Remarks on Kato's inequality when $\Delta_p u$ is a measure

Xiaojing LIU* and Toshio HORIUCHI**

Abstract

Let Ω be a bounded domain of $\mathbf{R}^N$ ($N \geq 1$). In this article, we shall study Kato's inequality when $\Delta_p u$ is a measure, where $\Delta_p u$ denotes a p -Laplace operator with $1 < p < \infty$. The classical Kato's inequality for a Laplacian asserts that given any function $u \in L^1_{\text{loc}}(\Omega)$ such that $\Delta u \in L^1_{\text{loc}}(\Omega)$, then $\Delta(u^+)$ is a Radon measure and the following holds: $\Delta(u^+) \geq \chi_{[u \geq 0]} \Delta u$ in $D'(\Omega)$. Our main result extends Kato's inequality to the case where $\Delta_p u$ is a Radon measures on Ω . We also establish the inverse maximum principle for Δ_p .

1. Introduction

Let Ω be a bounded domain of $\mathbf{R}^N$, ($N \geq 1$). In this article, we shall study Kato's inequality when $\Delta_p u$ is a measure. By $\Delta_p u$ we denote a p -Laplace operator

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u), \quad (1.1)$$

where $1 < p < \infty$ and $\nabla u = (\partial u / \partial x_1, \partial u / \partial x_2, \dots, \partial u / \partial x_N)$.

Now let us recall some fundamental results on Kato's inequality (see [12]; see also [5]). The classical Kato's inequality for a Laplacian asserts that given any function $u \in L^1_{\text{loc}}(\Omega)$ such that $\Delta u \in L^1_{\text{loc}}(\Omega)$, then $\Delta(u^+)$ is a Radon measure and the following holds:

$$\Delta(u^+) \geq \chi_{[u \geq 0]} \Delta u \quad \text{in } D'(\Omega), \quad (1.2)$$

where $u^+ = \max[u, 0]$. A similar inequality holds when Δu is replaced by $\Delta_p u$ under additional assumptions on distributional derivatives of $u \in L^1_{\text{loc}}(\Omega)$ (see e.g. [9, 10, 13]).

Our main result (see Theorem 1.1 and Corollary 1.1 below) further extends Kato's inequality involving Δ_p to the case where $\Delta_p u \in M(\Omega)$, where $M(\Omega)$ denotes the space of Radon measures on Ω . In other words, $\mu \in M(\Omega)$ if and only if, for every $\omega \subset\subset \Omega$, there exists $C_\omega > 0$ such that $|\int_\omega \varphi d\mu| \leq C_\omega \|\varphi\|_{L^\infty(\omega)}$ for any $\varphi \in C_0^\infty(\omega)$.

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*Ibaraki University, Mito, Ibaraki, 310-8512, Japan (liuxiaomao1234@hotmail.com)

**Ibaraki University, Mito, Ibaraki, 310-8512, Japan (toshio.horiuchi.math@vc.ibaraki.ac.jp)

We begin with recalling that for any $\mu \in M(\Omega)$ can be uniquely decomposed as a sum of two Radon measures on Ω (see e.g. [5, 8]) : $\mu = \mu_d + \mu_c$, where

$$\begin{cases} \mu_d(A) = 0 & \text{for any Borel measurable set } A \subset \Omega \text{ such that } C_p(A, \Omega) = 0, \\ |\mu_c|(\Omega \setminus F) = 0 & \text{for some Borel measurable set } F \subset \Omega \text{ such that } C_p(F, \Omega) = 0. \end{cases} \quad (1.3)$$

Here by $C_p(K, \Omega)$ we denote a p -capacity of a Borel set K relative to Ω (for the precise definition see Definition 1.7 in §2). We note that $(\mu_d)^\pm = (\mu^\pm)_d$, $(\mu_c)^\pm = (\mu^\pm)_c$, $|\mu_d| = |\mu|_d$ and $|\mu_c| = |\mu|_c$ hold by the definition, where $\nu^- = \max[-\nu, 0] = (-\nu)^+$ and $|\nu| = \nu^+ + \nu^-$ for a Radon measure ν on Ω .

Then we recall an admissible class of functions for the strong maximum principle in [13]:

Definition 1.1. (**Admissible class in $W_{\text{loc}}^{1,p^*}(\Omega)$**) Let $1 < p < \infty$ and $p^* = \max[1, p-1]$. A function $u \in W_{\text{loc}}^{1,p^*}(\Omega)$ is said to be admissible if $\Delta_p u \in M(\Omega)$ and there exists a sequence $\{u_n\}_{n=1}^\infty \subset W_{\text{loc}}^{1,p}(\Omega) \cap L^\infty(\Omega)$ such that:

1. $u_n \rightarrow u$ a.e. in Ω and $u_n \rightarrow u$ in $W_{\text{loc}}^{1,p^*}(\Omega)$ as $n \rightarrow \infty$.
2. $\Delta_p u_n \in L_{\text{loc}}^1(\Omega)$ ($n = 1, 2, \dots$) and

$$\sup_n |\Delta_p u_n|(\omega) = \sup_n \int_\omega |\Delta_p u_n| < \infty \quad \text{for every } \omega \subset\subset \Omega. \quad (1.4)$$

Remark 1.1.

If either $\{(\Delta_p u_n)^+\}$ or $\{(\Delta_p u_n)^-\}$ is bounded in $M(\Omega)$, then (1.4) is satisfied. Otherwise one may assume that $\{(\Delta_p u_n)^-\}$ is bounded and $\liminf_{n \rightarrow \infty} (\Delta_p u_n)^+(\omega) = \infty$ for some $\omega \subset\subset \Omega$, and hence there exists a continuous function $\psi \geq 0$ with compact support in ω such that $\liminf_{n \rightarrow \infty} \int (\Delta_p u_n)^+ \psi = \infty$. Hence, $\liminf_{n \rightarrow \infty} \int (\Delta_p u_n)^+ \varphi = \infty$ for any $\varphi \in C_0^\infty(\omega)$ such as $0 \leq \psi \leq \varphi$. Since $\Delta_p u_n \rightarrow \Delta_p u$ in $D'(\Omega)$ as $n \rightarrow \infty$, this is a contradiction.

Here we collect some known results concerning the admissibility. For the sake of reader's convenience we give brief proofs to them in Appendix.

Proposition 1.1. *Let $N \geq 1$, $1 < p < \infty$, $p^* = \max[1, p-1]$ and Ω be a bounded domain of $\mathbf{R}^N$.*

1. *Assume that a function $u \in W_{\text{loc}}^{1,p^*}(\Omega)$ is admissible. Then $u^+ = \max[u, 0]$ and $u^- = \max[-u, 0]$ are also admissible.*
2. *Assume that $p = 2$. Then a function $u \in W_{\text{loc}}^{1,1}(\Omega)$ is admissible if $\Delta u \in M(\Omega)$.*
3. *A function $u \in W_0^{1,p}(\Omega)$ is admissible if $\Delta_p u \in M(\Omega)$.*

Now we state our main results:

Theorem 1.1. *Let $N \geq 1$, $1 < p < \infty$, $p^* = \max[1, p - 1]$ and Ω be a bounded domain of $\mathbf{R}^N$. Let Φ be a C^1 convex function in $\mathbf{R}$ such that $\Phi' \geq 0$ in $\mathbf{R}$ and $\Phi' \in L^\infty(\mathbf{R})$.*

When $p = 2$, assume that $u \in L^1_{\text{loc}}(\Omega)$ and Δu is a Radon measure on Ω .

