Criterion toward understanding non-constant solutions to $p$-Laplace Neumann boundary value problem

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Abstract
We consider a $p$-Laplace equation $\Delta_p V + h(V) = 0$, with an arbitrary $C^1$-nonlinearity $h$, in a bounded domain and supplemented with the Neumann boundary condition. We prove a necessary condition for zeros of $h = h(V)$ to be touched by non-constant solutions to this problem.

1. Introduction

In this note, we present an elementary proof of a certain property of constant solutions to the following Neumann boundary value problem for the general nonlinear $p$-Laplace equation

$$\Delta_p V + h(V) = 0 \quad \text{in } \Omega, \quad (1.1)$$

$$\frac{\partial V}{\partial \nu} = 0 \quad \text{on } \partial \Omega, \quad (1.2)$$

where $\Omega \subset \mathbb{R}^n$ with $n \geq 1$ is a bounded domain with smooth boundary $\partial \Omega$, and $\nu$ denotes the unit outer normal vector to $\partial \Omega$. Here, we consider an arbitrary function $h \in C^1(\mathbb{R})$. $\Delta_p$ is the $p$-Laplace operator defined by

$$\Delta_p V = \text{div} \left( |\nabla V|^{p-2} \nabla V \right) \quad \text{for } V \in W^{1,p}(\Omega) \quad (1.3)$$

with $p \in (1, \infty)$.

We note that the equation (1.1) is the Euler-Lagrange equation for the variational integral

$$J_p(v) = \frac{1}{p} \int_{\Omega} \{ |\nabla v|^p - H(v) \} \, dx, \quad H(v) = \int_0^v h(s) \, ds.$$  

Hence, $V \in W^{1,p}(\Omega)$ is a weak solution of the equation (1.1) if

$$\int_{\Omega} \left( |\nabla V|^{p-2} \nabla V \cdot \nabla \eta \right) \, dx = \int_{\Omega} h(V) \eta \, dx \quad (1.4)$$

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is satisfied for all $\eta \in C^1_0(\Omega) = \{\eta \in C^1(\Omega) \mid \eta = 0 \text{ on } \partial \Omega\}$. If $V \in C^1(\Omega)$ satisfies (1.1) in the distribution sense, then it is called a classical solution.

For the case $h \equiv 0$ in (1.1), weak solutions of (1.1) become members of $C^1_{\text{loc}}(\Omega)$, which is the set of all locally Hölder continuous functions with exponent $\alpha = \alpha(n,p)$. Moreover, there are stronger regularity results, that is, the gradient is locally Hölder continuous, see [26, 2, 3, 25, 13, 24].

Concerning the boundary value problem for a $p$-Laplace equation, there are many existence results of solutions for the problem with homogeneous Dirichlet boundary condition, see for example [12, 11, 10, 21, 23, 22]. For the problem with Neumann boundary condition, there are a few systematic studies and we can find some results in series of papers [4, 5, 6, 7, 8, 9]. In [9], the existence of a positive solution $V \in C^1(\Omega)$ of (1.1) has been obtained if the nonlinear term $h$ satisfies the following hypotheses (A$_i$)-(A$_{iii}$):

(A$_i$) there exists $c > 0$ such that
$$h(\xi) \leq c(1 + \xi^{p-1}), \quad \text{for all } \xi \geq 0;$$

(A$_{ii}$) the function $\xi \mapsto \frac{h(\xi)}{\xi^{p-1}}$ is strictly decreasing on $(0, \infty)$;

(A$_{iii}$) $\lim_{\xi \to +\infty} \frac{h(\xi)}{\xi^{p-1}} < 0 < \lim_{\xi \to +0} \frac{h(\xi)}{\xi^{p-1}}$.

