The index and co-index of the twisted tangent bundle over projective spaces

RYUICHI TANAKA*

ABSTRACT. We determine the index and co-index of the twisted tangent bundle of projective spaces. We also discuss the stability of them, and determine the set of integers that can be realized as the stable co-index of a vector bundle over the projective space.

1. Introduction.

Let α be a finite-dimensional real vector bundle over a CW complex B , and let $S(\alpha)$ be its sphere bundle with respect to some metric on α . We regard $S(\alpha)$ as a $\mathbb{Z}/2$ -space by the antipodal map on each fibre. Then, the index of α , denoted $\text{ind } \alpha$, is defined to be the largest integer k for which there exists a $\mathbb{Z}/2$ -map from S^{k-1} to $S(\alpha)$. Here, S^{k-1} also is equipped with a $\mathbb{Z}/2$ -action by the antipodal map. As a dual notion to the index, the co-index of α , denoted $\text{co-ind } \alpha$, is defined to be the smallest integer k for which there exists a $\mathbb{Z}/2$ -map from $S(\alpha)$ to S^{k-1} . If there exist no such maps, we define $\text{co-ind } \alpha$ as ∞ . When B is a finite complex, $\text{co-ind } \alpha$ is finite for any α . By the Borsuk-Ulam theorem, $\text{ind } \alpha$ is not greater than $\text{co-ind } \alpha$.

In this paper, we calculate the index and co-index of the twisted tangent bundle of the real projective space $\mathbb{R}P^n$. Let τ denote the tangent bundle of $\mathbb{R}P^n$, and let ξ denote the canonical line bundle over $\mathbb{R}P^n$. We call $\tau \otimes \xi$ the twisted tangent bundle of $\mathbb{R}P^n$. For the tangent bundle τ , we already know that $\text{ind } \tau = n$, since the index of the tangent bundle of any closed manifold is equal to its dimension [CF1, Theorem 6.11]. As for the co-index of τ , it is known that $\text{co-ind } \tau$ is equal to $l(n)$ ($n \neq 1, 3, 7$) or $l(n) - 1$ ($n = 1, 3, 7$), where $l(n)$ denotes the smallest integer m such that $\mathbb{R}P^n$ can be immersed in $\mathbb{R}^m$ [AGJ, T1].

We shall prove the following results in which the k -dimensional trivial bundle is denoted simply by k .

Theorem 1.1. *For $\tau \otimes \xi$, the twisted tangent bundle of $\mathbb{R}P^n$, we have $\text{co-ind } (\tau \otimes \xi + k) = n + 1 + k$ for any $k \geq 0$.*

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* Department of Liberal Arts, Faculty of Science and Technology, Tokyo University of Science, Noda, Chiba, 278-8510, Japan.

e-mail: XXXXXXXXXX

Theorem 1.2. *For $\tau \otimes \xi$, the twisted tangent bundle of $\mathbb{R}P^n$, we have the following.*

- (1) *If n is odd, then $\text{ind}(\tau \otimes \xi) = n + 1$ and $\text{ind}(\tau \otimes \xi + k) = n + k$ for $k \geq 1$.*
- (2) *If n is even, then $\text{ind}(\tau \otimes \xi + k) = n + k$ for any $k \geq 0$.*

For a vector bundle α , in general, we call α to be *CI-stable* if the equality $\text{co-ind}(\alpha + k) = \text{co-ind } \alpha + k$ holds for any positive integer k . Likewise we call α to be *I-stable* if $\text{ind}(\alpha + k) = \text{ind } \alpha + k$ for any positive integer k . We have the following corollaries.

Corollary 1.3. *$\tau \otimes \xi$ is CI-stable.*

Corollary 1.4.

- (1) *If n is odd, then $\tau \otimes \xi$ is not I-stable but $\tau \otimes \xi + 1$ turns to be I-stable.*
- (2) *If n is even, then $\tau \otimes \xi$ is I-stable.*

As for τ , the tangent bundle of $\mathbb{R}P^n$, it is not known whether τ is CI-stable, though the equality $\text{co-ind}(\tau + 1) = \text{co-ind } \tau + 1$ holds as shown in [T1, Theorem 3.7]. On the other hand, τ is I-stable, since the tangent bundle of any closed manifold is known to be I-stable [T2, Theorem 4.6]. Also, it is to be noted that ξ is not I-stable, but CI-stable [T1, Theorem 4.1].