When $p \neq 2$, assume that $u \in W^{1,p^*}_{\text{loc}}(\Omega)$ and u is admissible in the sense of Definition 1.1.

Then we have

$$\Delta_p \Phi(u) \geq \Phi'(u)^{p-1} (\Delta_p u)_d - \|\Phi'\|_{L^\infty(\mathbf{R})}^{p-1} (\Delta_p u)_c^- \quad \text{in } D'(\Omega). \quad (1.5)$$

From this theorem it follows that we have

Corollary 1.1. *Assume the same assumptions in Theorem 1.1. Then it holds that*

$$\Delta_p(u^+) \geq \chi_{[u \geq 0]} (\Delta_p u)_d - (\Delta_p u)_c^- \quad \text{in } D'(\Omega), \quad (1.6)$$

$$\Delta_p |u| \geq \text{sgn}(u) (\Delta_p u)_d - |\Delta_p u|_c \quad \text{in } D'(\Omega), \quad (1.7)$$

where $\text{sgn}(t) = 1$ for $t > 0$, $\text{sgn}(t) = -1$ for $t < 0$, and $\text{sgn}(0) = 0$.

Note that the right-hand sides of (1.6) and (1.7) are well-defined because u is quasicontinuous. More precisely it follows from Lemma 1 that there exists $\tilde{u} : \Omega \rightarrow \mathbf{R}$ quasicontinuous and $u = \tilde{u}$ a.e. in Ω . In (1.6) and (1.7), we identify u with its quasicontinuous representative. Then we see that $\chi_{[u \geq 0]}$ and $\text{sgn}(u)$ are locally integrable in Ω with respect to $|(\Delta_p u)_d|$ (see also [5, 13, 17]).

The following is an extension of inverse maximum principle for $p = 2$ obtained in [8] and the proof can be done in a quite similar way (see also [6, 5]).

Theorem 1.2. (**Inverse maximum principle**) *Let $N \geq 1$, $1 < p < \infty$, $p^* = \max[1, p - 1]$ and Ω be a bounded domain of $\mathbf{R}^N$.*

When $p = 2$, assume that $u \in L^1_{\text{loc}}(\Omega)$, $u \geq 0$ a.e. in Ω and Δu is a Radon measure on Ω .

When $p \neq 2$, assume that $u \in W^{1,p^*}_{\text{loc}}(\Omega)$, $u \geq 0$ a.e. in Ω and u is admissible in the sense of Definition 1.1.

Then we have

$$(-\Delta_p u)_c \geq 0 \quad \text{in } \Omega. \quad (1.8)$$

From this theorem together with the previous one it follows that we have:

Corollary 1.2. *Let $N \geq 1$, $1 < p < \infty$, $p^* = \max[1, p - 1]$ and Ω be a bounded domain of $\mathbf{R}^N$.*

When $p = 2$, assume that $u \in L^1_{\text{loc}}(\Omega)$ and Δu is a Radon measure on Ω .

When $p \neq 2$, assume that $u \in W^{1,p^*}_{\text{loc}}(\Omega)$ and u is admissible in the sense of Definition 1.1.

Then we have

$$(-\Delta_p(u^+))_c = (-\Delta_p u)_c^+ \quad \text{in } \Omega. \quad (1.9)$$

$$(-\Delta_p(u^-))_c = (-\Delta_p u)_c^-, \quad (-\Delta_p |u|)_c = |\Delta_p u|_c \quad \text{in } \Omega. \quad (1.10)$$

Example 1. Let Ω be a bounded domain of $\mathbf{R}^N$ containing the origin. Let $u = |x|^\alpha$ for $\alpha = (p - N)/(p - 1)$.

1. u satisfies

$$\Delta_p u = \alpha |\alpha|^{p-2} c_N \delta \quad \text{in } D'(\Omega),$$

where δ denotes a Dirac mass and c_N denotes the surface area of the N -dimensional unit ball B_1 . It is easy to see that $u \in W^{1,p^*}(\Omega)$ if and only if $p > 2 - 1/N$. When $1 < p \leq 2 - 1/N$ holds, we can consider u as a renormalized solution. We recall that if $p \leq 2 - 1/N$, then we cannot expect the solution of an equation of the form $\Delta_p u = f \in M(\Omega)$ to be in $W_{\text{loc}}^{1,1}(\Omega)$. For the detail, see e.g. [1, 2, 7, 15, 16].

2. When $p \geq N$ with $N \geq 2$ or $p > 1$ with $N = 1$, $u \in W^{1,p}(\Omega)$ is admissible, and when $2 - \frac{1}{N} < p < N$ with $N \geq 2$, u is admissible in $W_{\text{loc}}^{1,p^*}(\Omega)$. In fact, $u = |x|^\alpha$ is approximated by a sequence of admissible functions $v_n = |x|^{\alpha(n)} \in L^1(\Omega)$ where $\alpha(n) = \alpha + 1/(n(p - 1))$ ($n = 1, 2, \dots$). Then, in the sense of measures we have

$$\Delta_p v_n = \frac{1}{n} |\alpha(n)|^{p-2} \alpha(n) |x|^{1/n-N} \rightarrow \Delta_p u = \alpha |\alpha|^{p-2} c_N \delta \quad \text{in } D'(\Omega) \quad \text{as } n \rightarrow \infty.$$

Therefore there exists a sequence $\{n_j\}$ such that $\{n_j\} \rightarrow \infty$ as $j \rightarrow \infty$ and a sequence of mollification $\{(v_j)_{\rho}^{n_j}\}$ satisfies the conditions in Definition 1.1

3. When $1 < p \leq N$ with $N \geq 2$, by $\alpha \leq 0$ we have $\Delta_p u = \alpha |\alpha|^{p-2} c_N \delta \leq 0$ in $D'(\Omega)$, and so $(\Delta_p u)_c \leq 0$ in $D'(\Omega)$, which implies

$$\Delta_p u = (\Delta_p u)_d + (\Delta_p u)_c = (\Delta_p u)_d - (\Delta_p u)_c^- = (\Delta_p u)_d - |\Delta_p u|_c \quad \text{in } D'(\Omega). \quad (1.11)$$

When $p > N$, by $\alpha > 0$ we have $-\Delta_p u = -\alpha |\alpha|^{p-2} c_N \delta \leq 0$ in $D'(\Omega)$, and $-(\Delta_p u)_c \leq 0$ in $D'(\Omega)$. Since $-(\Delta_p u)_c \geq 0$ in $D'(\Omega)$ by Theorem 1.2, which is applicable to this case because $u \in W^{1,p}(\Omega)$ is admissible, it follows that $(-\Delta_p u)_c = 0$ in $D'(\Omega)$. Hence we see $(\Delta_p u)_c = 0 = (\Delta_p u)_c^- = |\Delta_p u|_c$ in $D'(\Omega)$, which implies

$$\Delta_p u = (\Delta_p u)_d + (\Delta_p u)_c = (\Delta_p u)_d - (\Delta_p u)_c^- = (\Delta_p u)_d - |\Delta_p u|_c \quad \text{in } D'(\Omega). \quad (1.12)$$

Each of (1.11) and (1.12) shows that the equality holds in both inequalities (1.6) and (1.7).