We note that 0 in (A$_{iii}$) is the first eigenvalue of the nonlinear eigenvalue problem:
$$-\Delta_p V(x) = \lambda |V(x)|^{p-2}V(x) \quad \text{in } \Omega, \quad \frac{\partial V}{\partial \nu} = 0 \quad \text{on } \partial \Omega.$$  

The main motivation for this work comes from the following observation. The problem (1.1), particularly in the case $p = 2$, arises in an analysis of models from biology, physics, and other different fields of sciences. If $V \in C^2(\Omega) \cap C^1(\overline{\Omega})$ is a solution of problem (1.1) with $p = 2$, then integrating the equation and using the boundary condition, we obtain $\int_{\Omega} h(V(x)) \, dx = 0$. Hence, there exists $x_0 \in \Omega$ such that $h(V(x_0)) = 0$. In other words, the number $a_0 = V(x_0)$ is a constant solution of problem (1.1). In such a case, we shall say that the non-constant solution $V = V(x)$ touches the constant solution $\overline{V} \equiv a_0$. Hence, the property described above says that each non-constant solution of problem (1.1) with $p = 2$ has to touch at least one constant solution. We would like to consider that the same property holds true or not for classical solutions of the problem (1.1) with $p \in (1, \infty)$.

In this work, we present a necessary condition for certain constant solutions of problem (1.1) to be touched by a non-constant classical solution $V \in C^1(\overline{\Omega})$. As a consequence, we obtain a simple method which leads to a priori estimates of solutions to problem (1.1).

2. Results and examples

We begin by formulating our standing assumptions:
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1. \( \Omega \) satisfies an interior sphere condition, that is, for any \( y \in \partial \Omega \), there exists a ball \( B \subset \Omega \) with \( y \in \partial B \);
2. the function \( h \in C^1(\mathbb{R}) \) is arbitrary such that (1.1) has a classical solution in \( C^1(\overline{\Omega}) \);
3. we consider non-constant classical solutions \( V \in C^1(\Omega) \).

In the following, we say that \( a \in \mathbb{R} \) is a non-degenerate zero of \( h \) if \( h(a) = 0 \) and \( h'(a) \neq 0 \).

First, we state our main theorem.

**Theorem 1.** Let \( V \in C^1(\overline{\Omega}) \) be a non-constant classical solution of problem (1.1). Denote by \( a_0 \in \mathbb{R} \) a zero of \( h \) which is the biggest one touched by \( V \), and assume that \( a_0 \) is non-degenerate.

(i) If \( \max_{x \in \overline{\Omega}} V(x) > a_0 \), then \( h'(a_0) > 0 \);

(ii) If \( \max_{x \in \overline{\Omega}} V(x) = a_0 \), then \( h'(a_0) > 0 \) provided that \( 1 < p \leq 2 \).

We postpone a proof of Theorem 1 to the next section. For the case of \( p > 2 \) and \( \max_{x \in \overline{\Omega}} V(x) = a_0 \), which is a remaining case of (ii) in Theorem 1, we show the existence of solutions satisfying that \( V(x_0) = a_0 \) for \( x_0 \in \overline{\Omega} \) and \( h'(a_0) < 0 \) in Section 5.

In the following corollary, we consider solutions of (1.1) which touch more than one zero of the function \( h \).

**Corollary 2.** Let \( b \in \mathbb{R} \) be a zero of \( h \) which is the smallest one touched by \( V \in C^1(\overline{\Omega}) \). Assume that \( b \) is non-degenerate zero. Then we obtain \( h'(b) > 0 \) under each assumption of (i) and (ii) in Theorem 1.

**Proof.** Here, it suffices to apply Theorem 1 with the function \( \tilde{V}(x) = -V(x) \) which is a solution of equation

\[
\Delta_p \tilde{V} + \tilde{h}(\tilde{V}) = 0,
\]

where \( \tilde{h}(s) = -h(s) \). In this case, the number \( \tilde{b} = -b \) is the biggest zero of \( \tilde{h} \) which is touched by \( \tilde{V} \). Moreover, \( \frac{d}{ds} \tilde{h} \big|_{s=\tilde{b}} = \frac{d}{ds} h \big|_{s=b} \).

We conclude this section with examples which illustrate the theorem and the corollary stated above.

