Some theorems concerning anti-integral, super-primitive and ultra-primitive elements

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Let R be a Noetherian domain and $R[X]$ a polynomial ring. Let α be a non-zero element of an algebraic extension field L of the quotient field K and let $\pi : R[X] \rightarrow R[\alpha]$ be the R -algebra homomorphism sending X to α . Let $\varphi_\alpha(X)$ be the monic polynomial of α over K with $\deg \varphi_\alpha(X) = d$ and write

$$\varphi_\alpha(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d.$$

Then η_i ($1 \leq i \leq d$) are uniquely determined by α . Let $I_{\eta_i} = R :_R \eta_i$ and $I_{[\alpha]} := \bigcap_{i=1}^d I_{\eta_i}$. If $\text{Ker}(\pi) = I_{[\alpha]} \varphi_\alpha(X) R[X]$, we say that α is an *anti-integral element of degree d* over R . Put $J_{[\alpha]} = I_{[\alpha]}(1, \eta_1, \dots, \eta_d)$. Then $J_{[\alpha]} = c(I_{[\alpha]} \varphi_\alpha(X))$, where $c(\)$ denotes the ideal generated by the coefficients of the polynomials in $(\)$, that is, the content ideal of $(\)$. If $J_{[\alpha]} \not\subseteq p$ for all $p \in \text{Dp}_1(R) := \{p \in \text{Spec}(R) \mid \text{depth}(R_p) = 1\}$, then we say that α is a *super-primitive element of degree d* over R . It is known that a super-primitive element over R is anti-integral over R (cf. [3, Theorem 1.12]). It is also known that any algebraic element over a Krull domain R is anti-integral over R (cf. [3, Theorem 1.13]). When α is an element in K , $\varphi_\alpha(X) = X - \alpha$. So we have

$$J_{[\alpha]} = I_{[\alpha]}(1, \alpha) = I_\alpha + \alpha I_\alpha = I_\alpha + I_{\alpha^{-1}}.$$

where $I_\alpha := R :_R \alpha$. In this case, $\alpha \in K$ is anti-integral over R if and only if $R[\alpha] \cap R[\alpha^{-1}] = R$ (cf. [3]).

N. Onoda, T. Sugatani and K. Yoshida, for any $a \in R$, asked whether equality $R[\alpha] \cap R[1/\alpha] = R[\alpha] \cap R[1/(\alpha - a)]$ is always true and gave an affirmative answer in $d \leq 4$. Hence, in this case, the set $\{R[\alpha] \cap R[1/(\alpha - a)] \mid a \in R\}$ is a one point set.

In this paper, we considered the set $\{B_a \mid a \in R^*\}$, where $R^* = R - \{0\}$ and $B_a = R[a\alpha] \cap R[1/a\alpha]$. Under some conditions, we shall show that $\{B_a \mid a \in R^*\} \cong R^*/U(R)$, where $U(R)$ denotes the unit group of R .

Received September 14, 1998.

1991 *Mathematics Subject Classification*. 13B02.

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Throughout this paper, we use the following notation unless otherwise specified:

Let R be an Noetherian domain with quotient field K . Let $\overline{R}$ denote the integral closure of R in K . Then $\overline{R}$ is a Krull domain (cf. [2]). Let $c(\overline{R}/R)$ denote the conductor ideal, i.e. $\{a \in R \mid a\overline{R} \subset R\}$. Let L be an algebraic extension field of K and let α be an element in L which is degree d over K . Let $\varphi_\alpha(X) := X^d + \eta_1 X^{d-1} + \cdots + \eta_d$ denote the minimal polynomial of α over K (that is $\eta_i \in K$). Let I be an ideal of R . When $I \neq R$, then we define $\text{grade}(I) :=$ the length of a maximal R -sequence in I . When $I = R$, then we define $\text{grade}(I) := \infty$ by convention (cf. [2]).

An element $\alpha \in L$ is called to be *ultra-primitive* over R if α satisfies $\text{grade}(I_{[\alpha]} + c(\overline{R}/R)) > 1$. Note that $c(\overline{R}/R)$ may be (0) . In this case, if $\alpha \in K$ is ultra-primitive over R , then the condition $\text{grade}(I_{[\alpha]} + c(\overline{R}/R)) > 1$ implies that $I_{[\alpha]} = R$ because $\text{grade}(I_{[\alpha]}) = 1$ or $I_{[\alpha]} = R$, and hence that $\alpha \in R$. It is known that an ultra-primitive element is a super-primitive element and hence an anti-integral element (cf. [4]).

Our general reference for unexplained technical terms is [2].

We start with the following lemma.

