Caloric morphisms with respect to radial metrics on $\mathbb{R}^n \setminus \{0\}$

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1. Introduction.

In our previous paper [5], we defined the notion of caloric morphism, the transformation which preserve the solutions of heat equation, between Riemannian manifolds, and obtained a general characterization theorem. The Appell transformation is the typical example in Euclidean spaces. We also gave some examples of caloric morphism defined on $\mathbb{R} \times (\mathbb{R}^n \setminus \{0\})$ ($n \geq 2$) with respect to radial metrics. In this paper, we shall study such caloric morphisms in more detail. Let $M = \mathbb{R}^n \setminus \{0\}$ be the Riemannian manifold with the radial metric of form

$$(*) \quad g_{ij} = \rho(|x|)\delta_{ij}, \quad i, j = 1, 2, \dots, n,$$

where ρ is a positive C^∞ function defined on $(0, \infty)$ and δ_{ij} is Kronecker's delta. The goal of this paper is to determine the caloric morphism defined on a domain in $\mathbb{R} \times M$ of form

$$(a) \quad f(t, x) = (f_0(t), \nu(t)R(t)x)$$

or

$$(b) \quad f(t, x) = (f_0(t), \frac{\nu(t)R(t)x}{|x|^2}),$$

where $\nu(t)$ is a positive C^∞ -function and $R(t)$ is a C^∞ -function of orthogonal matrix value. We note that the existence of the caloric morphism of above forms puts strong restrictions on the metric in the case that the function ν is not constant. In fact, It is shown that ρ has the form $\rho(r) = pr^q$ in such cases. Our main result is formulated in the following theorem.

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Theorem 1. *Let $M = \mathbb{R}^n \setminus \{0\}$ be the Riemannian manifold with metric of form (*) and (f, φ) be a caloric morphism of form (a) or (b) defined on a domain $D \subset \mathbb{R} \times (\mathbb{R}^n \setminus \{0\})$. Then we can determine the forms of (f, φ) and ρ as follows.*

Case 1-a. $n = 2$ and ρ is of form $\rho(r) = pr^{-2}$ ($p > 0$) and f is of form (a),

$$f(t, x) = (t + d, ce^{at} R_0 \begin{pmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{pmatrix} x),$$

$$\varphi(t, r, \theta) = Cr^{\frac{1}{2}ap} \exp\left(\frac{p}{2}b\theta + \frac{p}{4}(a^2 + b^2)t\right).$$

Case 1-b. $n = 2$ and ρ is of form $\rho(r) = pr^{-2}$ ($p > 0$) and f is of form (b),

$$f(t, x) = (t + d, \frac{ce^{at}}{|x|^2} R_0 \begin{pmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{pmatrix} x),$$

$$\varphi(t, r, \theta) = Cr^{-\frac{1}{2}ap} \exp\left(\frac{p}{2}b\theta + \frac{p}{4}(a^2 + b^2)t\right).$$

In the cases 1-a and 1-b, $a, b, c, d, C \in \mathbb{R}$ ($c, C > 0$), R_0 is an orthogonal matrix and (r, θ) is the polar coordinate of $\mathbb{R}^2$.

Case 2-a. $n \geq 3$ and ρ is of form $\rho(r) = pr^{-2}$ ($p > 0$) and f is of form (a),

$$f(t, x) = (t + d, ce^{at} R_0 x), \quad \varphi(t, x) = C|x|^{\frac{1}{2}ap} \exp\left(\frac{1}{4}pa^2t\right).$$

Case 2-b. $n \geq 3$ and ρ is of form $\rho(r) = pr^{-2}$ ($p > 0$) and f is of form (b),

$$f(t, x) = (t + d, \frac{ce^{at}}{|x|^2} R_0 x), \quad \varphi(t, x) = C|x|^{-\frac{1}{2}ap} \exp\left(\frac{1}{4}pa^2t\right).$$

In the cases 2-a and 2-b, $a, c, d, C \in \mathbb{R}$ ($c, C > 0$) and R_0 is an orthogonal matrix.

Case 3. ρ is of form $\rho(r) = pr^q$ ($q \in \mathbb{R}$, $q \neq -2$),

$$f(t, x) = \left(\frac{ct + d}{at + b}, \frac{R_0 x}{|at + b|^{2/(q+2)}}\right),$$

$$\varphi(t, x) = \frac{C}{|at + b|^{n/2}} \exp\left[-\frac{pa|x|^{q+2}}{(q+2)^2(at+b)}\right],$$

where $a, b, c, d, C \in \mathbb{R}$ ($bc - ad = 1$, $C > 0$) and R_0 is an orthogonal matrix.

Case 4. $\rho(r)$ is any positive C^∞ -function satisfying $\rho(\nu r) = \lambda \rho(r)$ with some positive constants ν and λ ,

$$f(t, x) = (\lambda t + d, \nu R_0 x), \quad \varphi(t, x) = C,$$

where $C > 0$, $d \in \mathbb{R}$ and R_0 is an orthogonal matrix.

Case 5. $\rho(r)$ is any positive C^∞ -function satisfying $\rho\left(\frac{\nu}{r}\right) = \frac{\lambda r^4}{\nu^2} \rho(r)$ with some positive constants ν and λ ,

$$f(t, x) = \left(\lambda t + d, \frac{\nu R_0 x}{|x|^2}\right), \quad \varphi(t, x) = C,$$

where $C > 0$, $d \in \mathbb{R}$ and R_0 is an orthogonal matrix.

Case 6. ρ is any positive C^∞ -function,

$$f(t, x) = (t + d, R_0 x), \quad \varphi(t, x) = C,$$

where $C > 0$, $d \in \mathbb{R}$ and R_0 is an orthogonal matrix.

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2. Notations and Preliminaries.

Let (M, g) be an n -dimensional Riemannian manifold. In this paper, we always consider Riemannian manifolds to be connected. We denote Δ_g the Laplace-Beltrami operator on (M, g) which is given in local coordinates $(x_i)_{i=1}^n$ by

$$\Delta_g u = \sum_{i,j=1}^n \frac{1}{\sqrt{g}} \frac{\partial}{\partial x_i} \left(\sqrt{g} g^{ij} \frac{\partial u}{\partial x_j} \right),$$

where $g = \det(g_{ij})$ and (g^{ij}) denotes the inverse matrix of (g_{ij}) as usual.

We define the gradient vector field $\nabla_g u$ of a C^∞ -function u by

$$\nabla_g u = \sum_{i,j=1}^n g^{ij} \frac{\partial u}{\partial x_i} \frac{\partial}{\partial x_j}.$$

Then the gradient of the product function uv of two smooth functions u, v is

$$\nabla_g(uv) = u \nabla_g v + v \nabla_g u.$$

The inner product of two gradients $\nabla_g u$ and $\nabla_g v$ is given by

$$g(\nabla_g u, \nabla_g v) = \sum_{i,j=1}^n g^{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j}.$$

We denote the heat operator $\frac{\partial}{\partial t} - \Delta_g$ on $\mathbb{R} \times M$ by H_g .

