatible with each other.

(3.4) Example Let D be a domain with $\mu = 0$ and $\nu = 3$. Then we have $|\mathcal{S}(D)| = 45$ and $|\Sigma'(D)| = 61$.

Proof We have $\Sigma(D) = \{d\}$ by (1.2), and $Y = \{xy_1, xy_2, xy_3, xy_{12}, xy_{13}, xy_{23}, K \mid 0 \neq x \in K\}$. Every sigma operation has a unique maximal y-representation by (1.6). All the sigma operations on D are as follows. We confer (1.5). In the following, every one is a maximal y-representation, and every two are distinct with each other:

- $$\begin{aligned} \mathcal{S}(D) = \{ & \\ (0) & \sigma(K), \\ (1) & \sigma(y_1, K), \sigma(y_2, K), \sigma(y_3, K), \\ & \sigma(y_{12}, K), \sigma(y_{13}, K), \sigma(y_{23}, K), \\ (2) & \sigma(y_1, y_2, K), \sigma(y_1, y_3, K), \sigma(y_2, y_3, K), \\ & \sigma(y_1, y_{12}, K), \sigma(y_1, y_{13}, K), \sigma(y_1, y_{23}, K), \\ & \sigma(y_2, y_{12}, K), \sigma(y_2, y_{13}, K), \sigma(y_2, y_{23}, K), \\ & \sigma(y_3, y_{12}, K), \sigma(y_3, y_{13}, K), \sigma(y_3, y_{23}, K), \\ (3) & \sigma(y_1, y_2, y_3, K), \sigma(y_1, y_2, y_{12}, K), \sigma(y_1, y_2, y_{13}, K) \\ & \sigma(y_1, y_2, y_{23}, K), \sigma(y_1, y_3, y_{12}, K), \sigma(y_1, y_3, y_{13}, K), \\ & \sigma(y_1, y_3, y_{23}, K), \sigma(y_2, y_3, y_{12}, K), \sigma(y_2, y_3, y_{13}, K), \\ & \sigma(y_2, y_3, y_{23}, K), \sigma(y_1, y_{12}, y_{13}, K), \sigma(y_2, y_{12}, y_{23}, K), \\ & \sigma(y_3, y_{13}, y_{23}, K), \\ (4) & \sigma(y_1, y_2, y_3, y_{12}, K), \sigma(y_1, y_2, y_3, y_{13}, K), \sigma(y_1, y_2, y_3, y_{23}, K), \\ & \sigma(y_1, y_2, y_{12}, y_{13}, K), \sigma(y_1, y_2, y_{12}, y_{23}, K), \sigma(y_1, y_3, y_{12}, y_{13}, K), \\ & \sigma(y_1, y_3, y_{13}, y_{23}, K), \sigma(y_2, y_3, y_{12}, y_{23}, K), \sigma(y_2, y_3, y_{13}, y_{23}, K), \\ (5) & \sigma(y_1, y_2, y_3, y_{12}, y_{13}, K), \sigma(y_1, y_2, y_3, y_{12}, y_{23}, K), \sigma(y_1, y_2, y_3, y_{13}, y_{23}, K), \\ (6) & \sigma(y_1, y_2, y_3, y_{12}, y_{13}, y_{23}, K) \}. \end{aligned}$$

Therefore, we have $|\mathcal{S}(D)| = 45$. We have that $\{D, V_i, R_{ij}, K \mid i \neq j\}$ is the set of all overrings of D , where $R_{ij} := V_i \cap V_j$. By (1.1), we have

$$|\Sigma'(D)| = |\Sigma'_0(D)| + \sum_i |\Sigma'_0(V_i)| + \sum_{i \neq j} |\Sigma'_0(R_{ij})| + |\Sigma'_0(K)|.$$

For the d-star operation on D , and for every sigma operation σ , obviously, the mapping (d, σ) from $\bar{F}(D)$ to $\bar{F}(D)$ satisfies the condition (C). Therefore, $|\Sigma'_0(D)| = 45$ by (3.3). Obviously, $|\Sigma'_0(V_i)| = 1$ for every i . Confering [2, Proposition 6 (2)], we have $|\Sigma'_0(R_{ij})| = 4$ for every $i \neq j$, and clearly $|\Sigma'_0(K)| = 1$. It follows that $|\Sigma'(D)| = 45 + 1 \times 3 + 4 \times 3 + 1 = 61$.

