On Conditions for Denominator Ideals to Diffuse and Conditions for Elements to Be Exclusive in Anti-Integral Extensions

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Let R be a Noetherian integral domain and let α be an element in an algebraic extension field of the quotient field of R . Our objective is to show that a generalized denominator ideal $I_{[\alpha]}$ satisfies $I_{[\alpha]}R[\alpha] = R[\alpha]$ if and only if $I_{[\alpha]} + I_{[\alpha^{-1}]} = R$ if and only if either α or α^{-1} is integral over R_p for each $p \in \text{Spec}(R)$, provided that α is an anti-integral element over R . The other is to show that for the canonical map $\phi : \text{Spec}(R[\alpha]) \rightarrow \text{Spec}(R)$, $\text{Dp}_1(R)$ is contained in the image of ϕ if and only if α is exclusive. The former result is closely concerned to [KY1] and [OY3], and the latter result is related to [OY2].

Notation and Conventions

Throughout this paper, we use the following notation unless otherwise specified:

Let R be a Noetherian domain (which is commutative and has a unit), let $R[X]$ be a polynomial ring, let α be an element of an algebraic extension field of the quotient field K of R and let $\pi : R[X] \rightarrow R[\alpha]$ be the R -algebra homomorphism sending X to α . Let $\varphi_\alpha(X)$ be the monic minimal polynomial of α over K with $\deg \varphi_\alpha(X) = d$ and write $\varphi_\alpha(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d$. Then $\eta_i \in K$ ($1 \leq i \leq d$) are uniquely determined by α . Put $d = [K(\alpha) : K]$, $I_{\eta_i} := R :_R \eta_i$ and $I_{[\alpha]} := \bigcap_{i=1}^d I_{\eta_i}$. If $\text{Ker}(\pi) = I_{[\alpha]}\varphi_\alpha(X)R[X]$, we say that α is *anti-integral* over R (cf. [OSY]). Put $J_{[\alpha]} := I_{[\alpha]}(1, \eta_1, \dots, \eta_d)$. For $\beta \in K$, we put $J_\beta := I_\beta(1, \beta)$. Then $J_{[\alpha]} = c(I_{[\alpha]}\varphi_\alpha(X))$, where $c(\)$ denotes the ideal generated by the coefficients of the polynomials in $(\)$, that is, the content ideal of $(\)$. If $J_{[\alpha]} \not\subseteq p$ for all $p \in \text{Dp}_1(R) := \{p \in \text{Spec}(R) \mid \text{depth} R_p = 1\}$, the element α is called a *super-primitive element* over R . A super-primitive element over R is anti-integral over R (cf. [OSY, Theorem 1.12]). It is also known that any algebraic element over a Krull domain R is super-primitive over R (cf. [OSY, Theorem 1.13]), and hence α is anti-integral over R . We also note here that $I_{[\alpha]} = R \Leftrightarrow R[\alpha]$ is integral over R and that $J_{[\alpha]} = R \Leftrightarrow R[\alpha]$ is flat over R , provided that α is anti-integral over R .

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Our general reference for unexplained terminology is [M].

§1. Conditions for generalized denominator ideals to diffuse in anti-integral extensions

We start with the following proposition. The implication (i) $\Rightarrow$ (ii) is seen in [OY3, Proposition 7], but we give a proof for convenience.

PROPOSITION 1.1. *Assume that α is an anti-integral element over R . Then the following statements are equivalent:*

- (i) $I_{[\alpha]}R[\alpha] = R[\alpha]$;
- (ii) $\eta_d \in R[\alpha]$, $I_{[\alpha]} = I_{\eta_d}$ and $J_{\eta_d} = R$.

PROOF. If $I_{[\alpha]} = R$, then the implications (i) $\Leftrightarrow$ (ii) are trivially valid because $\eta_d \in R$ by the definition of $I_{[\alpha]}$. So we may assume that $I_{[\alpha]} \neq R$.