For α , a vector bundle over a finite complex B , we can define the stable co-index of α , denoted $\text{co-ind}^s(\alpha)$, as

$$\text{co-ind}^s(\alpha) = \varinjlim_k \{ \text{co-ind}(\alpha + k) - \dim(\alpha + k) \}.$$

This limit always exists since $0 \leq \text{co-ind}(\alpha + 1) - \dim(\alpha + 1) \leq \text{co-ind } \alpha - \dim \alpha$ (see [T1, Lemma 3.1]). Likewise we define the stable index $\text{ind}^s(\alpha)$. For example, if α is stably trivial, then, $\text{co-ind}^s(\alpha) = \text{ind}^s(\alpha) = 0$. Also, we know that $\text{co-ind}^s(\xi) = n$ and $\text{ind}^s(\xi) = 0$ from [T1, Theorem 4.1]. The above theorems imply $\text{co-ind}^s(\tau \otimes \xi) = 1$ and $\text{ind}^s(\tau \otimes \xi) = 0$. For the m -fold sum of the twisted tangent bundle $\tau \otimes \xi$, we shall prove the following theorem.

Theorem 1.5. *If $m \leq n$, then $\text{co-ind}^s(m \cdot \tau \otimes \xi) = m$.*

If α is stably equivalent to α' , it is clear that $\text{co-ind}^s(\alpha) = \text{co-ind}^s(\alpha')$. So, co-ind^s can be considered as a function from $\widetilde{KO}(B)$ to $\mathbb{Z}^+$, the set of non-negative integers. Since $\text{co-ind } \alpha \leq \dim \alpha + \dim B$ ([T1, Lemma 3.1]), we have $\text{co-ind}^s(\alpha) \leq \dim B$ in general. Thus, $\dim B$ is an upper bound of the image of the function co-ind^s . From Theorem 1.5, we obtain the following corollary.

Corollary 1.6. *The image of the function $\text{co-ind}^s : \widetilde{KO}(\mathbb{R}P^n) \rightarrow \mathbb{Z}^+$ is equal to the set $\{0, 1, 2, \dots, n\}$.*

Thus, over $\mathbb{R}P^n$, any non-negative integer not greater than $\dim \mathbb{R}P^n$ can be realized as the stable co-index of a vector bundle.

2. Proof of the theorems.

Let B be a finite complex and α be a real vector bundle over B . We define the ν -dimension of α to be the smallest integer k for which there is a k -dimensional

vector bundle β with $\alpha + \beta$ trivial, and we denote it by $\nu\text{-dim } \alpha$. The ν -dimension of α is closely related to the geometric dimension of $-\alpha$, but they slightly differ in general (see examples in [T2]). The inequalities $\text{g-dim}(-\alpha) \leq \nu\text{-dim } \alpha \leq \dim B$ hold generally. We easily see that $\text{co-ind } \alpha \leq \dim \alpha + \nu\text{-dim } \alpha$.

For a virtual vector bundle α , we denote by $\omega\text{-dim } \alpha$ the largest integer k for which the k th Stiefel-Whitney class $w_k(\alpha)$ is not zero. Then it is known that $\text{co-ind } \alpha \geq \dim \alpha + \omega\text{-dim}(-\alpha)$ and the following theorem holds.

Theorem 2.1([T2, Theorem 2.5]). *Let α be a real vector bundle over a finite complex B , and suppose that the equality $\omega\text{-dim}(-\alpha) = \nu\text{-dim } \alpha$ holds. Then,*

- (1) $\text{co-ind } \alpha = \dim \alpha + \nu\text{-dim } \alpha$.
- (2) α is CI-stable.

Now, we prove Theorem 1.1 and Theorem 1.5.

Proof of Theorem 1.1 and 1.5.

Since $\tau + 1 \cong (n + 1)\xi$, we have $\tau \otimes \xi + \xi \cong n + 1$. Hence, $\nu\text{-dim}(\tau \otimes \xi) \leq 1$. We also have $W(-\tau \otimes \xi) = W(\xi) = 1 + t$ where W is the total Stiefel-Whitney class and t is the generator of $H^*(\mathbb{R}P^n; \mathbb{Z}/2)$. Hence, $\omega\text{-dim}(-\tau \otimes \xi) = 1$. Since $\omega\text{-dim}(-\alpha) \leq \nu\text{-dim } \alpha$ in general, we have $\omega\text{-dim}(-\tau \otimes \xi) = \nu\text{-dim}(\tau \otimes \xi) = 1$. Therefore, from Theorem 2.1, we find that $\text{co-ind}(\tau \otimes \xi) = n + 1$, and also, $\tau \otimes \xi$ is CI-stable. Thus Theorem 1.1 follows. Our argument goes as well for the m -fold sum of the twisted tangent bundle $\tau \otimes \xi$. In fact, since $m \cdot \tau \otimes \xi + m\xi \cong m(n + 1)$, we have $\nu\text{-dim}(m \cdot \tau \otimes \xi) \leq m$. On the other hand, since $W(-m \cdot \tau \otimes \xi) = W(m\xi) = (1 + t)^m$, we have $\omega\text{-dim}(-m \cdot \tau \otimes \xi) = m$ if $m \leq n$. Thus, if $m \leq n$, we see that $\omega\text{-dim}(-m \cdot \tau \otimes \xi) = \nu\text{-dim}(m \cdot \tau \otimes \xi) = m$, and, from Theorem 2.1, it follows that $m \cdot \tau \otimes \xi$ is CI-stable and $\text{co-ind}^s(m \cdot \tau \otimes \xi) = m$. $\square$