A C^∞ -function $u(t, x)$ defined on a open set $D \subset \mathbb{R} \times M$ is said to be caloric on D , if u satisfies the heat equation

$$H_g u = 0$$

on D .

Definition. Let (M, g) be an n -dimensional Riemannian manifold. A pair (f, φ) of a C^∞ -mapping f from a domain $D \subset \mathbb{R} \times M$ to $\mathbb{R} \times M$ and a strictly positive C^∞ -function φ defined on D is said to be a caloric morphism, if f and φ satisfy the following conditions;

(1) $f(D)$ is a domain in $\mathbb{R} \times M$.

(2) For any subdomain $E \subset f(D)$ and any caloric function u defined on E , the function $\varphi(t, x)(u \circ f)(t, x)$ is caloric on $f^{-1}(E)$.

Let (f, φ) and (F, Φ) be caloric morphisms defined on D and E , respectively. If $f(D) \subset E$, then it is easily seen that the composition $(F \circ f, \varphi \cdot (\Phi \circ f))$ is a caloric morphism on D .

In our previous paper [5], we proved the following characterization theorem for caloric morphism.

Theorem A. Let (f, φ) be the pair of a smooth mapping f from $D \subset \mathbb{R} \times M$ to a domain $f(D)$ of $\mathbb{R} \times M$ and a positive smooth function φ defined on D .

Then the following three conditions are equivalent.

(1) (f, φ) is a caloric morphism.

(2) There exists a positive smooth function λ on D such that

$$H_g(\varphi \cdot u \circ f)(t, x) = \lambda^2(t, x)\varphi(t, x)((H_g u) \circ f)(t, x)$$

holds for any C^2 -function u defined on a subdomain of $f(D)$.

(3) Let $(x_\alpha)_{\alpha=1}^n$ be any local coordinate system of (M, g) , and let

$$f = (f_0, f_1, \dots, f_n)$$

be the expression of the mapping f in the coordinate system $(t, x_1, \dots, x_n)$ of the product manifold $\mathbb{R} \times M$. Then (f, φ) satisfies the following equations:

(E-1)

$$H_g \varphi = 0,$$

(E-2)

$$\frac{\partial f_\alpha}{\partial t} = \Delta_g f_\alpha + 2g(\nabla_g \log \varphi, \nabla_g f_\alpha) + \sum_{\beta, \gamma=1}^n g(\nabla_g f_\beta, \nabla_g f_\gamma)(\Gamma_{\beta\gamma}^\alpha \circ f),$$

$$\alpha = 1, \dots, n,$$

(E-3)

$$\nabla_g f_0 = 0,$$

(E-4)

$$g(\nabla_g f_\alpha, \nabla_g f_\beta) = f'_0(t)(g^{\alpha\beta} \circ f), \quad \alpha, \beta = 1, \dots, n,$$

where $\Gamma_{\beta\gamma}^\alpha = \sum_{l=1}^n \frac{1}{2} g^{\alpha l} \left(\frac{\partial g_{\gamma l}}{\partial x_\beta} + \frac{\partial g_{\beta l}}{\partial x_\gamma} - \frac{\partial g_{\beta\gamma}}{\partial x_l} \right)$ ($\alpha, \beta, \gamma = 1, \dots, n$) is the Christoffel symbol of (M, g) .

Remark. In the above theorem, we can replace (E-2) by

$$(E-2') \quad \frac{\partial f_\alpha}{\partial t} = \Delta_g f_\alpha + 2g(\nabla_g \log \varphi, \nabla_g f_\alpha) - f'_0(t)[(\Delta_g x_\alpha) \circ f], \quad \alpha = 1, \dots, n.$$

3. Radial metric on $\mathbb{R}^n \setminus \{0\}$.

In the following, we consider the caloric morphism on the Euclidean space with a radial metric. Let $M = \mathbb{R}^n \setminus \{0\}$ ($n \geq 2$) and

$$g_{ij} = \rho(|x|)\delta_{ij}, \quad i, j = 1, 2, \dots, n,$$

where ρ is a positive C^∞ -function defined on $(0, \infty)$. Then we have

$$g^{ij} = \frac{1}{\rho(|x|)}\delta_{ij}, \quad g = \rho(|x|)^n, \quad \sqrt{g} = \rho(|x|)^{n/2}.$$

We can choose the usual cartesian coordinate system as a local coordinate system of M . Then the Laplacian and the gradient of a function u are given by

$$\begin{aligned} \Delta_g u &= \sum_{i=1}^n \frac{1}{\rho(|x|)^{n/2}} \frac{\partial}{\partial x_i} \left(\rho(|x|)^{n/2-1} \frac{\partial u}{\partial x_i} \right) \\ (1) \quad &= \frac{1}{\rho(|x|)} \Delta_x u + \frac{n-2}{2} \frac{\rho'(|x|)}{\rho(|x|)^2} \sum_{i=1}^n \frac{x_i}{|x|} \frac{\partial u}{\partial x_i}, \end{aligned}$$

and

$$\nabla_g u = \frac{1}{\rho(|x|)} \sum_{i=1}^n \frac{\partial u}{\partial x_i} \frac{\partial}{\partial x_i},$$

respectively. Hence, the inner product of the gradients of two functions u and v is given by

$$(2) \quad g(\nabla_g u, \nabla_g v) = \frac{1}{\rho(|x|)} \sum_{i=1}^n \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i}.$$

The Laplacian of the coordinate function x_α is given by

$$(3) \quad \Delta_g x_\alpha = \frac{n-2}{2} \frac{\rho'(|x|)}{\rho(|x|)^2} \frac{x_\alpha}{|x|}, \quad \alpha = 1, 2, \dots, n.$$

Let (f, φ) be a caloric morphism such that f is of form (a):

$$f(t, x) = (f_0(t), \nu(t)R(t)x).$$

Then f is expressed as

$$f(t, x) = (f_0(t), f_1(t, x), \dots, f_n(t, x)),$$

$$f_\alpha(t, x) = \sum_{j=1}^n \nu(t) r_{\alpha j}(t) x_j, \quad \alpha = 1, 2, \dots, n,$$

where $r_{\alpha j}(t)$ ($\alpha, j = 1, 2, \dots, n$) is the (α, j) -element of the orthogonal matrix $R(t)$. Their derivatives are given by

$$(4) \quad \begin{aligned} \frac{\partial f_\alpha}{\partial t} &= \sum_{j=1}^n (\nu'(t)r_{\alpha j}(t) + \nu(t)r'_{\alpha j}(t))x_j, \\ \frac{\partial f_\alpha}{\partial x_i} &= \nu(t)r_{\alpha i}(t) \end{aligned}$$

for $\alpha, i = 1, 2, \dots, n$.