**Example 3.** We consider the boundary value problem

\[
\varepsilon^2 \Delta_p V - V + |V|^{q-1}V = 0 \quad \text{in } \Omega, \\
\frac{\partial V}{\partial \nu} = 0 \quad \text{on } \partial \Omega,
\]

with \( \varepsilon > 0 \) and \( q > 1 \). The problem (2.1) with \( p = 2 \) was considered e.g. in a series of papers [20, 14, 17, 18]. When \( p = 2 \) and \( 1 < q < \frac{n+2}{n-2} \) for \( n \geq 3 \), and \( p = 2 \) and
1 < q < ∞ for n = 1, 2, it was shown by using the variational method that problem (2.1) has a positive solution $u_ε$, so called a least-energy solution, for sufficiently small $ε > 0$. Moreover, it was proved that this least energy solution has its only maximum point located on $∂Ω$. We refer the reader to [16] for more comments and references on this problem with $p = 2$. For $1 < p < 2$, the nonlinear term $h(V) = -V + |V|^{q-1}V$ satisfies the hypotheses (A$_i$)-(A$_{iii}$) with $0 < q < p - 1$. Then, the problem (2.1) has a positive classical solution $V \in C^1(\overline{Ω})$. Assume that there exists a non-constant solution $V \in C^1(\overline{Ω})$, which is not necessarily positive. Here, the functions $h(V) = -V + |V|^{q-1}V$ has three non-degenerate zeros $V(x) \in \{-1, 0, 1\}$. It is clear from Theorem 1 and Corollary 2 that the solution has to touch either 1 or -1, because $h'(±1) = -1 + p > 0$. Since $h'(0) = -1$, if $V(x)$ touches 0, then it has to touch both numbers -1 and 1. If $V(x)$ is a positive non-constant solution of (2.1), then $\max_{x \in \overline{Ω}} V(x) > 1$.

Example 4. Next, we consider a $p$-Laplace equation with a bistable nonlinearity

$$ ε^2 \Delta_p V + V(1 - |V|^{q-1}) = 0 \quad \text{in } Ω, $$

$$ \frac{∂V}{∂ν} = 0 \quad \text{on } ∂Ω, $$

where $ε > 0$ and $q > 1$. If $p = 2$ and $q = 3$, this is the boundary value problem for the Allen-Cahn equation, for which questions on the existence of non-constant solutions has been answered in [1, 15] and in references therein.

Let $p \in [2, ∞)$. Then, the nonlinear term $h(V) = V(1 - |V|^{q-1})$ satisfies the hypotheses (A$_i$)-(A$_{iii}$). In this case, the problem (2.2) has a positive classical solution $V \in C^1(\overline{Ω})$. On the other hand, at the roots of the function $h(V) = 0$, we have

$$ h'(-1) < 0, \quad h'(0) > 0, \quad h'(1) < 0. $$

Thus, every non-constant solution $V \in C^1(\overline{Ω})$ of problem (2.2) for any $p \in (1, ∞)$ has to satisfy

$$ -1 \leq V(x) \leq 1 \quad \text{for all } x \in \overline{Ω} \quad \text{and} \quad V(x_0) = 0 \quad \text{for some } x_0 \in \overline{Ω}. $$

Therefore, if there exists a positive solution, then it should be a constant solution $V(x) \equiv 1$.

3. Preliminaries

It is sometimes useful to consider weak supersolutions and weak subsolutions of a $p$-Laplace equation.

Definition 5. We say $u \in W^{1,p}_{loc}(Ω)$ is a weak supersolution of a $p$-Laplace equation if $u$ satisfies

$$ \int_{Ω} |\nabla u|^{p-2}\nabla u \cdot \nabla η dx \geq 0 $$

(3.1)
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for all $\eta \in C_0^1(\Omega)$ with $\eta \geq 0$. If $u$ satisfies the reversed inequality of (3.1), that is,

$$
\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla \eta \, dx \leq 0,
$$

(3.2)

then it is called a weak subsolution.