Next, we turn to the index. To prove Theorem 1.2, we use the following result in [T1].

Proposition 2.2([T1, Corollary 2.5]). *Let α be a real vector bundle over $\mathbb{R}P^n$.*

- (1) *If $\dim \alpha > n$, then $\text{ind } \alpha = \dim \alpha$.*
- (2) *If $\dim \alpha \leq n$ and α contains ξ as a subbundle, then $\text{ind } \alpha = n + 1$.*

Proof of Theorem 1.2.

In view of Proposition 2.2 (1), we have only to prove that $\text{ind}(\tau \otimes \xi)$ is equal to $n + 1$ if n is odd, and n if n is even. Since $\text{ind } \alpha \leq \text{co-ind } \alpha$ in general, we have $\text{ind}(\tau \otimes \xi) \leq n + 1$ from Theorem 1.1. Also, since $\text{ind } \alpha \geq \dim \alpha$ in general, we clearly have $\text{ind}(\tau \otimes \xi) \geq n$. Now, suppose that there is a $\mathbb{Z}/2$ -map $f : S^n \rightarrow S(\tau \otimes \xi)$. Let $\tilde{f} : \mathbb{R}P^n \rightarrow P(\tau \otimes \xi)$ be the map induced from f . Here, $P(\tau \otimes \xi)$ is the projective bundle associated to $\tau \otimes \xi$. Let $p : P(\tau \otimes \xi) \rightarrow \mathbb{R}P^n$ be the bundle projection. Let e denote the $\mathbb{Z}/2$ -Euler class of the bundle $S(\tau \otimes \xi) \rightarrow P(\tau \otimes \xi)$. We consider the homomorphism $\tilde{f}^* : H^*(P(\tau \otimes \xi); \mathbb{Z}/2) \rightarrow H^*(\mathbb{R}P^n; \mathbb{Z}/2)$. From the naturality of the Euler class, we have $\tilde{f}^*(e) = t$, and so, $\tilde{f}^*(e^n) = t^n$, which is not zero. On the other hand, in $H^*(P(\tau \otimes \xi); \mathbb{Z}/2)$, we have the relation

$$e^n = w_n + w_{n-1}e + w_{n-2}e^2 + \cdots + w_1e^{n-1},$$

where w_i 's are the Stiefel-Whitney classes of $\tau \otimes \xi$ and $H^*(P(\tau \otimes \xi); \mathbb{Z}/2)$ is considered as a $H^*(\mathbb{R}P^n; \mathbb{Z}/2)$ -module in an obvious way. Since $\tau \otimes \xi + \xi \cong n+1$, we have $W(\tau \otimes \xi) = W(\xi)^{-1} = (1+t)^{-1}$. Hence, we have the relation

$$e^n = t^n + t^{n-1}e + t^{n-2}e^2 + \cdots + te^{n-1}.$$

Applying $\tilde{f}^*$, we have

$$t^n = (p \circ \tilde{f})^*(t^n) + (p \circ \tilde{f})^*(t^{n-1})t + (p \circ \tilde{f})^*(t^{n-2})t^2 + \cdots + (p \circ \tilde{f})^*(t)t^{n-1}.$$

But the homomorphism $(p \circ \tilde{f})^* : H^*(\mathbb{R}P^n; \mathbb{Z}/2) \rightarrow H^*(\mathbb{R}P^n; \mathbb{Z}/2)$ is either identity or zero. The latter case is impossible since t^n is not zero. Therefore, we have $t^n = nt^n$. If n is even, this contradicts $t^n \neq 0$. Thus we have proved that $\text{ind}(\tau \otimes \xi)$ is equal to n if n is even.

Now, let n be odd. Then τ has a non-zero cross-section since it is orientable and its Euler class is zero. In other words, τ has a one-dimensional trivial bundle as a subbundle. Hence $\tau \otimes \xi$ has ξ as a subbundle. From Proposition 2.2 (2), it follows that $\text{ind}(\tau \otimes \xi) = n+1$. This completes the proof. $\square$

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