This article is organized in the following way. In §2 we collect basic notations and definitions. In §3 we prove Corollary 1.1 and Corollary 1.2. In §4 we prepare some lemmas which are needed in this article. We prove Theorem 1.1 and Theorem 1.2 in §5 and §6 respectively. In §7 the proof of Proposition 1.1 is given.

2. Preliminaries

In this subsection we collect fundamental definitions in the present article together with some remarks.

Let $L^p(\Omega)$, $1 \leq p < \infty$, denote the space of Lebesgue measurable functions, defined on Ω , for which

$$\|f\|_{L^p(\Omega)} = \left(\int_{\Omega} |f|^p \right)^{1/p} < \infty.$$

By $L^p_{loc}(\Omega)$ we mean the space of functions locally integrable with power p in Ω , and by $L^\infty(\Omega)$ we mean the space of essentially bounded Lebesgue measurable functions. As a norm of f in $L^\infty(\Omega)$ we take its essential supremum, i.e.,

$$\|f\|_{L^\infty(\Omega)} = \inf\{c > 0 : |f(x)| \leq c \text{ for almost all } x \in \Omega\}.$$

Then we define the following Sobolev spaces:

Definition 2.1. ($W^{1,p}(\Omega)$, $W^{1,p}_{loc}(\Omega)$ and $W^{1,p}_0(\Omega)$)

For each $1 \leq p < \infty$, we set

$$W^{1,p}(\Omega) = \{f : \Omega \rightarrow \mathbb{R} : f \in L^p(\Omega), \partial_i f \in L^p(\Omega) \text{ for } i = 1, \dots, N\}, \quad (2.1)$$

$$W^{1,p}_{loc}(\Omega) = \{f : \Omega \rightarrow \mathbb{R} : f \in L^p_{loc}(\Omega), \partial_i f \in L^p_{loc}(\Omega) \text{ for } i = 1, \dots, N\}, \quad (2.2)$$

where $\partial_i f$ is taken as a distributional derivative of f for x_i with $i = 1, \dots, N$. The space $W^{1,p}(\Omega)$ is equipped with the norm

$$\|u\|_{W^{1,p}(\Omega)} = \|\nabla u\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}. \quad (2.3)$$

By $W^{1,p}_0(\Omega)$ we denote the completion of $C^\infty_0(\Omega)$ in the norm $\|\cdot\|_{W^{1,p}(\Omega)}$.

Definition 2.2. (A p -capacity relative to Ω)

Let $1 < p < \infty$. For each compact set $K \subset \Omega$ we define a p -capacity of K relative to Ω by

$$C_p(K, \Omega) = \inf \left\{ \int_{\Omega} |\nabla \varphi|^p : \varphi \in C^\infty_0(\Omega), \varphi \geq 1 \text{ in some neighborhood of } K \right\}.$$

Definition 2.3. (Quasi-everywhere and Quasicontinuity)

1. If a property holds everywhere except possibly on a set of p -capacity zero, then it is said to hold quasi-everywhere.
2. We say that a function $u : \Omega \rightarrow \mathbb{R}$ is quasicontinuous if there exists a sequence of open subsets ω_n of Ω such that $u|_{\omega_n}$ is continuous for $n \geq 1$ and $C_p(\omega_n, \Omega) \rightarrow 0$ as $n \rightarrow \infty$.

3. Proof of Corollary 1.1 and Corollary 1.2

Proof of Corollary 1.1:

Let Φ be a C^1 convex function in $\mathbf{R}$ such that $\Phi(t) = t$ if $t \geq 0$, $|\Phi(t)| < 1$ if $t < 0$, $0 \leq \Phi' \leq 1$ in $\mathbf{R}$ and $\lim_{t \rightarrow -\infty} \Phi'(t) = 0$. Set $\Phi_n(t) = \Phi(nt)/n$ for $t \in \mathbf{R}$ and $n = 1, 2, \dots$. Then we see that $\{\Phi_n\}$ is a sequence of C^1 convex functions in $\mathbf{R}$ such that $\Phi_n(t) = t$ if $t \geq 0$, $|\Phi_n(t)| < \frac{1}{n}$ if $t < 0$, $0 \leq \Phi'_n \leq 1$ in $\mathbf{R}$ and $\lim_{n \rightarrow \infty} \Phi'_n(t) < 0$. It follows from Theorem 1.1 that

$$\Delta_p \Phi_n(u) \geq \Phi'_n(u)^{p-1} (\Delta_p u)_d - (\Delta_p u)_c^- \quad \text{in } D'(\Omega). \quad (3.1)$$

First we see that $\Delta_p \Phi_n(u) \rightarrow \Delta_p(u^+)$ in $D'(\Omega)$ as $n \rightarrow \infty$. It follows from the remark just after Corollary 1.1 that $\chi_{[u \geq 0]}$ is locally integrable in Ω with respect to $|(\Delta_p u)_d|$. Therefore, as $n \rightarrow \infty$ in (3.1) we have

$$\Delta_p(u^+) \geq \chi_{[u \geq 0]} (\Delta_p u)_d - (\Delta_p u)_c^- \quad \text{in } D'(\Omega), \quad (3.2)$$

which is (1.6). Since for any $\varphi \in C_0^\infty(\Omega)$

$$\begin{aligned} \langle \Delta_p |u|, \varphi \rangle &= \langle \operatorname{div} (|\nabla |u||^{p-2} \nabla (u^+ + u^-)), \varphi \rangle \\ &= - \int_{\Omega} (|\nabla |u||^{p-2} (\nabla u^+ + \nabla u^-) \cdot \nabla \varphi) \\ &= - \int_{\Omega} |\nabla u^+|^{p-2} \nabla u^+ \cdot \nabla \varphi - \int_{\Omega} |\nabla u^-|^{p-2} \nabla u^- \cdot \nabla \varphi \\ &= \langle \Delta_p(u^+) + \Delta_p(u^-), \varphi \rangle, \end{aligned}$$

we have

$$\Delta_p |u| = \Delta_p(u^+) + \Delta_p(u^-) \quad \text{in } D'(\Omega). \quad (3.3)$$

Noting $u^- = (-u)^+$ and applying (3.2) to $-u$ instead of u , we have

$$\begin{aligned} \Delta_p(u^-) &= \Delta_p((-u)^+) \geq \chi_{[-u \geq 0]} (\Delta_p(-u))_d - (\Delta_p(-u))_c^- \\ &= -\chi_{[u \leq 0]} (\Delta_p u)_d - (\Delta_p u)_c^+ \quad \text{in } D'(\Omega). \end{aligned} \quad (3.4)$$

From (3.2), (3.3) and (3.4) it follows that

$$\Delta_p |u| \geq \operatorname{sgn}(u) (\Delta_p u)_d - |\Delta_p u|_c \quad \text{in } D'(\Omega),$$

which shows (1.7). $\square$

Proof of Corollary 1.2: Since u^+ and u^- are admissible by Proposition 1.1, we can apply Theorem 1.2 to u^+ and u^- respectively, so that we have

$$(-\Delta_p(u^+))_c \geq 0, \quad (-\Delta_p(u^-))_c \geq 0 \quad \text{in } D'(\Omega). \quad (3.5)$$