Next, if (f, φ) be a caloric morphism such that f is of form (b):

$$f(t, x) = (f_0(t), \frac{\nu(t)R(t)x}{|x|^2}),$$

then f_α is given by

$$f_\alpha(t, x) = \sum_{j=1}^n \nu(t)r_{\alpha j}(t) \frac{x_j}{|x|^2}, \quad \alpha = 1, 2, \dots, n.$$

Their derivatives are given by

$$(5) \quad \begin{aligned} \frac{\partial f_\alpha}{\partial t} &= \sum_{j=1}^n (\nu'(t)r_{\alpha j}(t) + \nu(t)r'_{\alpha j}(t)) \frac{x_j}{|x|^2}, \\ \frac{\partial f_\alpha}{\partial x_i} &= \nu(t)r_{\alpha i}(t) \frac{1}{|x|^2} - 2 \sum_{j=1}^n \nu(t)r_{\alpha j}(t) \frac{x_i x_j}{|x|^4} \end{aligned}$$

for $\alpha, i = 1, 2, \dots, n$.

In the rest of this section, we prepare some lemmas. The first lemma gives the condition for the metric to have a caloric morphism of (a) or (b).

Lemma 1. (1) Let (f, φ) be a caloric morphism such that f is of form (a). Then the equation

$$(6) \quad \rho(\nu(t)|x|) = \frac{f'_0(t)}{\nu(t)^2} \rho(|x|)$$

holds for all $(t, x) \in D$.

(2) Let (f, φ) be a caloric morphism such that f is of form (b). Then the equation

$$(7) \quad \rho\left(\frac{\nu(t)}{|x|}\right) = \frac{f'_0(t)|x|^4}{\nu(t)^2} \rho(|x|)$$

holds for all $(t, x) \in D$.

Proof. Let (f, φ) be a caloric morphism such that f is of form (a). Then by (2) and (4), the equalities

$$\begin{aligned} g(\nabla_g f_\alpha, \nabla_g f_\beta) &= \frac{1}{\rho(|x|)} \nu(t)^2 \sum_{i,j,k=1}^n r_{\alpha j}(t) \delta_{ji} r_{\beta k}(t) \delta_{ki} \\ &= \frac{\nu(t)^2}{\rho(|x|)} \delta_{\alpha\beta} \end{aligned}$$

and

$$f'_0(t)(g^{\alpha\beta} \circ f)(t, x) = f'_0(t) \delta_{\alpha\beta} \frac{1}{\rho(\nu(t)|x|)}$$

hold for $\alpha, \beta = 1, \dots, n$, because $R(t)$ is an orthogonal matrix for each t . The desired equation

$$\rho(\nu(t)|x|) = \frac{f'_0(t)}{\nu(t)^2} \rho(|x|), \quad (t, x) \in D$$

follows from (E-4).

Next let (f, φ) be a caloric morphism such that f is of form (b). Then by (2) and (5), the equalities

$$\begin{aligned} g(\nabla_g f_\alpha, \nabla_g f_\beta) &= \frac{1}{\rho(|x|)} \frac{\nu(t)^2}{|x|^4} \sum_{i,j,k=1}^n r_{\alpha j}(t) \left(\delta_{ji} - 2 \frac{x_i x_j}{|x|^2} \right) r_{\beta k}(t) \left(\delta_{ki} - 2 \frac{x_i x_k}{|x|^2} \right) \\ &= \frac{\nu(t)^2}{|x|^4 \rho(|x|)} \delta_{\alpha\beta} \end{aligned}$$

and

$$f'_0(t)(g^{\alpha\beta} \circ f) = f'_0(t) \delta_{\alpha\beta} \frac{1}{\rho(\nu(t)/|x|)}$$

hold for $\alpha, \beta = 1, \dots, n$, because $R(t)$ is an orthogonal matrix for each t . The desired equation

$$\rho\left(\frac{\nu(t)}{|x|}\right) = \frac{f'_0(t)|x|^4}{\nu(t)^2} \rho(|x|), \quad (t, x) \in D$$

follows from (E-4). $\square$

The equations (6) and (7) put strong restrictions on ρ in the case that the function ν is not constant. In fact, the second lemma shows that ρ has the form $\rho(r) = pr^q$ in such cases.

Lemma 2. *Let ρ be a positive smooth function defined on an open interval $I \subset (0, \infty)$ and E be a connected open subset of $\mathbb{R} \times \{x \in \mathbb{R}^n; |x| \in I\}$.*

(1) *The equation $\rho(\nu(t)|x|) = \frac{\lambda(t)}{\nu(t)^2} \rho(|x|)$ holds for all $(t, x) \in E$, where λ is a positive smooth function and ν is a non-constant positive smooth function, if and*

only if there exists a constant q and a positive constant p satisfying $\rho(|x|) = p|x|^q$ and $\lambda(t) = \nu(t)^{q+2}$ for all $(t, |x|) \in E$.

(2) The equation $\rho\left(\frac{\nu(t)}{|x|}\right) = \frac{\lambda(t)|x|^4}{\nu(t)^2}\rho(|x|)$ holds for all $(t, x) \in E$, λ is a positive smooth function and ν is a non-constant positive function, if and only if $\lambda(t) = 1$ and there exists a positive constant p satisfying $\rho(|x|) = p|x|^{-2}$ for all $(t, x) \in E$.

Proof. First, we prove (1). Substitute $r = |x|$ into the given equation and then differentiate the equation by r and by t . We have the equations

$$\rho'(\nu(t)r)\nu(t) = \frac{\lambda(t)}{\nu(t)^2}\rho'(r), \quad \rho'(\nu(t)r)\nu'(t)r = \left(\frac{\lambda'(t)}{\nu(t)^2} - 2\frac{\lambda(t)\nu'(t)}{\nu(t)^3}\right)\rho(r),$$

and hence

$$\frac{r\rho'(r)}{\rho(r)} = \frac{\lambda'(t)\nu(t)}{\lambda(t)\nu'(t)} - 2.$$

Therefore, both $\frac{r\rho'(r)}{\rho(r)}$ and $\frac{\lambda'(t)\nu(t)}{\lambda(t)\nu'(t)} - 2$ are equal to a constant q . Then we have

$$\frac{\rho'(r)}{\rho(r)} = \frac{q}{r}, \quad \frac{\lambda'(t)}{\lambda(t)} = (q+2)\frac{\nu'(t)}{\nu(t)}.$$

These two equations imply

$$\rho(r) = pr^q, \quad \lambda(t) = c\nu(t)^{q+2}$$

with some positive constants p and c . Substituting these into the given equation, we obtain $c = 1$.

To prove (2), substitute $r = |x|$ into the given equation and then differentiate the equation by r and by t , we have

$$\rho'\left(\frac{\nu(t)}{r}\right) = -\frac{\lambda(t)r^5}{\nu(t)^3}(4\rho(r) + r\rho'(r)), \quad \rho'\left(\frac{\nu(t)}{r}\right) = \frac{\lambda(t)r^5}{\nu(t)^3}\left(\frac{\lambda'(t)\nu(t)}{\lambda(t)\nu'(t)} - 2\right)\rho(r).$$

By the same argument as above, we have at first

$$\frac{r\rho'(r)}{\rho(r)} = -\frac{\lambda'(t)\nu(t)}{\lambda(t)\nu'(t)} - 2 = q$$

with some constant q . Then

$$\rho(r) = pr^q, \quad \lambda(t) = c\nu(t)^{-(q+2)}$$

with some positive constants p and c . Substituting these into the given equation, we obtain

$$\rho(r) = pr^{-2}, \quad \lambda(t) = 1.$$

□

The next lemma enables us to reduce the case (b) to the case (a) in the next section.

