On Some Results Deduced From the Equality

$$R[1/\alpha] = R[1/(\alpha - a)]$$

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Introduction.

Throughout this paper, all rings and algebras are assumed to be commutative and have identity. Let R be a Noetherian domain with quotient field K and let L be a finite algebraic field-extension of K . Let α be a non-zero element of L . Let $\pi : R[X] \rightarrow R[\alpha]$ be the R -algebra homomorphism sending X to α . Let $\varphi_\alpha(X)$ be the monic minimal polynomial of α over K with $\deg \varphi_\alpha(X) = d$ and $\varphi_\alpha(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d$. Put $I_{[\alpha]} = \cap_{i=1}^d (R :_R \eta_i)$. When $\alpha \in K$, $I_{[\alpha]} = (R :_R \alpha) =: I_\alpha$ because $\varphi_\alpha(X) = X - \alpha$. The element α is called an *anti-integral element* of degree d over R if $\text{Ker}(\pi) = I_{[\alpha]} \varphi_\alpha(X) R[X]$. When α is an anti-integral element over R , $R[\alpha]$ is called an *anti-integral extension* of R .

Assume that α is an anti-integral element over R of degree d . It is known that $R[\alpha] \cap R[\frac{1}{\alpha}]$ is integral over R (cf. [3, Exercise 4, p. 12]) and that $R[\alpha] \cap R[\frac{1}{\alpha}] = R \oplus I_{[\alpha]} \zeta_1 \oplus \cdots \oplus I_{[\alpha]} \zeta_{d-1}$, where $\zeta_i = \alpha^i + \eta_1 \alpha^{i-1} + \cdots + \eta_i$ ([5, Lemma 1.2], [2, Theorem 1] and [2, Example p.4503]).

In Ratliff-Mirbagheri [4], it is shown that if a ring satisfies $R[\alpha] \cap R[\frac{1}{\alpha}] = R$ with $\alpha \in K$, then $\text{Ker}(\pi)$ is generated by linear polynomials, that is, $\text{Ker}(\pi) = I_{[\alpha]}(X - \alpha)R[X]$. So if a ring satisfies $R[\alpha] \cap R[\frac{1}{\alpha}] = R$, then α is an anti-integral element over R of degree one (cf. [5, Remark 1.6 i)] and [4, Theorem 2.5]).

N. Onoda and T. Sugatani asked the first author when the equality $R[\alpha] \cap R[\frac{1}{\alpha}] = R[\alpha - a] \cap R[\frac{1}{\alpha - a}]$ is valid for all $a \in R$ ($\alpha \neq a$), and they showed him an affirmative answer in the case $d \leq 4$. The case $d > 5$ has remained unsettled. If $R[\frac{1}{\alpha - a}] = R[\frac{1}{\alpha}]$ for $a \in R$ with $a \neq \alpha$, then $R[\alpha - a] \cap R[\frac{1}{\alpha - a}] = R[\alpha] \cap R[\frac{1}{\alpha}]$. Hence it seems natural to ask what can be shown if the equality $R[\frac{1}{\alpha - a}] = R[\frac{1}{\alpha - b}]$ holds for $a, b \in R$.

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The condition $R[\frac{1}{\alpha-a}] = R[\frac{1}{\alpha-b}]$ for $a, b \in R$ is equivalent to $R[\frac{1}{\beta}] = R[\frac{1}{\beta-(b-a)}]$, where $\beta = \alpha - a$. Note that if α is anti-integral over R then so is β (cf. Proposition 3).

Our objective of this paper is to show some results deduced from the equality $R[\frac{1}{\alpha}] = R[\frac{1}{\alpha-a}]$.

