Torsion-Free Abelian Semigroup Rings X

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Let R be a commutative ring, and let S be a submonoid of a torsion-free abelian (additive) group (Then S is called a g -monoid). We consider the semigroup ring $R[S]$ of S over R . The aim of §1 of this note is to prove two Theorems on seminormality of $R[S]$. The aim of §2 of this note is to show that an MC-semigroup need not be a Macaulay semigroup. Torsion-free abelian semigroup rings IX is [M1].

§1 Seminormal Semigroup Rings.

We have the following results.

Theorem 1([AA]). *Let D be an integral domain. Then D is seminormal and S is seminormal if and only if $D[S]$ is seminormal.*

D is called seminormal if $\alpha^2 \in D$ and $\alpha^3 \in D$ imply $\alpha \in D$ for all $\alpha \in q(D)$ ($q(D)$ denotes the quotient field of D). S is called seminormal if $2\alpha \in S$ and $3\alpha \in S$ imply $\alpha \in S$ for all $\alpha \in q(S)$ ($q(S)$ denotes the quotient group of S ; $q(S) = \{s - s' \mid s, s' \in S\}$).

Theorem 2([M4, Theorem 3]). *D is u -closed if and only if $D[S]$ is u -closed.*

D is called u -closed if $\alpha^2 - \alpha \in D$ and $\alpha^3 - \alpha^2 \in D$ imply $\alpha \in D$ for all $\alpha \in q(D)$.

Proposition(cf. [OSY]). *If D is t -closed, then $D[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$ is t -closed, where $X_1, \dots, X_n$ are indeterminates.*

D is called t -closed if $\alpha^2 - a\alpha \in D$ and $\alpha^3 - a\alpha^2 \in D$ imply $\alpha \in D$ for all $a \in D$ and $\alpha \in q(D)$.

Lemma 1. *Let G be a torsion-free abelian group. Then D is t -closed if and only if $D[G]$ is t -closed.*

Proof. The sufficiency is straightforward.

The necessity: Assume that $F^2 - fF \in D[G]$ and $F^3 - fF^2 \in D[G]$ for elements $f \in D[G]$ and $F \in q(D[G])$. There exists a finitely generated subgroup H of G such that $f \in D[H]$, $F \in q(D[H])$, $F^2 - fF \in D[H]$ and $F^3 - fF^2 \in D[H]$. $D[H]$ is isomorphic onto $D[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$ for some n . By Proposition, we have $F \in D[H]$.

Received June 1, 2000.

1991 *Mathematics Subject Classification*. Primary 13A15, Secondary 20M14.

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Lemma 1 implies the following,

Lemma 2. *D is t -closed and $K[S]$ is t -closed if and only if $D[S]$ is t -closed, where $K = q(D)$.*

Let n be a positive integer > 1 . The following conditions are equivalent:

- (1) $n\alpha, (n+1)\alpha, \dots \in S$ imply $\alpha \in S$ for all $\alpha \in q(S)$.
- (2) S is seminormal.

Example 1. Example of $D[S]$ which is u -closed and which is not seminormal: Let n be a positive integer > 1 , $S = \{0, n, n+1, n+2, \dots\}$, and let D be a u -closed domain. Then $D[S]$ is u -closed and $D[S]$ is not seminormal.

Example 2. Assume that $q(S)$ has torsion-free rank one.

- (1) D is seminormal and S is integrally closed if and only if $D[S]$ is seminormal.
- (2) D is t -closed and S is integrally closed if and only if $D[S]$ is t -closed.

A g -monoid S is called integrally closed if $n\alpha \in S$ implies $\alpha \in S$ for all positive integers n and $\alpha \in q(S)$.

Let D be a domain, and let S be a g -monoid. First, we consider what are conditions for the ring $D[S]$ to satisfy in order to D being t -closed and S being seminormal. Our result is the following,

Theorem 3. *The following conditions are equivalent for the domain D and the g -monoid S :*

- (1) D is t -closed and S is seminormal.
- (2) $F^2 - aF \in D[S]$ and $F^3 - aF^2 \in D[S]$ imply $F \in D[S]$ for all elements $a \in D$ and $F \in q(D[S])$.

Condition (2) is clearly satisfied if $D[S]$ is t -closed.

Secondly, we consider to extend Theorem 1 for a commutative ring R with zero-divisors.