§4 Domains with $\mu = 1$ and $\nu = 3$

A non-zero ideal T of D is said to have type t_i if there is a subset $\{p_\lambda \mid \lambda \in \Lambda\}$ of $M_i \setminus \bigcup_{j \neq i} M_j$ such that T is generated by $\{p_\lambda \mid \lambda \in \Lambda\}$ and there does not exist $\inf v_i(T)$.

In this section, we study domains D with $\mu = 1$ and $\nu = 3$, and calculate $|\mathcal{S}(D)|$ and $|\Sigma'(D)|$.

(4.1) Example Let D be a domain with $\mu = 1$ and $\nu = 3$. Then $|\mathcal{S}(D)| = 126$, $|\Sigma'_0(D)| = 171$ and $|\Sigma'(D)| = 200$.

Proof We have that $|\Sigma(D)| = 2$ and $\Sigma(D) = \{d, v\}$ by (1.2), and that

$Y := \bar{F}(D) \setminus F(D) = \{xy_1, xy_2, xy_3, xy_{12}, xy_{13}, xy_{23}, xy_2^1, xy_3^1, xy_{23}^1, xT_1y_2, xT_1y_3, xT_1y_{23}, K \mid 0 \neq x \in K, T_1 \text{ is an ideal of } D \text{ with type } t_1\}$.

Every sigma operation has a unique maximal y -representation by (1.6). All the sigma operations on D are as follows. We confer (1.5). In the following, every one is a maximal y -representation, and every two are distinct with each other:

- $$\begin{aligned} \mathcal{S}(D) = \{ & \\ (0) & \sigma(K), \\ (1) & \sigma(y_1, K), \sigma(y_2, K), \sigma(y_3, K), \\ & \sigma(y_{12}, K), \sigma(y_{13}, K), \sigma(y_{23}, K), \\ (2) & \sigma(y_1, y_2, K), \sigma(y_1, y_3, K), \sigma(y_2, y_3, K), \\ & \sigma(y_1, y_{12}, K), \sigma(y_1, y_{13}, K), \sigma(y_1, y_{23}, K), \\ & \sigma(y_2, y_{12}, K), \sigma(y_2, y_{13}, K), \sigma(y_2, y_{23}, K), \\ & \sigma(y_3, y_{12}, K), \sigma(y_3, y_{13}, K), \sigma(y_3, y_{23}, K), \\ & \sigma(y_2, y_2^1, K), \sigma(y_3, y_3^1, K), \sigma(y_{23}, y_{23}^1, K), \\ (3) & \sigma(y_1, y_2, y_3, K), \sigma(y_1, y_2, y_{12}, K), \sigma(y_1, y_2, y_{13}, K), \\ & \sigma(y_1, y_2, y_{23}, K), \sigma(y_1, y_3, y_{12}, K), \sigma(y_1, y_3, y_{13}, K), \\ & \sigma(y_1, y_3, y_{23}, K), \sigma(y_2, y_3, y_{12}, K), \sigma(y_2, y_3, y_{13}, K), \\ & \sigma(y_2, y_3, y_{23}, K), \sigma(y_1, y_{12}, y_{13}, K), \sigma(y_2, y_{12}, y_{23}, K), \\ & \sigma(y_3, y_{13}, y_{23}, K), \sigma(y_1, y_2, y_2^1, K), \sigma(y_2, y_3, y_2^1, K), \\ & \sigma(y_2, y_{12}, y_2^1, K), \sigma(y_2, y_{13}, y_2^1, K), \sigma(y_2, y_{23}, y_2^1, K), \\ & \sigma(y_1, y_3, y_3^1, K), \sigma(y_2, y_3, y_3^1, K), \sigma(y_3, y_{12}, y_3^1, K), \\ & \sigma(y_3, y_{13}, y_3^1, K), \sigma(y_3, y_{23}, y_3^1, K), \sigma(y_1, y_{23}, y_{23}^1, K), \\ (4) & \sigma(y_1, y_2, y_3, y_{12}, K), \sigma(y_1, y_2, y_3, y_{13}, K), \sigma(y_1, y_2, y_3, y_{23}, K), \\ & \sigma(y_1, y_2, y_{12}, y_{13}, K), \sigma(y_1, y_2, y_{12}, y_{23}, K), \sigma(y_1, y_3, y_{12}, y_{13}, K), \\ & \sigma(y_1, y_3, y_{13}, y_{23}, K), \sigma(y_2, y_3, y_{12}, y_{23}, K), \sigma(y_2, y_3, y_{13}, y_{23}, K), \\ & \sigma(y_1, y_2, y_3, y_2^1, K), \sigma(y_1, y_2, y_{12}, y_2^1, K), \sigma(y_1, y_2, y_{13}, y_2^1, K), \\ & \sigma(y_1, y_2, y_{23}, y_2^1, K), \sigma(y_2, y_3, y_{12}, y_2^1, K), \sigma(y_2, y_3, y_{13}, y_2^1, K), \\ & \sigma(y_2, y_3, y_{23}, y_2^1, K), \sigma(y_2, y_{12}, y_{23}, y_2^1, K), \sigma(y_1, y_2, y_3, y_3^1, K), \\ & \sigma(y_1, y_3, y_{12}, y_3^1, K), \sigma(y_1, y_3, y_{13}, y_3^1, K), \sigma(y_1, y_3, y_{23}, y_3^1, K), \\ & \sigma(y_2, y_3, y_{12}, y_3^1, K), \sigma(y_2, y_3, y_{13}, y_3^1, K), \sigma(y_2, y_3, y_{23}, y_3^1, K), \\ & \sigma(y_3, y_{13}, y_{23}, y_3^1, K), \sigma(y_2, y_3, y_2^1, y_3^1, K), \sigma(y_2, y_{23}, y_2^1, y_3^1, K), \\ & \sigma(y_3, y_{23}, y_3^1, y_{23}^1, K), \\ (5) & \sigma(y_1, y_2, y_3, y_{12}, y_{13}, K), \sigma(y_1, y_2, y_3, y_{12}, y_{23}, K), \sigma(y_1, y_2, y_3, y_{13}, y_{23}, K), \end{aligned}$$