Lemma 3. Assume that the metric ρ satisfies $\rho(\frac{\nu}{r}) = \frac{\lambda r^4}{\nu^2} \rho(r)$ with some positive constants ν and λ . Then the inversion

$$f(t, x) = (\lambda t, \frac{\nu x}{|x|^2}), \quad \varphi(t, x) = 1$$

is a caloric morphism from $\mathbb{R} \times (\mathbb{R}^n \setminus \{0\})$ onto itself. Especially, if $\rho(r) = pr^{-2}$, then the simple inversion

$$f(t, x) = (t, \frac{x}{|x|^2}), \quad \varphi(t, x) = 1$$

is a caloric morphism from $\mathbb{R} \times (\mathbb{R}^n \setminus \{0\})$ onto itself.

Proof. Clearly, (f, φ) satisfies the equations (E-1) and (E-3). To show (E-4), we have by (2) and (5),

$$\begin{aligned} g(\nabla_g f_\alpha, \nabla_g f_\beta) &= \frac{1}{\rho(|x|)} \frac{\nu^2}{|x|^4} \sum_{i=1}^n \left(\delta_{\alpha i} - 2 \frac{x_i x_\alpha}{|x|^2} \right) \left(\delta_{\beta i} - 2 \frac{x_i x_\beta}{|x|^2} \right) \\ &= \frac{\nu^2}{|x|^4 \rho(|x|)} \delta_{\alpha\beta}, \quad \alpha, \beta = 1, 2, \dots, n. \end{aligned}$$

By the assumption on ρ , we have

$$\begin{aligned} \frac{\nu^2}{|x|^4 \rho(|x|)} \delta_{\alpha\beta} &= \lambda \delta_{\alpha\beta} \frac{1}{\rho(\nu/|x|)} \\ &= f'_0 \cdot g^{\alpha\beta} \circ f, \quad \alpha, \beta = 1, 2, \dots, n. \end{aligned}$$

Finally, we shall show (E-2). By (1) and (3), we have for $\alpha = 1, 2, \dots, n$,

$$\begin{aligned} \Delta_g f_\alpha + 2g(\nabla_g \log \varphi, \nabla_g f_\alpha) - \frac{df_0}{dt} [(\Delta_g x_\alpha) \circ f] \\ = \frac{1}{\rho(r)} 2(2-n) \frac{\nu x_\alpha}{r^4} + \frac{n-2}{2} \frac{\rho'(r)}{\rho(r)^2} \sum_{i=1}^n \frac{x_i}{r} \left(\frac{\nu \delta_{i\alpha}}{r^2} - 2 \frac{\nu x_i x_\alpha}{r^4} \right) \\ + \lambda \frac{2-n}{2} \frac{\rho'(\nu/r)}{\rho(\nu/r)^2} \frac{\nu x_\alpha / r^2}{\nu/r} \\ = \frac{2-n}{2} \left(\frac{4}{r^4 \rho(r)} + \frac{\rho'(r)}{r^3 \rho(r)^2} \right) \nu x_\alpha + \lambda \frac{2-n}{2} \frac{\rho'(\nu/r)}{\rho(\nu/r)^2 r} x_\alpha, \end{aligned}$$

where we put $r = |x|$ for simplicity. On the other hand, the assumption on ρ implies

$$\frac{\rho'(\nu/r)}{\rho(\nu/r)^2 r} = \frac{\nu r}{\lambda} \left(\frac{1}{r^4} \frac{1}{\rho(r)} \right)' = -\frac{\nu}{\lambda} \left(\frac{4}{r^4 \rho(r)} + \frac{\rho'(r)}{r^3 \rho(r)^2} \right)$$

because $\frac{\rho'(\nu/r)}{\rho(\nu/r)^2 r} = \frac{r}{\nu} \left(\frac{1}{\rho(\nu/r)} \right)'$. Thus we have

$$\Delta_g f_\alpha + 2g(\nabla_g \log \varphi, \nabla_g f_\alpha) - \frac{df_0}{dt} [(\Delta_g x_\alpha) \circ f] = 0 = \frac{\partial f_\alpha}{\partial t}.$$

Therefore (f, φ) is a caloric morphism. $\square$

We close this section with the following lemma concerning the equation for the function $\log \varphi$.

Lemma 4. *Let (f, φ) be a caloric morphism such that f is of form (a).*

(1) φ satisfies the following equation:

$$(8) \quad \frac{\partial \log \varphi}{\partial x_j} = \frac{\rho(|x|)}{2} \left(\frac{\nu'(t)}{\nu(t)} x_j + \sum_{k=1}^n s_{jk}(t) x_k \right), \quad j = 1, 2, \dots, n,$$

where $s_{jk}(t)$ ($j, k = 1, \dots, n$) is the (j, k) -element of the skew symmetric matrix

$$S(t) = R(t)^{-1} R'(t).$$

(2) Unless both the conditions $n = 2$ and $\rho(r) = pr^{-2}$ hold, $R'(t) = O$ holds for all t . In this case, the equation (8) becomes

$$(9) \quad \frac{\partial \log \varphi}{\partial x_j} = \frac{\rho(|x|)}{2} \frac{\nu'(t)}{\nu(t)} x_j, \quad j = 1, 2, \dots, n.$$

Proof. Since $f(t, x) = (f_0(t), \nu(t)R(t)x)$, where $\nu(t) > 0$ and $R(t)$ is an orthogonal matrix, we have

$$(10) \quad \begin{aligned} \Delta_g f_\alpha &= \frac{1}{\rho(|x|)} \Delta_x f_\alpha + \frac{n-2}{2} \frac{\rho'(|x|)}{\rho(|x|)^2} \sum_{i=1}^n \frac{x_i}{|x|} \frac{\partial f_\alpha}{\partial x_i} \\ &= \frac{n-2}{2} \frac{\rho'(|x|)}{|x|\rho(|x|)^2} \nu(t) \sum_{i=1}^n r_{\alpha i}(t) x_i, \end{aligned}$$

and

$$(11) \quad \begin{aligned} 2g(\nabla_g \log \varphi, \nabla_g f_\alpha) &= \frac{2}{\rho(|x|)} \sum_{i=1}^n \frac{\partial \log \varphi}{\partial x_i} \frac{\partial f_\alpha}{\partial x_i} \\ &= \frac{2}{\rho(|x|)} \sum_{i=1}^n \frac{\partial \log \varphi}{\partial x_i} \nu(t) r_{\alpha i}(t), \end{aligned}$$