Let $J_{[\alpha]} = I_{[\alpha]}c(\varphi_\alpha(X))$, where $c(\varphi_\alpha(X))$ denotes the fractional ideal in K generated by all coefficients $1, \eta_1, \dots, \eta_d$ of $\varphi_\alpha(X)$. When α is an anti-integral element over R , it is known that $R[\alpha]$ is flat over R if and only if $J_{[\alpha]} = R$ ([5, Proposition 2.6]). From this result, when α is an anti-integral element over R , we show an extension $R[\alpha]$ is flat over R if $R[\alpha, \frac{1}{\alpha}] = R[\alpha, \frac{1}{\alpha-a}]$ and $\sqrt{\alpha R[\alpha]} = \sqrt{a R[\alpha]}$ (cf. Proposition 9). We also give a condition which is equivalent to $R[\alpha, \frac{1}{\alpha}] = R[\alpha, \frac{1}{\alpha-a}]$ (Theorem 8).

Symbols and Conventions. We fix the following symbols and conventions unless otherwise specified:

R : a Noetherian integral domain

K : the quotient field of R

L : an algebraic extension of K

α : a non-zero element of L

$d := [K(\alpha) : K]$

$A := R[\alpha]$

$B := R[\alpha] \cap R[\frac{1}{\alpha}]$

$\varphi_\alpha(X) := X^d + \eta_1 X^{d-1} + \dots + \eta_d$, the minimal polynomial of α over K

$\pi : R[X] \longrightarrow R[\alpha]$, the R -algebra homomorphism sending X to α

$I_{[\alpha]} := \cap_{i=1}^d (R :_R \eta_i)$

$J_{[\alpha]} := I_{[\alpha]}c(\varphi_\alpha(X))$, where $c(\varphi_\alpha(X))$ denotes the fractional ideal in K generated by all coefficients $1, \eta_1, \dots, \eta_d$ of $\varphi_\alpha(X)$.

$I_\alpha^B := \{b \in B \mid b\alpha \in B\}$

$J_\alpha^B := I_\alpha^B + \alpha I_\alpha^B$

$I_a := (R :_R aR)$ for $a \in K$ and $J_a = I_a + aI_a$.

$\zeta_i = \alpha^i + \eta_1 \alpha^{i-1} + \dots + \eta_i$

$\sqrt{I}$: a radical of an ideal I

$\text{grade}_R(I)$: the grade of an ideal I of R

For an integral domain D , let $U(D)$ denote the set of the units in D .

Assume that $\alpha \in K$ and that α is an anti-integral element over R . Let $a \in R$ with $a \neq \alpha$. We assume that $R[\frac{1}{\alpha}] = R[\frac{1}{\alpha-a}]$. Then there exists $x_i \in R$ such that $\frac{1}{\alpha} = x_0 + x_1(\frac{1}{\alpha-a}) + x_2(\frac{1}{\alpha-a})^2 + \dots + x_n(\frac{1}{\alpha-a})^n$. Multiplying this expression by $\alpha - a$, we get that $\frac{\alpha-a}{\alpha} = x_0(\alpha - a) + x_1 + x_2(\frac{1}{\alpha-a}) + \dots + x_n(\frac{1}{\alpha-a})^{n-1}$, and so $1 - \frac{a}{\alpha} - (x_2(\frac{1}{\alpha-a}) + \dots + x_n(\frac{1}{\alpha-a})^{n-1}) = x_0(\alpha - a) + x_1 \in R[\alpha] \cap R[\frac{1}{\alpha}] = R$. Set $x'_1 = x_0(\alpha - a) + x_1 \in R$. $\frac{1}{\alpha} = x'_1(\frac{1}{\alpha-a}) + x_2(\frac{1}{\alpha-a})^2 + \dots + x_n(\frac{1}{\alpha-a})^n$. Thus it is enough to assume that $x_0 = 0$, $x_1 = x'_1$. Therefore $\frac{1}{\alpha} = x_1(\frac{1}{\alpha-a}) + x_2(\frac{1}{\alpha-a})^2 + \dots + x_n(\frac{1}{\alpha-a})^n$. Again, multiplying this expression by $\alpha - a$, we get