LEMMA 1. *Let a, b be elements of R . Assume that a and b are non-zero divisors on $R/I_{[\alpha]}$. Then $I_{[\alpha]} = I_{[a\alpha]} = I_{[b\alpha]}$ hold. If, in addition, α is super-primitive, then $aI_{[\alpha]} = bI_{[\alpha]}$ implies that $aR = bR$.*

PROOF. First we shall show that $I_{[\alpha]} = I_{[a\alpha]}$. Since $\varphi_{a\alpha}(X) := X^d + a\eta_1 X^{d-1} + a^2\eta_2 X^{d-2} + \cdots + a^d\eta_d$ is the minimal polynomial of $a\alpha$ of degree d over K , we have $I_{[\alpha]} \subset \cap_{i=1}^d I_{a^i\eta_i} = I_{[a\alpha]}$ and $I_{[a\alpha]} = \cap_{i=1}^d I_{a^i\eta_i} = \cap_{i=1}^d (I_{\eta_i} : a^i) \subset \cap_{i=1}^d (I_{\eta_i} : a^d) = (\cap_{i=1}^d I_{\eta_i}) : a^d = I_{[\alpha]} : a^d$.

Since a is a non-zero divisor on $R/I_{[\alpha]}$, it follows $I_{[\alpha]} = I_{[a\alpha]}$.

Next we assume that α is super-primitive. We have only to show that $b/a \in R$. In order to prove that $b/a \in R$, it suffices that $b/a \in R_p$ for every $p \in \text{Dp}_1(R)$. If $p \not\supset I_{[\alpha]}$, then $(I_{[\alpha]})_p = R_p$, so $aR_p = a(I_{[\alpha]})_p = b(I_{[\alpha]})_p = bR_p$. Hence $b/a \in R_p$. If $p \supset I_{[\alpha]}$, since α is super-primitive, then $I_{[\alpha]}R_p = xR_p$ for some $x \in I_{[\alpha]}$ by [3, Theorem 2.11]. Thus $aI_{[\alpha]}R_p = axR_p = bxR_p = bI_{[\alpha]}R_p$, and hence $aR_p = bR_p$. Therefore $b/a \in R_p$. Thus we have $b/a \in \cap_{p \in \text{Dp}_1(R)} R_p = R$ by [6], as desired. ■

LEMMA 2. *Let a be an element of R . Assume that a is a non-zero divisor on $R/I_{[\alpha]}$. Let α be a super-primitive element of degree d over R . Then $a\alpha$ is a super-primitive element over R .*

PROOF. We have

$$J_{[a\alpha]} = I_{[a\alpha]}(1, a\eta_1, \dots, a^d\eta_d) = I_{[\alpha]}(1, a\eta_1, \dots, a^d\eta_d)$$

by Lemma 1. Suppose that $p \supset J_{[a\alpha]}$ for some $p \in \text{Dp}_1(R)$. Then $I_{[\alpha]} \subset I_{[\alpha]}(1, a\eta_1, \dots, a^d\eta_d)$. Since α is super-primitive over R , p is a prime divisor

of $I_{[\alpha]}$. Hence $a \notin p$ by assumption. Since $a^i \eta_i I_{[\alpha]} \subset p$ ($1 \leq i \leq d$), it holds $\eta_i I_{[\alpha]} \subset p$. Therefore we obtain $J_{[\alpha]} = I_{[\alpha]}(1, \eta_1, \dots, \eta_d) \subset p$. This contradicts that α is super-primitive over R . Hence we see that $p \not\supset J_{[\alpha]}$ for any $p \in \text{Dp}_1(R)$. This yields that $a\alpha$ is super-primitive over R . ■

Let α be an anti-integral element of degree d over R . Then it is well-known that

$R[\alpha] \cap R[\alpha^{-1}] = R \oplus I_{[\alpha]}\zeta_1 \oplus \dots \oplus I_{[\alpha]}\zeta_{d-1}$, where $\zeta_i = \alpha^i + \eta_1 \alpha^{i-1} + \dots + \eta_i$ ($1 \leq i \leq d-1$), by [1, Theorem 1].

THEOREM 3. *Let R be a Noetherian domain with quotient field K . Let α be a super-primitive element of degree d (≥ 2) over R and let $a, b \in R$. Assume that a and b are non-zero divisors on $R/I_{[\alpha]}$. Let $B_a = R[a\alpha] \cap R[1/(a\alpha)]$ and $B_b = R[b\alpha] \cap R[1/(b\alpha)]$. Then the following statements are equivalent.*

- (1) $B_a = B_b$.
- (2) $a = ub$ for some $u \in U(R)$.