If we write

$$
\text{div}(|\nabla u|^{p-2}\nabla u) \leq 0,
$$

then we promise that it denotes the inequality (3.1). For the reversed inequality above, it corresponds to (3.2).

We prepare some notations. The positive and negative parts of a function are defined by

$$
f^+(x) = \max\{f(x), 0\}, \quad f^-(x) = \max\{-f(x), 0\}.
$$

It is clear that $f = f^+ - f^-$ and $|f| = f^+ + f^-$. For $x_0 \in \mathbb{R}^n$ and $r > 0$, $B_r(x_0)$ denotes a ball defined by

$$
B_r(x_0) = \{x \in \mathbb{R}^n \mid |x - x_0| < r\}.
$$

If $x_0 = 0$, then we simply write $B_r$.

Next, we introduce the maximum principle, the Hopf boundary lemma and the Harnack inequality for solutions of an elliptic equation in divergence form.

We consider the inequality

$$
\text{div}(|\nabla u|^{p-2}\nabla u) + G(x, u, \nabla u) \leq 0 \quad \text{in} \ \Omega,
$$

(3.3)

where $G(x, z, \xi) \in L^\infty(\Omega \times \mathbb{R}^+ \times \mathbb{R})$ satisfies, for $\kappa > 0$, that

$$
G(x, z, \xi) \geq -\kappa|\xi|^{p-1} - f(z)
$$

(3.4)

for $x \in \Omega$, $z \geq 0$ and all $\xi \in \mathbb{R}^n$ with $|\xi| \leq 1$. The function $f$ is in $C(\mathbb{R}^+ \cup \{0\})$ and assumed to satisfy

$$
f(0) = 0, \quad \text{and} \quad f \text{ is non-decreasing on some interval } (0, \delta), \delta > 0.
$$

(3.5)

We define functions $F$ and $G$ by

$$
F(s) = \int_0^s f(u) \, du,
$$

and

$$
\Phi(s) = \frac{p-1}{p} s^p.
$$

**Theorem 6** (Theorem 5.3.1 in [19]). Let (3.4) and (3.5) be satisfied. If $u \in C^1(\Omega)$ with $u \geq 0$ in $\Omega$ satisfies (3.3) and $u(x_0) = 0$ for some $x_0 \in \Omega$, then $u \equiv 0$ provided that either $f \equiv 0$ in $[0, d]$ with $d > 0$ or the following holds:

$$
\lim_{\varepsilon \to +0} \int_{\varepsilon}^{\delta} \frac{ds}{\Phi^{-1}(F(s))} = \infty.
$$

(3.6)
\textbf{Theorem 7} (Theorem 5.5.1 in [19]). \textit{Let (3.4) and (3.5) be satisfied, and assume that either }f \equiv 0 \text{ in } [0, d] \text{ with } d > 0 \text{ or (3.6) is satisfied. If } u \in C^1(\overline{\Omega}) \text{ satisfies (3.3) with } u > 0 \text{ in } \Omega \text{ and } u(y) = 0 \text{ for some } y \in \partial \Omega, \text{ then}
\frac{\partial u}{\partial \nu}(y) < 0. (3.7)

4. Proof of Theorem 1

In this section, we denote by \( x_0 \in \overline{\Omega} \) a point such that \( \Omega \) such that \( \Omega \) and \( 7 \) that

\[ \text{Since } U \text{ does not touch any zeros bigger than } a \]

are continuous, there exists \( a \) such that \( a < 0 \). In the following, we let \( U(x) = a_M - V(x) \). Then, we see that \( U \) is a weak solution of the problem

\[ \Delta_p U + k(U) = 0 \text{ in } \Omega, \quad \frac{\partial U}{\partial \nu} = 0 \text{ on } \partial \Omega, \] (4.1)

where \( k(U) = -h(a_M - U) \). Moreover, we have that \( U(x) \geq 0 \) for \( x \in \overline{\Omega} \) and \( U(x_M) = 0 \).