that $1 - \frac{a}{\alpha} = x_1 + x_2\left(\frac{1}{\alpha-a}\right) + x_3\left(\frac{1}{\alpha-a}\right)^2 + \cdots + x_n\left(\frac{1}{\alpha-a}\right)^{n-1}$, that is,

$$(1) \quad \frac{a}{\alpha} = (1 - x_1) + (-x_2)\left(\frac{1}{\alpha-a}\right) + (-x_3)\left(\frac{1}{\alpha-a}\right)^2 + \cdots + (-x_n)\left(\frac{1}{\alpha-a}\right)^{n-1}$$

Multiplying the expression (1) by $\alpha - a$, we have

$$a(1 - \frac{a}{\alpha}) = (1 - x_1)(\alpha - a) + (-x_2) + (-x_3)\left(\frac{1}{\alpha-a}\right) + \cdots + (-x_n)\left(\frac{1}{\alpha-a}\right)^{n-2}$$

and

$$\frac{a^2}{\alpha} = (x_1 - 1)(\alpha - a) + a + x_2 + x_3\left(\frac{1}{\alpha-a}\right) + \cdots + x_n\left(\frac{1}{\alpha-a}\right)^{n-2}.$$

So we have

$$(x_1 - 1)(\alpha - a) + a + x_2 = \frac{a^2}{\alpha} - x_3\left(\frac{1}{\alpha-a}\right) - \cdots - x_n\left(\frac{1}{\alpha-a}\right)^{n-2} \in R[\alpha] \cap R\left[\frac{1}{\alpha}\right] = R.$$

Replacing left side by x_2 , we have the following expression :

$$\frac{a^2}{\alpha} = x_2 + x_3\left(\frac{1}{\alpha-a}\right) + x_4\left(\frac{1}{\alpha-a}\right)^2 + \cdots + x_n\left(\frac{1}{\alpha-a}\right)^{n-2}.$$

Continuing this process, it follows that

$$(2) \quad \frac{a^{n-1}}{\alpha} = x_{n-1} + x_n\left(\frac{1}{\alpha-a}\right).$$

Multiplying the expression (2) by $\alpha - a$, we have $a^{n-1}(1 - \frac{a}{\alpha}) = x_{n-1}(\alpha - a) + x_n \in R[\frac{1}{\alpha}] \cap R[\alpha] = R$, since α is anti-integral over R . Set $x = x_{n-1}(\alpha - a) + x_n$. Then $\frac{a^{n-1}(\alpha - a)}{\alpha} = x$, and so we give

$$(3) \quad \frac{a^{n-1}}{\alpha} = \frac{x}{\alpha - a}.$$

Now, there exists $y_i \in R$ ($i = 1, 2, \dots, n$) such that $\frac{1}{\alpha-a} = y_0 + y_1\left(\frac{1}{\alpha}\right) + \cdots + y_m\left(\frac{1}{\alpha}\right)^m$. Put $\beta = \alpha - a$. Then $\alpha = \beta + a$. Hence we have that $\frac{1}{\beta} = y_0 + y_1\left(\frac{1}{\beta+a}\right) + \cdots + y_m\left(\frac{1}{\beta+a}\right)^m$. Therefore

$$(4) \quad \frac{a^{m-1}}{\alpha - a} = \frac{y}{\alpha} \quad \text{where } y \in R.$$

By (3) and (4), we get that $a^{n-1}(\alpha - a) = x\alpha$, $a^{n-1}\alpha - a^n = x\alpha$. Therefore

$$(5) \quad (a^{n-1} - x)\alpha = a^n, \quad \alpha = \frac{a^n}{a^{n-1} - x}.$$

Also, we have that $a^{m-1}\alpha = y(\alpha - a)$, $a^{m-1}\alpha = y\alpha - ay$. Thus we have that $(a^{m-1} - y)\alpha = -ay$. Hence

$$(6) \quad \alpha = \frac{-ay}{a^{m-1} - y}.$$

By (5) and (6), we have that $\frac{a^n}{a^{n-1}-x} = \frac{-ay}{a^{m-1}-y}$. Therefore $a^{n+m-1} - a^n y = -a^n y + axy$. Hence

$$(7) \quad xy = a^{n+m-2}.$$

From the above argument, we have the following result.