In a similar way as (3.3), we see

$$\Delta_p u = \Delta_p(u^+) - \Delta_p(u^-) \quad \text{in } D'(\Omega), \quad (3.6)$$

which implies

$$(-\Delta_p u)_c = (-\Delta_p(u^+))_c - (-\Delta_p(u^-))_c \quad \text{in } D'(\Omega). \quad (3.7)$$

By (3.5) and (3.7) we have $(-\Delta_p u)_c \leq (-\Delta_p(u^+))_c$ and $(-\Delta_p(u^+))_c \geq 0$ in $D'(\Omega)$, therefore we obtain

$$(-\Delta_p u)_c^+ \leq (-\Delta_p(u^+))_c \quad \text{in } D'(\Omega). \quad (3.8)$$

On the other hand, from (1.6) it follows that

$$(-\Delta_p(u^+))_c \leq (\Delta_p u)_c^- = (-\Delta_p u)_c^+ \quad \text{in } D'(\Omega). \quad (3.9)$$

(3.8) and (3.9) show (1.9).

From (3.7) and (1.9) we have

$$(-\Delta_p(u^-))_c = (-\Delta_p(u^+))_c - (-\Delta_p u)_c = (-\Delta_p u)_c^+ - (-\Delta_p u)_c = (-\Delta_p u)_c^- \quad \text{in } D'(\Omega), \quad (3.10)$$

by (3.10), (1.9) and (3.3) we have

$$(-\Delta_p |u|)_c = (-\Delta_p(u^+))_c + (-\Delta_p(u^-))_c = (-\Delta_p u)_c^+ + (-\Delta_p u)_c^- = |-\Delta_p u|_c \quad \text{in } D'(\Omega), \quad (3.11)$$

which shows (1.10). □

4. Lemmas

Let $N \geq 1$, $1 < p < \infty$, $p^* = \max[1, p-1]$ and Ω be a bounded domain of $\mathbf{R}^N$. We prepare a fundamental lemma (for the proof see e.g. [17], Chapter 2).

Lemma 1.

1. If $u \in W_{loc}^{1,p}(\Omega)$, then u can be redefined almost everywhere in Ω so as to be quasicontinuous.
2. If $u \in W_{loc}^{1,p}(\Omega)$, then there exists a subsequence $\{\varphi_{n_j}\}$ of $\{\varphi_n\}$ such that $\varphi_{n_j} \rightarrow u$ point wise quasi-everywhere in Ω as $j \rightarrow \infty$.

Let us prepare more lemmas. Given $k > 0$, we denote a truncated function $T_k: \mathbf{R} \rightarrow \mathbf{R}$ by

$$T_k(s) := \begin{cases} k & \text{if } s \geq k, \\ s & \text{if } -k < s < k, \\ -k & \text{if } s \leq -k. \end{cases} \quad (4.1)$$

Since $T_k|_{R_+}$ is concave, we have the following lemma in the spirit of the standard L^1 -version of Kato's inequality (see [12]).

Lemma 2. Assume that $v \in W_{\text{loc}}^{1,p}(\Omega)$, $\Delta_p v \in L_{\text{loc}}^1(\Omega)$ and $v \geq 0$ a.e. in Ω . Then, for any $k > 0$ we have

$$\Delta_p T_k(v) \leq t_k(v) \Delta_p v \quad \text{in } D'(\Omega), \quad (4.2)$$

where the function $t_k : \mathbf{R}_+ \rightarrow \{0, 1\}$ is given by

$$t_k(s) := \begin{cases} 1 & \text{if } 0 \leq s \leq k, \\ 0 & \text{if } s > k. \end{cases}$$

Proof: Let Φ be a C^2 concave function in $\mathbf{R}$ such that $\Phi(t) = t$ if $t \leq 0$, $|\Phi(t)| < 1$ if $t > 0$, $0 \leq \Phi' \leq 1$ in $\mathbf{R}$ and $\lim_{t \rightarrow \infty} \Phi'(t) = 0$. Set $\Phi_n(t) = k + \Phi(n(t - k))/n$ for $t \in \mathbf{R}$ and $n = 1, 2, \dots$. Then we see that $\{\Phi_n\}$ is a sequence of C^2 concave functions in $\mathbf{R}$ such that $\Phi_n(t) = t$ if $t \leq k$, $|\Phi_n(t) - k| < \frac{1}{n}$ if $t > k$, $0 \leq \Phi'_n \leq 1$ in $\mathbf{R}$ and $\lim_{n \rightarrow \infty} \Phi'_n(t) = 0$ if $t > k$. It follows from Theorem 1.1 that Then we define

$$\Phi_{n,\eta}(t) = \Phi_n(t) + \eta t \quad \text{for } \eta > 0 \text{ and } t \in \mathbf{R}. \quad (4.3)$$

We may assume that v is smooth by the approximation argument. By a direct calculation

$$\begin{aligned} \Delta_p \Phi_{n,\eta}(v) &= \Phi'_{n,\eta}(v)^{p-1} \Delta_p v + (p-1) \Phi'_{n,\eta}(v)^{p-2} \Phi''_{n,\eta}(v) |\nabla v|^p \\ &\leq \Phi'_{n,\eta}(v)^{p-1} \Delta_p v \quad (\Phi''_{n,\eta} = \Phi''_n \leq 0 \text{ by concavity of } \Phi_n) \end{aligned} \quad (4.4)$$

Letting $\eta \rightarrow 0$, we clearly have

$$\Delta_p \Phi_n(v) \leq \Phi'_n(v)^{p-1} \Delta_p v \quad \text{in } \Omega. \quad (4.5)$$

As $n \rightarrow \infty$, we finally get the inequality (4.2). $\square$

Recall the following standard inequality (see, e.g., Lemma 1 in [4], Lemma 2 and Lemma 3 in [13]):

Lemma 3. Let Φ be a C^1 convex function such that $0 \leq \Phi' < \infty$ on $\mathbf{R}$.

When $p = 2$, assume that $u \in L_{\text{loc}}^1(\Omega)$ and Δu is a Radon measure on Ω .

When $p \neq 2$, assume that $u \in W_{\text{loc}}^{1,p^*}(\Omega)$ and u is admissible in the sense of Definition 1.1.

Then we have the followings:

1. $T_k(u) \in W_{\text{loc}}^{1,p}(\Omega)$ for every $k > 0$. Moreover, given $\omega \subset \subset \omega' \subset \subset \Omega$, there exist positive constant C independent of u and k such that

$$\begin{cases} \int_{\omega} |\nabla T_k(u)|^2 \leq k \left(|\Delta u|(\omega') + C \int_{\omega'} |u| \right) & \text{if } p = 2, \\ \int_{\omega} |\nabla T_k(u)|^p \leq Ck \left(|\Delta_p u|(\omega') + \int_{\omega'} |\nabla u|^{p-1} \right) & \text{if } p \neq 2. \end{cases} \quad (4.6)$$

2. If $u \geq 0$ a.e. in Ω , then $\Delta_p(T_k(u))$ is a Radon measure for every $k > 0$. Moreover, we have

$$\Delta_p T_k(u) \leq (\Delta_p u)^+ \quad \text{in } \Omega. \quad (4.7)$$

The next lemma is seen in [3]; Theorem 2.1.