The proof of Theorem 1 is based on Theorems 6 and 7. We discuss the cases \( a_0 < a_M \) and \( a_0 = a_M \), separately. In the following, we suppose \( h'(a_0) < 0 \) and show this hypothesis leads to a contradiction.

\textbf{Case I: } \( a_0 < a_M \)

First, we assume that \( x_M \in \Omega \). If \( h(V(x_M)) \geq 0 \), then there exists a \( x_1 \in \overline{\Omega} \) such that \( a_1 = V(x_1) > a_0 \) and \( h(a_1) = 0 \). This is a contradiction because \( V(x) \) does not touch any zeros bigger than \( a_0 \). Thus, we have \( h(V(x_M)) < 0 \). Since \( V \) and \( h \) are continuous, there exists \( r > 0 \) such that \( h(V(x)) < 0 \) for all \( x \in B_r(x_M) \) where

\[ B_r(x_M) = \{ x \in \Omega \mid |x - x_M| < r \} \subset \Omega. \]

Since \( k(U(x)) > 0 \) for all \( x \in B_r(x_M) \) in (4.1), it is easily to see that \( U \) becomes a weak supersolution of a \( p \)-Laplace equation in \( B_r(x_M) \), that is, \( U \) satisfies

\[ \int_{B_r(x_M)} |\nabla U|^{p-2} \nabla U \cdot \nabla \eta \, dx = \int_{B_r(x_M)} k(U) \eta \, dx \geq 0 \]

for all \( \eta \in C_0^1(B_r(x_M)) \) with \( \eta \geq 0 \). Since \( U(x_M) = 0 \), we use Theorem 6 with \( G \equiv 0 \) to obtain that \( U \equiv 0 \) in \( B_r(x_M) \). By the standard argument, we see that \( U \equiv 0 \) in \( \Omega \). This is a contradiction.

Next, we assume that \( x_M \in \partial \Omega \). From the assumption, there exists a ball \( B_r \subset \Omega \) with \( x_M \in \partial B_r \) such that

\[ U(x) > U(x_M) = 0 \quad \text{for } x \in B_r. \]

Since \( U \) is a weak supersolution of a \( p \)-Laplace equation in \( B_r \), it follows from Theorem 7 that

\[ \frac{\partial U}{\partial \nu}(x_M) < 0. \]
This is a contradiction because $U$ satisfies a homogeneous Neumann boundary condition.

**Case II: $a_0 = a_M$**

Let $1 < p \leq 2$. For the function $k$ in (4.1), we have that

$$k(0) = 0 \quad \text{and} \quad k'(0) < 0. \quad (4.2)$$

We assume that $x_0 \in \Omega$. By the continuity of $k(U(x))$ and by (4.2), there exists an open neighborhood $B \subset \Omega$ of $x_0$ such that

$$k'(U(x)) \leq 0 \quad \text{for all} \quad x \in B.$$

Now, we use the well-known formula to obtain that

$$k(U(x)) = k(U(x)) - k(0) = \int_0^1 \frac{d}{ds} k(sU(x)) \, ds = U(x) \cdot \int_0^1 k'(sU(x)) \, ds \quad (4.3)$$

and this implies that

$$c(x) \equiv \int_0^1 k'(sU(x)) \, ds \leq 0 \quad \text{for all} \quad x \in B. \quad (4.4)$$

Thus, $U$ satisfies that

$$\text{div} \left( |\nabla U|^{p-2} \nabla U \right) + c(x)U = 0. \quad (4.5)$$

Since there exists $\delta > 0$ such that $-\delta < c(x) \leq 0$ because $U \in C^1(\overline{\Omega})$ and $k \in C^1(\mathbb{R})$ from the assumption, the condition (3.4) holds. Hence, we can apply Theorem 6 to (4.5) and obtain that $U \equiv 0$ provided that (3.6) is satisfied. Here, we see, for $1 < p \leq 2$, that (3.6) is

$$\lim_{\varepsilon \to 0} \int_{\varepsilon}^{\delta} s^{-2/p} \, ds = \infty. \quad (4.6)$$

This derives a contradiction.