(i) $\Rightarrow$ (ii) : Since $\eta_1, \dots, \eta_d \in I_{[\alpha]}^{-1}R[\alpha] = I_{[\alpha]}^{-1}I_{[\alpha]}R[\alpha] \subseteq R[\alpha]$, we have $\eta_d \in R[\alpha]$. Put $C := R[\eta_1, \dots, \eta_d]$. Then $R[\alpha]$ is a free C -module and $R[\alpha]$ is integral over C . So $I_{[\alpha]}R[\alpha] = R[\alpha]$ yields that $I_{[\alpha]}C = C$. Thus C is flat over R . Since $R[\alpha]$ is flat over C , $R[\alpha]$ is flat over R , and hence $J_{[\alpha]} = R$. Now we shall show that $I_{[\alpha]} = I_{\eta_d}$. It is easy to see that $I_{[\alpha]} \subseteq I_{\eta_d}$. Take $p \in \text{Spec}(R)$ with $p \supseteq I_{[\alpha]}$. Since $I_{[\alpha]}R[\alpha] = R[\alpha]$, we have $1 = a_0 + a_1\alpha + \dots + a_n\alpha^n$, where $a_i \in I_{[\alpha]} \subseteq p$. Since $1 - a_0$ is a unit in R_p , α^{-1} is integral over R . Since α is anti-integral over R , α^{-1} is also anti-integral over R . Thus $\varphi_{\alpha^{-1}}(X) = X^d + (\eta_d^{-1}\eta_{d-1})X^{d-1} + \dots + (\eta_d^{-1}\eta_1)X + \eta_d^{-1} \in R_p[X]$. So there exists $x \in R_p$ such that $\eta_d = 1/x$, $\eta_i = y_i/x$ with some $y_i \in R_p$. Therefore we have $(I_{[\alpha]})_p = xR_p = (I_{\eta_d})_p$. Since p is arbitrary, we conclude that $I_{[\alpha]} = I_{\eta_d}$. Next we shall show that $J_{\eta_d} = I_{\eta_d}(1, \eta_d) = R$. Suppose that $I_{\eta_d}(1, \eta_d) \subseteq p$ for some $p \in \text{Spec}(R)$. Then $I_{[\alpha]} \subseteq p$. Thus by using the above notation, we have $(I_{\eta_d}(1, \eta_d))_p = x(1, 1/x)R_p = R_p$, which contradicts the assumption $I_{\eta_d}(1, \eta_d) \subseteq p$. Therefore we conclude that $J_{\eta_d} = R$.

(ii) $\Rightarrow$ (i) follows the implications : $R[\alpha] \supseteq I_{[\alpha]}R[\alpha] \supseteq I_{[\alpha]}(1, \eta_d) = I_{\eta_d}(1, \eta_d) = R \ni 1$, and hence $R[\alpha] \supseteq I_{[\alpha]}R[\alpha] \supseteq R[\alpha]$. $\square$

COROLLARY 1.2. *Assume that α is an anti-integral element over R . If $I_{[\alpha]}R[\alpha] = R[\alpha]$, then $R[\alpha] \cap K = R[\eta_d]$.*

PROOF. Put $C := R[\eta_1, \dots, \eta_d]$. Then by the proof of Proposition 1.1, we have $C \subseteq R[\alpha]$. Since $R[\alpha]$ is a free C -module $C + C\alpha + \dots + C\alpha^{d-1}$, we conclude that $R[\alpha] \cap K = C$. Besides, as in the proof of Proposition 1, if $I_{[\alpha]} \subseteq p$, then there exists x in R_p such that $\eta_d = 1/x$ and $\eta_i = y_i/x \in R_p[\eta_d]$ for some $y_i \in R_p$. If $I_{[\alpha]} \not\subseteq p$, then $C_p = R_p = R_p[\eta_d]$. Therefore we have $C = R[\eta_d]$. $\square$

LEMMA 1.3. $I_{[\alpha^{-1}]} = \eta_d I_{[\alpha]}$.