for $\alpha = 1, 2, \dots, n$ by (1), (2) and (4). The formula (3) implies

$$(12) \quad \begin{aligned} f'_0[(\Delta_g x_\alpha) \circ f] &= f'_0(t) \frac{n-2}{2} \frac{\rho'(|f|)}{\rho(|f|)^2} \frac{f_\alpha}{|f|} \\ &= f'_0(t) \frac{n-2}{2} \frac{\rho'(|f|)}{\rho(|f|)^2} \frac{\sum_{i=1}^n \nu(t) r_{\alpha i}(t) x_i}{\nu(t)|x|} \\ &= f'_0(t) \frac{n-2}{2} \frac{\rho'(\nu(t)|x|)}{|x|\rho(\nu(t)|x|)^2} \sum_{i=1}^n r_{\alpha i}(t) x_i \end{aligned}$$

for $\alpha = 1, 2, \dots, n$. On the other hand, differentiating (6) by $|x|$, we have

$$(13) \quad \frac{f'_0(t) \rho'(\nu(t)|x|)}{\rho(\nu(t)|x|)^2} = \frac{\nu(t) \rho'(|x|)}{\rho(|x|)^2}.$$

Substituting (4), (10), (11), (12) and (13) into (E-2'):

$$\frac{\partial f_\alpha}{\partial t} = \Delta_g f_\alpha + 2g(\nabla_g \log \varphi, \nabla_g f_\alpha) - f'_0[(\Delta_g x_\alpha) \circ f],$$

we have

$$\sum_{k=1}^n (\nu'(t)r_{\alpha k}(t) + \nu(t)r'_{\alpha k}(t))x_k = \frac{2}{\rho(|x|)} \sum_{i=1}^n \frac{\partial \log \varphi}{\partial x_i} \nu(t)r_{\alpha i}(t).$$

Therefore, the equation

$$(14) \quad \frac{\partial \log \varphi}{\partial x_j} = \frac{\rho(|x|)}{2} \left(\frac{\nu'(t)}{\nu(t)} x_j + \sum_{k=1}^n s_{jk}(t)x_k \right), \quad j = 1, 2, \dots, n,$$

holds, where $s_{jk}(t)$ is the (j, k) element of the matrix $S(t) = R(t)^{-1}R'(t)$.

Next we proceed to prove the statement (2). Differentiating the equation (14) by x_i ($i \neq j$), we have

$$\frac{\partial}{\partial x_i} \frac{\partial \log \varphi}{\partial x_j} = \frac{\rho'(|x|)}{2|x|} \left(\frac{\nu'(t)}{\nu(t)} x_i x_j + \sum_{l=1}^n s_{jl}(t)x_i x_l \right) + \frac{\rho(|x|)}{2} s_{ji}(t).$$

Also

$$\frac{\partial}{\partial x_j} \frac{\partial \log \varphi}{\partial x_i} = \frac{\rho'(|x|)}{2|x|} \left(\frac{\nu'(t)}{\nu(t)} x_j x_i + \sum_{l=1}^n s_{il}(t)x_j x_l \right) + \frac{\rho(|x|)}{2} s_{ij}(t)$$

holds. Since $\frac{\partial}{\partial x_i} \frac{\partial \log \varphi}{\partial x_j} = \frac{\partial}{\partial x_j} \frac{\partial \log \varphi}{\partial x_i}$,

$$\frac{\rho'(|x|)}{|x|} \sum_{l=1}^n s_{jl}(t)x_i x_l + \rho(|x|)s_{ji}(t) = \frac{\rho'(|x|)}{|x|} \sum_{l=1}^n s_{il}(t)x_j x_l + \rho(|x|)s_{ij}(t),$$

holds for all $i, j = 1, 2, \dots, n$ with $i \neq j$. The matrix $S(t)$ is skew symmetric because $R(t)$ is orthogonal. Then we have

$$2s_{ij}(t) = \frac{\rho'(|x|)}{2|x|\rho(|x|)} \left(x_i \sum_{l=1}^n s_{jl}(t)x_l - x_j \sum_{l=1}^n s_{il}(t)x_l \right),$$

for all $i, j = 1, 2, \dots, n$ with $i \neq j$, and hence

$$(15) \quad s_{ij}(t) = \frac{\rho'(|x|)}{2|x|\rho(|x|)} (x_i y_j - y_i x_j),$$

where we put $y = Sx$.

First we consider the case that $n \geq 3$. Then for each fixed t and each triple indices i, j, k with $1 \leq i < j < l \leq n$, the equation (15) implies

$$s_{ij}(t)x_l + s_{jl}(t)x_i + s_{li}(t)x_j = 0$$

for all (x_i, x_j, x_l) in a open subset of $\mathbb{R}^3$. That implies $(s_{ij}(t), s_{jl}(t), s_{li}(t)) = 0$ for each $1 \leq i < j < l \leq n$. Therefore $S(t) = O$ and hence $R'(t) = O$ for all t , if $n \geq 3$.

Next, in the case that $n = 2$, the equation (16) implies

$$s_{12}(t) = -\frac{r\rho'(r)}{2\rho(r)}s_{12}(t).$$

Therefore $s_{12} = 0$ unless $\rho(r) = pr^{-2}$. Thus we have shown that $R'(t) = O$ for all t unless $n = 2$ and $\rho(r) = pr^{-2}$. Consequently, the equation (9) immediately follows from (8) and $S(t) = O$. $\square$

4. Some special cases.

Before the proof of Theorem 1, we deal with the case that ρ has the form $\rho(r) = pr^q$ in this section. The following theorem corresponds to the cases 1-a and 1-b of Theorem 1.

Theorem 2. Let $n = 2$ and $\rho(r) = \frac{p}{r^2}$ ($p > 0$).

(1) Let (f, φ) be a caloric morphism such that f is of form (a). Then

$$\begin{aligned} f(t, x) &= (t + d, ce^{at}R_0 \begin{pmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{pmatrix} x), \\ \varphi(t, r, \theta) &= Cr^{\frac{1}{2}ap} \exp \left(\frac{p}{2}b\theta + \frac{p}{4}(a^2 + b^2)t \right). \end{aligned}$$

(2) Let (f, φ) be a caloric morphism such that f is of form (b). Then

$$\begin{aligned} f(t, x) &= (t + d, \frac{ce^{at}}{|x|^2}R_0 \begin{pmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{pmatrix} x), \\ \varphi(t, r, \theta) &= Cr^{-\frac{1}{2}ap} \exp \left(\frac{p}{2}b\theta + \frac{p}{4}(a^2 + b^2)t \right). \end{aligned}$$

In both cases, $a, b, c, d, C \in \mathbb{R}$ ($c, C > 0$), R_0 is an orthogonal matrix and (r, θ) is the polar coordinate of $\mathbb{R}^2$. $a = 0$ if and only if $\nu'(t) = 0$ for some t , where ν is the function appeared in the forms (a) and (b).

