An example of a positive definite function which is not of positive type on $\mathbb{Z}^2$

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Let $S = (S, +, \cdot, *)$ be an abelian *-semigroup with the identity 0, $\mathcal{H}$ a complex Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ and $B(\mathcal{H})$ the set of bounded linear operators on $\mathcal{H}$. A function $\varphi : S \to B(\mathcal{H})$ is called of positive type if

$$\sum_{i,j=1}^{n} \langle \varphi(s_i + s_j^*) \xi_i, \xi_j \rangle \geq 0$$

for all $n \geq 1$, $s_1, s_2, \ldots, s_n \in S$ and $\xi_1, \xi_2, \ldots, \xi_n \in \mathcal{H}$. Moreover, $\varphi$ is called positive definite if

$$\sum_{i,j=1}^{n} c_i c_j^* \langle \varphi(s_i + s_j^*) \xi_i, \xi_j \rangle \geq 0$$

for all $n \geq 1$, $s_1, s_2, \ldots, s_n \in S$, $c_1, c_2, \ldots, c_n \in \mathbb{C}$ and $\xi \in \mathcal{H}$. Every function of positive type is positive definite, and every scalar-valued, positive definite function is of positive type. But a positive definite function is not necessarily of positive type. In fact, T. M. Bisgaard demonstrated that there exists an explicit example of a positive definite function which is not of positive type on $(\mathbb{N}_0, +, x^* = x)$ where $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ (see [1, Theorem 1]), and we did on $(\mathbb{Z}, +, x^* = x)$ (see [3, Theorem 3.7]). For abelian *-semigroups $(\mathbb{N}_0^2, +, x^* = x)$ and $(\mathbb{Z}^2, +, x^* = x)$, is there such an example? When $(\mathbb{N}_0^2, +, x^* = x)$, the answer is clear because we have the zero extension of Bisgaard's example (i.e. $\varphi(n, 0)$ is Bisgaard's and $\varphi(n, m) = 0$ for $m > 0$). In this paper, we shall show such an explicit example on $(\mathbb{Z}^2, +, x^* = x)$.

REMARK. A function $\varphi : S \to B(\mathcal{H})$ is called an operator moment function if there exists a $B(\mathcal{H})$-valued measure $F$ on $S^*$ such that

$$\langle \varphi(s) \xi, \eta \rangle = \int_{S^*} \rho(s) d\langle F(\rho) \xi, \eta \rangle \quad \text{for } s \in S, \xi, \eta \in \mathcal{H}.$$  

Every operator moment function is of positive type. But a function of positive type is not necessarily an operator moment function. In fact, even a scalar-valued, positive definite function is not necessarily a moment function. So an
abelian \(*\)-semigroup \(S\) is called operator semiperfect (resp. semiperfect) if every function of positive type (resp. scalar-valued, positive definite function) on \(S\) is a operator moment function (resp. moment function). Operator semiperfect \(*\)-semigroups have been analyzed by Bisgaard ([2]), Stochel and Szafraniec ([4]), and the authors ([3]). Semiperfect \(*\)-semigroups have been more analyzed by Bisgaard and the authors.

Let us first define a linear order \(\ll\) on \(\mathbb{Z}^2\) as follows:

\[
(n_1, m_1) \ll (n_2, m_2) :\iff |n_1| + |m_1| < |n_2| + |m_2| \quad \text{or} \\
|n_1| + |m_1| = |n_2| + |m_2|, \\
\quad m_1 > 0, m_2 > 0, n_1 < n_2 \quad \text{or} \\
|n_1| + |m_1| = |n_2| + |m_2|, m_1 > 0, m_2 \leq 0 \quad \text{or} \\
|n_1| + |m_1| = |n_2| + |m_2|, m_1 \leq 0, m_2 \leq 0, n_2 < n_1.
\]

Arrange points of \(\mathbb{Z}^2\) by this order, i.e.

\[
x_0 := (0,0), \quad x_1 := (0,1), \quad x_2 := (1,0), \quad x_3 := (0,-1), \\
x_4 := (-1,0), \quad x_5 := (-1,1), \quad x_6 := (0,2), \quad \ldots
\]

and let \(a_n := 2^{(n+2)n}, n \geq 1\). The following is our theorem, in which the choices of \(2 \times 2\) matrices is similar to those in Bisgaard’s example.

**Theorem.** Let \(\varphi : \mathbb{Z}^2 \to M_2(\mathbb{C})\) be a function defined by

\[
\varphi(x_0) := \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}, \quad \varphi(x_i) := \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \quad 1 \leq i \leq 5, \\
\varphi(x_6) := \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}, \quad \varphi(x_n) := \begin{pmatrix} a_n & 0 \\ 0 & a_n \end{pmatrix}, \quad n \geq 7.
\]

Then \(\varphi\) is positive definite and not of positive type on \(\mathbb{Z}^2, +, x^* = x\).