Proposition 1. *Assume that $\alpha \in K$, and let a be an element in R with $a \neq \alpha$. Assume also that α is an anti-integral element over R ($\alpha \neq 0$). If $R[\frac{1}{\alpha}] = R[\frac{1}{\alpha-a}]$, then there exist positive integers n , m and x , $y \in R$ such that*

$$xy = a^{n+m-2} \quad \text{and} \quad \alpha = \frac{a^n}{a^{n-1} - x} = \frac{ay}{y - a^{m-1}}.$$

Lemma 2. *Let $a \in R$. Then $I_{[\alpha]} = I_{[\alpha-a]}$.*

Proof. $\varphi_\alpha(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d$ ($\eta_i \in K$) be the minimal polynomial of α over K . Set $X = Y + a$. Then $\varphi_{\alpha-a}(Y) = Y^d + \eta'_1 Y^{d-1} + \cdots + \eta'_d$ and each η'_i is an R -linear combination of $\eta_1, \eta_2, \dots, \eta_d$. Therefore $I_{[\alpha]} \subset I_{[\alpha-a]}$. Put $\beta = \alpha - a$. Let $\varphi_\beta(X) = X^d + \xi_1 X^{d-1} + \cdots + \xi_d$ be the minimal polynomial of β over K . Set $X = Y - a$. Then $\varphi_\alpha(Y) = Y^d + \xi'_1 Y^{d-1} + \cdots + \xi'_d$ where each ξ'_i is an R -linear combination of $\xi_1, \dots, \xi_d$. Since $\xi'_1, \xi'_2, \dots, \xi'_d$ is an R -linear combination of $\xi_1, \dots, \xi_d$, we have that $I_{[\alpha]} = I_{[\alpha-a]}$. $\square$

Proposition 3. *Let $a \in R$. If α is an anti-integral element of degree d , then $\alpha - a$ is also an anti-integral element of degree d over R .*

Proof. Assume that $f(Y) \in R[Y]$ such that $f(\alpha - a) = 0$ and $f(Y) = a_n Y^n + a_{n-1} Y^{n-1} + \cdots + a_0$ ($a_i \in R$), $X = Y + a$ and $g(X) = f(Y - a)$. Since $g(\alpha) = 0$, we have that $g(X) = h(X)\varphi_\alpha(X)$ where $h(X) \in I_{[\alpha]}R[X]$. Then $f(X) = g(Y + a) = h(Y + a)\varphi_\alpha(Y + a) = h(Y + a)\varphi_{\alpha-a}(Y)$ where $h(Y + a) \in I_{[\alpha]}R[Y]$. By Lemma 2, $h(Y + a) \in I_{[\alpha-a]}R[Y]$. Therefore $f(Y) \in I_{[\alpha-a]}\varphi_{\alpha-a}(Y)R[Y]$. Therefore $\text{Ker}(\pi) \subset I_{[\alpha-a]}\varphi_{\alpha-a}(X)R[X]$. $\square$

Remark. Assume that an element α is anti-integral of degree d over R . Set $\zeta_i := \alpha^i + \eta_1 \alpha^{i-1} + \cdots + \eta_i$.