$$\begin{aligned}
& \sigma(y_1, y_2, y_3, y_{12}, y_2^1, K), \sigma(y_1, y_2, y_3, y_{13}, y_2^1, K), \sigma(y_1, y_2, y_3, y_{23}, y_2^1, K), \\
& \sigma(y_1, y_2, y_{12}, y_{13}, y_2^1, K), \sigma(y_1, y_2, y_{12}, y_{23}, y_2^1, K), \sigma(y_2, y_3, y_{12}, y_{23}, y_2^1, K), \\
& \sigma(y_2, y_3, y_{13}, y_{23}, y_2^1, K), \sigma(y_1, y_2, y_3, y_{12}, y_3^1, K), \sigma(y_1, y_2, y_3, y_{13}, y_3^1, K), \\
& \sigma(y_1, y_2, y_3, y_{23}, y_3^1, K), \sigma(y_1, y_3, y_{12}, y_{13}, y_3^1, K), \sigma(y_1, y_3, y_{13}, y_{23}, y_3^1, K), \\
& \sigma(y_2, y_3, y_{12}, y_{23}, y_3^1, K), \sigma(y_2, y_3, y_{13}, y_{23}, y_3^1, K), \sigma(y_1, y_2, y_3, y_2^1, y_3^1, K), \\
& \sigma(y_2, y_3, y_{12}, y_2^1, y_3^1, K), \sigma(y_2, y_3, y_{13}, y_2^1, y_3^1, K), \sigma(y_2, y_3, y_{23}, y_2^1, y_3^1, K), \\
& \sigma(y_1, y_2, y_{23}, y_2^1, y_{23}^1, K), \sigma(y_2, y_{12}, y_{23}, y_2^1, y_{23}^1, K), \sigma(y_1, y_3, y_{23}, y_3^1, y_{23}^1, K), \\
& \sigma(y_3, y_{13}, y_{23}, y_3^1, y_{23}^1, K), \\
(6) & \sigma(y_1, y_2, y_3, y_{12}, y_{13}, y_{23}, K), \sigma(y_1, y_2, y_3, y_{12}, y_{13}, y_2^1, K), \sigma(y_1, y_2, y_3, y_{12}, y_{23}, y_2^1, K), \\
& \sigma(y_1, y_2, y_3, y_{13}, y_{23}, y_2^1, K), \sigma(y_1, y_2, y_3, y_{12}, y_{13}, y_3^1, K), \sigma(y_1, y_2, y_3, y_{12}, y_{23}, y_3^1, K), \\
& \sigma(y_1, y_2, y_3, y_{13}, y_{23}, y_3^1, K), \sigma(y_1, y_2, y_3, y_{12}, y_2^1, y_3^1, K), \sigma(y_1, y_2, y_3, y_{13}, y_2^1, y_3^1, K), \\
& \sigma(y_1, y_2, y_3, y_{23}, y_2^1, y_3^1, K), \sigma(y_2, y_3, y_{12}, y_{23}, y_2^1, y_3^1, K), \sigma(y_2, y_3, y_{13}, y_{23}, y_2^1, y_3^1, K), \\
& \sigma(y_1, y_2, y_{12}, y_{23}, y_2^1, y_{23}^1, K), \sigma(y_1, y_3, y_{13}, y_{23}, y_3^1, y_{23}^1, K), \sigma(y_2, y_3, y_{23}, y_2^1, y_3^1, y_{23}^1, K), \\
& K), \\
(7) & \sigma(y_1, y_2, y_3, y_{12}, y_{13}, y_{23}, y_2^1, K), \sigma(y_1, y_2, y_3, y_{12}, y_{13}, y_{23}, y_3^1, K), \sigma(y_1, y_2, y_3, y_{12}, y_{23}, y_3^1, K), \\
& y_{13}, y_2^1, y_3^1, K), \\
& \sigma(y_1, y_2, y_3, y_{12}, y_{23}, y_2^1, y_3^1, K), \sigma(y_1, y_2, y_3, y_{13}, y_{23}, y_2^1, y_3^1, K), \sigma(y_1, y_2, y_3, y_{23}, y_2^1, y_3^1, K), \\
& y_{23}^1, K), \\
& \sigma(y_2, y_3, y_{12}, y_{23}, y_2^1, y_3^1, y_{23}^1, K), \sigma(y_2, y_3, y_{13}, y_{23}, y_2^1, y_3^1, y_{23}^1, K), \\
(8) & \sigma(y_1, y_2, y_3, y_{12}, y_{13}, y_{23}, y_2^1, y_3^1, K), \sigma(y_1, y_2, y_3, y_{12}, y_{23}, y_2^1, y_3^1, y_{23}^1, K), \sigma(y_1, y_2, y_3, y_{13}, y_{23}, y_2^1, y_3^1, y_{23}^1, K), \\
& y_{13}, y_{23}, y_2^1, y_3^1, y_{23}^1, K), \\
(9) & \sigma(y_1, y_2, y_3, y_{12}, y_{13}, y_{23}, y_2^1, y_3^1, y_{23}^1, K)\}.
\end{aligned}$$