Next, we assume that $x_0 \in \partial \Omega$. As before, we find a ball $B \subseteq \Omega$ such that $x_0 \in \partial B$ and

$$k'(U(x)) < 0 \quad \text{for all} \quad x \in B.$$  

Hence, again using (4.4), we have that $U$ is a solution of (4.5) in $B$ with $-\delta < c(x) \leq 0$.  

Now, we apply Theorem 7 to the equation (4.5). Since (4.6) is satisfied, it follows from Theorem 7 that, for a non-constant solution $U$ of (4.5) satisfying $U(x) > U(x_0) = 0$ in $B$, we have

$$\frac{\partial U(x_0)}{\partial \nu} < 0.$$  

Therefore, we get a contradiction.
5. Monotone solutions for bistable nonlinear case

We consider the following problem which is the problem (2.2) in one-dimensional case:

\[
\begin{aligned}
\varepsilon^2 (|V'|^{p-2}V')' + V (1 - |V|^q) &= 0, \quad x \in (-1, 1) \\
V'(-1) &= V'(1) = 0,
\end{aligned}
\]  

(5.1)

where \( p > 2 \) and \( q \geq 2 \). Let \( h(V) = V (1 - |V|^q) \). Then, we have already seen in Example 4 that the roots of \( h(V) = 0 \) are \(-1, 0, 1\) and it is satisfied

\[
\begin{aligned}
h'(-1) < 0, \quad h'(0) > 0, \quad h'(1) < 0.
\end{aligned}
\]

From Theorem 1, a non-constant classical solution of (5.1) with \( p \geq 2 \) cannot touch \(-1\) and \( 1\). However, for the case \( p > 2 \), the problem (5.1) can have a solution \( V(x) \) satisfying \( V(x_0) = 1 \) for some \( x_0 \in [-1, 1] \). In order to prove the existence of such solutions, we will construct a solution of the problem (5.1) satisfying

\[
-1 < V(x) < 1 \quad \text{for} \quad x \in (-1, 1), \quad \text{and} \quad V(-1) = -1, \quad V(1) = 1,
\]

which attains 1 and \(-1\) at the boundary points of the domain.

In the following, we use some ideas from [23] which treats Dirichlet boundary problems.

Letting \( \psi = |w'|^{p-2}w' \), we consider the following problem:

\[
\begin{aligned}
\varepsilon \psi' + h(w) &= 0, \quad x \in (0, \infty), \\
w(0) &= 0, \quad \psi(0) = \alpha.
\end{aligned}
\]

(5.2)  

(5.3)

Here \( \alpha \) is a parameter. We will find a solution satisfying \( w(1) = 1 \) and \( \psi(1) = 0 \) for some \( \alpha \).

Integrating both sides of (5.2) with respect to \( x \) after multiplying them by \( w' \), we obtain that

\[
\varepsilon \frac{p-1}{p} \int_0^x (|\psi|^{p/(p-1)})' dx + \int_0^w h(s) ds = 0,
\]

where we have used \( w' = \psi|\psi|^{-(p-2)/(p-1)} \). Therefore, noting (5.3), we see that

\[
|\psi|^{p/(p-1)} = |\alpha|^{p/(p-1)} - \frac{p}{\varepsilon(p-1)} H(w), \quad H(w) = \int_0^w h(s) ds.
\]

(5.4)

Since

\[
H(w) = \int_0^w s(1 - |s|^q) ds = \frac{1}{q+2}w^2 \left( \frac{q+2}{2} - |w|^q \right),
\]

it follows from Figure 1 that there exists a \( x^* \in (0, \infty) \) such that we can have a solution of (5.2)–(5.3) with \( w(x^*) = 1 \) and \( \psi(x^*) = 0 \) if and only if \( \alpha = \alpha_\pm \), where

\[
\alpha_\pm = \pm \left( \frac{p}{\varepsilon(p-1)} H(1) \right)^{(p-1)/p} = \pm \left( \frac{pq}{2\varepsilon(p-1)(q+2)} \right)^{(p-1)/p}.
\]

(5.5)
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Figure 1: Graph of $-H(w)$. $E_{\pm} = \pm ((q + 2)/2)^{1/q}$ and $-H(\pm 1) = -q/(2(q + 2))$.