- (i) It is known that $B = R + I_{[\alpha]}\zeta_1 \oplus \cdots \oplus I_{[\alpha]}\zeta_{d-1}$ by [2, Theorem 1].
- (ii) Let $a \in R$ with $a \neq \alpha$ and assume that $R[\frac{1}{\alpha}] = R[\frac{1}{\alpha-a}]$. Then $R[\alpha - a] \cap R[\frac{1}{\alpha-a}] = R[\alpha] \cap R[\frac{1}{\alpha}] = B$. Hence $R[\frac{1}{\alpha}] = B[\frac{1}{\alpha}] = B[\frac{1}{\alpha-a}]$.
- (iii) α is integral over R if and only if $\alpha \in B$. For, assume that α is integral over R . Then we have $\alpha^n + a_1 \alpha^{n-1} + \cdots + a_n = 0$ for some n , where $a_i \in R$. Therefore $\alpha \in R[\frac{1}{\alpha}]$ and so $\alpha \in B = R[\alpha] \cap R[\frac{1}{\alpha}]$. Conversely, assume that $\alpha \in B$. Since B is integral over R , α is integral over R .

Theorem 4. Let $\alpha \in K$. Assume that α is an anti-integral element over R and $a \in R$ with $a \neq \alpha$. If $R[\frac{1}{\alpha}] = R[\frac{1}{\alpha-a}]$, there exists a unit u of $R[\frac{1}{\alpha}]$ such that $\frac{1}{\alpha} = \frac{u}{\alpha-a}$, that is, $a = (1-u)\alpha$.

Proof. Using the same manner in the proof of Proposition 1, we get that $\frac{1}{\alpha} = \frac{x}{a^n-1} = \frac{1}{\alpha-a}$ for some $x \in R$. Therefore $\frac{\alpha-a}{\alpha} = 1 - \frac{a}{\alpha} = \frac{ax}{a^n-1} \in R[\frac{1}{\alpha-a}] = R[\frac{1}{\alpha}]$. Since $\frac{\alpha}{\alpha-a}, \frac{\alpha-a}{\alpha} \in R[\frac{1}{\alpha}]$, we have that $\frac{\alpha-a}{\alpha} = u \in U(R[\frac{1}{\alpha}])$. Hence $\frac{1}{\alpha} = \frac{u}{\alpha-a}$. $\square$

If $R[\frac{1}{\alpha}] = R[\frac{1}{\alpha-a}]$, then $B[\frac{1}{\alpha}] = B[\frac{1}{\alpha-a}]$. Note that $B[\frac{1}{\alpha}]$ and B are birational.

Theorem 5. Assume that α is an anti-integral element of degree d over R . Let $a \in R$ with $a \neq \alpha$. If $R[\frac{1}{\alpha}] = R[\frac{1}{\alpha-a}]$, there exists a unit γ in $R[\frac{1}{\alpha}]$ such that $\frac{1}{\alpha} = \frac{\gamma}{\alpha-a}$, that is, $a = (1-\gamma)\alpha$.

Proof. If $R[\frac{1}{\alpha}] = R[\frac{1}{\alpha-a}]$, then $B = R[\alpha] \cap R[\frac{1}{\alpha-a}]$. So $B[\frac{1}{\alpha}] = B[\frac{1}{\alpha-a}]$. Therefore α is an anti-integral element over B . Since $\alpha \in K[\alpha]$ ($K[\alpha]$ is the quotient field of B), we can apply the Theorem 4 to B instead of R . $\square$

Lemma 6. Let $f \in B$. Then $f\alpha \in R$ if and only if $f = x\zeta_{d-1}$ for some $x \in I_{[\alpha]}$.

Proof. ($\Leftarrow$) Since $\zeta_{d-1} = \alpha^{d-1} + \eta_1\alpha^{d-2} + \cdots + \eta_{d-1}$, we have that $\alpha\zeta_{d-1} = (\alpha^d + \eta_1\alpha^{d-1} + \cdots + \eta_{d-1}\alpha + \eta_d) - \eta_d = -\eta_d$. Therefore $f\alpha = x\zeta_{d-1}\alpha = x(\alpha\zeta_{d-1}) = -x\eta_d \in R$ because $x \in I_{[\alpha]}$.