Lemma 4. *Let $\nu \in M(\Omega)$. Then $\nu \in L^1(\Omega) + W^{-1,p'}(\Omega)$ if and only if ν is a diffuse measure with respect to p -capacity (i.e. $\nu_c = 0$). Here $p' = p/(p-1)$ and $W^{-1,p'}(\Omega)$ is the dual space of $W_0^{1,p}(\Omega)$.*

When $p = 2$, the next lemma is seen in [5]; Lemma 2.1.

Lemma 5. *Assume that $\nu \in M(\Omega)$ is a diffuse measure with respect to p -capacity (i.e. $\nu_c = 0$). Let $\{v_n\}$ be a sequence in $L^\infty(\Omega) \cap W_0^{1,p}(\Omega)$ such that $\sup_n \|v_n\|_{L^\infty(\Omega)} < \infty$ and $v_n \rightarrow v$ weakly in $W_0^{1,p}(\Omega)$ as $n \rightarrow \infty$. Then*

$$v_n \rightarrow v \quad \text{in } L_{\text{loc}}^1(\Omega; d\nu) \text{ as } n \rightarrow \infty.$$

Equivalently, there exists a subsequence $\{v_{n_k}\}$ converging to v $|\nu|$ -a.e. in Ω . Here $L_{\text{loc}}^1(\Omega; d\nu) = \{f : \int_\omega |f| d\nu < \infty \text{ for any } \omega \subset \subset \Omega\}$.

Proof: We may assume that $|\nu|(\Omega) < \infty$. By Lemma 4, we can assume that $\nu = f - \text{div } G$ in $\mathcal{D}'(\Omega)$ for some $f \in L^1(\Omega)$ and $G = (g_1, g_2, \dots, g_N) \in (L^{p'}(\Omega))^N$. By a density argument, we have

$$\int_\Omega w \varphi d\nu = \int_\Omega w \varphi f + \int_\Omega G \cdot \nabla(w \varphi) \quad \text{for any } \varphi \in C_0^\infty(\Omega), w \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega). \quad (4.8)$$

Since the sequence $\{|v_n - v|\}$ is bounded in $W_0^{1,p}(\Omega)$, by Rellich's theorem we have $|v_n - v| \rightarrow 0$ in $L^p(\Omega)$ as $n \rightarrow \infty$. Therefore $|v_n - v| \rightarrow 0$ weakly in $W_0^{1,p}(\Omega)$ as $n \rightarrow \infty$. For a given $\varepsilon > 0$, we choose $\omega \subset \subset \Omega$ such that $|\nu|(\Omega \setminus \omega) < \varepsilon$. Then we fix $\varphi_0 \in C_0^\infty(\Omega)$ such that $0 \leq \varphi_0 \leq 1$ in Ω and $\varphi_0 = 1$ in ω . By using (4.8) with $w = |v_n - v|$ and $\varphi = \varphi_0$, we have

$$\begin{aligned} \int_\Omega |v_n - v| d\nu &= \int_\omega |v_n - v| d\nu + \int_{\Omega \setminus \omega} |v_n - v| d\nu \\ &\leq \int_\omega |v_n - v| d\nu + 2C|\nu|(\Omega \setminus \omega) \leq \int_\Omega |v_n - v| \varphi_0 d\nu + 2C\varepsilon \\ &= \int_\Omega |v_n - v| \varphi_0 f + \int_\Omega \nabla(|v_n - v| \varphi_0) \cdot G + 2C\varepsilon. \end{aligned}$$

By weak convergence of $|v_n - v| \varphi_0$ in $W_0^{1,p}(\Omega)$, we see immediately that $\int_\Omega \nabla(|v_n - v| \varphi_0) \cdot G \rightarrow 0$ as $n \rightarrow \infty$. In the next we show the first term goes to 0 as $n \rightarrow \infty$. For any $\varepsilon > 0$, there exists a $\delta > 0$ such that if $S \subset \Omega$ and $\text{meas}(S) < \delta$, then we have $\int_S |f| < \varepsilon$, since $f \in L^1(\Omega)$. Let us set $K_\varepsilon^n = \{x \in \Omega : |v_n - v| \geq \varepsilon\}$. Since $|v_n - v| \rightarrow 0$

in measure, there exists some integer $n_\varepsilon > 0$ such that we have $\text{meas}(K_\varepsilon^n) < \delta$ for all $n \geq n_\varepsilon$. Thus

$$\begin{aligned} \int_{\Omega} |v_n - v| \varphi_0 f &= \int_{K_\varepsilon^n} |v_n - v| \varphi_0 f + \int_{\Omega \setminus K_\varepsilon^n} |v_n - v| \varphi_0 f \\ &\leq 2C \int_{K_\varepsilon^n} |f| + \varepsilon \int_{\Omega \setminus K_\varepsilon^n} |f| \\ &\leq 2C\varepsilon + \varepsilon \|f\|_{L^1(\Omega)} = C'\varepsilon. \end{aligned}$$

After all we conclude that $\limsup_{n \rightarrow \infty} \int_{\Omega} |v_n - v| d\nu \leq (2C + C')\varepsilon$, and the desired property is now clear. $\square$

5. Proof of Theorem 1.1

Let us recall the assumption in Theorem 1.1 that Φ is a C^1 convex function in $\mathbf{R}$, $\Phi' \geq 0$ in $\mathbf{R}$ and $\Phi' \in L^\infty(\mathbf{R})$. Without loss of generality, we may assume that $\Phi \in C^2(\mathbf{R})$, $0 \leq \Phi' \leq 1$ and Φ'' has compact support in $\mathbf{R}$ because the general case can be deduced by approximation.

First we prepare the next lemma.

Lemma 6. *Assume that $u \in W_{\text{loc}}^{1,p^*}(\Omega)$ is admissible. Assume*

$$\sup_{t \in \mathbf{R}} (\Phi'(t))^{p-2} \Phi''(t) < \infty \quad \text{if } 1 < p < 2. \quad (5.1)$$

Let $\{u_n\}$ be an approximate sequence of u in the sense of Definition 1.1.

Then $\{\Phi'(u_n)^{p-1}\}$ is bounded in $W_{\text{loc}}^{1,p}(\Omega)$ and $\Phi'(u_n)^{p-1} \varphi \rightarrow \Phi'(u)^{p-1} \varphi$ weakly in $W_0^{1,p}(\Omega)$ as $n \rightarrow \infty$.

Proof: Since $\Phi'(u_n)^{p-1} \varphi \rightarrow \Phi'(u)^{p-1} \varphi$ in $L^1(\Omega)$ as $n \rightarrow \infty$ and φ has compact support in Ω , it suffices to show that $\{\Phi'(u_n)^{p-1}\}$ is bounded in $W_{\text{loc}}^{1,p}(\Omega)$. Let $M > 0$ be such that $\text{supp } \Phi'' \subset [-M, M]$.