It is remained to show that $x^* = 1$. Now, we consider the case $\alpha_+ > 0$ and $x > 0$ is small. Then, $\psi(x) > 0$. Moreover, we assume $w(x) > 0$. Then, for $0 < \alpha \leq \alpha_+$, there exists $b_\alpha$ such that each solution of (5.2)–(5.3) satisfies $w(x_\alpha) = b_\alpha > 0$ and $\psi'(x_\alpha) = 0$ for some $x_\alpha \in (0, \infty)$. Note that $b_\alpha \to 1$ as $\alpha \to \alpha_+$. Differentiating both sides of (5.4) with respect to $x$, we obtain that

$$\frac{p}{p - 1} \psi^{1/(p - 1)} \frac{d\psi}{dx} = -\frac{p}{\varepsilon(p - 1)} h(w) \frac{dw}{dx}. $$

Since, using (5.4) again, we have

$$\psi^{1/(p - 1)} = \left( \frac{\alpha^{p/(p - 1)}}{\varepsilon(p - 1)} - \frac{p}{\varepsilon(p - 1)} H(w) \right)^{1/p},$$

the equation (5.2) becomes

$$\frac{dw}{dx} = \left( \frac{\alpha^{p/(p - 1)}}{\varepsilon(p - 1)} - \frac{p}{\varepsilon(p - 1)} H(w) \right)^{1/p}. $$

Here we see that $\alpha^{p/(p - 1)} = \varepsilon(p - 1)/H(b_\alpha)$ from (5.4). Integrating the both sides of (5.6) with respect to $x$ from 0 to $x_\alpha$, we obtain that

$$x_\alpha = \left( \frac{\varepsilon(p - 1)}{p} \right)^{1/p} \int_0^{b_\alpha} (H(b_\alpha) - H(w))^{-1/p} dw. $$

Since the function $\alpha \mapsto b_\alpha$ is one to one, we define

$$I(a) = \int_0^a (H(a) - H(w))^{-1/p} dw, \quad a \in (0, 1] $$

(5.8)
and show that there exists \( \varepsilon > 0 \) such that

\[
I(1) = \left( \frac{\varepsilon(p - 1)}{p} \right)^{-1/p}.
\]

(5.9)

Letting \( w = au \), we calculate \( I(a) \) and obtain

\[
I(a) = \int_0^1 (H(a) - H(au))^{-1/p} a \, du
\]

\[
= a \int_0^1 \left\{ \int_{au}^a w(1 - w^q) \, dw \right\}^{-1/p} du
\]

\[
= a^{1 - \frac{2}{p}} \int_0^1 \left\{ \int_0^1 (1 - (as)^q) \, ds \right\}^{-1/p} du
\]

\[
= a^{1 - \frac{2}{p}} \int_0^1 \left\{ \frac{q + 2(1 - a^q)}{2(q + 2)} - \frac{u^2}{2} + \frac{a^q}{q + 2}u^{q+2} \right\}^{-1/p} du.
\]

Define the function \( g \) as

\[
g(u, a) = \frac{q + 2(1 - a^q)}{2(q + 2)} - \frac{u^2}{2} + \frac{a^q}{q + 2}u^{q+2}.
\]

Then, it satisfies

\[
g(0, a) \geq \frac{q}{2(q + 2)} > 0 \quad \text{and} \quad g(1, a) = 0 \quad \text{for all} \quad a \in (0, 1].
\]

