An example of a positive definite function which is not of positive type on $\mathbb{Z}^2$

KOJI FURUTA* and NOBUHISA SAKAKIBARA**

Let $S = (S, +, *)$ be an abelian $*$ -semigroup with the identity 0, $\mathcal{H}$ a complex Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ and $B(\mathcal{H})$ the set of bounded linear operators on $\mathcal{H}$. A function $\varphi : S \rightarrow B(\mathcal{H})$ is called of *positive type* if

$$\sum_{i,j=1}^n \langle \varphi(s_i + s_j^*) \xi_i, \xi_j \rangle \geq 0$$

for all $n \geq 1$, $s_1, s_2, \dots, s_n \in S$ and $\xi_1, \xi_2, \dots, \xi_n \in \mathcal{H}$. Moreover, φ is called *positive definite* if

$$\sum_{i,j=1}^n c_i \overline{c_j} \langle \varphi(s_i + s_j^*) \xi, \xi \rangle \geq 0$$

for all $n \geq 1$, $s_1, s_2, \dots, s_n \in S$, $c_1, c_2, \dots, c_n \in \mathbb{C}$ and $\xi \in \mathcal{H}$. Every function of positive type is positive definite, and every scalar-valued, positive definite function is of positive type. But a positive definite function is not necessarily of positive type. In fact, T. M. Bisgaard demonstrated that there exists an explicit example of a positive definite function which is not of positive type on $(\mathbb{N}_0, +, x^* = x)$ where $\mathbb{N}_0 := \{0, 1, 2, \dots\}$ (see [1, Theorem 1]), and we did on $(\mathbb{Z}, +, x^* = x)$ (see [3, Theorem 3.7]). For abelian $*$ -semigroups $(\mathbb{N}_0^2, +, x^* = x)$ and $(\mathbb{Z}^2, +, x^* = x)$, is there such an example? When $(\mathbb{N}_0^2, +, x^* = x)$, the answer is clear because we have the zero extension of Bisgaard's example (i.e. $\varphi(n, 0)$ is Bisgaard's and $\varphi(n, m) = 0$ for $m > 0$). In this paper, we shall show such an explicit example on $(\mathbb{Z}^2, +, x^* = x)$.

REMARK. A function $\varphi : S \rightarrow B(\mathcal{H})$ is called a *operator moment function* if there exists a $B(\mathcal{H})_+$ -valued measure F on S^* such that

$$\langle \varphi(s) \xi, \eta \rangle = \int_{S^*} \rho(s) d\langle F(\rho) \xi, \eta \rangle \quad \text{for } s \in S, \xi, \eta \in \mathcal{H}.$$

Every operator moment function is of positive type. But a function of positive type is not necessarily an operator moment function. In fact, even a scalar-valued, positive definite function is not necessarily a moment function. So an

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* Department of Mathematics, Musashi Institute of Technology, Tamazutsumi, Setagaya, Tokyo 158-8557, Japan.

email: [REDACTED]

** Faculty of Engineering, Ibaraki University, Nakanarusawa 4-12-1, Hitachi 316-8511, Japan.

email: [REDACTED]

abelian $*$ -semigroup S is called *operator semiperfect* (resp. *semiperfect*) if every function of positive type (resp. scalar-valued, positive definite function) on S is a operator moment function (resp. moment function). Operator semiperfect $*$ -semigroups have been analyzed by Bisgaard ([2]), Stochel and Szafraniec ([4]), and the authors ([3]). Semiperfect $*$ -semigroups have been more analyzed by Bisgaard and the authors.

Let us first define a linear order $\ll$ on $\mathbf{Z}^2$ as follows:

$$\begin{aligned} (n_1, m_1) \ll (n_2, m_2) : & \iff \cdot |n_1| + |m_1| < |n_2| + |m_2| \quad \text{or} \\ & \cdot |n_1| + |m_1| = |n_2| + |m_2|, \\ & \qquad \qquad \qquad m_1 > 0, m_2 > 0, n_1 < n_2 \quad \text{or} \\ & \cdot |n_1| + |m_1| = |n_2| + |m_2|, m_1 > 0, m_2 \leq 0 \quad \text{or} \\ & \cdot |n_1| + |m_1| = |n_2| + |m_2|, m_1 \leq 0, m_2 \leq 0, n_2 < n_1. \end{aligned}$$

Arrange points of $\mathbf{Z}^2$ by this order, i.e.

$$\begin{aligned} x_0 &:= (0, 0), & x_1 &:= (0, 1), & x_2 &:= (1, 0), & x_3 &:= (0, -1), \\ x_4 &:= (-1, 0), & x_5 &:= (-1, 1), & x_6 &:= (0, 2), & & \dots \end{aligned}$$

and let $a_n := 2^{(n+2)!}$, $n \geq 1$. The following is our theorem, in which the choices of 2×2 matrices is similar to those in Bisgaard's example.

THEOREM. *Let $\varphi : \mathbf{Z}^2 \rightarrow M_2(\mathbf{C})$ be a function defined by*

$$\begin{aligned} \varphi(x_0) &:= \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}, & \varphi(x_i) &:= \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, & 1 \leq i \leq 5, \\ \varphi(x_6) &:= \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}, & \varphi(x_n) &:= \begin{pmatrix} a_n & 0 \\ 0 & a_n \end{pmatrix}, & n \geq 7. \end{aligned}$$

Then φ is positive definite and not of positive type on $(\mathbf{Z}^2, +, x^ = x)$.*