PROOF. Since $\varphi_{\alpha^{-1}}(X) = X^d + (\eta_d^{-1}\eta_{d-1})X^{d-1} + \dots + (\eta_d^{-1}\eta_1)X + \eta_d^{-1}$, we have $I_{[\alpha^{-1}]} = \bigcap_{i=1}^{d-1} I_{\eta_d^{-1}\eta_i} \cap I_{\eta_d^{-1}}$. Take $x = \eta_d y \in \eta_d I_{[\alpha]}$ with $y \in I_{[\alpha]}$. Then $x\eta_d^{-1}\eta_i = y\eta_i \in R$ and hence $x\eta_d^{-1} = y \in R$. Thus $I_{[\alpha^{-1}]} \supseteq \eta_d I_{[\alpha]}$. Conversely,

take $x \in I_{[\alpha^{-1}]}$. Then $x\eta_d^{-1} = y \in R$, $y\eta_d = x \in R$ and $x\eta_d^{-1}\eta_i = y\eta_i \in R$. hence $y \in I_{[\alpha]}$. Therefore we have $I_{[\alpha^{-1}]} = \eta_d I_{[\alpha]}$. $\square$

REMARK 1.4. (i) An element α is anti-integral over R if and only if so is α^{-1} (cf. [KY]).

(ii) An element α is super-primitive over R if and only if so is α^{-1} . Indeed, the minimal monic polynomial $\varphi_{\alpha^{-1}}$ of α^{-1} is $X^d + (\eta_{d-1}/\eta_d)X^{d-1} + \cdots + (\eta_1/\eta_d)X + (1/\eta_d)$. Hence we have

$$\begin{aligned} J_{[\alpha^{-1}]} &= I_{[\alpha^{-1}]}(1, \eta_{d-1}/\eta_d, \dots, \eta_1/\eta_d) \\ &= \eta_d I_{[\alpha]}(1, \eta_{d-1}/\eta_d, \dots, \eta_1/\eta_d) \\ &= I_{[\alpha]}(1, \eta_1, \dots, \eta_d) = J_{[\alpha]}, \end{aligned}$$

here we use Lemma 1.3. So our assertion follows from the definition of super-primitiveness.

THEOREM 1.5. Assume that α is an anti-integral element over R and that $\eta_d \in R[\alpha]$. The following statements are equivalent :

- (i) $I_{[\alpha]}R[\alpha] = R[\alpha]$;
- (ii) $I_{[\alpha]} + I_{[\alpha^{-1}]} = R$;
- (iii) either α or α^{-1} is integral over R_p for each $p \in \text{Spec}(R)$.

PROOF. (i) $\Rightarrow$ (ii) : We have $\eta_d \in R[\alpha]$, $I_{[\alpha]} = I_{\eta_d}$ and $J_{\eta_d} = R$ by Proposition 1.1. So we obtain $R = J_{\eta_d} = I_{\eta_d}(1, \eta_d) = I_{[\alpha]}(1, \eta_d) = I_{[\alpha]} + \eta_d I_{[\alpha]} = I_{[\alpha]} + I_{[\alpha^{-1}]}$ (cf. Lemma 1.3).

(ii) $\Rightarrow$ (i) : We have $R = I_{[\alpha]} + I_{[\alpha^{-1}]} = I_{[\alpha]} + \eta_d I_{[\alpha]} = I_{[\alpha]}(1, \eta_d)$. Thus $R[\alpha] = I_{[\alpha]}(1, \eta_d)R[\alpha] = I_{[\alpha]}R[\alpha] + I_{[\alpha]}\eta_d R[\alpha] \subseteq I_{[\alpha]}R[\alpha]$ because $\eta_d \in R[\alpha]$. The converse inclusion is obvious. Thus $I_{[\alpha]}R[\alpha] = R[\alpha]$.

(ii) $\Rightarrow$ (iii) : Take $p \in \text{Spec}(R)$. Then $I_{[\alpha]} \not\subseteq p$ or $I_{[\alpha^{-1}]} \not\subseteq p$ because $I_{[\alpha]} + I_{[\alpha^{-1}]} = R$. If $I_{[\alpha]} \not\subseteq p$, we have $\eta_1, \dots, \eta_d \in R_p$. Hence $\alpha^d + \eta_1 \alpha^{d-1} + \cdots + \eta_d = 0$, which means that α is integral over R_p . Next, we assume that $I_{[\alpha^{-1}]} \not\subseteq p$. The $\eta_d^{-1}\eta_1, \dots, \eta_d^{-1}\eta_{d-1}, \eta_d^{-1} \in R_p$. Thus $(\alpha^{-1})^d + (\eta_d^{-1}\eta_{d-1})\alpha^{d-1} + \cdots + \eta_d^{-1}\eta_1\alpha + \eta_d^{-1} = 0$, which means that α^{-1} is integral over R_p .