R is called seminormal if $\alpha^2 \in R$ and $\alpha^3 \in R$ imply $\alpha \in R$ for all $\alpha \in q(R)$ ($q(R)$ denotes the total quotient ring of R ; $q(R) = \{a/b \mid a \in R \text{ and } b \text{ is a non-zero-divisor of } R\}$). R is called a von Neumann regular ring if, for each element a of R , there exists an element b of R such that $a = a^2b$.

We have the following results.

Theorem 4 ([M5, Theorem 20]). *Assume that $q(R)$ is a von Neumann regular ring. Then R is integrally closed and S is integrally closed if and only if $R[S]$ is integrally closed.*

Theorem 5 ([BCM]). *Assume that $q(R)$ is a von Neumann regular ring. If R is seminormal, then $R[X]$ is seminormal.*

Our result is the following,

Theorem 6. *Assume that $q(R)$ is a von Neumann regular ring. Then R is seminormal and S is seminormal if and only if $R[S]$ is seminormal.*

Proof of Theorem 3. Assume that condition (2) holds.

Assume that $\alpha^2 - a\alpha \in D$ and $\alpha^3 - a\alpha^2 \in D$ for elements $a \in D$ and $\alpha \in K$, where K is the quotient field of D . By condition (2), we have $\alpha \in D[S]$, and hence $\alpha \in D$. Therefore D is t-closed.

The condition (2) implies that $D[S]$ is seminormal. Then S is seminormal by Theorem 1.

Next, assume that condition (1) holds. Then $D[S]$ is seminormal by Theorem 1. Let G be the quotient group of S . Then $D[G]$ is t-closed by Proposition.

Assume that $F^2 - aF \in D[S]$ and $F^3 - aF^2 \in D[S]$ for elements $a \in D$ and $F \in q(D[S])$. Since $D[G]$ is t-closed, we have $F \in D[G]$.

The case $a = 0$: Since $D[S]$ is seminormal, we have $F \in D[S]$.

The case $a \neq 0$: We have $(F/a)^2 - F/a \in K[S]$ and $(F/a)^3 - (F/a)^2 \in K[S]$, where $K = q(D)$. $K[S]$ is u-closed by Theorem 2. It follows that $F/a \in K[S]$, and hence $F \in K[S]$. Since $F \in D[G]$, we have $F \in D[S]$.

Therefore condition (2) holds.

Proof of Theorem 6. The sufficiency is straightforward.

The necessity: Let $X_1, X_2, X_3, \dots$ be indeterminates. $R[X_1]$ is seminormal by Theorem 5. Suppose that $R[X_1, \dots, X_{k-1}]$ is seminormal. $q(R[X_1, \dots, X_{k-1}])$ is a von Neumann regular ring by [M5, Lemma 17]. Therefore $(R[X_1, \dots, X_{k-1}])[X_k]$ is seminormal. It follows that $R[X_1, X_1^{-1}, \dots, X_k, X_k^{-1}]$ is seminormal for any k .

Assume that $F^2 \in R[G]$ and $F^3 \in R[G]$ for $F \in q(R[G])$. There exists a finitely generated subgroup H of G such that $F^2 \in R[H]$, $F^3 \in R[H]$ and $F \in q(R[H])$. $R[H]$ is isomorphic onto $R[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$ for some n . Since $R[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$ is seminormal, we have $F \in R[H] \subset R[G]$. Therefore $R[G]$ is seminormal. We note that $q(R[H])$ is canonically regarded as a subring of $q(R[G])$.

Let $F^2 \in R[S]$ and $F^3 \in R[S]$ for $F \in q(R[S])$. Then $F \in R[G]$. Put $F = \sum a_i X^{\alpha_i}$, where $0 \neq a_i \in R$ and $\alpha_i \neq \alpha_j$ for $i \neq j$. Suppose that $\alpha_k \notin S$ for some k . Since R has no non-zero nilpotents, there exists a prime ideal P of R which does not contain a_k . Set $D = R/P$. Then, in $D[G]$, we have $\bar{F}^2 \in D[S]$, $\bar{F}^3 \in D[S]$ and $\bar{F} \notin q(D)[S]$. This contradicts to Theorem 1. Therefore $R[S]$ is seminormal.

§2. An MC-semigroup need not be a Macaulay semigroup.

In [SM1] we proved a semigroup version of Chapters 1, 2 and 3 of Narita's Ideal Theory [N]. In [SM3] we proved almost all propositions in Chapters 4 and 5 of [N] for g-monoids.