Noting that $\nabla u_n = \nabla T_M(u_n)$ for $u_n \in \text{supp } \Phi''$, we have

$$\nabla(\Phi'(u_n)^{p-1}) = (p-1)\Phi'(u_n)^{p-2} \Phi''(u_n) \nabla T_M(u_n) \quad \text{in } \Omega. \quad (5.2)$$

Let $\omega \subset \subset \omega' \subset \subset \Omega$. From (4.6) it follows that for some constant $C > 0$ independent of n and M

$$\int_{\omega} |\nabla T_M(u_n)|^p \leq \begin{cases} CM \int_{\omega'} (|\Delta u_n| + |u_n|) & \text{if } p = 2, \\ CM \int_{\omega'} (|\Delta_p u_n| + |\nabla u_n|^{p-1}) & \text{if } p \neq 2, \end{cases} \quad (5.3)$$

which implies

$$\sup_n \int_{\omega} |\nabla T_M(u_n)|^p < \infty \quad (5.4)$$

because $\{u_n\}$, $\{|\nabla u_n|^{p-1}\}$ and $\{|\Delta_p u_n|\}$ are bounded in $L^1(\omega')$ by the assumption on u_n . Hence by (5.2), (5.4) and (5.1) we obtain

$$\sup_n \int_{\omega} |\nabla(\Phi'(u_n)^{p-1})|^p \leq (p-1)^p \left(\sup_{t \in \mathbf{R}} \Phi'(t)^{p-2} \Phi''(t) \right)^p \int_{\omega} |\nabla T_M(u_n)|^p < \infty.$$

That concludes the proof. $\square$

Proof of Theorem 1.1:

Let $\{u_n\}$ be an approximate sequence of u in the sense of Definition 1.1. Then by direct calculation we have

$$\Delta_p \Phi(u_n) = \Phi'(u_n)^{p-1} \Delta_p u_n + (p-1) \Phi'(u_n)^{p-2} \Phi''(u_n) |\nabla u_n|^p \quad \text{in } \Omega. \quad (5.5)$$

First we assume that $p \geq 2$. Since $\Phi' > 0$, $\Phi'' \geq 0$ in $\mathbf{R}$ and $\Phi' \in L^\infty(\mathbf{R})$, from (5.5) it follows that

$$\begin{aligned} \Delta_p \Phi(u_n) &\geq \Phi'(u_n)^{p-1} \Delta_p u_n = \Phi'(u_n)^{p-1} \Delta_p u + \Phi'(u_n)^{p-1} (\Delta_p u_n - \Delta_p u) \\ &\geq \Phi'(u_n)^{p-1} (\Delta_p u)_d - \|\Phi'\|_{L^\infty(\mathbf{R})}^{p-1} (\Delta_p u)_c^- + \Phi'(u_n)^{p-1} (\Delta_p u_n - \Delta_p u) \\ &= I_n - \|\Phi'\|_{L^\infty(\mathbf{R})}^{p-1} (\Delta_p u)_c^- + J_n \quad \text{in } D'(\Omega), \end{aligned} \quad (5.6)$$

where $I_n = \Phi'(u_n)^{p-1} (\Delta_p u)_d$ and $J_n = \Phi'(u_n)^{p-1} (\Delta_p u_n - \Delta_p u)$. Let $\varphi \in C_0^\infty(\Omega)$ with $\varphi \geq 0$ in Ω . Since $\varphi' \in C(\mathbf{R}) \cap L^\infty(\mathbf{R})$, $\varphi'(u_n) \rightarrow \varphi'(u)$ a.e. in Ω and $\| |\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u \| \rightarrow 0$ in $L_{\text{loc}}^1(\Omega)$ as $n \rightarrow \infty$ from the assumption on u_n , we have

$$\begin{aligned} \langle \Delta_p \Phi(u_n), \varphi \rangle &= - \int_{\Omega} \Phi'(u_n)^{p-1} |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla \varphi \\ &\rightarrow - \int_{\Omega} \Phi'(u)^{p-1} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi = \langle \Delta_p \Phi(u), \varphi \rangle \quad \text{as } n \rightarrow \infty \end{aligned} \quad (5.7)$$

by the dominated convergence theorem. Noting that $\Phi'(u_n)^{p-1} \varphi \rightarrow \Phi'(u)^{p-1} \varphi$ weakly in $W_0^{1,p}(\Omega)$ as $n \rightarrow \infty$ by Lemma 6, we can apply Lemma 5 with $v_n = \Phi'(u_n)^{p-1} \varphi$, $v = \Phi'(u)^{p-1} \varphi$ and $\nu = (\Delta_p u)_d$, so that we get

$$\langle I_n, \varphi \rangle = \langle \Phi'(u_n)^{p-1} (\Delta_p u)_d, \varphi \rangle \rightarrow \langle \Phi'(u)^{p-1} (\Delta_p u)_d, \varphi \rangle \quad \text{as } n \rightarrow \infty \quad (5.8)$$

by Lemma 4 and the dominated convergence theorem. As for J_n we have

$$\begin{aligned} \langle J_n, \varphi \rangle &= \langle \Phi'(u_n)^{p-1} (\Delta_p u_n - \Delta_p u), \varphi \rangle \\ &= - \int_{\Omega} \Phi'(u_n)^{p-1} (|\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u) \cdot \nabla \varphi \\ &\quad - \int_{\Omega} \nabla(\Phi'(u_n)^{p-1}) \cdot (|\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u) \varphi \end{aligned} \quad (5.9)$$

From $\Phi' \in L^\infty(\mathbf{R})$ and $\| |\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u \| \rightarrow 0$ in $L_{\text{loc}}^1(\Omega)$ as $n \rightarrow \infty$ by the assumption on u_n it follows that

$$\int_{\Omega} \Phi'(u_n)^{p-1} (|\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u) \cdot \nabla \varphi \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (5.10)$$

We show

$$\int_{\Omega} \nabla(\Phi'(u_n)^{p-1}) \cdot (|\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u) \cdot \varphi \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (5.11)$$

Noting that Φ'' has compact support, we see that $\{|\nabla u_n|\}$ is bounded in $L^p(\omega)$ with $\omega = \text{supp } \varphi$ by virtue of Lemma 3. In fact, since $\nabla u_n = \nabla T_M(u_n)$ for $u_n \in \text{supp } \Phi''$, with a constant $M > 0$ satisfying $\text{supp } \Phi'' \subset [-M, M]$, it follows from (5.4) that $\sup_n \int_{\omega} |\nabla u_n|^p < \infty$. Hence we may suppose $|\nabla u| \in L^p(\omega)$ and $|\nabla u_n - \nabla u| \rightarrow 0$ in $L^p(\omega)$ as $n \rightarrow \infty$ by Fatou's lemma with the assumption on u_n and taking a subsequence if necessary, so that we get $\| |\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u \| \rightarrow 0$ in $L^{p'}(\omega)$ as $n \rightarrow \infty$ by Lemma 7 stated later in Appendix. Therefore (5.11) follows, since $\{|\nabla(\Phi'(u_n)^{p-1})|\}$ is bounded in $L^p(\omega)$ by Lemma 6. From (5.7), (5.8) and (5.11) it follows that

$$\langle J_n, \varphi \rangle \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (5.12)$$

Consequently, using (5.7), (5.8) and (5.12), from (5.6) we obtain

$$\langle \Delta_p \Phi(u), \varphi \rangle \geq \langle \Phi'(u)^{p-1} (\Delta_p u)_d, \varphi \rangle - \|\Phi'\|_{L^\infty(\mathbf{R})}^{p-1} \langle (\Delta_p u)_c^-, \varphi \rangle, \quad (5.13)$$

which implies (1.5).