(iii) $\Rightarrow$ (ii) : Suppose that there exists a prime ideal p of R such that $I_{[\alpha]} + I_{[\alpha^{-1}]} \subseteq p$. Note that α or α^{-1} is integral over R_p . Assume that α is integral over R_p . Then α satisfies a monic relation of degree d over R_p because α is anti-integral over R_p . But since $\varphi_{\alpha}(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d \in R_p[X]$, we have $I_{[\alpha]} \not\subseteq p$, a contradiction. Similarly we come to a contradiction when we assume α^{-1} is integral over R_p . $\square$

PROPOSITION 1.6. Assume that α is an anti-integral element of degree d over R . If $\eta_d \in R[\alpha]$, then $\sqrt{I_{[\alpha]}R[\alpha]} \cap R = \sqrt{I_{[\alpha]} + I_{[\alpha^{-1}]}}$.

PROOF. Take $p \in \text{Spec}(R)$. Then $I_{[\alpha]}R[\alpha] \cap R \not\subseteq p \Leftrightarrow I_{[\alpha]}R_p[\alpha] = R_p[\alpha] \Leftrightarrow (I_{[\alpha]} + I_{[\alpha^{-1}]})_p = R_p \Leftrightarrow I_{[\alpha]} + I_{[\alpha^{-1}]} \not\subseteq p$. Thus we come to our conclusion by Theorem 1.5. $\square$

§2. Conditions for anti-integral elements to be exclusive.

Recall first that an algebraic element α over R is called to be *exclusive* if $R[\alpha] \cap K = R$. Put $\tilde{J}_{[\alpha]} := I_{[\alpha]}(1, \eta_1, \dots, \eta_{d-1})$, an ideal of R .

REMARK 2.1. Assume that R contains a field k . Let x be an indeterminate and put $S := R \otimes_k k(x)$. Then S contains the infinite field $k(x)$. Put $I_{[\alpha]}^S := \bigcap_{i=1}^d (S :_S \eta_i)$, $J_{[\alpha]}^S := I_{[\alpha]}^S(1, \eta_1, \dots, \eta_d)S$ and $\tilde{J}_{[\alpha]}^S := I_{[\alpha]}^S(1, \eta_1, \dots, \eta_{d-1})S$. Since S is faithfully flat over R , $I_{[\alpha]}^S = I_{[\alpha]}S$, $J_{[\alpha]}^S = J_{[\alpha]}S$ and $\tilde{J}_{[\alpha]}^S = \tilde{J}_{[\alpha]}S$. So if α is super-primitive (resp. anti-integral) over R , then so is α over S . Moreover α is exclusive over S if and only if so is α over R ; $\bigcap_{i=1}^{d-1} I_{\eta_i} \subseteq I_{\eta_d}$ if and only if $\bigcap_{i=1}^{d-1} I_{\eta_i}^S \subseteq I_{\eta_d}^S$; $\text{grade}(\tilde{J}_{[\alpha]}) > 1$ if and only if $\text{grade}(\tilde{J}_{[\alpha]}^S) > 1$; and α is exclusive over S , i.e., $S[\alpha] \cap K(x) = S[\alpha]$ if and only if α is exclusive over R , i.e., $R[\alpha] \cap K = R$.

LEMMA 2.2(cf. [OY2, Theorem 5]) Assume that α is super-primitive over R . Then the following statements (i) and (ii) are equivalent :

- (i) $\bigcap_{i=1}^{d-1} I_{\eta_i} \subseteq I_{\eta_d}$;
- (ii) $\text{grade}(\tilde{J}_{[\alpha]}) > 1$ or $\tilde{J}_{[\alpha]} = R$.

Furthermore if R contains a field, then the following (iii) is equivalent to (i)

- (iii) α is exclusive over R .

PROOF. (i) $\Leftrightarrow$ (ii) follows from [OY2, Lemma 3].