Proof. First of all, $f'_0(t) = 1$ follows from the equality (6) in Lemma 1. Therefore $f_0(t) = t + d$ with $d \in \mathbb{R}$.

Next, by Lemma 4 (1), $\log \varphi$ satisfies the equation

$$(16) \quad \frac{\partial \log \varphi}{\partial x_j} = \frac{p|x|^{-2}}{2} \left(\frac{\nu'(t)}{\nu(t)} x_j + \sum_{k=1}^2 s_{jk}(t)x_k \right)$$

for each $j = 1, 2$. Since $S(t)$ is skew symmetric and $n = 2$, $S(t) = \begin{pmatrix} 0 & -s(t) \\ s(t) & 0 \end{pmatrix}$ where we put $s(t) = s_{21}(t)$ for simplicity. Then (16) becomes

$$(17) \quad \begin{aligned} \frac{\partial \log \varphi}{\partial x_1} &= \frac{p}{2|x|^2} \left(\frac{\nu'(t)}{\nu(t)} x_1 - s(t) x_2 \right), \\ \frac{\partial \log \varphi}{\partial x_2} &= \frac{p}{2|x|^2} \left(\frac{\nu'(t)}{\nu(t)} x_2 + s(t) x_1 \right). \end{aligned}$$

In the polar coordinate $x_1 = r \cos \theta$, $x_2 = r \sin \theta$, (17) becomes

$$\begin{aligned} \frac{\partial \log \varphi}{\partial r} &= \frac{x_1}{r} \frac{\partial \log \varphi}{\partial x_1} + \frac{x_2}{r} \frac{\partial \log \varphi}{\partial x_2} = \frac{p \nu'(t)}{2 \nu(t)} \frac{1}{r}, \\ \frac{\partial \log \varphi}{\partial \theta} &= -x_2 \frac{\partial \log \varphi}{\partial x_1} + x_1 \frac{\partial \log \varphi}{\partial x_2} = \frac{p}{2} s(t). \end{aligned}$$

Integrating these equations, we have

$$(18) \quad \log \varphi(t, r, \theta) = \frac{p \nu'(t)}{2 \nu(t)} \log r + \frac{p}{2} s(t) \theta + \psi(t).$$

On the other hand, φ satisfies the equation (E-1): $H_g \varphi = 0$. We put $w = \log \varphi$ for simplicity. Since

$$(19) \quad \frac{\partial \varphi}{\partial t} - \Delta_g \varphi = \varphi \left\{ \frac{\partial w}{\partial t} - \frac{1}{\rho} \left[\Delta_x w + |\nabla_x w|^2 + \frac{n-2}{2} \frac{\rho'}{\rho} \frac{\partial w}{\partial r} \right] \right\}$$

holds, w satisfies the equation

$$(20) \quad \frac{\partial w}{\partial t} - \frac{1}{\rho} \left[\frac{\partial^2 w}{\partial r^2} + \left(\frac{\partial w}{\partial r} \right)^2 + \left(\frac{1}{r} \frac{\partial w}{\partial \theta} \right)^2 + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} \right] = 0.$$

Differentiating (18), we have

$$\begin{aligned} \frac{\partial w}{\partial t} &= \frac{p}{2} \left(\frac{\nu'}{\nu} \right)' \log r + \frac{p}{2} s' \theta + \psi', \\ \frac{\partial w}{\partial r} &= \frac{p}{2r} \frac{\nu'}{\nu}, \quad \left(\frac{\partial w}{\partial r} \right)^2 = \frac{p^2}{4r^2} \left(\frac{\nu'}{\nu} \right)^2, \quad \frac{\partial^2 w}{\partial r^2} = -\frac{p}{2r^2} \frac{\nu'}{\nu}, \\ \left(\frac{1}{r} \frac{\partial w}{\partial \theta} \right)^2 &= \frac{p^2}{4r^2} s^2 \quad \text{and} \quad \frac{\partial^2 w}{\partial \theta^2} = 0. \end{aligned}$$

Substitute these calculations into (20). Then we have the equation

$$(21) \quad \frac{p}{2} \left(\frac{\nu'}{\nu} \right)' \log r + \frac{p}{2} s' \theta + \psi' - \frac{p}{4} \left[\left(\frac{\nu'}{\nu} \right)^2 + s^2 \right] = 0.$$

Therefore we obtain a system of differential equations

$$\begin{cases} \left(\frac{\nu'}{\nu}\right)' = 0, \\ s' = 0, \\ \psi' = \frac{p}{4} \left[\left(\frac{\nu'}{\nu}\right)^2 + s^2 \right], \end{cases}$$

by (21). The solution of this system is

$$(22) \quad \begin{cases} \nu(t) = ce^{at}, \\ s(t) = b, \\ \psi(t) = \frac{p}{4}(a^2 + b^2)t + C_0, \end{cases}$$

where $a, b, c, C_0 \in \mathbb{R}$, $a \neq 0$ and $c > 0$. Clearly, $a = 0$ if and only if $\nu'(t) = 0$ for all t . Substituting (22) into (18), we have

$$\log \varphi(t, r, \theta) = \frac{ap}{2} \log r + \frac{p}{2} b\theta + \frac{p}{4}(a^2 + b^2)t + C,$$

and

$$(23) \quad S(t) = \begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix}.$$

Therefore

$$\varphi(t, r, \theta) = Cr^{\frac{1}{2}ap} \exp \left(\frac{p}{2} b\theta + \frac{p}{4}(a^2 + b^2)t \right).$$

Now choose a number $t_0 \in \mathbb{R}$ so that $\{t = t_0\} \cap D \neq \emptyset$, and put $R_0 = R(t_0)$. Since $R(t)$ is continuous, we can put

$$R(t) = R_0 \begin{pmatrix} \cos v(t) & -\sin v(t) \\ \sin v(t) & \cos v(t) \end{pmatrix}$$

where $v(t)$ is a C^∞ -function such that $v(t_0) = 0$. Substituting this into the definition of $S(t)$, we have

$$\begin{aligned} S(t) &= \begin{pmatrix} \cos v(t) & \sin v(t) \\ -\sin v(t) & \cos v(t) \end{pmatrix} {}^tR_0 R_0 \begin{pmatrix} -\sin v(t) & -\cos v(t) \\ \cos v(t) & -\sin v(t) \end{pmatrix} v'(t) \\ &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} v'(t). \end{aligned}$$

Then by (23), we have

$$\begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} v'(t).$$

and hence $v(t) = bt$. Thus

$$f(t, x) = (t + d, ce^{at} R_0 \begin{pmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{pmatrix} x),$$

$$\varphi(t, r, \theta) = Cr^{\frac{1}{2}ap} \exp\left(\frac{p}{2}b\theta + \frac{p}{4}(a^2 + b^2)t\right),$$

for all $(t, x) \in D$. We have shown the first statement (1). The proof of the second statement is reduced to the first statement by the composition with an inversion. In fact, $\rho(r) = pr^{-2}$ satisfies $\rho(\frac{1}{r}) = r^4 \rho(r)$. Then the inversion $(j, 1)$, where

$$j(t, x) = (t, \frac{x}{|x|^2}),$$

is a caloric morphism by Lemma 3. The composition $(f \circ j, \varphi \circ j)$ of (f, φ) and $(j, 1)$ is

$$(f_0(t), \nu(t)R(t)x), \quad \varphi(t, \frac{x}{|x|^2}),$$

which is reduced to the case that $f \circ j$ is of form (a). Thus we have shown Theorem 2. $\square$

The next theorem corresponds to the cases 2-a and 2-b of Theorem 1.