PROOF. (1) $\implies$ (2): $I_{[\alpha]} = I_{[a\alpha]} = I_{[b\alpha]}$ by Lemma 1. Since α is super-primitive and a, b are non-zero divisors on $R/I_{[\alpha]}$, it follows that $a\alpha, b\alpha$ are super-primitive over R by Lemma 2, and hence anti-integral over R . Therefore we have the direct sum decompositions of B_a, B_b as follows:

$$\begin{aligned} B_a &= R \oplus I_{[a\alpha]}a\zeta_1 \oplus \dots \oplus I_{[a\alpha]}a^{d-1}\zeta_{d-1} \\ &= R \oplus I_{[\alpha]}a\zeta_1 \oplus \dots \oplus I_{[\alpha]}a^{d-1}\zeta_{d-1} \quad \dots (*) \\ B_b &= R \oplus I_{[b\alpha]}b\zeta_1 \oplus \dots \oplus I_{[b\alpha]}b^{d-1}\zeta_{d-1} \\ &= R \oplus I_{[\alpha]}b\zeta_1 \oplus \dots \oplus I_{[\alpha]}b^{d-1}\zeta_{d-1}. \end{aligned}$$

Note that $\zeta_i = \alpha^i + \eta_1 \alpha^{i-1} + \dots + \eta_i$ ($1 \leq i \leq d-1$) by [1]. Since $B_a = B_b$ and $d \geq 2$, we conclude $I_{[\alpha]}b\zeta_1 = I_{[\alpha]}a\zeta_1$, and hence $aI_{[\alpha]} = bI_{[\alpha]}$. Therefore, by Lemma 1, it holds $a/b \in R$, and thus $a = ub$ for some $u \in U(R)$, as desired.

(2) $\implies$ (1): Since $a = ub$ for some $u \in U(R)$, we conclude $I_{[\alpha]}a^i\zeta_i = I_{[\alpha]}b^i\zeta_i$ for all i ($1 \leq i \leq d-1$). Hence $B_a = B_b$ by (*). ■

COROLLARY 1. *Let $S = R - \cup p_i$ ($p_i \in \text{Ass}(R/I_{[\alpha]})$). We define the mapping $\Phi : S \longrightarrow \{B_a \mid a \in S\}$ sending $a \in S$ to B_a . Then $\text{Ker}(\Phi) = S/U(R)$ by Theorem 3, and hence we have the bijection $\overline{\Phi} : S/U(R) \cong \{B_a \mid a \in S\}$.*

LEMMA 4. *Let α be an ultra-primitive element over R . Then, for any $a \in R$, $a\alpha$ is ultra-primitive, therefore $a\alpha$ is anti-integral over R .*

PROOF. Since α is ultra-primitive, then α is anti-integral. Hence α is anti-integral over R_p for any $p \in \text{Dp}_1(R)$ by [5]. If $p \not\supset c(\overline{R}/R)$, then R_p is normal, and hence $a\alpha$ is anti-integral by [3]. If $p \not\supset I_{[\alpha]}$, then α is integral over R_p ([3, Corollary 2.3]), and thus $a\alpha$ is integral over R_p . Note that $a\alpha$ has monic relation of degree d . Therefore $a\alpha$ is anti-integral over R_p . We are done. ■

Next we study the set $\{R[1/a\alpha] \mid a \in R, a\alpha \text{ is anti-integral over } R\}$. Under some conditions, we shall show that the above set correspondent to $R^*/U(R)$ bijectively.

THEOREM 5. *Let R be a Noetherian domain with quotient field K . Let α be a ultra-primitive element of degree d (≥ 3) over R . Let $a, b \in R^*$ and let $\bar{R}$ be the integral closure of R . Suppose that $\bar{R}$ is a finite R -module. Let $B_a = R[a\alpha] \cap R[1/(a\alpha)]$ and $B_b = R[b\alpha] \cap R[1/(b\alpha)]$. Then the following statements are equivalent.*

- (1) $B_a = B_b$.
- (2) $a = ub$ for some $u \in U(R)$.

PROOF. (2) $\implies$ (1): Since α is ultra-primitive over R , $a\alpha$ and $b\alpha$ are anti-integral over R by Lemma 4. The rest of this implication is proved in the same way as the proof of Theorem 3 ((2) $\implies$ (1)).

(1) $\implies$ (2): Since $a\alpha$ and $b\alpha$ are anti-integral by Lemma 4, we have the decompositions

$$B_a = R \oplus I_{[a\alpha]}a\zeta_1 \oplus I_{[a\alpha]}a^2\zeta_2 \oplus \cdots \oplus I_{[a\alpha]}a^{d-1}\zeta_{d-1}$$

$$B_b = R \oplus I_{[b\alpha]}b\zeta_1 \oplus I_{[b\alpha]}b^2\zeta_2 \oplus \cdots \oplus I_{[b\alpha]}b^{d-1}\zeta_{d-1}.$$

Since $d \geq 3$, we have $aI_{[a\alpha]} = bI_{[b\alpha]}$ and $a^2I_{[a\alpha]} = b^2I_{[b\alpha]}$. It follows that $b^2I_{[b\alpha]} = a^2I_{[a\alpha]} = abI_{[b\alpha]}$, and hence $bI_{[b\alpha]} = aI_{[b\alpha]}$. Therefore $a = ub$ for some $u \in U(R)$ by Lemma 1, as desired. ■

COROLLARY 2. *By the above theorem, if α is ultra-primitive, then we have the bijection $\Phi : R^*/U(R) \cong \{ B_a \mid a \in R^* \}$ defined by $\Phi(a \bmod U(R)) = B_a$ for any $a \in R^*$.*

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