Next assume that R contains a field. We may assume that R contains an infinite field by Remark 2.1. Hence our conclusion (ii) $\Leftrightarrow$ (iii) follows from [OY2, Theorem 5]. $\square$

REMARK 2.3. Assume that R contains a field.

(i) When α is a super-primitive element over R , α is exclusive over R if and only if $\text{grade}(\tilde{J}_{[\alpha]}) > 1$ by Lemma 2.2.

(ii) $\tilde{J}_{[\alpha^{-1}]} = I_{[\alpha]}(\eta_1, \dots, \eta_d)$. This follows from the similar argument of Remark 1.4(ii).

(iii) By Remark 1.4, α is super-primitive over R if and only if so is α^{-1} . So by (i) and (ii) above, we have α^{-1} is exclusive over R if and only if $\text{grade}(I_{[\alpha]}(\eta_1, \dots, \eta_d)) > 1$.

(iv) If $\text{grade}(I_{[\alpha]}(\eta_1, \dots, \eta_{d-1})) > 1$, then both α and α^{-1} are exclusive over R by (i), (ii) and (iii).

PROPOSITION 2.4. Assume that α is an anti-integral element of degree d over R . If $I_{[\alpha]}R[\alpha]_P = R[\alpha]_P$ for every $P \in \text{Dp}_1(R[\alpha])$, then $R[\alpha] \cap K = R[\eta_1, \dots, \eta_d]$.

PROOF. It follows that $R[\alpha]_P \supseteq I_{[\alpha]}^{-1}I_{[\alpha]}R[\alpha]_P = I_{[\alpha]}^{-1}R[\alpha]_P \supseteq I_{[\alpha]}^{-1} \ni \eta_1, \dots, \eta_d$ because $\bigcap_{P \in \text{Dp}_1(R)} R[\alpha]_P = R[\alpha]$. Thus $R[\alpha] \cap K \supseteq R[\eta_1, \dots, \eta_d] =: C$, α is integral over C and α is anti-integral over C . Since

$K(C) = K$ and $[K(\alpha) : K] = d$, $R[\alpha]$ is a free C -module $C + C\alpha + \cdots + C\alpha^{d-1}$. Hence $R[\alpha] \cap K = C$, as was to be shown. $\square$

THEOREM 2.5. *Assume that α is a super-primitive element of degree d over R and that R contains a field. Let $\phi : \text{Spec}(R[\alpha]) \rightarrow \text{Spec}(R)$ be the canonical map induced from the inclusion $R \subseteq R[\alpha]$. Then the following statements are equivalent:*

- (i) $\text{Dp}_1(R)$ is contained in the image of ϕ , i.e., $\text{Im}(\phi) \supseteq \text{Dp}_1(R)$;
- (ii) α is exclusive over R .

PROOF. $\text{Im}(\phi) \supseteq \text{Dp}_1(R)$ if and only if $\tilde{J}_{[\alpha]p} = J_{[\alpha]p} = R_p$ for every $p \in \text{Dp}_1(R)$ by [OY2, Lemma 2]. Since α is super-primitive over R , we see that $\text{grade}(J_{[\alpha]}) > 1$. Hence we have $\text{grade}(\tilde{J}_{[\alpha]}) > 1$, that is, $\tilde{J}_{[\alpha]p} = R_p$ for every $p \in \text{Dp}_1(R)$. So we conclude that α is exclusive over R by Lemma 2.2.

Conversely, the set of diffusing points (i.e., $p \in \text{Spec}(R)$ such that $pR[\alpha]_p = R[\alpha]_p$) is given by $\bigcap_{i=1}^{d-1} V(I_{[\alpha]}\eta_i) \setminus V(I_{[\alpha]}\eta_d) = V(\tilde{J}_{[\alpha]}) \setminus V(I_{[\alpha]}\eta_d)$ by [OY2, Lemma 2]. So take $p \in \text{Dp}_1(R)$ such that $pR[\alpha]_p = R[\alpha]_p$. Then $p \supseteq \tilde{J}_{[\alpha]}$ and $p \not\supseteq I_{[\alpha]}\eta_d$. Thus we have $\text{grade}(\tilde{J}_{[\alpha]}) = 1$, which yields that α is not exclusive over R by Lemma 2.2. Hence $qR[\alpha]_q \neq R[\alpha]_q$ for all $q \in \text{Im}(\phi)$. $\square$