Theorem 3. Let $n \geq 3$ and ρ is of form $\rho(r) = \frac{p}{r^2}$ ($p > 0$).

(1) Let (f, φ) be a caloric morphism such that f is of form (a). Then

$$f(t, x) = (t + d, ce^{at} R_0 x), \quad \varphi(t, x) = C|x|^{\frac{1}{2}ap} \exp\left(\frac{1}{4}pa^2t\right).$$

(2) Let (f, φ) be a caloric morphism such that f is of form (b). Then

$$f(t, x) = (t + d, \frac{ce^{at}}{|x|^2} R_0 x), \quad \varphi(t, x) = C|x|^{-\frac{1}{2}ap} \exp\left(\frac{1}{4}pa^2t\right).$$

In both cases, $a, c, d, C \in \mathbb{R}$ ($c, C > 0$), R_0 is an orthogonal matrix. $a = 0$ if and only if $\nu'(t) = 0$ for some t , where ν is the function appeared in the forms (a) and (b).

Proof. First of all, $f'_0(t) = 1$ follows from the equality (6) in Lemma 1. Therefore $f_0(t) = t + d$ with $d \in \mathbb{R}$.

By Lemma 4 (2), R is constant and $\log \varphi$ satisfies the equation

$$\frac{\partial \log \varphi}{\partial x_j} = \frac{p}{2|x|^2} \frac{\nu'(t)}{\nu(t)} x_j, \quad j = 1, \dots, n,$$

because $n > 2$. It follows that φ is a radial function of x , i.e.

$$\varphi(t, x) = \varphi(t, |x|),$$

and that

$$\frac{\partial \log \varphi}{\partial r} = \frac{p\nu'(t)}{2\nu(t)} \frac{1}{r},$$

and hence

$$(24) \quad \log \varphi(t, r) = \frac{p\nu'(t)}{2\nu(t)} \log r + \psi(t).$$

By (E-1): $H_g \varphi = 0$ and (19), $w = \log \varphi$ satisfies the equation

$$(25) \quad \frac{\partial w}{\partial t} - \frac{1}{\rho} \left[\frac{\partial^2 w}{\partial r^2} + \left(\frac{\partial w}{\partial r} \right)^2 + \frac{1}{r} \frac{\partial w}{\partial r} \right] = 0.$$

Substituting the following

$$\begin{aligned} \frac{\partial w}{\partial t} &= \frac{p}{2} \left(\frac{\nu'}{\nu} \right)' \log r + \psi', \\ \frac{\partial w}{\partial r} &= \frac{p}{2r} \frac{\nu'}{\nu}, \quad \left(\frac{\partial w}{\partial r} \right)^2 = \frac{p^2}{4r^2} \left(\frac{\nu'}{\nu} \right)^2, \quad \frac{\partial^2 w}{\partial r^2} = -\frac{p}{2r^2} \frac{\nu'}{\nu} \end{aligned}$$

into (25), we have the equation

$$\frac{p}{2} \left(\frac{\nu'}{\nu} \right)' \log r + \psi' - \frac{p}{4} \left(\frac{\nu'}{\nu} \right)^2 = 0.$$

Therefore we obtain a system of differential equations

$$\begin{cases} \left(\frac{\nu'}{\nu} \right)' = 0, \\ \psi' = \frac{p}{4} \left(\frac{\nu'}{\nu} \right)^2. \end{cases}$$

The solution is

$$(26) \quad \begin{cases} \nu(t) = ce^{at}, \\ \psi(t) = \frac{pa^2}{4}t + C_0, \end{cases}$$

where $a, c, C_0 \in \mathbb{R}$, $a \neq 0$ and $c > 0$. Clearly, $a = 0$ if and only if $\nu'(t) = 0$ for some t . Substituting (26) into (24), we have

$$\log \varphi(t, r) = \frac{ap}{2} \log r + \frac{pa^2}{4}t + C_0.$$

Thus we have,

$$f(t, x) = (t + d, ce^{at}Rx), \quad \varphi(t, x) = C|x|^{\frac{1}{2}ap} \exp\left(\frac{1}{4}pa^2t\right),$$

for all $(t, x) \in D$. We have shown the first statement (1). As in Theorem 2, the proof of the second statement is reduced to the first statement by composing the inversion $(j, 1)$. $\square$

The next theorem corresponds to the case 3 of Theorem 1. We note that the case of $d/a = -1$, $b = c = 0$, $p = 1$ and $q = 0$ is the Appell transformation.

Theorem 4. Let ρ be of form $\rho(r) = pr^q$ ($p > 0$, $q \in \mathbb{R}$, $q \neq -2$) and let (f, φ) be a caloric morphism such that f is of form (a) or (b). Then

$$f(t, x) = \left(\frac{ct + d}{at + b}, \frac{R_0 x}{|at + b|^{2/(q+2)}} \right),$$

$$\varphi(t, x) = \frac{C}{|at + b|^{n/2}} \exp \left[- \frac{pa|x|^{q+2}}{(q+2)^2(at+b)} \right],$$

where $a, b, c, d, C \in \mathbb{R}$ ($bc - ad = 1$, $C > 0$) and R_0 is an orthogonal matrix. $a = 0$ if and only if $\nu'(t) = 0$ for some t , where ν is the function appeared in the forms (a) and (b).

Proof. Let $q \neq -2$. Then by Lemma 4 (2), R is constant and

$$\frac{\partial \log \varphi}{\partial x_j} = \frac{p|x|^q}{2} \frac{\nu'(t)}{\nu(t)} x_j, \quad j = 1, \dots, n$$

holds. As in the proof of Theorem 4, φ is a radial function of x , $\varphi(t, x) = \varphi(t, |x|)$ and hence

$$(27) \quad \log \varphi(t, r) = \frac{pr^{q+2}}{2(q+2)} \frac{\nu'(t)}{\nu(t)} + \psi(t), \quad r = |x|.$$

By (E-1) and (19), $w = \log \varphi$ satisfies the equation

$$(28) \quad \frac{\partial w}{\partial t} - \frac{1}{\rho} \left[\frac{\partial^2 w}{\partial r^2} + \left(\frac{\partial w}{\partial r} \right)^2 + \frac{nq + 2n - 2q - 2}{2r} \frac{\partial w}{\partial r} \right] = 0.$$