Let I be an ideal of S , and $s \in S$. If $s + a \in I$ implies $a \in I$, then s is called a non-zerodivisor on S modulo I . If s is not a non-zerodivisor on S modulo I , then s is called a zerodivisor. The set of zerodivisors on S modulo I is denoted by $Z(S/I)$.

The ordered sequence $x_1, \dots, x_n$ of elements of S is called a regular sequence if $(x_1, \dots, x_n) \subsetneq S$ and if $x_2 \notin Z(S/(x_1)), \dots, x_n \notin Z(S/(x_1, \dots, x_{n-1}))$.

The maximum of lengths of all regular sequences in I on S is called the grade of I on S , and is denoted by $G(I, S)$.

If S is a Noetherian semigroup with maximal ideal M , and if $G(M, S)$ equals the dimension of S , then S is called a Macaulay semigroup ([M3]).

Now, let S be a Noetherian semigroup, and I an ideal of S . If all prime divisors of I have the same height, then I is called unmixed.

Let S be a Noetherian semigroup. Assume that, if I is an ideal of S which is generated by the number height(I) of elements, then I is unmixed. Then S is called an MC-semigroup.

In [SM3] we posed the following,

Problem. Is a g-monoid S a Macaulay semigroup if and only if S is an MC-semigroup?

It is well-known that the corresponding notions are equivalent for Noetherian rings.

The aim of this section is to answer to the Problem.

Lemma 1 (A part of [M3, Theorem 4]). *Let S be a Noetherian semigroup, and let $x_1, \dots, x_n$ be a regular sequence on S . Then any permutation of the x 's is a regular sequence on S .*

Lemma 2 ([M2, Theorem 3]). *In an integrally closed Noetherian semigroup S , each prime divisor of any proper principal ideal has height one.*

Example ([SM2]). Let $x_1 + x_2 = x_3 + x_4$ be a unique relation of letters x_1, x_2, x_3 and x_4 . Set $S = \mathbf{Z}_0 x_1 + \mathbf{Z}_0 x_2 + \mathbf{Z}_0 x_3 + \mathbf{Z}_0 x_4$, where $\mathbf{Z}_0$ is the non-negative integers. Then S is a g-monoid with maximal ideal $M = (x_1, x_2, x_3, x_4)$. Set $Q_1 = (x_2, x_3, x_4)$, $Q_2 = (x_1, x_3, x_4)$, $Q_3 = (x_1, x_2, x_4)$, $Q_4 = (x_1, x_2, x_3)$, $P_1 = (x_1, x_3)$, $P_2 = (x_1, x_4)$, $P_3 = (x_2, x_3)$ and $P_4 = (x_2, x_4)$. Then Q_i and P_i are prime ideals of S for each i . There do not exist other prime ideals of S than $M, Q_1, Q_2, Q_3, Q_4, P_1, P_2, P_3$ and P_4 . It follows that S is a Noetherian semigroup of dimension three, and M is not a minimal prime divisor of any ideal generated by three elements.

We note that, if M is a maximal ideal of a two-dimensional Noetherian semigroup, then M is a minimal prime divisor of an ideal generated by two elements ([M2, Theorem 5]).

We will show that the g-monoid S of our Example is not a Macaulay semigroup and that S is an MC-semigroup. Throughout the section, S denotes the g-monoid in Example.

Lemma 3. (1) *Let l_1, l_2, l_3 be integers. Then $l_1 x_1 + l_2 x_2 + l_3 x_3 = 0$ if and only if $l_1 = l_2 = l_3 = 0$.*

(2) *Let $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4$ be non-negative integers. Then*

$$a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4 = b_1 x_1 + b_2 x_2 + b_3 x_3 + b_4 x_4$$

if and only if

$$b_1 - a_1 = b_2 - a_2 = a_3 - b_3 = a_4 - b_4.$$

Lemma 4. *S is integrally closed.*

Proof. Let $\alpha = l_1 x_1 + l_2 x_2 + l_3 x_3$ be an element of the quotient group $q(S)$ of S with $l_i \in \mathbf{Z}$. We will determine conditions for $\alpha \in S$.

We have $l_1x_1 + l_2x_2 + l_3x_3 = k_1x_1 + k_2x_2 + k_3x_3 + k_4x_4$ with $k_i \geq 0$. By Lemma 3, we have $l_1 = k_1 + k_4$, $l_2 = k_2 + k_4$ and $l_3 = k_3 - k_4$. Hence $k_4 \leq \min(l_1, l_2)$ and $l_3 \geq -k_4 \geq -\min(l_1, l_2)$. That is, $l_1 \geq 0$, $l_2 \geq 0$ and $l_3 \geq -\min(l_1, l_2)$.