We proceed to the case where $1 < p < 2$. We set $\Phi^\eta(t) := \Phi(t) + \eta t$ for $t \in \mathbf{R}$ with $\eta > 0$. Then we see that for each $\eta > 0$

$$\sup_{t \in \mathbf{R}} (\Phi^\eta)'(t)^{p-2} (\Phi^\eta)''(t) = \sup_{t \in \mathbf{R}} (\Phi'(t) + \eta)^{p-2} \Phi''(t) \leq \eta^{p-2} \sup_{t \in \mathbf{R}} \Phi''(t) < \infty. \quad (5.14)$$

Hence we can apply Lemma 6 with Φ^η instead of Φ , so that in a quite similar way we reach to the inequality (5.13) replaced Φ by Φ^η , that is, we have

$$\begin{aligned} \langle \Delta_p \Phi^\eta(u), \varphi \rangle &\geq \langle (\Phi^\eta)'(u)^{p-1} (\Delta_p u)_d, \varphi \rangle - \|(\Phi^\eta)'\|_{L^\infty(\mathbf{R})}^{p-1} \langle (\Delta_p u)_c^-, \varphi \rangle \\ &= \langle (\Phi'(u) + \eta)^{p-1} (\Delta_p u)_d, \varphi \rangle - (\|\Phi'\|_{L^\infty(\mathbf{R})} + \eta)^{p-1} \langle (\Delta_p u)_c^-, \varphi \rangle \end{aligned} \quad (5.15)$$

for any $\varphi \in C_0^\infty(\Omega)$ with $\varphi \geq 0$ in Ω . Letting $\eta \rightarrow 0$, we have

$$\begin{aligned} \langle \Delta_p \Phi^\eta(u), \varphi \rangle &= - \int_{\Omega} (\Phi^\eta)'(u)^{p-1} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \\ &= - \int_{\Omega} (\Phi'(u) + \eta)^{p-1} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \\ &\rightarrow - \int_{\Omega} \Phi'(u)^{p-1} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi = \langle \Delta_p \Phi(u), \varphi \rangle. \end{aligned} \quad (5.16)$$

by the dominated convergence theorem owing to $\| |\nabla u|^{p-2} \nabla u \| = |\nabla u|^{p-1} \in L_{\text{loc}}^1(\Omega)$, and also we have

$$\langle (\Phi'(u) + \eta)^{p-1} (\Delta_p u)_d, \varphi \rangle \rightarrow \langle \Phi'(u)^{p-1} (\Delta_p u)_d, \varphi \rangle \quad (5.17)$$

because $(\Phi'(u) + \eta)^{p-1}$ is locally integrable in Ω with respect to $|(\Delta_p u)_d|$ uniformly for $\eta > 0$. Hence (5.13) follows from (5.15) and (5.17). That completes the proof. $\square$

6. Proof of Theorem 1.2

Proof: It follows from Lemma 3 that $T_k(u) \in W_{\text{loc}}^{1,p}(\Omega)$ and $\Delta_p T_k(u) \in M(\Omega)$ for any $k > 0$. Moreover $\Delta_p T_k(u) \leq (\Delta_p u)^+$ in $D'(\Omega)$. Let us simply denote $\Delta_p u$ by $\mu \in M(\Omega)$. Let E be a Borel measurable subset of Ω such that $C_p(E, \Omega) = 0$ and $|\mu_c|(\Omega \setminus E) = 0$, and let K be a compact subset of E . Then we immediately have

$$\Delta_p T_k(u) \leq \mu^+ \quad \text{in } D'(\Omega \setminus K). \quad (6.1)$$

Take and fix an arbitrary $\varphi \in C_0^\infty(\Omega)$ with $\varphi \geq 0$ in Ω . Let $\{\varphi_n\} \subset C_0^\infty(\Omega \setminus K)$ be such that $0 \leq \varphi_n \leq \varphi$ in Ω and $\varphi_n \rightarrow \varphi$ in $W_0^{1,p}(\Omega)$ as $n \rightarrow \infty$. Then we see as $n \rightarrow \infty$ that

$$\begin{aligned} \langle \Delta_p T_k(u), \varphi_n \rangle &= - \int |\nabla T_k(u)|^{p-2} \nabla T_k(u) \cdot \nabla \varphi_n \\ &\rightarrow - \int |\nabla T_k(u)|^{p-2} \nabla T_k(u) \cdot \nabla \varphi = \langle \Delta_p(T_k(u)), \varphi \rangle. \end{aligned} \quad (6.2)$$

Since it follows from (6.1) that for any $n \geq 1$

$$\langle \Delta_p T_k(u), \varphi_n \rangle \leq \int_{\Omega \setminus K} \varphi_n d\mu^+ \leq \int_{\Omega \setminus K} \varphi d\mu^+, \quad (6.3)$$

by (6.2) and (6.3) we get

$$\langle \Delta_p T_k(u), \varphi \rangle \leq \int_{\Omega \setminus K} \varphi d\mu^+,$$

which implies

$$\Delta_p T_k(u) \leq \chi_{\Omega \setminus K} \mu^+ \quad \text{in } D'(\Omega).$$

As $k \rightarrow \infty$ we get

$$\mu = \Delta_p u \leq \chi_{\Omega \setminus K} \mu^+ \quad \text{in } D'(\Omega).$$

Therefore,

$$\mu_c|_K = \mu|_K \leq 0 \quad \text{in } D'(\Omega).$$

Since K is an arbitrary compact subset of E and μ is (Borel) regular as a Radon measure, we conclude that

$$\mu_c \leq 0 \quad \text{in } D'(\Omega).$$

$\square$

7. Appendix

Let us recall Proposition 1.1 as Proposition 7.1:

Proposition 7.1 *Let $N \geq 1$, $1 < p < \infty$, $p^* = \max[1, p-1]$ and Ω be a bounded domain of $\mathbf{R}^N$.*

1. *Assume that $u \in W_{\text{loc}}^{1,p^*}(\Omega)$ is admissible. Then $u^+ = \max[u, 0]$ and $u^- = \max[-u, 0]$ are also admissible.*
2. *Assume that $p = 2$. Then $u \in W_{\text{loc}}^{1,1}(\Omega)$ is admissible if $\Delta u \in M(\Omega)$.*
3. *$u \in W_0^{1,p}(\Omega)$ is admissible if $\Delta_p u \in M(\Omega)$.*

Proof of 1: Since $u \in W_{\text{loc}}^{1,p^*}(\Omega)$, $\Delta_p u$, $\Delta_p(u^+)$ and $\Delta_p(u^-)$ are well-defined in $D'(\Omega)$. Let $\{u_n\}$ be an approximate sequence of u in the sense of Definition 1.1. It follows from the condition 1 that $\Delta_p u_n = \Delta_p(u_n^+) - \Delta_p(u_n^-)$ and $\Delta_p u_n \rightarrow \Delta_p u$ (i.e. $\Delta_p(u_n^\pm) \rightarrow \Delta_p(u^\pm)$) in $D'(\Omega)$ as $n \rightarrow \infty$. Moreover, it follows from the condition 2 and the weak compactness of measures that we have $\Delta_p u_n \rightarrow \Delta_p u$ (i.e. $\Delta_p(u_n^\pm) \rightarrow \Delta_p(u^\pm)$) in the sense of measures as $n \rightarrow \infty$. Therefore if u is admissible, then u^+ and u^- are admissible as well.