PROOF. To see that φ is positive definite on $(\mathbf{Z}^2, +, x^* = x)$, we shall prove

$$D_n(\xi) := \begin{vmatrix} \langle \varphi(x_0 + x_0)\xi, \xi \rangle & \cdots & \langle \varphi(x_0 + x_n)\xi, \xi \rangle \\ \vdots & \ddots & \vdots \\ \langle \varphi(x_n + x_0)\xi, \xi \rangle & \cdots & \langle \varphi(x_n + x_n)\xi, \xi \rangle \end{vmatrix} > 0$$

for $n \geq 0$ and $\xi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in \mathbf{C}^2$ with $|\alpha|^2 + |\beta|^2 = 1$. We have

$$\begin{aligned} D_0(\xi) &= 4|\alpha|^2 + |\beta|^2 \geq 1, \\ D_1(\xi) &= (4|\alpha|^2 + |\beta|^2)(|\alpha|^2 + 4|\beta|^2) - 16(\operatorname{Re}(\alpha\bar{\beta}))^2 \\ &= 4|\alpha|^4 + 4|\beta|^4 + 17|\alpha\beta|^2 - 16(\operatorname{Re}(\alpha\bar{\beta}))^2 \\ &\geq 4|\alpha|^4 + 4|\beta|^4 + |\alpha|^2|\beta|^2 \\ &= 4 - 7|\alpha|^2|\beta|^2 \geq 4 - \frac{7}{4} > 1. \end{aligned}$$

Let $n \geq 2$ and suppose that $D_{n-1}(\xi) \geq 1$. We show that $D_n(\xi) \geq 1$. Suppose the following:

(*) For every $n \geq 2$, there exists a natural number $m \geq n$, $m \geq 8$ such that

$$\begin{aligned} \|\varphi(x_n + x_n)\| &= a_m, \\ \|\varphi(x_j + x_k)\| &\leq a_{m-1}, \quad 0 \leq j \leq n, \quad 0 \leq k \leq n-1. \end{aligned}$$

Then

$$\begin{aligned} D_n(\xi) &= \langle \varphi(x_n + x_n)\xi, \xi \rangle D_{n-1}(\xi) \\ &\quad + \sum_{k=0}^{n-1} (-1)^{n+1+k+1} \langle \varphi(x_k + x_n)\xi, \xi \rangle D'_k(\xi) \\ &\geq a_m - na_{m-1}^{n+1}n! \\ &\geq 2^{(m+2)!} - (n+1)!2^{(m+1)!(n+1)} \\ &\geq 2^{(m+1)!(m+1)}(2^{(m+1)!} - (n+1)!) \\ &\geq 2^{(m+1)!} - (m+1)! \\ &\geq 1, \end{aligned}$$

where

$$D'_k(\xi) := \begin{vmatrix} \langle \varphi(x_0 + x_0)\xi, \xi \rangle & \cdots & \langle \varphi(x_0 + x_{n-1})\xi, \xi \rangle \\ \vdots & \ddots & \vdots \\ \langle \varphi(x_{k-1} + x_0)\xi, \xi \rangle & \cdots & \langle \varphi(x_{k-1} + x_{n-1})\xi, \xi \rangle \\ \langle \varphi(x_{k+1} + x_0)\xi, \xi \rangle & \cdots & \langle \varphi(x_{k+1} + x_{n-1})\xi, \xi \rangle \\ \vdots & \ddots & \vdots \\ \langle \varphi(x_n + x_0)\xi, \xi \rangle & \cdots & \langle \varphi(x_n + x_{n-1})\xi, \xi \rangle \end{vmatrix},$$

$k = 0, 1, 2, \dots, n-1$. Thus, by induction, we get $D_n(\xi) \geq 1$ for $n \geq 0$. Suppose that

(**) for every $n \geq 2$, $x_n + x_k \ll x_n + x_n$, $k = 0, 1, 2, \dots, n-1$.

Then we can easily prove (*). Therefore let us prove (**).

Put $x_n = (p, q)$ and $x_k = (r, s)$. Then $x_n + x_n = (2p, 2q)$ and $x_n + x_k = (p+r, q+s)$. In case that $|r| + |s| < |p| + |q|$ or $pr < 0$ or $qs < 0$, it is easily seen that $|p+r| + |q+s| < |2p| + |2q|$. Hence $x_n + x_k \ll x_n + x_n$. Suppose that $|r| + |s| = |p| + |q|$, $pr \geq 0$ and $qs \geq 0$. When $p > 0$ and $q \geq 0$, the condition $x_k \ll x_n$ implies $p > r \geq 0$ and $s \geq 0$. Hence $(p+r) + (q+s) = 2p+2q$, $0 \leq 2q < q+s$ and $p+r < 2p$. Therefore $x_n + x_k \ll x_n + x_n$. When $p \geq 0$ and $q < 0$, we have $r > p \geq 0$ and $0 \geq s > q$. Hence $(p+r) + |q+s| = 2p + |2q|$, $2q < q+s < 0$ and $p+r > 2p$. Therefore $x_n + x_k \ll x_n + x_n$. When $p \leq 0$

and $q > 0$, since $0 \geq p > r$ and $q > s > 0$, we have $|p + r| + (q + s) = |2p| + 2q$, $0 < q + s < 2q$ and $p + r < 2p$. Therefore $x_n + x_k \ll x_n + x_n$. When $p < 0$ and $q \leq 0$, since $0 \geq r > p$ and $s < q \leq 0$, we have $|p + r| + |q + s| = |2p| + |2q|$, $q + s < 2q \leq 0$ and $2p < p + r$. Therefore $x_n + x_k \ll x_n + x_n$. We finished the proof of (**).

Put $\xi_0 := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\xi_1 := \begin{pmatrix} -1 \\ 0 \end{pmatrix}$. Then

$$\sum_{i,j=0}^1 \langle \varphi(x_i + x_j) \xi_i, \xi_j \rangle = -2 < 0,$$

which implies that φ is not of positive type. This completes the proof. ■

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