If $\alpha = l_1x_1 + l_2x_2 + l_3x_3$ and $n\alpha \in S$ with $n > 0$, then $nl_1 \geq 0$, $nl_2 \geq 0$ and $nl_3 \geq -\min(nl_1, nl_2)$. Hence $\alpha \in S$. That is, S is integrally closed.

Lemmas 2 and 4 imply that each principal ideal I of S is unmixed.

Proposition 1. *S is not a Macaulay semigroup.*

Proof. Suppose that f_1, f_2, f_3 is a regular sequence on S . We must derive a contradiction. We may assume that f_1 is of the form $x_1 + g_1$ with $g_1 \in S$. Then x_1, f_2, f_3 is a regular sequence on S . We use Lemma 1. Then f_2, x_1, f_3 is a regular sequence. Suppose that f_2 is of the form $x_3 + g_2$ with $g_2 \in S$. Then x_3, x_1, f_3 is a regular sequence. But $x_1 + x_2 \in (x_3)$ and $x_2 \notin (x_3)$; a contradiction. Hence f_2 is of the form $x_2 + g_4$ with $g_4 \in S$. Then x_2, x_1, f_3 is a regular sequence. Thus we have a contradiction.

Lemma 5. *Let I be an ideal of S with height two which is generated by two elements. Then I is of the form (nx_1, mx_2) or (nx_3, mx_4) with positive integers n and m .*

Proof. Let $f_1 = a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4$ and $f_2 = b_1x_1 + b_2x_2 + b_3x_3 + b_4x_4$ be generators for I with non-negative integers $a_1, a_2, a_3, a_4, b_1, b_2, b_3$ and b_4 .

Assume that $a_1 > 0$. Since I is not contained in any of P_1, P_2, P_3 and P_4 , we have $b_1 = b_3 = b_4 = 0$. Hence I is of the form (nx_1, mx_2) .

The other cases are similar.

Lemma 6. *Let n and m be positive integers. Then we have,*

- (1) *The ideal $nP_1 \cup mP_3$ is an Q_4 -primary ideal.*
- (2) *$nP_2 \cup mP_4$ is an Q_3 -primary ideal.*

Proof. Let $A \in Q_4$ and $B \in M - Q_4$, and assume that $A + B \in nP_1 \cup mP_3$. It is sufficient to show that $A \in nP_1 \cup mP_3$.

A is of the form $c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4$ where the c_i are non-negative integers, B is of the form px_4 with $p > 0$, $A + B$ is of the form

$$(1) ax_1 + a'x_3 + a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4$$

or

$$(2) bx_2 + b'x_3 + b_1x_1 + b_2x_2 + b_3x_3 + b_4x_4,$$

where $a, a', a_1, a_2, a_3, a_4, b, b', b_1, b_2, b_3$ and b_4 are non-negative integers with $a + a' = n$ and $b + b' = m$.

Case (1):

$$A + B = c_1x_1 + c_2x_2 + c_3x_3 + (c_4 + p)x_4 = (a + a_1)x_1 + a_2x_2 + (a' + a_3)x_3 + a_4x_4.$$

By Lemma 3, there exists an integer l such that

$$c_1 = a + a_1 + l, c_2 = a_2 + l, c_3 = a' + a_3 - l$$

and

$$c_4 + p = a_4 - l.$$

Since $c_1 + c_3 \geq n$, we have $A \in nP_1$.

Case (2): The similar proof shows that $A \in mP_3$.

The proof is complete.

Lemma 7. *Let n and m be positive integers. Then we have,*

$$(1) (nx_1, mx_2) = (nP_1 \cup mP_3) \cap (nP_2 \cup mP_4).$$

$$(2) (nx_3, mx_4) = (nP_1 \cup mP_2) \cap (nP_3 \cup mP_4).$$

Proof. It is sufficient to show that each element x of $(nP_1 \cup mP_3) \cap (nP_2 \cup mP_4)$ belongs to $I = (nx_1, mx_2)$.