($\Rightarrow$) Since $f = f_0 + f_1\zeta_1 + \cdots + f_{d-1}\zeta_{d-1}$ where $f_i \in I_{[\alpha]}$ ($i \geq 1$) and $f_0 \in R$, we have $f\alpha = (f_0\alpha) + (f_1\zeta_1\alpha) + \cdots + (f_{d-1}\zeta_{d-1}\alpha) = f_0\zeta_1 + f_1\zeta_2 + \cdots + f_{d-2}\zeta_{d-1} - (f_0\eta_1 + f_1\eta_2 + \cdots + f_{d-1}\eta_d)$. Therefore $f\alpha \in R$ if and only if $f_0 = f_1 = \cdots = f_{d-2} = 0$ by Remark (i) above. Hence $f = x\zeta_{d-1}$ ($x \in I_{[\alpha]}$). $\square$

Remark. Assume that α is an anti-integral element of degree d over R and that $R[\frac{1}{\alpha}] = R[\frac{1}{\alpha-a}]$. If $U(R[\frac{1}{\alpha}]) = U(B)$, then there exists a unit γ in $R[\frac{1}{\alpha}]$ such that $(1-\gamma)\alpha = a \in R$ and $\gamma \in (1 + I_{[\alpha]}\zeta_{d-1}) \cap U(B)$.

In fact, assume that $U(R[\frac{1}{\alpha}]) = U(B)$. Since $\gamma \in B$ and $(1-\gamma)\alpha = a$ in Theorem 5, we have that $1-\gamma = x\zeta_{d-1}$ where $x \in I_{[\alpha]}$ by Lemma 6. Therefore $\gamma = 1 - x\zeta_{d-1} \in (1 + I_{[\alpha]}\zeta_{d-1}) \cap U(B)$.

The following proposition will be required later.

Proposition 7. Let D be an integral domain. Assume that $x, a \in D$ with $x \neq a$. Then the following conditions are equivalent :

- (1) $D[\frac{1}{x}] = D[\frac{1}{x-a}]$;
- (2) $\sqrt{xD} = \sqrt{(x-a)D}$.

Proof. (1) $\Rightarrow$ (2). Let $p \in \text{Spec}(D)$ and $p \not\ni x$. Then $\frac{1}{x} \in D_p$ and so $D_p \supset D[\frac{1}{x}] = D[\frac{1}{x-a}]$. Therefore $x-a \notin p$. Conversely, if $x-a \notin p$, then $D_p \supset D[\frac{1}{x-a}] = D[\frac{1}{x}]$. Hence $x \notin p$. Consequently $p \ni x$ if and only if $p \ni x-a$ for $p \in \text{Spec}(D)$. Hence $\sqrt{xD} = \sqrt{(x-a)D}$.

(2) $\Rightarrow$ (1). Since $D[\frac{1}{x}] = \cap D_p$ where $p \in \text{Spec}(D)$ satisfying $x \notin p$, this ring is $D[\frac{1}{x-a}]$ because $\sqrt{xD} = \sqrt{(x-a)D}$. Therefore (1) is valid. $\square$

Remark. Assume that $x \in R$ and $a \in R$. If $\sqrt{xR} \supset \sqrt{(x-a)R}$, then $\sqrt{xR} \supset \sqrt{aR}$. In fact, if $p \ni x$ then $p \ni x-a$ and so $a \in p$.

Theorem 8. Assume that α is an anti-integral element of degree d over R . Let $a \in R$ with $a \neq \alpha$. Consider the following statements :

- (1) $R[\alpha, \frac{1}{\alpha}] = R[\alpha, \frac{1}{\alpha-a}]$;
- (2) $\sqrt{\alpha A} = \sqrt{(\alpha-a)A}$;
- (3) $\sqrt{\alpha A} \supset \sqrt{aA}$;
- (4) $a \in \sqrt{I_{[\alpha^{-1}]}}$.