Therefore, we have $|\mathcal{S}(D)| = 126$. By (1.1), we have

$$|\Sigma'(D)| = |\Sigma'_0(D)| + \Sigma_i |\Sigma'_0(V_i)| + \Sigma_{i \neq j} |\Sigma'_0(R_{ij})| + |\Sigma'_0(K)|.$$

Let σ be a sigma operation on D . For the d-star operation on D , obviously, the mapping (d, σ) from $\bar{F}(D)$ to $\bar{F}(D)$ satisfies the condition (C). By (3.2), we have the following: (v, σ) does not satisfy the condition (C) if and only if $y_2^1 \in \sigma$ or $y_3^1 \in \sigma$ or $y_{23}^1 \in \sigma$. By (3.3), we have that $|\Sigma'_0(D)| = 171$.

We have that $|\Sigma'_0(V_1)| = 2$ (cf., [1, Proposition 12]) and $|\Sigma'_0(V_2)| = |\Sigma'_0(V_3)| = 1$.

Further, conferring [3, Theorem], we have that $|\Sigma'_0(R_{23})| = 4$ and $|\Sigma'_0(R_{12})| = |\Sigma'_0(R_{13})| = 10$, where $R_{ij} := V_i \cap V_j$ for $i \neq j$. It follows that $|\Sigma'(D)| = 171 + (2 + 1 + 1) + (10 + 10 + 4) + 1 = 200$.

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