Moreover, we see

\[
\frac{\partial g}{\partial u}(u, a) = -u(1 - a^q u^q) < 0 \quad \text{for} \quad u \in (0, 1), \ a \in (0, 1],
\]

and

\[
\frac{\partial^2 g}{\partial u^2}(u, a) = -1 + (q + 1)a^q u^q > 0 \quad \text{for} \quad u \in (0, 1),
\]

provided that \( a > 1/(q + 1)^{1/q} \). Since

\[
\lim_{u \to 1^-} \frac{g(u, 1)}{(1 - u)^2} = \frac{q}{2} > 0,
\]

it follows that \( g(u, 1) = O((1 - u)^2) \) as \( u \to 1 \). Therefore, we obtain that there exists \( C > 0 \) such that

\[
g(u, a)^{-1/p} \leq C(1 - u)^{-2/p} \quad \text{for} \quad \left( \frac{1}{q + 1} \right)^{1/q} < a \leq 1
\]

in consideration of concavity and convexity of functions. Hence, it is satisfied that

\[
\lim_{a \to 1^-} I(a) = I(1) = \int_0^1 g(u, 1)^{-1/p} \, du < \infty.
\]
Now, we will show (5.9). The function $m(\varepsilon) = \varepsilon^{-1/p}$ is a continuous and monotone decreasing function of $\varepsilon \in (0, \infty)$ so that $\lim_{\varepsilon \to +0} m(\varepsilon) = \infty$ and $\lim_{\varepsilon \to \infty} m(\varepsilon) = 0$. Therefore, there exists $\varepsilon_0 > 0$ such that

$$I(1) = \left( \frac{\varepsilon_0(p - 1)}{p} \right)^{-1/p}.$$  

Consequently, we have obtained a solution of (5.2)–(5.3) satisfying

$$w(0) = 0, \quad \psi(0) = \alpha_+,$$
$$w(1) = 1, \quad \psi(1) = 0. \quad (5.10)$$

If $w$ is the solution of (5.2)–(5.3), then $z(x) = -w(-x)$ is a solution of the problem

$$\begin{cases}
\varepsilon_0 \frac{d\tilde{\psi}}{dx} + h(z) = 0, & x \in (-\infty, 0), \\
z(0) = 0, \quad \tilde{\psi}(0) = \alpha_+, \\
z(-1) = -1, \quad \tilde{\psi}(-1) = 0.
\end{cases} \quad (5.11)$$

Letting

$$V(x) = \begin{cases}
w(x) & x \in [0, 1], \\
z(x) & x \in [-1, 0],
\end{cases}$$

we see that this is a solution of the problem (5.1) with $\varepsilon = \varepsilon_0$ satisfying $V(1) = 1$ and $V(-1) = -1$, which is the desired solution.

5.1. Flatness of solutions at the boundary

We see that there exists a solution of (5.1) which has flat parts around the boundary.

Fix $0 < \xi < 1$ arbitrarily. We consider the existence of a solution of (5.2)–(5.3) satisfying $w(\xi) = 1$ and $\psi(\xi) = 0$. According to the same procedure as that in the previous case, we obtain $\alpha = \alpha_0$ which is given by (5.5). Moreover, the existence of such solution can be shown if there exists $\varepsilon > 0$ such that

$$I(1) = \xi \left( \frac{\varepsilon(p - 1)}{p} \right)^{-1/p},$$

where the function $I(a)$ is defined by (5.8) for $a \in (0, 1]$. It is easily seen that this is satisfied if $\varepsilon = \varepsilon_0 \xi^p$. Hence, we have a solution of (5.2)–(5.3) satisfying

$$w(0) = 0, \quad \psi(0) = \alpha_+, \quad w(\xi) = 1, \quad \psi(\xi) = 0.$$  

Letting $z(x) = -w(-x)$, which is defined for $-\xi \leq x \leq 0$, we obtain a solution of (5.1) with $\varepsilon = \varepsilon_0 \xi^p$:

$$V(x) = \begin{cases}
1 & (\xi < x \leq 1), \\
w(x) & (0 < x \leq \xi), \\
z(x) & (-\xi < x \leq 0), \\
-1 & (-1 \leq x \leq -\xi).
\end{cases}$$
This is monotone increasing function on $[-1, 1]$ and the maximum of $V$ is attained not only at the boundary but also at inner parts of the domain $(-1, 1)$.

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References


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