Proof of 2: Let $\rho \in C_0^\infty(B_1)$ be a radial, nonnegative and decreasing mollifier. By extending $u \in L^1(\Omega)$ to the whole space $\mathbf{R}^N$ so that $u \equiv 0$ outside Ω , we define a mollification of u by

$$u_\rho^n(x) := \rho_{1/n} * u(x) = \int_{\Omega} \rho_{1/n}(x-y)u(y)dy \quad \text{for } x \in \Omega \text{ and } n = 1, 2, \dots \quad (7.1)$$

Set $u_n(x) = u_\rho^n(x)$ for $n = 1, 2, \dots$. Then the condition 1 is clearly satisfied. Moreover if $p = 2$, (1.4) is also satisfied if Δu is a Radon measure. In fact, letting $\omega \subset \subset \Omega$, we see that $\Delta u_n = (\Delta u)_n$ and $|\Delta u_n| = (\Delta u_n)^+ + (\Delta u_n)^- = ((\Delta u)_n)^+ + ((\Delta u)_n)^- = |(\Delta u)_n|$ in ω for n sufficiently large, hence (1.4) follows from the definition of the Radon measure.

Proof of 3: We prove that $u \in W_0^{1,p}(\Omega)$ is admissible if $\Delta_p u$ is a Radon measure on Ω . Let $w_n \in W_0^{1,p}(\Omega)$ be the unique weak solution of the boundary value problem for the monotone operator Δ_p (see e.g. [14]): For $n = 1, 2, \dots$,

$$\begin{cases} \Delta_p w_n = \text{div } F_\rho^n & \text{in } \Omega, \\ w_n = 0 & \text{on } \partial\Omega, \end{cases} \quad (7.2)$$

where $F = (F_1, \dots, F_N) = |\nabla u|^{p-2} \nabla u \in (L^{p'}(\Omega))^N$ with $p' = p/(p-1)$, $F_\rho^n = ((F_1)_\rho^n, \dots, (F_N)_\rho^n) \in (C^\infty(\mathbf{R}^N))^N$ and $(F_i)_\rho^n$ is a mollification of F_i defined by (7.1) for each $i = 1, \dots, N$. Let us set $\omega \subset \subset \Omega$ and $\Delta_p u = \text{div } F = \mu$. Since $|\mu|(\omega) < \infty$, we see $\text{div } F_\rho^n = (\text{div } F)_\rho^n = (\Delta_p u)_\rho^n = \mu_\rho^n$ in ω provided that n is sufficiently large. Hence we clearly have

$$|\Delta_p w_n|(\omega) = |\text{div } F_\rho^n|(\omega) = |\mu_\rho^n|(\omega) \rightarrow |\mu|(\omega) \quad \text{as } n \rightarrow \infty.$$

This proves the condition 2. Next we show

$$w_n \rightarrow u \text{ in } W_0^{1,p}(\Omega) \quad \text{as } n \rightarrow \infty. \quad (7.3)$$

Then the condition 1 is clearly satisfied, since one may choose a subsequence. We need the following elementary lemma, see e.g. [2, 11].

Lemma 7. *Let $1 < p < \infty$.*

1. *There exist positive constants $c_1(p)$ and $c_2(p)$ depending on p such that for every $\xi, \eta \in \mathbf{R}^N$ we have*

$$|\xi - \eta|^2(|\xi| + |\eta|)^{p-2} \leq c_1(p)(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta) \cdot (\xi - \eta). \quad (7.4)$$

In particular if $p > 2$ we have

$$|\xi - \eta|^p \leq c_1(p)(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta) \cdot (\xi - \eta), \quad (7.5)$$

and if $1 < p < 2$ we have for any $\varepsilon \in (0, 1)$

$$|\xi - \eta|^p \leq c_2(p)\varepsilon^{(p-2)/p}(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta) \cdot (\xi - \eta) + \varepsilon(|\xi| + |\eta|)^p. \quad (7.6)$$

2. *There exist positive constants $d_1(p)$, $d_2(p)$ and $d_3(p)$ depending on p such that for every $\xi, \eta \in \mathbf{R}^N$ we have*

$$\begin{cases} |\xi|^{p-2}\xi - |\eta|^{p-2}\eta \leq d_1(p)|\xi - \eta|(|\xi| + |\eta|)^{p-2} & \text{if } p \geq 2, \\ |\xi|^{p-2}\xi - |\eta|^{p-2}\eta \leq d_2(p)|\xi - \eta|^{p-1} & \text{if } 1 < p < 2. \end{cases} \quad (7.7)$$

In particular if $p > 2$, then we have for any $\varepsilon \in (0, 1)$

$$|\xi|^{p-2}\xi - |\eta|^{p-2}\eta \leq d_3(p)\varepsilon^{-1/(p-2)}|\xi - \eta|^{p-1} + \varepsilon(|\xi| + |\eta|)^{p-1}. \quad (7.8)$$

First we treat the case where $p \geq 2$: By using $w_n - u \in W_0^{1,p}(\Omega)$ as a test function, we have

$$\begin{aligned} -\langle \Delta_p w_n - \Delta_p u, w_n - u \rangle &= \int_{\Omega} (|\nabla w_n|^{p-2}\nabla w_n - |\nabla u|^{p-2}\nabla u) \cdot \nabla(w_n - u) \\ &\geq c_1(p)^{-1} \int_{\Omega} |\nabla(w_n - u)|^p \quad (\text{by (7.5)}). \end{aligned} \quad (7.9)$$

In the left-hand side, using Young's inequality for $\delta > 0$ we have

$$\begin{aligned} -\langle \Delta_p w_n - \Delta_p u, w_n - u \rangle &= \int_{\Omega} (F_{\rho}^n - F) \cdot \nabla(w_n - u) \\ &\leq C(\delta) \int_{\Omega} |F_{\rho}^n - F|^{p'} + \delta \int_{\Omega} |\nabla(w_n - u)|^p, \end{aligned} \quad (7.10)$$

where $C(\delta) > 0$ is a constant depending only on δ .

Since $|F_{\rho}^n|^{p'} \rightarrow |F|^{p'}$ in $(L^{p'}(\Omega))^N$ as $n \rightarrow \infty$, it follows from (7.9) and (7.10) that $\nabla w_n \rightarrow \nabla u$ in $(L^p(\Omega))^N$ as $n \rightarrow \infty$, which implies (7.3). Then, taking a subsequence if necessary, $\{w_n\}$ satisfies the property $w_n \rightarrow u$ a.e. in Ω as $n \rightarrow \infty$.

Lastly we treat the case where $1 < p < 2$. In this case the proof can be done using (7.6) instead of (7.5) in a quite similar way. Hence we omit the detail.

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