Then the implications $(1) \Leftrightarrow (2) \Rightarrow (3) \Leftrightarrow (4)$ hold.

Proof. $(1) \Leftrightarrow (2)$: $R[\alpha, \frac{1}{\alpha}] = R[\alpha, \frac{1}{\alpha-a}]$ if and only if $A[\frac{1}{\alpha}] = A[\frac{1}{\alpha-a}]$. Apply Proposition 7 to the case $D = A$, and $x = \alpha$, we have that $A[\frac{1}{\alpha}] = A[\frac{1}{\alpha-a}]$ is equivalent to $\sqrt{\alpha A} = \sqrt{(\alpha-a)A}$.

$(2) \Rightarrow (3)$ is trivial.

$(3) \Rightarrow (4)$: There exists a positive integer n such that $a^n \in \alpha A$. Thus it follows that

$$a^n = \alpha(x_0 + x_1\alpha + \cdots + x_m\alpha^m)$$

for each $x_i \in R$. Put $f(X) = x_m X^{m+1} + \cdots + x_1 X^2 + x_0 X - a^n$. Then $f(X) \in \text{Ker } \pi$. Since α is anti-integral over R , we have $a^n \in I_{[\alpha^{-1}]}$. [In fact, since $f(X) \in \text{Ker } \pi = I_{[\alpha]}\varphi_\alpha(X)R[X]$, we have $f(X) = \varphi_\alpha(X)(\sum x_i g_i(X))$, where each $x_i \in I_{[\alpha]}$. Put $g_i(0) =: g_i \in R$. By [1, Lemma 3 (2)], $-a^n = \eta_d(\sum x_i g_i) \in \eta_d I_{[\alpha]} = I_{[\alpha^{-1}]}$.] Hence $a \in \sqrt{I_{[\alpha^{-1}]}}$.

$(4) \Rightarrow (3)$: If $a^n \in I_{[\alpha^{-1}]}$, then $1 + \eta_d^{-1}\eta_{d-1}\alpha + \cdots + \eta_d^{-1}\alpha^d = 0$. So $a^n + (a^n\eta_d^{-1}\eta_{d-1})\alpha + \cdots + (a^n\eta_d^{-1})\alpha^d = 0$. Also, $(\frac{1}{\alpha})^d + (\eta_d^{-1}\eta_{d-1})(\frac{1}{\alpha})^{d-1} + \cdots + \eta_d^{-1} = 0$. Since $a^n \in I_{[\alpha^{-1}]}$, we have that $a^n\eta_d^{-1}\eta_i \in R$. Therefore $a^n \in \alpha A$. $\square$

Remark. Assume that $R[\frac{1}{\alpha}] = R[\frac{1}{\alpha-a}]$ with $a \in R, a \neq \alpha$. Then $\sqrt{\alpha A} = \sqrt{aA}$ does not necessarily hold. In fact, we considered that $a = 0$.

Proposition 9. Assume that α is an anti-integral element of degree d over R . Let $a \in R$ with $a \neq \alpha$. If $R[\alpha, \frac{1}{\alpha}] = R[\alpha, \frac{1}{\alpha-a}]$ and $\sqrt{\alpha A} = \sqrt{aA}$, then $J_{[\alpha]} = R$ and hence A is flat over R .