PROPOSITION 2.6. *Assume that $\alpha_1, \dots, \alpha_n$ are super-primitive elements over R and that R contains a field. Put $A := R[\alpha_1, \dots, \alpha_n]$ and let $\phi : \text{Spec}(A) \rightarrow \text{Spec}(R)$ be the canonical map induced from the inclusion $R \subseteq A$. If $\text{Dp}_1(R)$ is contained in the image of ϕ , then each α_i ($1 \leq i \leq n$) is exclusive over R .*

PROOF. Let $\psi_i : \text{Spec}(A) \rightarrow \text{Spec}(R[\alpha_i])$ and $\phi_i : \text{Spec}(R[\alpha_i]) \rightarrow \text{Spec}(R)$ be the canonical maps induced from the inclusion $R[\alpha_i] \subseteq A$ and $R \subseteq R[\alpha_i]$, respectively. Then $\phi = \phi_i \circ \psi_i$ induces the inclusions $\text{Im}(\phi_i) \supseteq \text{Im}(\phi) \supseteq \text{Dp}_1(R)$. Thus our conclusion follows Theorem 2.5. $\square$

PROBLEM. Is the converse statement of Proposition 2.6 valid ?

PROPOSITION 2.7. *Assume that α is a super-primitive element of degree d over R . If $\eta_d \in R$, then the canonical map $\phi : \text{Spec}(R[\alpha]) \rightarrow \text{Spec}(R)$ is surjective.*

PROOF. Note that $\phi : \text{Spec}(R[\alpha]) \rightarrow \text{Spec}(R)$ is surjective if and only if $V(\tilde{J}_{[\alpha]}) = V(J_{[\alpha]}) \subseteq V(I_{[\alpha]}\eta_d)$ (cf. [KY1, Theorem 7], [OY2, Lemma 2]). Since $\eta_d \in R$, we have $\eta_d I_{[\alpha]} \subseteq I_{[\alpha]}$, which implies that $V(\tilde{J}_{[\alpha]}) = V(J_{[\alpha]})$ by the construction of $J_{[\alpha]}$. So we have our conclusion. $\square$

Note that $\phi : \text{Spec}(R[\alpha]) \rightarrow \text{Spec}(R)$ is surjective and ϕ is flat if and only if ϕ is faithfully flat. So we obtain the following corollary.

COROLLARY 2.8. *Assume that $\eta_d \in R$. If $R[\alpha]$ is flat over R , then $R[\alpha]$ is faithfully flat over R .*

The following proposition gives rise to a condition for $\eta_d \in R$.

PROPOSITION 2.9. Assume that α is a super-primitive element of degree d over R and that R contains a field. Then the following statement are equivalent :

- (i) $\eta_d \in R$;
- (ii) α is exclusive over R and $I_{\eta_d}R[\alpha] = R[\alpha]$.

PROOF. (i) $\Rightarrow$ (ii) follows from Lemma 2.2.

(ii) $\Rightarrow$ (i) : Since $\eta_d \in I_{\eta_d}^{-1}R[\alpha] = I_{\eta_d}^{-1}I_{\eta_d}R[\alpha] \subseteq R[\alpha]$, we conclude that $\eta_d \in R[\alpha] \cap K = R$. $\square$

REMARK 2.10. Let I denote an ideal of R . Then

$$\text{grade}(I) > 1 \Leftrightarrow I^{-1} = R,$$

where $I^{-1} := R :_K I$.

THEOREM 2.11. Assume that α is an anti-integral element of degree d over R . Let $f : \text{Spec}(R[\alpha]) \rightarrow \text{Spec}(R)$ and $g : \text{Spec}(R[\eta_d]) \rightarrow \text{Spec}(R)$ be the canonical maps obtained from the inclusions $R \subseteq R[\alpha]$ and $R \subseteq R[\eta_d]$, respectively. If $\text{grade}_{R[\alpha]}(J_{[\alpha]}R[\alpha]) > 1$, $\text{grade}_{R[\alpha]}(J_{\eta_d}R[\alpha]) > 1$ and $\text{grade}_{R[\eta_d]}(J_{[\alpha]}R[\eta_d]) > 1$, then $R[\alpha] \cap K = R[\eta_d]$ and $\text{Im}(f) = \text{Im}(g)$.