Differentiating (27), we have

$$\frac{\partial w}{\partial t} = \frac{pr^{q+2}}{2(q+2)} \left(\frac{\nu'}{\nu} \right)' + \psi', \quad \frac{\partial w}{\partial r} = \frac{pr^{q+1}}{2} \frac{\nu'}{\nu},$$

$$\left(\frac{\partial w}{\partial r} \right)^2 = \frac{p^2 r^{2q+2}}{4} \left(\frac{\nu'}{\nu} \right)^2, \quad \frac{\partial^2 w}{\partial r^2} = \frac{(q+1)pr^q}{2} \frac{\nu'}{\nu}.$$

Substitute these into (28). Then we have

$$\frac{pr^{q+2}}{2(q+2)} \left(\frac{\nu'}{\nu} \right)' + \psi' - \left[\frac{pr^{q+2}}{4} \left(\frac{\nu'}{\nu} \right)^2 + \frac{nq + 2n}{4} \frac{\nu'}{\nu} \right] = 0,$$

and hence

$$\frac{pr^{q+2}}{2(q+2)} \left[\left(\frac{\nu'}{\nu} \right)' - \frac{q+2}{2} \left(\frac{\nu'}{\nu} \right)^2 \right] + \psi' - \frac{n(q+2)}{4} (\log \nu)' = 0.$$

Therefore we obtain a system of differential equations,

$$\begin{cases} \left(\frac{\nu'}{\nu} \right)' - \frac{q+2}{2} \left(\frac{\nu'}{\nu} \right)^2 = 0, \\ \psi' - \frac{n(q+2)}{4} (\log \nu)' = 0. \end{cases}$$

The solution is

$$(29) \quad \begin{cases} \nu(t) = \frac{1}{|at + b|^{2/(q+2)}}, \\ \psi(t) = \log C \nu^{n(q+2)/4} = \log C |at + b|^{-n/2}, \end{cases}$$

where $a, b \in \mathbb{R}$ and $C > 0$. Clearly, $a = 0$ if and only if $\nu'(t) = 0$ for some t . Substituting (29) into (27), we have

$$\log \varphi(t, r) = -\frac{par^{q+2}}{(q+2)^2(at+b)} + \log C |at + b|^{-n/2}.$$

On the other hand, (6) in Lemma 1 implies

$$f'_0(t) = \nu^{q+2}(t) = \frac{1}{|at + b|^2}.$$

Therefore $f_0(t) = \frac{ct + d}{at + b}$ where $a, b, c, d \in \mathbb{R}$ and $bc - ad = 1$. Consequently,

$$\begin{aligned} f(t, x) &= \left(\frac{ct + d}{at + b}, \frac{R_0 x}{|at + b|^{2/(q+2)}} \right), \\ \varphi(t, x) &= \frac{C}{|at + b|^{n/2}} \exp \left[-\frac{pa|x|^{q+2}}{(q+2)^2(at+b)} \right], \end{aligned}$$

for all $(t, x) \in D$. Thus we have shown Theorem 4. $\square$

4. Proof of Theorem 1.

Now we can prove our main theorem.

Proof of Theorem 1. First we deal with the case that the function $\nu(t)$ is not constant. Let I_0 be the open interval consisting of all t such that $(t, x) \in D$ for some $x \in \mathbb{R}^n$. Let I be a connected component of the open set $\{t \in I_0; \nu'(t) \neq 0\}$ and let J be the open interval consisting of all $|x|$ such that $(t, x) \in D$ for some $t \in I$. Then $\rho(r) = pr^q$ on J by Lemma 2, and all the cases have already been treated in the theorems, Theorem 2, 3 and 4. Since we assume that ν is not constant, if $q = -2$,

$$\nu'(t) = cae^{at}, \quad (a \neq 0)$$

on I from Theorem 2 and Theorem 3, and if $q \neq -2$,

$$\nu'(t) = \frac{-2a}{(q+2)|at + b|^{2/(q+2)}}, \quad (a \neq 0)$$

on I from Theorem 4. In both cases, $\nu'(t) \neq 0$ on I , the closure of I in I_0 . Hence, $I = I_0$ because I_0 is connected. Therefore $D \subset I \times J$ and $\rho(r)$ has the form pr^q . Thus we have shown that (f, φ) and ρ are completely determined as the cases 1-a, 1-b, 2-a, 2-b, 3 in Theorem 1, if the function $\nu(t)$ is not constant.

Next we deal with the case that ν is constant. Because of the preceding argument, we may exclude the case that $\rho(r)$ has the form $\rho(r) = pr^q$. We first consider the case (a) as before. By Lemma 4, $R'(t) = O$ and

$$\frac{\partial \log \varphi}{\partial x_j} = 0$$

because $\nu'(t) = 0$. Therefore R is a constant matrix R_0 and φ depends only on t . Since φ satisfies (E-1): $H_g \varphi = \frac{\partial \varphi}{\partial t} = 0$, φ is a positive constant C . On the other hand, (6) in Lemma 1 implies $\rho(\nu r) = f'_0(t)\rho(r)$. Therefore $f'_0(t) = \rho(\nu r)/\rho(r)$ and hence $f'_0 = \lambda$ with some positive constant λ . Thus we have

$$f(t, x) = (\lambda t + d, \nu R_0 x), \quad \varphi(t, x) = C.$$

This is Case 4.

Finally, we consider the case (b). Since ν is constant, f'_0 is equal to a constant λ such that

$$\rho\left(\frac{\nu}{r}\right) = \frac{\lambda r^4}{\nu^2} \rho(r)$$

by (7) in Lemma 1. Then, the inversion $(h, 1)$, where

$$h(t, x) = \left(\lambda t, \frac{\nu x}{|x|^2}\right),$$

is a caloric morphism by Lemma 3. The composition of $(h, 1)$ and (f, φ) is

$$(\lambda f_0(t), Rx), \quad \varphi(t, x),$$

which is reduced to the case that $f \circ h$ is of form (a). Thus we have completed the proof of Theorem 1. $\square$

REFERENCES

- [1] B. Fuglede, *Harmonic Morphisms Between Riemannian Manifolds*, Ann. Inst. Fourier, Grenoble **28** (1978), 107–144.
- [2] T. Ishihara, *A mapping of Riemannian manifolds which preserves harmonic functions*, J. Math. Kyoto Univ. **19** (1979), 215–229.
- [3] H. Leutwiler, *On the Appell Transformation*, Potential Theory (J.Král, J.Lukeš, I.Netuka, J.Veselý, eds.), Plenum, New York, 1988, pp. 215–222.
- [4] M. Nishio and K. Shimomura, *Caloric morphisms on semi-euclidean space*, Rev. Roumaine. Math. Pures Appl. **47** (2002), 727–736.
- [5] M. Nishio and K. Shimomura, *A characterization of caloric morphisms between manifolds*, Ann. Acad. Sci. Fenn. Math. **28** (2003), 111–122.
- [6] K. Shimomura, *The determination of caloric morphisms on Euclidean domains*, Nagoya Math. J. **158** (2000), 133–166.