Proof. Suppose that $J_{[\alpha]} \neq R$. Since $\sqrt{\alpha A} = \sqrt{aA}$, we have that $\alpha^n = ax_0 + ax_1\alpha + \cdots + ax_m\alpha^m$ for each $x_i \in R$. Note that $A/J_{[\alpha]}A \cong (R/J_{[\alpha]})[T]$, a polynomial ring by [5]. Let $\bar{\alpha}$ denote the rssidue class of α in $A/J_{[\alpha]}A$. Then $\bar{\alpha}$ is algebraically independent over $R/J_{[\alpha]}$ and $A/J_{[\alpha]}A = (R/J_{[\alpha]})[\bar{\alpha}]$. We have that

$$\bar{\alpha}^n = (\overline{a_0x_0}) + (\overline{ax_1})\bar{\alpha} + \cdots + (\overline{ax_m})\bar{\alpha}^m.$$

Hence $\overline{ax_n} = 1$, that is, $1 - ax_n \in J_{[\alpha]}$. On the other hand, by Theorem 8 and [1, Lemma 3 (2)], we have that $a \in \sqrt{I_{[\alpha^{-1}]}} = \sqrt{\eta_d I_{[\alpha]}} \subset \sqrt{J_{[\alpha]}}$. This contradicts to the fact that $1 \in \sqrt{J_{[\alpha]}}$. Therefore $J_{[\alpha]} = R$. $\square$

Example. Let k be a field and x, y be two variables over k . Set $R = k[x, y, \frac{1}{1-x}]$, $\alpha = \frac{y}{x}$ and $A = R[\alpha] = k[x, \frac{1}{1-x}, \alpha]$. Then $\alpha A \ni y$ and $\alpha - y = \frac{y-x}{x} = (1-x)\alpha$. Since $1-x$ is unit in R , we have that $\sqrt{(\alpha-y)A} = \sqrt{\alpha A}$. By Theorem 8, it follows that $R[\alpha, \frac{1}{\alpha}] = R[\alpha, \frac{1}{\alpha-y}]$. But $J_{[\alpha]} = (x, y)R$. In fact, x, y is a regular

sequence of $k[x, y]$ and also it is a regular sequence of $R = k[x, y, \frac{1}{1-x}]$. Hence $I_\alpha = xR$ and $J_\alpha = I_\alpha + \alpha I_\alpha = (x, y)R$ (cf. [6]). Therefore A is not flat over R . By Proposition 9, we get that $\sqrt{\alpha A} \supsetneq \sqrt{yA}$. Thus $\sqrt{yA} \not\supset \alpha$. Therefore $I_{\alpha^{-1}} = \alpha I_\alpha = \alpha(xR) = yR$.

Proposition 10. *Assume that α is an anti-integral element of degree d over R , that $a \in I_{[\alpha^{-1}]}$ and assume that $\text{grade}(I_{[\alpha]} + I_{[\alpha^{-1}]}) > 1$. Then there exists $g(\alpha) \in A = R[\alpha]$ which satisfies $a = \alpha g(\alpha)$ and $1 - g(\alpha) \in U(A)$ if and only if $\alpha A = (\alpha - a)A$.*

Proof. ($\Leftarrow$) Since $\alpha \in (\alpha - a)A$, we have $\alpha = (\alpha - a)f(\alpha)$ for some $f(\alpha) \in A$. By [7, Proposition 6], we have that $\alpha A \cap R = I_{[\alpha^{-1}]}$. Thus $a \in I_{[\alpha^{-1}]} = \alpha A \cap R \subset \alpha A$. Since $a \in I_{[\alpha^{-1}]} \subset \alpha A$, we have that $a = \alpha g(\alpha)$ for some $g(\alpha) \in A$. Hence $\alpha = \alpha f(\alpha) - a f(\alpha) = \alpha f(\alpha) - \alpha g(\alpha) f(\alpha)$ and so $1 = f(\alpha)(1 - g(\alpha))$. Thus $f(\alpha) \in U(A)$ and $1 - g(\alpha) \in U(A)$.

($\Rightarrow$) Since $1 - g(\alpha) \in U(A)$, there exists $f(\alpha) \in U(A)$ such that $1 - g(\alpha) = f(\alpha)$. So $\alpha f(\alpha) = \alpha - \alpha g(\alpha) = \alpha - a$. Since $f(\alpha)$ is a unit in A , we conclude that $\alpha A = (\alpha - a)A$. $\square$

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