PROOF. First note that $J_{[\alpha]}R[\eta_d] \subseteq J_{[\alpha]}^{R[\eta_d]}$ (here we use the notation as in Remark 2.1; put $S = R[\eta_d]$). So $\text{grade}_{R[\eta_d]}(J_{[\alpha]}R[\eta_d]) > 1$ yields that $\text{grade}_{R[\eta_d]}(J_{[\alpha]}^{R[\eta_d]}) > 1$, so that α is super-primitive over $R[\eta_d]$. Take $P \in \text{Dp}_1(R[\alpha])$ and put $p := P \cap R$. Since $J_{[\alpha]}R[\alpha] \not\subseteq P$, we have either $J_{[\alpha]} \not\subseteq p$ or $I_{\eta_d} \not\subseteq p$. Thus $R_p[\alpha]$ is flat over R_p (cf. [OSY]). So we have $I_{\eta_d}R_p[\alpha] = (R :_R \eta_d)R_p[\alpha] = R_p[\alpha] :_{R_p[\alpha]} \eta_d = R_p[\alpha]$. Thus $\eta_d \in R_p[\alpha]$ by Proposition 1.1. Therefore $\eta_d \in \bigcap_{P \in \text{Dp}_1(R[\alpha])} R[\alpha]_P = R[\alpha]$. As mentioned above, α is super-primitive over $R[\eta_d]$. Since $\eta_d \in R[\eta_d]$, α is exclusive over $R[\eta_d]$. Hence the canonical map $\psi : \text{Spec}(R[\alpha]) \rightarrow \text{Spec}(R[\eta_d])$ is surjective by Proposition 2.7. Consider the following commutative diagram :

$$\begin{array}{ccc} \text{Spec}(R[\eta_d][\alpha]) & \xlongequal{\quad} & \text{Spec}(R[\alpha]) \\ \psi \downarrow & & g \downarrow \\ \text{Spec}(R[\eta_d]) & \xrightarrow{\quad f \quad} & \text{Spec}(R), \end{array}$$

here we use that $\eta_d \in R[\alpha]$. Since ψ is surjective and $f \cdot \psi = g$, we conclude that $\text{Im}(f) = \text{Im}(g)$. $\square$

We say that α is an *ultra-primitive* element of degree d over R if $\text{grade}(I_{[\alpha]} + \mathcal{C}(\overline{R}/R)) > 1$, where $\overline{R}$ denotes the integral closure of R in K and $\mathcal{C}(\overline{R}/R)$ denotes the conductor between R and $\overline{R}$ (cf. [OY3]).

PROPOSITION 2.12. Assume that an ultra-primitive element of degree d . If $\text{grade}(I_{[\alpha]} :_R I_{[\alpha^{-1}]}) > 1$, then $\eta_d \in R$.

PROOF. Take $p \in \text{Dp}_1(R)$. Then either $I_{[\alpha]} \not\subseteq p$ or $\mathcal{C}(\overline{R}/R) \not\subseteq p$. If $I_{[\alpha]} \not\subseteq p$, then $I_{[\alpha]} \subseteq I_{\eta_d} \not\subseteq p$, that is, $\eta_d \in R_p$. If $I_{[\alpha]} \subseteq p$, then $\mathcal{C}(\overline{R}/R) \not\subseteq p$ and hence

R_p is a normal domain. Note that $I_{[\alpha^{-1}]}_p = \eta_d I_{[\alpha]}_p \subseteq I_{[\alpha]}_p$ (cf. Lemma 1.3). The latter inclusion $\eta_d I_{[\alpha]}_p \subseteq I_{[\alpha]}_p$, that is, $\eta_d \in (I_{[\alpha]} :_K I_{[\alpha]})_p$ implies that η_d is integral over R_p , noting that $I_{[\alpha]}_p$ is finitely generated over R_p . So we have $\eta_d \in R_p$. Therefore $\eta_d \in \bigcap_{p \in \text{Dp}_1(R)} R_p = R$. $\square$

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