**Proof.** To see that \(\varphi\) is positive definite on \((\mathbb{Z}^2, +, x^* = x)\), we shall prove

\[
D_n(\xi) := \begin{vmatrix} \langle \varphi(x_0 + x_0)\xi, \xi \rangle & \cdots & \langle \varphi(x_0 + x_n)\xi, \xi \rangle \\ \vdots & \ddots & \vdots \\ \langle \varphi(x_n + x_0)\xi, \xi \rangle & \cdots & \langle \varphi(x_n + x_n)\xi, \xi \rangle \end{vmatrix} > 0
\]

for \(n \geq 0\) and \(\xi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in \mathbb{C}^2\) with \(|\alpha|^2 + |\beta|^2 = 1\). We have

\[
D_0(\xi) = 4|\alpha|^2 + |\beta|^2 \geq 1,
\]

\[
D_1(\xi) = (4|\alpha|^2 + |\beta|^2)(|\alpha|^2 + 4|\beta|^2) - 16(\text{Re}(\alpha\overline{\beta}))^2
\]

\[
= 4|\alpha|^4 + 4|\beta|^4 + 17|\alpha\beta|^2 - 16(\text{Re}(\alpha\overline{\beta}))^2
\]

\[
\geq 4|\alpha|^4 + 4|\beta|^4 + |\alpha|^2|\beta|^2
\]

\[
= 4 - 7|\alpha|^2|\beta|^2 \geq 4 - \frac{7}{4} > 1.
\]
Let \( n \geq 2 \) and suppose that \( D_{n-1}(\xi) \geq 1 \). We show that \( D_n(\xi) \geq 1 \). Suppose the following:

\( (*) \) For every \( n \geq 2 \), there exists a natural number \( m \geq n, m \geq 8 \) such that

\[
\begin{align*}
||\varphi(x_n + x_n)|| &= a_m, \\
||\varphi(x_j + x_k)|| &\leq a_{m-1}, \quad 0 \leq j \leq n, \quad 0 \leq k \leq n-1.
\end{align*}
\]

Then

\[
D_n(\xi) = \langle \varphi(x_n + x_n), \xi \rangle D_{n-1}(\xi) \\
\quad + \sum_{k=0}^{n-1} (-1)^{n+1+k+1} \langle \varphi(x_k + x_n), \xi \rangle D'_k(\xi)
\]

\[
\geq a_m - n a_{m-1} n!
\]

\[
\geq 2^{(m+2)!} - (n + 1)! 2^{(m+1)!} (n+1)
\]

\[
\geq 2^{(m+1)!} (m+1)! (2^{(m+1)!} - (n + 1) !)
\]

\[
\geq 2^{(m+1)!} - (m+1) !
\]

\[
\geq 1,
\]

where

\[
D'_k(\xi) := \begin{vmatrix}
\langle \varphi(x_0 + x_0), \xi \rangle & \cdots & \langle \varphi(x_0 + x_{n-1}), \xi \rangle \\
\vdots & \ddots & \vdots \\
\langle \varphi(x_{k-1} + x_0), \xi \rangle & \cdots & \langle \varphi(x_{k-1} + x_{n-1}), \xi \rangle \\
\langle \varphi(x_{k+1} + x_0), \xi \rangle & \cdots & \langle \varphi(x_{k+1} + x_{n-1}), \xi \rangle \\
\vdots & \ddots & \vdots \\
\langle \varphi(x_n + x_0), \xi \rangle & \cdots & \langle \varphi(x_n + x_{n-1}), \xi \rangle
\end{vmatrix},
\]

\( k = 0, 1, 2, \ldots, n-1 \). Thus, by induction, we get \( D_n(\xi) \geq 1 \) for \( n \geq 0 \). Suppose that

\( (**) \) for every \( n \geq 2 \), \( x_n + x_k \ll x_n + x_n \), \( k = 0, 1, 2, \ldots, n-1 \).

Then we can easily prove \( (*) \). Therefore let us prove \( (**) \).

Put \( x_n = (p, q) \) and \( x_k = (r, s) \). Then \( x_n + x_n = (2p, 2q) \) and \( x_n + x_k = (p + r, q + s) \). In case that \( |r| + |s| < |p| + |q| \) or \( pr < 0 \) or \( qs < 0 \), it is easily seen that \( p + r + q + s < 2p + 2q \). Hence \( x_n + x_k \ll x_n + x_n \). Suppose that \( |r| + |s| = |p| + |q|, \ pr \geq 0 \) and \( qs \geq 0 \). When \( p \geq 0 \) and \( q \geq 0 \), the condition \( x_k \ll x_n \) implies \( p > r \geq 0 \) and \( s \geq 0 \). Hence \( (p + r) + (q + s) = 2p + 2q, 0 \leq 2q < q + s \) and \( p + r < 2p \). Therefore \( x_n + x_k \ll x_n + x_n \). When \( p \geq 0 \) and \( q < 0 \), we have \( r > p \geq 0 \) and \( 0 \geq s > q \). Hence \( (p + r) + (q + s) = 2p + 2q \), \( 2q < q + s < 0 \) and \( p + r > 2p \). Therefore \( x_n + x_k \ll x_n + x_n \). When \( p \leq 0 \)}
and \( q > 0 \), since \( 0 \leq p > r \) and \( q > s > 0 \), we have \( |p + r| + (q + s) = |2p| + 2q \), \( 0 < q + s < 2q \) and \( p + r < 2p \). Therefore \( x_n + x_k \ll x_n + x_n \). When \( p < 0 \) and \( q \leq 0 \), since \( 0 \geq r > p \) and \( s < q \leq 0 \), we have \( |p + r| + |q + s| = |2p| + |2q| \), \( q + s < 2q \leq 0 \) and \( 2p < p + r \). Therefore \( x_n + x_k \ll x_n + x_n \). We finished the proof of (**).

Put \( \xi_0 := \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) and \( \xi_1 := \begin{pmatrix} -1 \\ 0 \end{pmatrix} \). Then

\[
\sum_{i,j=0}^{1} (\varphi(x_i + x_j)\xi_i, \xi_j) = -2 < 0,
\]

which implies that \( \varphi \) is not of positive type. This completes the proof.

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**References**