Each element of $nP_1 \cup mP_3$ is of the form

$$A = ax_1 + a'x_3 + a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4$$

or

$$B = bx_2 + b'x_3 + b_1x_1 + b_2x_2 + b_3x_3 + b_4x_4$$

where $a, a', a_1, a_2, a_3, a_4, b, b', b_1, b_2, b_3$ and b_4 are non-negative integers with $a + a' = n$ and $b + b' = m$.

Similarly, each element of $nP_2 \cup mP_4$ is of the form

$$C = cx_1 + c'x_4 + c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4$$

or

$$D = dx_2 + d'x_4 + d_1x_1 + d_2x_2 + d_3x_3 + d_4x_4,$$

where $c, c', c_1, c_2, c_3, c_4, d, d', d_1, d_2, d_3$ and d_4 are non-negative integers with $c + c' = n$ and $d + d' = m$.

There arise four cases: (1) $x = A = C$, (2) $x = A = D$, (3) $x = B = C$ and (4) $x = B = D$.

Case (1):

$$x = (a + a_1)x_1 + a_2x_2 + (a' + a_3)x_3 + a_4x_4 = (c + c_1)x_1 + c_2x_2 + c_3x_3 + (c' + c_4)x_4.$$

By Lemma 3, there exists an integer l such that

$$c + c_1 = a + a_1 + l, c_2 = a_2 + l, c_3 = a' + a_3 - l \text{ and } c' + c_4 = a_4 - l.$$

If $a' + a_3 \leq a_4$, then

$$x = (a + a_1)x_1 + a_2x_2 + (a' + a_3)(x_3 + x_4) + (a_4 - a' - a_3)x_4 = (a + a_1)x_1 + a_2x_2 + (a' + a_3)(x_1 + x_2) + (a_4 - a' - a_3)x_4 \in (nx_1).$$

If $a' + a_3 \geq a_4$, then $c_3 \geq c' + c_4$.

$$x = (c + c_1)x_1 + c_2x_2 + (c_3 - c' - c_4)x_3 + (c' + c_4)(x_3 + x_4) = (c + c_1)x_1 + c_2x_2 + (c_3 - c' - c_4)x_3 + (c' + c_4)(x_1 + x_2) \in (nx_1).$$

Hence $x \in I$.

Case (2):

$$x = (a + a_1)x_1 + a_2x_2 + (a' + a_3)x_3 + a_4x_4 = d_1x_1 + (d + d_2)x_2 + d_3x_3 + (d' + d_4)x_4.$$

By Lemma 3, there exists an integer l such that

$$d_1 = a + a_1 + l, d + d_2 = a_2 + l, d_3 = a' + a_3 - l$$

and $d' + d_4 = a_4 - l$.

If $a' + a_3 \leq a_4$, then

$$x = (a + a_1)x_1 + a_2x_2 + (a' + a_3)(x_3 + x_4) + (a_4 - a' - a_3)x_4 = (a + a_1)x_1 + a_2x_2 + (a' + a_3)(x_1 + x_2) + (a_4 - a' - a_3)x_4 \in (nx_1).$$

If $a' + a_3 \geq a_4$, then $d' + d_4 \leq d_3$.

$$x = d_1x_1 + (d + d_2)x_2 + (d_3 - d' - d_4)x_3 + (d' + d_4)(x_3 + x_4) = d_1x_1 + (d + d_2)x_2 + (d_3 - d' - d_4)x_3 + (d' + d_4)(x_1 + x_2) \in (mx_2).$$

Hence $x \in I$.

Cases (3) and (4) are similar.

Proposition 2. *S is an MC-semigroup.*

Proof. Let I be an ideal of S which is generated by the number $\text{ht}(I)$ of elements. We must show that I is unmixed.

The case of $\text{ht}(I) = 1$: Then I is unmixed by Lemmas 2 and 4.

The case of $\text{ht}(I) = 2$: Then I is of the form (nx_1, mx_2) or (nx_3, mx_4) with positive integers n and m by Lemma 5.

If $I = (nx_1, mx_2)$, then $I = (nP_1 \cup mP_3) \cap (nP_2 \cup mP_4)$ by Lemma 7. $nP_1 \cup mP_3$ is Q_4 -primary, and $nP_2 \cup mP_4$ is Q_3 -primary by Lemma 6. Since $\text{ht}(Q_3) = \text{ht}(Q_4) = 2$, I is unmixed.

If $I = (nx_3, mx_4)$, the proof is similar.

The case of $\text{ht}(I) = 3$: Example shows that the case can not happen.

We have proved that an MC-semigroup need not be a Macaulay semigroup.

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